A study of $(x(q + 1), x; 2, q)$-minihypers

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Abstract. In this paper, we study the weighted $(x(q + 1), x; 2, q)$-minihypers. These are weighted sets of $x(q + 1)$ points in $\text{PG}(2, q)$ intersecting every line in at least $x$ points. We investigate the decomposability of these minihypers, and define a switching construction which associates to an $(x(q + 1), x; 2, q)$-minihyper, with $x \leq q^2 - q$, not decomposable in the sum of another minihyper and a line, a $(j(q + 1), j; 2, q)$-minihyper, where $j = q^2 - q - x$, again not decomposable into the sum of another minihyper and a line. We also characterize particular $(x(q + 1), x; 2, q)$-minihypers, and give new examples. Additionally, we show that $(x(q + 1), x; 2, q)$-minihypers can be described as rational sums of lines. In this way, this work continues the research on $(x(q + 1), x; 2, q)$-minihypers by Hill and Ward [9], giving further results on these minihypers.

1 Introduction

Let $\mathcal{P}$ be the set of points of the projective geometry $\text{PG}(t, q)$. A multiset in $\text{PG}(t, q)$ is a mapping $\mathcal{R}: \mathcal{P} \rightarrow \mathbb{N}$. This mapping is extended in a natural way to the subsets of $\mathcal{P}$: for any subset $\mathcal{Q}$ of $\mathcal{P}$, we set $\mathcal{R}(\mathcal{Q}) = \sum_{P \in \mathcal{Q}} \mathcal{R}(P)$. The integer $\mathcal{R}(P)$ is called the multiplicity of the point $P$ and $n = \sum_{P \in \mathcal{P}} \mathcal{R}(P)$ is called the cardinality of $\mathcal{R}$. The support $\text{supp}(\mathcal{R})$ of a multiset $\mathcal{R}$ is the set of all points of positive multiplicity. A multiset $\mathcal{R}$ is said to be projective if $\mathcal{R}(P) \in \{0, 1\}$ for all points $P$. Projective multisets can be considered as sets of points by identifying them with their supports.

Conversely, given a finite set $\mathcal{Q}$ of points in $\text{PG}(t, q)$, we define the characteristic multiset $\chi_\mathcal{Q}$ by:

$$\chi_\mathcal{Q}(P) = \begin{cases} 1 & \text{if } P \in \mathcal{Q}, \\ 0 & \text{if } P \notin \mathcal{Q}. \end{cases}$$

A multiset in $\text{PG}(t, q)$ is called an $(n, w; t, q)$-multiarc if

(a) $\mathcal{R}(\mathcal{P}) = n$;
(b) $\mathcal{R}(H) \leq w$ for any hyperplane $H$;

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(c) there exists a hyperplane $H_0$ with $\mathcal{R}(H_0) = w$.

A multiset $\mathcal{F}$ in $\text{PG}(t,q)$ is called an $(f,m;t,q)$-blocking multiset or $(f,m;t,q)$-minihyper, if

(a) $\mathcal{F}(P) = f$;

(b) $\mathcal{F}(H) \geq m > 0$ for any hyperplane $H$;

(c) there exists a hyperplane $H_0$ with $\mathcal{F}(H_0) = m$.

To avoid trivialities, we impose $m > 0$ in the definition of an $(f,m;2,q)$-minihyper.

We can speak of $(n,w)$-multiarcs or $(f,m)$-minihypers if the geometry we consider is clear from the context. The characteristic multiset of a subspace of dimension $u$ in $\text{PG}(t,q)$ is a minihyper with parameters $(v_{u+1},v_u)$, where $v_u = \frac{q^{u+1}-1}{q-1}$.

An $(f,m)$-minihyper $\mathcal{F}$ is called minimal if there exists no $(f-1,m)$-minihyper $\mathcal{F}'$ with $\mathcal{F}'(P) \leq \mathcal{F}(P)$ for all points $P$.

The sum $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$ of two minihypers $\mathcal{F}_1$ and $\mathcal{F}_2$, where $\mathcal{F}_i$ has parameters $(f_i,m_i;t,q)$, $i = 1,2$, is the $(f,m;t,q)$-minihyper, with $f = f_1 + f_2$, $m = m_1 + m_2$, and with the multiplicity of a point $P$ in $\mathcal{F}$ equal to the sum of its multiplicities in $\mathcal{F}_1$ and $\mathcal{F}_2$. As a particular example of a minihyper which is the sum of minihypers, we note that the sum of any $x$ (not necessarily distinct) lines is a $(x(q+1),x;2,q)$-minihyper.

An $(f,m;t,q)$-minihyper $\mathcal{F}$ is called indecomposable or irreducible [7, Definition 2.5] if it cannot be represented as a sum $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$ of two other minihypers $\mathcal{F}_i$, $i = 1,2$, where $\mathcal{F}_i$ has parameters $(f_i,m_i;t,q)$, $i = 1,2$. Note again that this implies that $m_1,m_2 > 0$.

Within this introduction, we also define two important substructures of a projective plane $\text{PG}(2,q)$.

A hyperoval $K$ in $\text{PG}(2,q)$, $q$ even, is a set of $q+2$ points, no three collinear. The classical example of a hyperoval in $\text{PG}(2,q)$, $q$ even, is the union of a conic and its nucleus; such a hyperoval is called a regular hyperoval. For $q$ even, $q \geq 16$, in $\text{PG}(2,q)$, there exist irregular hyperovals, i.e., hyperovals which are not the union of a conic and its nucleus. We refer to [11] for the list of the known infinite classes of hyperovals in $\text{PG}(2,q)$, $q$ even.

A maximal arc $K$ of $\text{PG}(2,q)$ is a set of points intersecting every line in zero or $n$ points. For $1 < n < q$, this necessarily implies that $n$ is a divisor of $q$. Ball, Blokhuis, and Mazzocca proved that such maximal arcs cannot exist for $q$ odd [1, 2]. For $q$ even, Denniston proved the existence of a maximal arc in $\text{PG}(2,2^h)$ intersecting every line in $2^i$ points, for every $i$ satisfying $1 \leq i \leq h-1$ [4].
We also refer to the standard reference of Hirschfeld [10] for more information on hyperovals and maximal arcs in \( \text{PG}(2, q) \), \( q \) even.

Consider a multiset \( \mathcal{K} \) of \( \text{PG}(2, q) \); then to \( \mathcal{K} \) corresponds the spectrum \( (a_i)_{i \geq 0} \) of \( \mathcal{K} \). The spectrum \( (a_i)_{i \geq 0} \) of \( \mathcal{K} \) is the sequence of numbers \( a_i \), \( i \geq 0 \), with \( a_i \) the number of lines of \( \text{PG}(2, q) \) intersecting \( \mathcal{K} \) in \( i \) points.

\section{Minihypers with parameters \((x(q + 1), x)\)}

In [9], Hill and Ward consider minihypers with parameters \((x(q + 1), x)\), \( x < q \), in \( \text{PG}(2, q) \). They try to characterize all indecomposable minihypers with the above parameters. Hill and Ward restrict the values of \( x \) to \( x < q \) so that the associated codes are Griesmer codes. In [3], this problem is studied for \( x = q \) in the terms of multiarcs. The following result describes divisibility properties of \((x(q + 1), x)\)-minihypers and is a straightforward generalization of Theorem 5.1 in [3].

\textbf{Theorem 1.} ([9, Theorem 18]) Let \( \mathfrak{F} \) be an \((x(q + 1), x)\)-minihyper in \( \Pi = \text{PG}(2, q) \), \( q = p^m \), \( p \) prime, \( m \geq 1 \), with \( p^l \) divides \( x \). Then for each line \( L \) in \( \Pi \), \( \mathfrak{F}(L) \equiv x \pmod{p^l+1} \).

This theorem implies two useful corollaries.

\textbf{Corollary 2.} ([9, Theorem 20]) Every \((x(q + 1), x)\)-minihyper in \( \text{PG}(2, q) \), \( q = p^m \), \( p \) prime, \( m \geq 1 \), with \( x \leq q - \frac{q}{p} \), is a sum of \( x \) lines. In particular, if \( \mathfrak{F} \) is an indecomposable \((x(q + 1), x)\)-minihyper, then \( x > q - \frac{q}{p} \) or \( \mathfrak{F} \) is a line.

We wish to stress that the preceding corollary implies that the only indecomposable \((x(q + 1), x; 2, q)\)-minihypers in \( \text{PG}(2, q) \), \( q \) prime, with \( x < q \), are the \((q + 1, 1; 2, q)\)-minihypers, so are the lines of \( \text{PG}(2, q) \), \( q \) prime.

\textbf{Corollary 3.} ([9, Theorem 23]) Let \( \mathfrak{K} \) be an indecomposable \((x(q + 1), x)\)-minihyper in \( \text{PG}(2, q) \), \( q = p^m \), \( p \) prime, \( m \geq 1 \), for which \( x \leq y < q \) and \( p^l \) divides \( y \).

\( (i) \) For each line \( L \), \( \mathfrak{K}(L) \leq x + q - p^l+1 \).

\( (ii) \) For each point \( P \), \( \mathfrak{K}(P) \leq x - p^l+1 \).

\( (iii) \) If \( q - p + 1 \leq x \leq q - 1 \) and \( \mathfrak{K}(P) > x - 2p \), then \( \mathfrak{K}(P) \) is divisible by \( \frac{q}{p} - 1 \).
A challenging problem is that of the classification of all indecomposable \((x(q+1),x)\)-minihypers, \(x < q\), in projective planes over arbitrary finite fields. This problem has been solved for \(\text{PG}(2,8)\) and \(\text{PG}(2,9)\) by Hill and Ward in [9]. In the same paper, they describe five families of indecomposable \((x(q+1),x)\)-minihypers in the planes of square order.

3 Ball’s construction

A nice class of indecomposable \((x(q+1),x)\)-minihypers was found by Ball in an unpublished note. The class of Ball’s minihypers is explained below.

Suppose that \(q = 2^m\) and consider two hyperovals \(F_1\) and \(F_2\) in \(\text{PG}(2,q)\) that meet in \(q + 2 - x\) points. Let \(F = F_1 + F_2 \mod 2\), i.e. \(\text{supp} F\) is the symmetric difference of the supports of the two hyperovals \(F_1\) and \(F_2\). Clearly, \(|F| = 2x\) and \(F(L) = 0, 2,\) or \(4\) for every line \(L\). If we dualize and regard 4-lines as 2-points, 2-lines as 1-points, and 0-lines as 0-points, we get an \((x(q+1),x)\)-minihyper with spectrum

\[
a_x = q^2 + q + 1 - 2x, \quad a_{x+q/2} = 2x, \quad a_i = 0, \quad \text{for } i \neq x, x + \frac{q}{2}.
\]

This construction works in a more general setting. Assume that we are given a multiset \(K\) with \(|K| = sx\) such that \(K(L) = is, \quad i \in \mathbb{N}\), for every line \(L\), i.e. the multiplicity of every line is a multiple of \(s\). We define a multiset \(\mathcal{F}\) in the dual plane in which lines of multiplicity \(is\) become points of multiplicity \(i\). Then \(\mathcal{F}\) is an \((x(q+1),x)\)-minihyper. We prove this as follows.

Let \((a_i)_{i \geq 0}\) be the spectrum of \(\mathcal{K}\). By counting the flags \((P,L), \quad P \in L\), we obtain

\[
sa_s + 2sa_{2s} + \cdots = sx(q+1).
\]

This implies that \(\sum ia_is = x(q+1)\), which means that \(\mathcal{F}\) has the desired cardinality. Let \(P\) be an arbitrary point. It has to be checked that \(\sum i b_is(P) \geq x\), where \(b_j = b_j(P)\) denotes the number of lines through \(P\) that have multiplicity \(j\). Assume that \(\mathcal{R}(P) = \varepsilon\). Then

\[
b_s + b_{2s} + \cdots = q + 1,
\]

\[
(s - \varepsilon)b_s + (2s - \varepsilon)b_{2s} + \cdots = |\mathcal{R}| - \varepsilon = sx - \varepsilon,
\]

which implies that

\[
b_s + 2b_{2s} + \cdots = x + \varepsilon \frac{q}{s}.
\]

This means that each line in the dual plane has multiplicity at least \(x\) since \(\varepsilon q/s \geq 0\).
Example 1. In PG(2, q), q even, let \( \mathcal{M} \) be a maximal arc of degree \( 2^i \) and let \( L \) be an external line to \( \mathcal{M} \). Define \( \mathcal{R} = \chi_F \setminus L - \mathcal{M} \). Clearly, \( \mathcal{R}(M) = 0 \), \( q - 2^i \), or \( q \) for every line \( M \) of PG(2, q) and

\[
|\mathcal{R}| = q^2 - q(2^i - 1) - 2^i = 2^i \left( \frac{q^2}{2^i} - q + \frac{q}{2^i} - 1 \right).
\]

If we set \( s = 2^i \), \( x = \frac{q^2}{2^i} - q + \frac{q}{2^i} - 1 \), using Ball’s construction, we get a minihyper with parameters

\[
\left( \frac{q^2}{2^i} - q + \frac{q}{2^i} - 1)(q + 1), \frac{q^2}{2^i} - q + \frac{q}{2^i} - 1; 2, q \right).
\]

Remark 4. In the general case, we do not know whether the minihybers constructed in Example 1 are indecomposable. This can however be proven in some special cases. For instance, if we start with a maximal arc of degree \( q/2 \), \( q \geq 4 \), we arrive at a minihyper with

\[
x = \frac{q^2}{q/2} - q + \frac{q}{q/2} - 1 = q + 1.
\]

Note that in this case, the minihyper has \( q + 1 \) 2-points, one 0-point, and \( q^2 - 1 \) 1-points. The 0-point corresponds to the fixed external line \( L \) to the maximal arc \( \mathcal{M} \), and the \( q + 1 \) 2-points correspond to the other \( q + 1 \) external lines to the maximal arc \( \mathcal{M} \). These \( q + 2 \) external lines to the original maximal arc \( \mathcal{M} \) form a dual hyperoval, and so consequently, the 0-point and the \( q + 1 \) 2-points form a hyperoval in the plane where the \( ((\frac{q + 1}{2})(q + 1), q + 1; 2, q) \)-minihyper \( \mathcal{F} \) is embedded.

Assume that \( \mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2 \), with \( \mathcal{F}_1 \) an \((f_1, x_1; 2, q)\)-minihyper and \( \mathcal{F}_2 \) an \((f_2, x_2; 2, q)\)-minihyper. Since there exists a 0-point to \( \mathcal{F} \), then \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) also have a 0-point. Considering all lines of PG(2, q) through this 0-point, implies that \( f_1 = x_1(q + 1) \) and \( f_2 = x_2(q + 1) \), with \( q + 1 = x_1 + x_2 \).

Since \( (q + 1)(q + 1) = f_1 + f_2 \), necessarily \( f_1 = x_1(q + 1) \) and \( f_2 = x_2(q + 1) \) if \( x_1 + x_2 = q + 1 \). So \( \mathcal{F}_1 \) is an \((x_1(q + 1), x_1; 2, q)\)-minihyper and \( \mathcal{F}_2 \) an \((x_2(q + 1), x_2; 2, q)\)-minihyper.

Moreover, since \( x_1 + x_2 = q + 1 \), necessarily \( x_1 \leq q/2 \) or \( x_2 \leq q/2 \). Assume that \( x_1 \leq q/2 \). Then Corollary 2 implies that \( \mathcal{F}_1 \) is the sum of \( x_1 \) lines. Let \( L_1 \) be one of these lines. Then reducing the weight of every point of \( \mathcal{F}_1 \) on \( L_1 \) by one, a new \((x_1 - 1)(q + 1), x_1 - 1; 2, q)\)-minihyper \( \mathcal{F}_1' \) is obtained. But then \( \mathcal{F}_1' + \mathcal{F}_2 \) is a \((q(q + 1), q; 2, q)\)-minihyper. This is only possible if originally for \( L_1 \), \( \mathcal{F}(L_1) = q + 1 + q = 2q + 1 \). But \( L_1 \) can contain at most two points of weight two, so \( \mathcal{F}(L_1) \leq q + 1 + 2 \). This contradicts \( q \geq 4 \). So \( \mathcal{F} \) is indecomposable.
Example 2. Take a \((q+t, t)\)-arc \(R'\) of type \((0, 2, t = 2^i)\). These \((q+t, t)\)-arcs of type \((0, 2, t = 2^i)\) have been studied in detail by Korchmáros and Mazzocca [12], and by Gács and Weiner [6]. For instance, a particular property of a \((q+t, t)\)-arc \(R'\) of type \((0, 2, t = 2^i)\) is that all the \(t\)-secants pass through a common point.

Let \(L\) be an external line to \(R'\). Set \(K = \chi_P \setminus L - R'\). Then

\[|K| = q^2 - q - 2^i = 2 \cdot \left(\frac{q^2}{2} - \frac{q}{2} - 2^{i-1}\right).\]

We have one 0-line and secants of multiplicity \(q, q - 2, \) and \(q - 2^i\). By the Ball construction, we get a minihyper with \(x = \frac{q^2}{2} - \frac{q}{2} - 2^{i-1}\).

Example 3. The complement of a unital. Consider \(PG(2, q)\), where \(q\) is a square. Let \(U\) be a unital and set \(R = \chi_P - U\). Here

\[|R| = \sqrt{q} (q\sqrt{q} - q + \sqrt{q})\]

with line multiplicities \(q - \sqrt{q}\) and \(q\). By Ball’s construction, we get a minihyper with parameters

\[((q\sqrt{q} - q + \sqrt{q})(q + 1), q\sqrt{q} - q + \sqrt{q}; 2, q).\]

In order to describe the fourth example, we need to define a linear blocking set in \(PG(2, q)\). We first of all introduce the notion of a Desarguesian spread.

By what is sometimes called field reduction, the points of \(PG(2, q), q = p^h, p\) prime, \(h \geq 1\), correspond to \((h - 1)\)-dimensional subspaces of \(PG(3h - 1, p)\), since a point of \(PG(2, q)\) is a 1-dimensional vector space over \(\mathbb{F}_q\), and so an \(h\)-dimensional vector space over \(\mathbb{F}_p\). In this way, we obtain a partition \(D\) of the point set of \(PG(3h - 1, p)\) by \((h - 1)\)-dimensional subspaces. In general, a partition of the point set of a projective space by subspaces of a given dimension \(k\) is called a spread, or a \(k\)-spread if we want to specify the dimension. The spread we have obtained here is called a Desarguesian spread. Note that the Desarguesian spread satisfies the property that each subspace spanned by two spread elements is again partitioned by spread elements.

Definition 1. Let \(D\) be the Desarguesian \((h - 1)\)-spread of \(PG(3h - 1, p)\), corresponding to the points of \(PG(2, p^h)\) and let \(U\) be a subset of \(PG(3h - 1, p)\), then \(B(U) = \{R \in D ||U \cap R \neq \emptyset\}\). We identify the spread elements of \(B(U)\) with the corresponding points of \(PG(2, p^h)\).
Definition 2. We denote the \((h-1)\)-dimensional spread element of \(\text{PG}(3h-1, p)\) corresponding to a point \(P\) of \(\text{PG}(2, p^h)\) by \(S(P)\). If \(U\) is a subspace of \(\text{PG}(2, p^h)\), then \(S(U) := \{ S(P) | P \in U \} \).

Analogously to the correspondence between the points of \(\text{PG}(2, q), q = p^h\), and the elements of a Desarguesian spread \(\mathcal{D}\) in \(\text{PG}(3h-1, p)\), we obtain the correspondence between the lines of \(\text{PG}(2, q)\) and the \((2h-1)\)-dimensional subspaces of \(\text{PG}(3h-1, p)\) spanned by two elements of \(\mathcal{D}\). With this in mind, it is clear that any subspace \(U\) of dimension at least \(h\) of \(\text{PG}(3h-1, p)\) defines a blocking set \(B(U)\) in \(\text{PG}(2, q)\). A blocking set constructed in this way is called a linear blocking set. Linear blocking sets were first introduced by Lunardon [14], although there a different approach is used. For more on the approach explained here, we refer to [13].

Example 4. The complement of a linear blocking set in \(\text{PG}(2, q), q = p^h, p\) prime, \(h \geq 1\).

Such a linear blocking set \(B(U)\) in \(\text{PG}(2, q)\) intersects every line in \(1 \mod p\) points. Suppose that \(e\) is the maximal integer such that every line of \(\text{PG}(2, q)\) intersects \(B(U)\) in \(1 \mod p^e\) points. Then the complement of \(B(U)\) intersects every line in \(0 \mod p^e\) points. If \(|B(U)| = q + k + 1\), then the complement has size \(q^2 - k = p^e(q^2/p^e - k/p^e)\), so defines a \(((q^2/p^e - k/p^e)(q+1), q^2/p^e - k/p^e; 2, q)\)-minihyper.

4 Rational sums of lines

It has been noted that an \((x(q + 1), x)\)-minihyper is not necessarily a sum of lines. However this is always the case if we assume rational multiplicities for the points.

Theorem 5. Let \(\mathfrak{F}\) be an \((x(q + 1), x)\)-minihyper in \(\text{PG}(2, q)\). Then there exist lines \(L_1, \ldots, L_s\) and positive rational numbers \(c_1, \ldots, c_s\), such that

\[
\mathfrak{F} = c_1 \chi_{L_1} + \cdots + c_s \chi_{L_s},
\]

with \(\sum_{i=1}^s c_i = x\).

Proof. Assume that there exists a line \(L\) with \(\mathfrak{F}(L) \geq x + q\). Then \(\mathfrak{F}' = \mathfrak{F} - \chi_L\) is an \(((x-1)(q+1), x - 1)\)-minihyper and we get the result by induction on \(x\). Hence, without loss of generality, we can assume that all lines have multiplicity at most \(x + q - 1\).
Let $P$ be a point of multiplicity $\varepsilon$. Denote by $a_i$ the number of lines through $P$ that have multiplicity $i$. We have
\[
\sum_{i=0}^{q-1} a_{x+i} = q + 1,
\]
\[
\sum_{i=0}^{q-1} (x + i - \varepsilon) a_{x+i} = x(q + 1) - \varepsilon,
\]
which implies
\[
\sum_{i=0}^{q-1} \frac{i}{q} a_{x+i} = \varepsilon.
\]
Therefore,
\[
\mathcal{F} = \sum_{i=0}^{q-1} \sum_{L: \mathcal{F}(L) = x+i} \frac{i}{q} \chi_L,
\]
which had to be proven.

**Example 5.** Consider $\text{PG}(2, q)$, $q$ even. Then this projective plane contains hyperovals.

Consider a dual hyperoval $\{L_1, \ldots, L_{q+2}\}$ in $\text{PG}(2, q)$, $q$ even. Then the rational sum
\[
\frac{1}{2} L_1 + \cdots + \frac{1}{2} L_{q+2}
\]
is a $((\frac{q}{2} + 1)(q + 1), \frac{q}{2} + 1; 2, q)$-minihyper in $\text{PG}(2, q)$, $q$ even.

We know from Theorem 1 that if $\mathcal{F}$ is an $(x(q + 1), x)$-minihyper in $\Pi = \text{PG}(2, q)$, $q = p^m$, $p$ prime, $m \geq 1$, with $x < q$ where $p^f$ divides $x$, then for each line $L$ in $\Pi$, $\mathcal{F}(L) \equiv x \pmod{p^{f+1}}$.

This allows us to describe more in detail the rational coefficients of the rational sum.

**Corollary 6.** Let $\mathcal{F}$ be an indecomposable $(x(q + 1), x)$-minihyper in $\Pi = \text{PG}(2, q)$, $q = p^m$, $p$ prime, $m \geq 1$, with $x < q$ where $p^f$ is the maximal power of $p$ that divides $x$, then
\[
\mathcal{F} = \sum_{L(1)} \frac{1}{p^{m-f-1}} \chi_{L(1)} + \sum_{L(2)} \frac{2}{p^{m-f-1}} \chi_{L(2)} + \cdots + \sum_{L(p^{m-f-1} - 1)} \frac{p^{m-f-1} - 1}{p^{m-f-1}} \chi_{L(p^{m-f-1} - 1)}.
\]
with \( L^{(1)} \) the lines intersecting \( \mathcal{F} \) in \( x + p^f + 1 \) points, with \( L^{(2)} \) the lines intersecting \( \mathcal{F} \) in \( x + 2p^f + 1 \) points, \ldots, and with \( L^{(p^m - f - 1)} \) the lines intersecting \( \mathcal{F} \) in \( x + (p^m - f - 1)p^f + 1 \) points.

5 A construction for \( x = \frac{3q}{4} \)

If \( \mathcal{K} \) is a \((q^2 + q + 2, q + 2)\)-arc in \( \text{PG}(2,q) \), then \( \mathcal{K}(P) \leq 2 \) for all points \( P \) [3]. Thus the correspondence \( \mathcal{K} \leftrightarrow 2\chi_P - \mathcal{K} \) establishes a one-to-one correspondence between such arcs and \((q(q + 1), q; 2, q)\)-minihypers \( \mathcal{F} \) for which \( \mathcal{F}(P) \leq 2 \) for all points \( P \). Thus the examples of [2, Section 2] provide \((q(q + 1), q; 2, q)\)-minihypers having this added restriction. In particular, the 3-line construction from [3, Theorem 2.3] leads to the following results.

**Theorem 7.** Let \( G \) be a subgroup of the additive group \((\mathbb{F}_q,+). \) Let \( A, B, \) and \( C \) be cosets of \( G \) in \((\mathbb{F}_q,+)* on \text{arc } PG(2,q) \) with \( A + B + C \neq G. \) Set
\[
A' = -B - C, \quad B' = -A - C, \quad \text{and } C' = -A - B.
\]

Define in the following way a multiset \( \mathcal{K} \) in \( \text{PG}(2,q) \):
- **2-points**\( (a,0,-1) \) for \( a \in A' \),
  \( (b,-1,0) \) for \( b \in B' \),
  \( (c,1,1) \) for \( c \in C' \),
- **0-points**\( (a,0,-1) \) for \( a \in A \),
  \( (b,-1,0) \) for \( b \in B \),
  \( (c,1,1) \) for \( c \in C \),
- **1-points** the remaining points.

Then the multiset \( \mathcal{K} \) is a \((q^2 + q,q)\)-minihyper. Conversely, every \((q^2 + q,q)\)-minihyper for which the 2-points lie on three lines meeting in a 0-point is isomorphic to \( \mathcal{K} \).

Now consider the special case \( q = 2^r, \ r \geq 2. \) Let \( G \) be a subgroup of \((\mathbb{F}_q,+)* \) of order \( 2^{r-1}. \) If \( \mathcal{K} \) is the minihyper defined in Theorem 7, the lines in \( \text{PG}(2,q) \) have the types described in Table 1 below.

<table>
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<th>multiplicity</th>
<th># of such lines</th>
<th># of 2-pts</th>
<th># of 1-pts</th>
<th># of 0-pts</th>
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<tr>
<td>(A) &amp; 3</td>
<td>3</td>
<td>q/2</td>
<td>0</td>
<td>q/2 + 1</td>
</tr>
<tr>
<td>(B) &amp; q</td>
<td>3q^2/4</td>
<td>1</td>
<td>q - 2</td>
<td>2</td>
</tr>
<tr>
<td>(C) &amp; q</td>
<td>q - 2</td>
<td>0</td>
<td>q</td>
<td>1</td>
</tr>
<tr>
<td>(D) &amp; q + 4</td>
<td>q^2/4</td>
<td>3</td>
<td>q - 2</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 1
The points of $\text{PG}(2,q)$ can be divided into four classes with respect to the types of the lines they are incident with. The possible types of the points are described in Table 2.

<table>
<thead>
<tr>
<th>multiplicity</th>
<th># of such points</th>
<th># lines of type</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\alpha)$</td>
<td>2</td>
<td>$3q/2$</td>
</tr>
<tr>
<td>$(\beta)$</td>
<td>1</td>
<td>$q^2 - 2q$</td>
</tr>
<tr>
<td>$(\gamma)$</td>
<td>0</td>
<td>$3q/2$</td>
</tr>
<tr>
<td>$(\delta)$</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 2

Now we define a new multiset $\mathcal{F}$ in the dual plane of $\text{PG}(2,q)$ by

$$\mathcal{F}(l) = \begin{cases} 
q/4 & \text{if } l \text{ is of type (A); } \\
1 & \text{if } l \text{ is of type (B); } \\
0 & \text{if } l \text{ is of type (C) or (D)}. 
\end{cases}$$

(1)

**Theorem 8.** The multiset $\mathcal{F}$ is a 2-weight minihyper with parameters $(3q/4(q + 1), \frac{3q}{4})$ and spectrum

$$a_{3q/4} = q^2 - \frac{q}{2} + 1, \quad a_{5q/4} = \frac{3q}{2}, \quad a_i = 0 \text{ for } i \neq \frac{3q}{4}, \frac{5q}{4}.$$  

Proof. The proof is immediate from Table 2. Points of type $(\alpha), (\beta),$ and $(\delta)$ become lines of multiplicity $3q/4$, while lines of type $(\gamma)$ become lines of multiplicity $5q/4$. Table 2 implies also the values of $a_{3q/4}$ and $a_{5q/4}$.  

For $r = 2$, this construction gives a projective $(15, 3)$-minihyper with 3- and 5-lines which is the complement of the hyperoval in $\text{PG}(2, 4)$. For $r = 3$, the theorem gives a $(54, 6)$-minihyper with 6- and 10-lines which is an “orphan” minihyper obtained in [9, Theorem 31]. For $r = 4$, the construction gives a $(204, 12)$-minihyper with lines of multiplicity 12 and 20. For $r \geq 4$, these minihypers do not come from Ball’s symmetric difference construction since they have points of multiplicity $2^{r-2} > 2$, while in Ball’s construction all points have multiplicity 0, 1, and 2.

This construction can be reversed: from a minihyper with three collinear points of multiplicity $q/4$ (the remaining points on the line being 0-points), $3q^2/4$ points of multiplicity 1 and $q^2/4 + q - 2$ points of multiplicity 0, one can obtain back the $(q(q + 1), q)$-minihyper and the $(q^2 + q + 2, q + 2)$-arc from the 3-line construction.
6 Indecomposable minihypers

In \cite{9}, Hill and Ward consider plane \((x(q + 1), x)-\)minihypers with \(x < q\) only. The reason is that such minihypers give rise to Griesmer codes via the well-known construction of Hamada \cite{5, 8}. If \(\mathcal{F}\) is an \((x(q + 1), x; 2, q)\)-minihyper and \(s\) is the maximal multiplicity of a point, then \(s\chi_P - \mathcal{F}\) is an \((s(q^2 + q + 1) - x(q + 1), s(q + 1) - x; 2, q)\)-arc. The code associated to this arc has parameters \([s(q^2 + q + 1) - x(q + 1), 3, sq^2 - xq]\) and is easily checked to meet the Griesmer bound for \(x < q\).

In what follows, we consider minimal, indecomposable \((x(q + 1), x; 2, q)\)-minihypers without imposing an explicit restriction on \(x\). It turns out that the indecomposability requirement implies an upper bound on \(x\).

**Theorem 9.** Every \((x(q + 1), x; 2, q)\)-minihyper, with \(x \geq q^2 - q + 1\), is decomposable.

**Proof.** Assume that \(\mathcal{F}\) is an indecomposable \((x(q + 1), x)\)-minihyper. There exists a point \(P\) in \(PG(2, q)\) that is of multiplicity 0. Otherwise \(\mathcal{F}\) can be represented as the sum of \(\chi_P\) and an \(((x - q - 1)(q + 1) + q, x - q - 1)\)-minihyper, a contradiction to the indecomposability condition.

Note that all lines through the 0-point \(P\) are \(x\)-lines. Moreover the multiplicity of any line \(L\) cannot be larger than \(x + q - 1\) (otherwise, the minihyper is represented as the sum of \(\chi_L\) and an \(((x - q - 1)(q + 1) + q, x - q - 1)\)-minihyper). Counting the flags \((P', L')\), with \(P' \in L'\), we get

\[x(q + 1)^2 \leq x(q + 1) + q^2(x + q - 1).\]

This implies \(x \leq q^2 - q\). \(\Box\)

**Theorem 10.** For an indecomposable \((x(q + 1), x; 2, q)\)-minihyper with \(x \leq q^2 - q\), all points of \(PG(2, q)\) have weight at most \(q - 1\).

**Proof.** Consider a point \(P\) of weight \(e\), and now consider all the lines through \(P\). Since they all have weight at most \(x + q - 1\), we obtain the inequality

\[qe + x(q + 1) \leq q(x + q - 1) + x,\]

where we used the fact that every line has weight at most \(x + q - 1\), and there is at least one \(x\)-secant through \(P\). Namely, there is at least one point \(P'\) having weight zero, and the line \(PP'\) is an \(x\)-secant to the minihyper.

This leads to \(e \leq q - 1\). \(\Box\)
Corollary 11. There is no indecomposable \((x(q+1), x; 2, q)\)-minihyper with \(x = q^2 - q\).

Proof. Assume otherwise and let \(\mathcal{F}\) be an indecomposable \(((q^2-q)(q+1), q^2-q; 2, q)\)-minihyper. Then by the counting argument from Theorem 9, we get that all points other than the 0-point \(P\) have multiplicity \(q - 1\) and \(\mathcal{F} = (q - 1)\mathcal{X}_{P\setminus\{P\}}\). But \(\mathcal{X}_{P\setminus\{P\}}\) itself is a \((q(q+1), q; 2, q)\)-minihyper, a contradiction to our initial assumption. \(\square\)

Remark 12. Using the same arguments, we can improve on the bound for \(x\) if we assume the existence of a certain number of 0-points. For example, if we assume that there are two 0-points, then the number of \(x\)-lines becomes at least \(2q + 1\) and the double counting argument gives

\[x(q+1)^2 \leq x(2q+1) + (q^2-q)(x+q-1),\]

whence \(x \leq q^2 - 2q + 1\).

7 A switching construction

Consider an indecomposable \((x(q+1), x; 2, q)\)-minihyper with \(x \leq q^2 - q - 1\). Then all points have weight smaller than or equal to \(q - 1\) (Theorem 10), and for every line \(L\), \(\mathcal{F}(L) \leq x + q - 1\). Then this minihyper has at least one 0-point; see the proof of Theorem 9. Let us fix such a 0-point, \(P\) say. All the lines through \(P\) are of multiplicity \(x\). Set \(x = q^2 - q - y\), \(0 < y\), and define a new minihyper \(\mathcal{F}'\) in the following way:

\[
\mathcal{F}'(Q) = \begin{cases} 
q - 1 - \mathcal{F}(Q) & \text{if } Q \neq P; \\
0 & \text{if } Q = P.
\end{cases}
\]

We say that \(\mathcal{F}'\) is obtained from \(\mathcal{F}\) by using switching with respect to \(P\). We have

\[
|\mathcal{F}'| = \sum_{Q:Q\neq P} (q - 1 - \mathcal{F}(Q)) = (q^2 + q)(q-1) - \sum_{Q:Q\neq P} \mathcal{F}(Q)
\]

\[= (q^2 + q)(q-1) - x(q+1) = y(q+1).\]

Furthermore, all lines through \(P\) have multiplicity \(y = q(q-1) - x\). For the remaining lines \(L\), one has

\[\mathcal{F}'(L) \geq (q+1)(q-1) - (x + q - 1) = q(q-1) - x = y.\]

Hence, \(\mathcal{F}'\) is a \((y(q+1), y)\)-minihyper.

It is clear that switching \(\mathcal{F}'\) with respect to \(P\), we again obtain \(\mathcal{F}\).
Lemma 13. Let $\mathcal{F}$ be an $(x(q+1), x; 2, q)$-minihyper, with $x \leq q^2 - q$, having a 0-point $P$ and such that $\mathcal{F}(L) \leq x + q - 1$ for every line $L$. Let $\mathcal{F}'$ be the $(y(q+1), y; 2, q)$-minihyper, $y = q^2 - q - x$, obtained from $\mathcal{F}$ by switching with respect to $P$. Then $\mathcal{F}'(L) \leq y + q - 1$ for every line $L$. In particular, $\mathcal{F}'$ is not a sum of lines.

Proof. For every point $Q$, $\mathcal{F}'(Q) \leq q - 1 - F(Q)$, so $\mathcal{F}(L) \leq q^2 - 1 - x = y + q - 1$.

\[ \square \]

Theorem 14. Every $(x(q+1), x; 2, q)$-minihyper, with $x \geq q^2 - 2q + \frac{q}{p}$, is decomposable.

Proof. Assume otherwise and let $\mathcal{F}$ be an indecomposable minihyper with parameters $(x(q+1), x)$. Assume first of all that $\mathcal{F}$ is indecomposable. This implies in particular that the weight $\mathcal{F}(L)$ of every line $L$ is at most $q^2 - 2q + \frac{q}{p}$. Then $x \leq q^2 - q$ by Theorem 9. By the switching construction, we get a $(y(q+1), y)$-minihyper $\mathcal{F}'$ with $y \leq (q^2 - q - (q^2 - 2q + \frac{q}{p})) = q - \frac{q}{p}$. Since $\mathcal{F}$ is indecomposable, $\mathcal{F}(L) \leq x + q - 1$ and, by Lemma 13, $\mathcal{F}'(L) \leq y + q - 1$. This contradicts Corollary 2.

\[ \square \]

8 Two characterization results

8.1 A first characterization result

Consider PG$(2, q)$, $q$ even: then there are two known ways to construct $((\frac{q}{2} + 1)(q+1), \frac{q}{2} + 1; 2, q)$-minihypers. First of all, there is the sum $L_1 + \cdots + L_{q/2+1}$ of $q/2 + 1$ lines $L_1, \ldots, L_{q/2+1}$, and secondly there is the rational sum \[
\sum_{i=1}^{q/2} \frac{1}{2}(L_1 + \cdots + L_{q+2}), \]
where $\{L_1, \ldots, L_{q+2}\}$ is a dual hyperoval of PG$(2, q)$.

We now show that all $((\frac{q}{2} + 1)(q+1), \frac{q}{2} + 1; 2, q)$-minihypers arise from these two constructions.

Theorem 15. Every $(((\frac{q}{2} + 1)(q+1), \frac{q}{2} + 1; 2, q)$-minihyper $\mathcal{F}$ in PG$(2, q)$, $q$ even, is either:

(1) a sum $L_1 + \cdots + L_{q/2+1}$ of $q/2 + 1$ lines $L_1, \ldots, L_{q/2+1}$, or

(2) a rational sum $\frac{1}{2}(L_1 + \cdots + L_{q+2})$, where $\{L_1, \ldots, L_{q+2}\}$ is a dual hyperoval.

Proof. Let $x = q/2 + 1$ and $y = 3q/4$. Assume first of all that $\mathcal{F}$ is indecomposable. This implies in particular that the weight $\mathcal{F}(L)$ of every line $L$ is at most $q/2 + 1 + q - 1$. Then the following properties are valid.
Add a sum of $q/4 - 1$ lines to $\mathcal{R}$ to obtain a $((3q/4)(q + 1), 3q/4; 2, q)$-minihyper $\mathcal{R}'$. Then Theorem 1 implies that for every line $L$, $\mathcal{R}'(L) \equiv 3q/4 \pmod{q/2}$. Subtracting the contribution of the sum of the $q/4 - 1$ lines in $\mathcal{R}' - \mathcal{R}$, this implies that for every line $L$, $\mathcal{R}(L) \equiv x \equiv q/2 + 1 \pmod{q/2}$.

Since for every line $L$, $\mathcal{R}(L) \equiv x \pmod{q/2}$, and since $\mathcal{R}(L) \leq x + q - 1$, this implies that $\mathcal{R}(L) \in \{q/2 + 1, q + 1\}$.

Again, since $x = q/2 + 1 \leq y = 3q/4 = q/2 + q/4$ and since $q/4$ divides $y$, for every point $P$, $\mathcal{R}(P) \leq q/2 + 1 - q/2 = 1$ (Corollary 3). So $\mathcal{R}$ only has points of weight one.

Let $P$ be a point of $\mathcal{R}$, let $P$ belong to $\alpha$ $(q/2 + 1)$-secants and to $\beta$ $(q + 1)$-secants to $\mathcal{R}$, then

\[
\begin{align*}
\alpha + \beta &= q + 1, \\
\alpha \cdot \frac{q}{2} + \beta \cdot q &= (\frac{q}{2} + 1)(q + 1) - 1 = \frac{q^2}{2} + \frac{3q}{2}.
\end{align*}
\]

This implies that $\beta = 2$. So every point of $\mathcal{R}$ belongs to two $(q + 1)$-secants to $\mathcal{R}$. This implies that there are in total $2(q/2 + 1)(q + 1)/(q + 1) = q + 2$ different $(q + 1)$-secants. Denote them by $L_1, \ldots, L_{q+2}$. Then since every point of $\mathcal{R}$ belongs to two of the lines $L_1, \ldots, L_{q+2}$, the lines $L_1, \ldots, L_{q+2}$ necessarily define a dual hyperoval of PG$(2, q)$, $q$ even.

Assume now that $\mathcal{R}$ is decomposable. Then the same arguments as in Remark 4 show that $\mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2$, with $\mathcal{R}_1$ an $(x_1(q + 1), x_1; 2, q)$-minihyper and with $\mathcal{R}_2$ an $(x_2(q + 1), x_2; 2, q)$-minihyper with $x_1 + x_2 = x = q/2 + 1$.

But since $x_1 \leq q/2$ and $x_2 \leq q/2$, $\mathcal{R}_1$ and $\mathcal{R}_2$ are respectively a sum of $x_1$ and $x_2$ lines (Corollary 2), so $\mathcal{R}$ is a sum of $x = q/2 + 1$ lines.

\section*{8.2 A second characterization result}

Consider again PG$(2, q)$, $q$ even. Then we have already three constructions for $((q/2 + 2)(q + 1), q/2 + 2; 2, q)$-minihypers. The first construction is via a sum of $q/2 + 2$ lines, the second construction via a $(q + 4, 4)$-arc of type $(0, 2, 4)$ (Example 2), and the third construction is via the sum of a line and a $((\frac{q}{2} + 1)(q + 1), \frac{q}{2} + 1; 2, q)$-minihyper arising from a dual hyperoval in PG$(2, q)$, $q$ even (Example 1).

We now prove that these are the only three constructions for $((q/2 + 2)(q + 1), q/2 + 2; 2, q)$-minihypers.

\textbf{Theorem 16.} Every $((\frac{q}{2} + 2)(q + 1), \frac{q}{2} + 2; 2, q)$-minihyper $\mathcal{R}$ in PG$(2, q)$, $q$ even, $q \geq 8$, is either:

(1) a sum $L_1 + \cdots + L_{q/2 + 2}$ of $q/2 + 2$ lines $L_1, \ldots, L_{q/2 + 2}$; or
(2) a \(((q/2+2)(q+1), q/2+2; q)\)-minihyper constructed via a \((q+4, 4)\)-arc of type \((0, 2, 4)\), or

(3) the sum of a line and a \(((q/2+1)(q+1), q/2+1; 2, q)\)-minihyper arising from a dual hyperoval in \(PG(2, q)\), \(q\) even.

Proof. Assume first of all that \(\mathcal{R}\) is indecomposable, then again in particular, the weight of every line is at most \(q/2+2+q−1\). Then the following properties are valid.

We again use that \(x = q/2+2 \leq y = 3q/4 = q/2+q/4\). Since \(q/4\) divides \(y\), for every point \(P\), \(\mathcal{R}(P) \leq q/2+2−q/2 = 2\) (Corollary 3). So \(\mathcal{R}\) only has points of weight one and two.

Using the same technique as in the proof of the preceding theorem, Theorem 1 implies that for every line \(L\), \(\mathcal{R}(L) \equiv q/2+2 \pmod{q/2}\), and since \(\mathcal{R}(L) \leq q/2+2+q−1\), this implies that \(\mathcal{R}(L) \in \{q/2+2, q+2\}\). We first determine the numbers \(a_{q/2+2}\) and \(a_{q+2}\) of \((q/2+2)\)-secants and \((q+2)\)-secants. The standard equations are:

\[ a_{q/2+2} + a_{q+2} = q^2 + q + 1, \]
\[ (q/2 + 2)a_{q/2+2} + (q + 2)a_{q+2} = (q/2 + 2)(q + 1)^2, \]

leading to \(a_{q/2+2} = q^2 − 3\) and \(a_{q+2} = q + 4\).

The third standard equation is [9]:

\[ (q/2 + 2)^2 a_{q/2+2} + (q + 2)^2 a_{q+2} = (q + 1)^2(q/2 + 2)^2 + q(p_1 + 4p_2), \]

where \(p_1\) is the number of points in \(\mathcal{R}\) of weight one and \(p_2\) the number of points in \(\mathcal{R}\) of weight two. This leads to \(p_1 + 4p_2 = q^2 + 2 + 3q + 4\).

But we also have the equations

\[ p_0 + p_1 + p_2 = q^2 + q + 1, \]
\[ p_1 + 2p_2 = (q/2 + 2)(q + 1), \]

leading to \(p_0 = q^2/2 - 5q/4\), \(p_1 = q^2/2 + 2q\), and \(p_2 = q/4 + 1\).

We now check how the secants pass through a point of weight zero, one, or two. A 0-point only lies on \((q/2+2)\)-secants. Suppose that a 1-point lies on \(x_{q/2+2}\) different \((q/2+2)\)-secants and on \(x_{q+2}\) different \((q+2)\)-secants. Then

\[ x_{q/2+2} + x_{q+2} = q + 1, \]
\[ (q/2 + 2)x_{q/2+2} + (q + 2)x_{q+2} = (q/2 + 2)(q + 1)+ q, \]
leading to $x_{q+2} = 2$ and $x_{q/2+2} = q - 1$. So a point of weight one lies on exactly two of the $(q + 2)$-secants.

Suppose that a 2-point lies on $x'_{q/2+2}$ different $(q/2 + 2)$-secants and on $x'_{q+2}$ different $(q + 2)$-secants. Then

$$x'_{q/2+2} + x'_{q+2} = q + 1,$$
$$\frac{q}{2} + 2)x'_{q/2+2} + (q + 2)x'_{q+2} = \frac{q}{2} + 2)(q + 1) + 2q,$$

leading to $x'_{q/2+2} = 4$ and $x'_{q/2+2} = q - 3$.

A $(q + 2)$-line is completely contained in $\mathcal{R}$, and hence contains one point of weight two and $q$ points of weight one.

This all leads to the conclusion that a point of weight zero lies on zero of the $(q + 2)$-lines, a point of weight one lies on exactly two of the $(q + 2)$-secants, and a point of weight two lies on exactly four of the $(q + 2)$-secants. This implies that the $(q+2)$-secants form a dual $(q+4,4)$-arc of type $(0,2,4)$. This shows that the minihyper arises from the construction of Example 2.

Assume now that $\mathcal{R}$ is decomposable, then $\mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2$, with $\mathcal{R}_1$ an $(x_1(q + 1), x_1; 2, q)$-minihyper and with $\mathcal{R}_2$ an $(x_2(q + 1), x_2; 2, q)$-minihyper with $x_1 + x_2 = x = q/2 + 2$. Assume that $x_1 \geq x_2$. If $x_2 \geq 2$, then $x_1 \leq q/2$ and $x_2 \leq q/2$, so $\mathcal{R}_1$ and $\mathcal{R}_2$ are a sum of respectively $x_1$ and $x_2$ lines, implying that $\mathcal{R}$ is a sum of $x = x_1 + x_2 = q/2 + 2$ lines. If $x_1 = q/2 + 1$ and $x_2 = 1$, then $\mathcal{R}_1$ is as described in the preceding theorem and $\mathcal{R}_2$ is a line.

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