Convergence result for the constraint preserving mid-point scheme for micromagnetism

Ivan Cimrák [1,2]

NfaM² Research Group, Department of Mathematical Analysis, Ghent University, Galglaan 2, B-9000 Ghent, Belgium

Abstract

An important progress was recently done in numerical approximation of weak solutions to a micromagnetic model equation. The problem with the nonconvex side-constraint of preserving the length of the magnetization was tackled by using reduced integration. Several schemes were proposed and their convergence to weak solutions was proved. All schemes were derived from the Landau-Lifshitz-Gilbert form of the micromagnetic equation. However, when the precessional term in the original Landau-Lifshitz (LL) form of the micromagnetic equation tends to zero, the above shemes become unusable.

We propose a scheme derived from the mid-point rule for the LL form of the micromagnetic equation combined with the reduced integration. We show convergence to a weak solution of the LL equation and demonstrate the usefulness of the proposed scheme to study the limit process when the precessional parameter of the micromagnetic equation goes to zero.

Key words: micromagnetism, weak solutions, reduced integration, weak convergence
2000 MSC: 65N12, 65N30

[1] E-mail: ivan.cimrak@ugent.be, URL: http://cage.ugent.be/~cimo/
[2] Author is supported by the Fund for Scientific Research - Flanders FWO (Belgium)

1 Introduction

In micromagnetism, the Landau-Lifshitz equation is used for the description of the ferromagnetic behaviour. It takes the form

$$\mathbf{m}_t = -\alpha_L \mathbf{m} \times (\mathbf{m} \times \mathbf{H}_{\text{eff}}) - \beta_L \mathbf{m} \times \mathbf{H}_{\text{eff}}, \tag{1}$$

equipped with Neumann boundary conditions and suitable initial conditions. The problem is considered in the domain $\Omega_T = \Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^d,\ d = 2, 3$. Here, $\mathbf{m}$ stands for the magnetization and $\mathbf{H}_{\text{eff}}$ denotes the effective field acting on the magnetization. In general, several terms describing different phenomena contribute to $\mathbf{H}_{\text{eff}}$. The case when thermal effects have been considered was recently discussed in [5]. We focus on the mathematically most challenging case where only the term representing the exchange field is considered. This term contains the highest derivatives of $\mathbf{m}$ represented by the Laplacian of $\mathbf{m}$. Thus from now on we have

$$\mathbf{H}_{\text{eff}} = \Delta \mathbf{m}.$$

The positive coefficient α_L influences the damping of the system while β_L controls the precession of $\mathbf{m}$ around $\mathbf{H}_{\text{eff}}$. The LL equation can be transformed into the Landau-Lifshitz-Gilbert (LLG) form

$$\mathbf{m}_t - \alpha_G \mathbf{m} \times \mathbf{m}_t = -\beta_G \mathbf{m} \times \mathbf{H}_{\text{eff}}, \tag{2}$$

where $\alpha_G = \alpha_L/\beta_L$ and $\beta_G = (\alpha_L^2 + \beta_L^2)/\beta_L$. Both, the LL and the LLG form are analytically equivalent, when $\beta_L \neq 0$. However, an analytical equivalence does not hold for the numerics and a numerical scheme approximating (1) behaves differently from the one approximating (2).

The nonconvex side-constraint $|\mathbf{m}| = 1$ holds for both forms of the micromagnetic equation. To see this, multiply the corresponding equations with $\mathbf{m}$. The nonconvexity of the side-constraint makes the construction of convergent schemes a nontrivial task. For the comprehensive overview of numerical schemes dealing with the LL equation we refer for a recent survey paper [9]. There is a comprehensive literature concerning the numerics of strong solutions to the LL equation [17,14,15,10]. We focus on more challenging case of the approximation of weak solutions.

As already mentioned in the abstract, several schemes have been introduced, all derived from the LLG form. The first finite element scheme with convergence result was introduced by Alouges and Jaisson [3], which was recently generalized [2]. Note that these schemes are linear. Bartels and Prohl in [8] suggested the following constraint preserving scheme

$$\delta\mathbf{m}^{i+1} - \alpha_G \mathbf{m}^i \times \delta\mathbf{m}^{i+1} = -\beta_G \mathbf{m}^{i+1/2} \times \Delta\mathbf{m}^{i+1/2}. \tag{3}$$

Here $\mathbf{m}^i$ approximates $\mathbf{m}$ at i-th time level, $\delta\mathbf{m}^{i+1}$ approximates the time derivative by backward Euler approximation, and quantity $\mathbf{m}^{i+1/2}$ is defined by $\mathbf{m}^{i+1/2} = (\mathbf{m}^{i+1} + \mathbf{m}^i)/2$. The method was recently generalized for the case of coupled Maxwell-Landau-Lifshitz system [4]. Scalar multiplication of (3) with $\mathbf{m}^{i+1/2}$ gives the pointwise identity $|\mathbf{m}^{i+1}| = |\mathbf{m}^i|$, provided (3) is solved exactly. Unfortunately, the right-hand side is nonlinear. In [8] the authors proposed a fixed point iteration strategy to solve the nonlinearity. However, these fixed point iterations preserve the magnitude of the magnetization only asymptotically. It would be desirable to design a scheme that solves that nonlinearity on one time level preserving the length of $\mathbf{m}$ exactly.

In their next work [7], Bartels and Prohl designed such a scheme for a different problem. It tackles the approximation of the harmonic map heat flow into spheres governed by

$$\mathbf{m}_t - \Delta\mathbf{m} = |\nabla\mathbf{m}|^2, \tag{4}$$

supplemented by the initial condition $\mathbf{m}(0) = \mathbf{m}_0 \in W^{1,2}(\Omega)$, Neumann boundary condition, and side-constraint $|\mathbf{m}| = 1$. Motivated by the formal identity $\mathbf{m} \times (\mathbf{m} \times \Delta\mathbf{m}) = -|\nabla\mathbf{m}|^2\mathbf{m} - \Delta\mathbf{m}$, we see similarity with (1). The authors proposed the following numerical scheme

$$\delta\mathbf{m}^{i+1} + \mathbf{m}^{i+1/2} \times (\mathbf{m}^{i+1/2} \times \Delta\mathbf{m}^{i+1/2}) = 0. \tag{5}$$

They analyze a larger class of schemes by allowing a (small) right-hand side with an appropriate vector product structure. This covers algorithms that solve the nonlinear system only approximately. They propose and analyze an iterative method which introduces a residual that does not significantly influence the properties of discrete solutions. Moreover, the approximate solutions obtained by this iterative method satisfy the sphere constraint exactly at the nodes of the triangulation and an approximate discrete energy law.

The strategy applied in [7] for the harmonic map heat flow was successfully adapted for the scheme (3) by Bartels in [6]. For sake of completeness we provide this algorithm at the end of Section 3, denoted by Algorithm 2. This means that Algorithm 2 derived from (3) becomes a competitive candidate for the computations of the Landau-Lifshitz equation for which the convergence analysis is done and the problem with the nonlinear term is solved.

There are however scenarios where Algorithm 2 is totally unusable. We provide one example:

Study of the limit process $\beta_L \to 0$. As already pointed out in [13], two limit cases of the Landau-Lifshitz equation lead to two known equations known in other fields: First, the limit $\alpha_L \to 0$ leads to the Hamiltonian (or symplectic) flow of harmonic maps to S^2, and second, the limit $\beta_L \to 0$ gives the heat flow of harmonic maps to S^2. Let us focus on the latter. Some properties of the well-understood harmonic heat flow can be transferred to the less-understood LL equation, and this by studying the limit process $\beta_L \to 0$. However, Algorithm 2 is not suitable for the study of a sequence of the problems with decreasing values of β_L to zero. Indeed, for $\beta_L \to 0$ we get $\alpha_G \to \infty$ and $\beta_G \to \infty$, thus the time step in Algorithm 2 must go to zero, which is impossible. We resolve this problem by introducing another algorithm for which no such a problem occurs and no refinement of the time discretization for $\beta_L \to 0$ is needed. The newly proposed scheme (6) is based on the LL form (1) of the LL equation.

After we showed the necessity for designing of an algorithm derived from the LL form of the LL equation, we propose a natural semi-discrete scheme based on the mid-point rule

$$\delta\mathbf{m}^{i+1} = -\alpha_L \mathbf{m}^{i+1/2} \times (\mathbf{m}^{i+1/2} \times \Delta\mathbf{m}^{i+1/2}) - \beta_L \mathbf{m}^{i+1/2} \times \Delta\mathbf{m}^{i+1/2}. \quad (6)$$

This scheme has frequently been used by several authors in the computations [11,12,16]. The scheme preserves the length of $\mathbf{m}^i$, which can be checked by multiplication of (6) with $\mathbf{m}^{i+1/2}$. So no projection strategy is needed. However, a rigorous justification of the convergence of this scheme is missing for the case of exchange field contributing to the effective field. We try to fill this gap in the literature. Moreover, taking scheme (6) and setting $\beta_L = 0$, we directly get (5) by which we see a natural link between the LL equation and the harmonic map heat flow.

As already claimed in [16], the accuracy of the scheme in time for smooth solutions is of order τ^2. Indeed, $\delta\mathbf{m}^{i+1}$ is the second order approximation of the time derivative evaluated in time $(t_i + t_{i+1})/2$. Similarly, $u^{i+1/2}$ is the second order approximation of $u((t_i + t_{i+1})/2)$.

The paper is organized as follows. In Section 2 we provide the basic notations and results needed for the further research. In Section 3 we define the newly proposed algorithm in a rigorous way. We also recall the definition of a weak solution to the LL equation. In Theorem 1 we establish a weak convergence of the approximate solutions obtained by Algorithm 1 to weak solutions of the LL equation. To prove this, we first derive the energy estimates for the approximate solutions in Lemma 1. In next lemma, we show the convergence of the approximate solutions to some weak limit, and finally in Lemma 3, we identify this limit to be a weak solution of the LL equation.

In Section 4, we discuss the solution on one time level. The result on the convergence is stated in Theorem 2. We do not prove this theorem because the proof is identical with that of Theorem 4.1 from [7]. At the end of the paper, we discuss the advantages of the newly proposed algorithm in more detail.

2 Notations and preliminaries

We take over the notations from [8] since they concisely cover our needs. We assume that $\mathcal{T}_h$ is a quasi-uniform regular triangulation of the polygonal or polyhedral bounded Lipschitz domain $\Omega \in \mathbb{R}^n$ into triangles or tetrahedra for $n = 2$ or $n = 3$, respectively. The assumption of quasi-uniformity of the mesh can be relaxed, see Remark 3. We define the lowest order finite element space $V_h \in W^{1,2}(\Omega)$ containing continuous functions that elementwise are polynomials of total degree less or equal to one. We denote $\mathcal{N}_h$ the set of all nodes z of the triangulation $\mathcal{T}_h$ and we introduce the nodal interpolation operator $\mathcal{I}_h : C(\overline{\Omega}, \mathbb{R}^3) \to V_h$ satisfying $\mathcal{I}_h\phi(z) = \phi(z)$ for all $z \in \mathcal{N}_h$. By $\langle\cdot,\cdot\rangle$ we denote the inner product of two vectors in $\mathbb{R}^m$ and we let $(\cdot,\cdot)$ denote the L^2 scalar product of two vectorial functions. By $\|\cdot\|_p$ we understand the L^p norm for $1 < p \leq \infty$. For continuous functions $\theta, \phi \in C(\overline{\Omega}, \mathbb{R}^3)$ we define

$$(\theta, \phi)_h = \int_\Omega \mathcal{I}_h\left(\langle\theta, \phi\rangle\right) \mathrm{d}x = \sum_{z\in\mathcal{N}_h} \beta_z \langle\theta(z), \phi(z)\rangle,$$

for certain weights β_z. More specific, we have $\beta_z = \int_\Omega \varphi_z \mathrm{d}x$, if for each $z \in \mathcal{N}_h$ we denote by $\varphi_z \in C(\overline{\Omega})$ the nodal basis function which is $\mathcal{T}_h$–elementwise affine and satisfies $\varphi_z(y) = \delta_{zy}$ for all $y \in \mathcal{N}_h$. We define $\|\phi\|_h^2 = (\phi, \phi)_h$.

Basic interpolation estimates yield

$$\left|\Big(\phi_h, \psi_h\Big)_h - \Big(\phi_h, \psi_h\Big)\right| \leq Ch\|\phi_h\|_2\|\nabla\psi_h\|_2, \tag{7}$$

for all $\phi_h, \psi_h \in V_h$, where $C > 0$ denotes an (h, τ, ε)-independent constant.

Remark 1 *Note that the value of $(\theta, \phi)_h$ depends on the values of θ and ϕ in the nodes only. Thus, (i) for continuous functions θ, ϕ the following relation holds*

$$\Big(\theta, \mathcal{I}_h\phi\Big)_h = \Big(\theta, \phi\Big)_h.$$

Moreover, (ii) if two continuous functions θ and ψ have the same values in the nodes of $\mathcal{T}_h$ then

$$\big(\theta, \phi\big)_h = \big(\psi, \phi\big)_h.$$

Further, we define a discrete Laplace operator $\tilde{\Delta}_h : W^{1,2}(\Omega) \to V_h$ by

$$-(\tilde{\Delta}_h \phi, \chi_h)_h = (\nabla \phi, \nabla \chi_h) \quad \text{for all } \chi_h \in V_h.$$

We list some properties of operator $\tilde{\Delta}_h$ still taken from [8]. Denote h the maximal mesh-size of $\mathcal{T}_h$ defined as a maximal diameter of all elements in $\mathcal{T}_h$. It holds

$$\|\nabla \phi_h\|_2 \le c_1 h^{-1} \|\phi_h\|_2, \tag{8}$$
$$\|\tilde{\Delta}_h \phi_h\|_h \le c_1 h^{-1} \|\nabla \phi_h\|_2, \tag{9}$$
$$|\tilde{\Delta}_h \phi_h(z)| \le c_2 h^{-2} \|\phi_h\|_\infty, \tag{10}$$

for some positive c_1, c_2.

We provide an equidistant discretization of time with the steplength denoted by τ; thus for the interval $(0, T)$ we get $N = T/\tau$ discretization points $t_j = j\tau$. By δu^{i+1} we denote backward Euler approximation of time derivative defined as $\delta u^{i+1} := (u^{i+1} - u^i)/\tau$ and by $u^{i+1/2}$ we denote the approximation in the middle of the interval defined as $u^{i+1/2} := (u^i + u^{i+1})/2$. Piecewise constant interpolations of u^i are defined for $0 \le t \le J\tau$ such that if $t \in [i\tau, (i+1)\tau)$ for some i then $\overline{u}(t) := u^{i+1/2}$ and $u^+(t) := u^{i+1}$. Piecewise linear approximation reads as

$$\hat{u}(t) := \frac{t - i\tau}{\tau} u^{i+1} + \frac{(i+1)\tau - t}{\tau} u^i.$$

3 Convergence result

This section is organized as follows: First, we formalize method (6) using the above notations. Second, we define the notion of a weak solution to the LL equation. Finally, we prove a theorem stating the convergence of the approximate solution to a weak solution of the LL equation.

Inspired by the work [7], we propose the following algorithm approximating the LL equation. Notice the perturbation term on the right-hand side allowing for incorporating approximate solutions of equation (6).

Algorithm 1 *Given* $\mathbf{m}^j \in V_h$ *and* $\mathbf{r}^j \in V_h$ *satisfying* $\|\mathbf{r}^j\|_h \le \varepsilon$ *find* $\mathbf{m}^{j+1} \in V_h$ *such that for all* $\phi_h \in V_h$ *there holds*

$$\begin{aligned}\Big(\delta\mathbf{m}^{i+1}, \phi_h\Big)_h &+ \alpha_L\Big(\mathbf{m}^{i+1/2} \times (\mathbf{m}^{i+1/2} \times \tilde{\Delta}_h\mathbf{m}^{i+1/2}), \phi_h\Big)_h \\ &+\beta_L\Big(\mathbf{m}^{i+1/2} \times \tilde{\Delta}_h\mathbf{m}^{i+1/2}, \phi_h\Big)_h = \Big(\mathbf{m}^{i+1/2} \times \mathbf{r}^{j+1}, \phi_h\Big)_h. \end{aligned} \tag{11}$$

Remark 2 *Real positive number* ε *from Algorithm 1 is typically a small perturbation parameter. In Algorithm 3, we specify the role of* ε *more closely.*

We recall the definition [1] of a weak solution to the LL equation.

Definition 1 *Given* $\mathbf{m}_0 \in W^{1,2}(\Omega)$ *such that* $|\mathbf{m}_0| = 1$ *almost everywhere in* Ω*, a function* $\mathbf{m}$ *is called* a weak solution of (1) *if for all positive* T *there holds (i)* $\mathbf{m} \in H^1(\Omega_T, \mathbb{R}^3)$ *with* $\mathbf{m}(0,\cdot) = \mathbf{m}_0$ *in the sense of traces, (ii)* $|\mathbf{m}| = 1$ *almost everywhere in* Ω_T*, (iii) for almost all* $T' \in (0,T)$ *there holds*

$$\frac{1}{2}\int_\Omega |\nabla\mathbf{m}(T',x)|^2 \mathrm{d}x + \beta_G^{-1}\int_0^{T'} \|\mathbf{m}_t\|_2^2 \mathrm{d}t \le \frac{1}{2}\int_\Omega |\nabla\mathbf{m}_0(x)|^2 \mathrm{d}x, \tag{12}$$

and, (iv) for all $\phi \in C^\infty(\overline{\Omega}_T, \mathbb{R}^3)$ *there holds*

$$\int_0^T \Big(\mathbf{m}_t, \phi\Big)\mathrm{d}t - \alpha_G\int_0^T \Big(\mathbf{m} \times \mathbf{m}_t, \phi\Big)\mathrm{d}t = \beta_G\int_0^T\Big(\mathbf{m} \times \nabla\mathbf{m}, \nabla\phi\Big)\mathrm{d}t. \tag{13}$$

At this stage we have prepared all necessary ingredients to state the main theoretical result in the following theorem.

Theorem 1 *Let* τ *and* ε *be positive numbers, and* $(\mathcal{T}_h)_{h>0}$ *be a family of quasi-uniform regular triangulations of* Ω *with maximal mesh-size* h*. Suppose that (i)* $\mathbf{r}^i \in V_h$ *satisfies* $\|\mathbf{r}^i\|_h \le \varepsilon$*, for* $0 \le i \le J$*, (ii)* $|\mathbf{m}^0(z)| = 1$ *for all nodes* $z \in \mathcal{N}_h$*, (iii) for all* $0 \le i \le J-1$ *and for all* $\phi_h \in V_h$ *the relation (11) holds.*

Then the modulus of $\mathbf{m}^i$ *is preserved, that is,* $|\mathbf{m}^i(z)| = 1$ *for all nodes* $z \in \mathcal{N}_h$ *and for* $0 \le i \le J$*. If* $J\tau > T$ *and* $\mathbf{m}^0 \to \mathbf{m}_0$ *in* $W^{1,2}(\Omega)$ *for* $h \to 0$ *then, taking* $\hat{\mathbf{m}}_{h,\tau,\varepsilon}$ *as a piecewise linear approximation of* $\mathbf{m}^i$*, there exists a subsequence of* $(\hat{\mathbf{m}}_{h,\tau,\varepsilon})$ *as* $(h,\tau,\varepsilon) \to 0$ *which converges weakly in* $W^{1,2}(\Omega_T)$ *to a weak solution of the LL equation.*

Proof. The outline of the proof is as follows: First, in Lemma 1, we prove stability results for $\mathbf{m}^i$. In Lemma 2 we show that $\hat{\mathbf{m}}_{h,\tau,\varepsilon}$ converges up to subsequence to some $\mathbf{m}$ in suitable function spaces. Finally, in Lemma 3, we verify that $\mathbf{m}$ actually satisfies the conditions of Definition 1. □

Lemma 1 *Let the assumptions of Theorem 1 be valid. Then* $|\mathbf{m}^i(z)| = 1$ *for all* $0 \le i \le J$ *and all nodes* $z \in \mathcal{N}_h$*. Moreover, if* $0 < \varepsilon \le \alpha_L$ *then for all* $0 \le J' \le J$ *there holds*

$$\frac{1}{2}\|\nabla \mathbf{m}^{J'}\|_2^2 + (\alpha_L - \varepsilon)\tau \sum_{i=0}^{J'-1} \|\mathbf{m}^{i+1/2} \times \tilde{\Delta}_h \mathbf{m}^{i+1/2}\|_h^2 \le \frac{1}{2}\|\nabla \mathbf{m}^0\|_2^2 + \frac{1}{4}J'\tau\varepsilon,$$

and

$$\begin{aligned}&\frac{1}{2}\|\nabla \mathbf{m}^{J'}\|_2^2 + \frac{\alpha_L - \varepsilon}{\alpha_L^2 + \beta_L^2}\frac{1-\varepsilon}{1+\varepsilon}\tau \sum_{i=0}^{J'-1} \|\delta \mathbf{m}^{i+1}\|_h^2 \\ &\quad \le \frac{1}{2}\|\nabla \mathbf{m}^0\|_2^2 + \frac{1}{2}J'\tau\varepsilon\left[\frac{1}{2} + \frac{\alpha_L}{\alpha_L^2 + \beta_L^2}\right],\end{aligned}$$

Proof. Take $\phi_h = \mathbf{m}^{i+1/2}(z)\varphi_z$ in (11) to obtain

$$0 = (\delta \mathbf{m}^{i+1}(z), \mathbf{m}^{i+1/2}(z)\varphi_z)_h = \beta_z \langle \delta \mathbf{m}^{i+1}(z), \mathbf{m}^{i+1/2}(z)\rangle = \frac{\beta_z}{2}\delta|\mathbf{m}^{i+1}|^2, \tag{14}$$

which results in $|\mathbf{m}^{i+1}(z)| = 1$, provided that $|\mathbf{m}^i(z)| = 1$. Next put $\phi_h = -\tilde{\Delta}_h \mathbf{m}^{i+1/2}$ in (11). This leads to

$$\begin{aligned}\frac{1}{2}\delta\|\nabla \mathbf{m}^{i+1}\|_2^2 + \alpha_L\|\mathbf{m}^{i+1/2} \times \tilde{\Delta}_h \mathbf{m}^{i+1/2}\|_h^2 &= -\Big(\mathbf{m}^{i+1/2} \times \mathbf{r}^{i+1}, \tilde{\Delta}_h \mathbf{m}^{i+1/2}\Big)_h \\ &= \Big(\mathbf{m}^{i+1/2} \times \tilde{\Delta}_h \mathbf{m}^{i+1/2}, \mathbf{r}^{i+1}\Big)_h,\end{aligned}$$

which, after using Young's inequality and the bound for $\mathbf{r}^{i+1}$, results in

$$\frac{1}{2}\delta\|\nabla \mathbf{m}^{i+1}\|_2^2 + (\alpha_L - \varepsilon)\|\mathbf{m}^{i+1/2} \times \tilde{\Delta}_h \mathbf{m}^{i+1/2}\|_h^2 \le \frac{1}{4}\varepsilon. \tag{15}$$

A summation over $j = 0, \ldots, J' - 1$ implies the first estimate.

Next, we prepare two ingredients, which are necessary for the proof of the second statement of the lemma. Choose $\phi_h = \mathcal{I}_h(\mathbf{m}^{i+1/2} \times \tilde{\Delta}_h \mathbf{m}^{i+1/2})$ in (11). According to (i) in Remark 1, we can delete the projection $\mathcal{I}_h$, so that we can state the first ingredient

$$\Big(\mathbf{m}^{i+1/2} \times \tilde{\Delta}_h \mathbf{m}^{i+1/2}, \delta \mathbf{m}^{i+1}\Big)_h = -\beta_L\|\mathbf{m}^{i+1/2} \times \tilde{\Delta}_h \mathbf{m}^{i+1/2}\|_h^2$$

$$+\Big(\mathbf{m}^{i+1/2}\times\mathbf{r}^{i+1},\mathbf{m}^{i+1/2}\times\tilde{\Delta}_h\mathbf{m}^{i+1/2}\Big)_h. \tag{16}$$

Choose $\phi_h=\mathcal{I}_h(|\mathbf{m}^{i+1/2}|^2\tilde{\Delta}_h\mathbf{m}^{i+1/2})$ in (11). Again, according to (i) in Remark 1, we can delete the projection $\mathcal{I}_h$, so that we can state the second ingredient

$$\begin{aligned}\Big(|\mathbf{m}^{i+1/2}|^2\delta\mathbf{m}^{i+1},\tilde{\Delta}_h\mathbf{m}^{i+1/2}\Big)_h &= \alpha_L\Big\||\mathbf{m}^{i+1/2}|\mathbf{m}^{i+1/2}\times\tilde{\Delta}_h\mathbf{m}^{i+1/2}\Big\|_h^2\\ &+\Big(\mathbf{m}^{i+1/2}\times\mathbf{r}^{i+1},\Big|\mathbf{m}^{i+1/2}\Big|^2\tilde{\Delta}_h\mathbf{m}^{i+1/2}\Big)_h,\end{aligned} \tag{17}$$

where we used cross-product property $(\mathbf{a}\times\mathbf{b},\mathbf{c})=(\mathbf{c}\times\mathbf{a},\mathbf{b})$. Further we put $\phi_h=\delta\mathbf{m}^{i+1}$ in (11). We get

$$\begin{aligned}\|\delta\mathbf{m}^{i+1}\|_h^2 &= \Big(\mathbf{m}^{i+1/2}\times\mathbf{r}^{i+1},\delta\mathbf{m}^{i+1}\Big)_h\\ &-\alpha_L\Big(\mathbf{m}^{i+1/2}\times(\mathbf{m}^{i+1/2}\times\delta\mathbf{m}^{i+1}),\tilde{\Delta}_h\mathbf{m}^{i+1/2}\Big)_h\\ &-\beta_L\Big(\mathbf{m}^{i+1/2}\times\tilde{\Delta}_h\mathbf{m}^{i+1/2},\delta\mathbf{m}^{i+1}\Big)_h.\end{aligned} \tag{18}$$

From (14) we know that $\delta\mathbf{m}^{i+1}(z)$ is perpendicular to $\mathbf{m}^{i+1/2}(z)$ for all $z\in\mathcal{N}_h$. This leads to the relation

$$\mathbf{m}^{i+1/2}(z)\times\Big(\mathbf{m}^{i+1/2}(z)\times\delta\mathbf{m}^{i+1}(z)\Big)=-|\mathbf{m}^{i+1/2}(z)|^2\delta\mathbf{m}^{i+1}(z), \tag{19}$$

for all $z\in\mathcal{N}_h$. Consequently, according to (ii) in Remark 1, the following identity is valid

$$-\alpha_L\Big(\mathbf{m}^{i+1/2}\times(\mathbf{m}^{i+1/2}\times\delta\mathbf{m}^{i+1}),\tilde{\Delta}_h\mathbf{m}^{i+1/2}\Big)_h=\alpha_L\Big(|\mathbf{m}^{i+1/2}|^2\delta\mathbf{m}^{i+1},\tilde{\Delta}_h\mathbf{m}^{i+1/2}\Big)_h,$$

which can be plugged into (18). We replace also the last term in (18) by relation (16), arriving at

$$\begin{aligned}\|\delta\mathbf{m}^{i+1}\|_h^2 &= \Big(\mathbf{m}^{i+1/2}\times\mathbf{r}^{i+1},\delta\mathbf{m}^{i+1}\Big)_h+\alpha_L\Big(|\mathbf{m}^{i+1/2}|^2\delta\mathbf{m}^{i+1},\tilde{\Delta}_h\mathbf{m}^{i+1/2}\Big)_h\\ &+\beta_L^2\|\mathbf{m}^{i+1/2}\times\tilde{\Delta}_h\mathbf{m}^{i+1/2}\|_h^2-\beta_L\Big(\mathbf{m}^{i+1/2}\times\mathbf{r}^{i+1},\mathbf{m}^{i+1/2}\times\tilde{\Delta}_h\mathbf{m}^{i+1/2}\Big)_h.\end{aligned}$$

Using (17) we get

$$\begin{aligned}\|\delta\mathbf{m}^{i+1}\|_h^2 &= \Big(\mathbf{m}^{i+1/2}\times\mathbf{r}^{i+1},\delta\mathbf{m}^{i+1}\Big)_h+\alpha_L^2\Big\||\mathbf{m}^{i+1/2}|\mathbf{m}^{i+1/2}\times\tilde{\Delta}_h\mathbf{m}^{i+1/2}\Big\|_h^2\\ &+\alpha_L\Big(\Big|\mathbf{m}^{i+1/2}\Big|^2\mathbf{m}^{i+1/2}\times\mathbf{r}^{i+1},\tilde{\Delta}_h\mathbf{m}^{i+1/2}\Big)_h\end{aligned}$$

$$+\beta_L^2\|\mathbf{m}^{i+1/2}\times\tilde{\Delta}_h\mathbf{m}^{i+1/2}\Big\|_h^2-\beta_L\Big(\mathbf{m}^{i+1/2}\times\mathbf{r}^{i+1},\mathbf{m}^{i+1/2}\times\tilde{\Delta}_h\mathbf{m}^{i+1/2}\Big)_h.$$

We rearrange some terms according to above mentioned cross-product property. Using standard integral inequalities together with $\|\mathbf{m}^{i+1/2}\|_\infty\le 1$, we obtain

$$\|\delta\mathbf{m}^{i+1}\|_h^2\le(\alpha_L^2+\beta_L^2)\|\mathbf{m}^{i+1/2}\times\tilde{\Delta}_h\mathbf{m}^{i+1/2}\Big\|_h^2+\|\mathbf{r}^{i+1}\|_h\|\delta\mathbf{m}^{i+1}\|_h$$
$$+\alpha_L\|\mathbf{r}^{i+1}\|_h\|\mathbf{m}^{i+1/2}\times\tilde{\Delta}_h\mathbf{m}^{i+1/2}\|_h+\beta_L\|\mathbf{r}^{i+1}\|_h\|\mathbf{m}^{i+1/2}\times\tilde{\Delta}_h\mathbf{m}^{i+1/2}\|_h.$$

For the last three terms on the right-hand side we use Young's inequality with weights $(2\varepsilon)^{1/2}$ and $(2\varepsilon)^{-1/2}$. Finally, we conclude that

$$(1-\varepsilon)\|\delta\mathbf{m}^{i+1}\|_h^2\le(1+\varepsilon)(\alpha_L^2+\beta_L^2)\|\mathbf{m}^{i+1/2}\times\tilde{\Delta}_h\mathbf{m}^{i+1/2}\Big\|_h^2+\frac{1}{2\varepsilon}\|\mathbf{r}^{i+1}\|_h^2.$$

We multiply the previous inequality with $(\alpha_L-\varepsilon)/[(\alpha_L^2+\beta_L^2)(1+\varepsilon)]$, and apply (15). Finally, summation over $i=0,\dots,J'-1$ finishes the proof of the lemma. □

Lemma 2 *Let the assumptions of Theorem 1 be valid. Then there exists a subsequence of* $(\hat{\mathbf{m}}_{h,\tau,\varepsilon})$ *as* $(h,\tau,\varepsilon)\to 0$ *and* $\mathbf{m}\in W^{1,2}(\Omega_T)$ *such that* $\hat{\mathbf{m}}_t\rightharpoonup\mathbf{m}_t$ *in* $L^2(\Omega_T)$, $\overline{\mathbf{m}}\to\mathbf{m}$ *in* $L^2(\Omega_T)$, $\nabla\overline{\mathbf{m}}\rightharpoonup\nabla\mathbf{m}$ *in* $L^2(\Omega_T)$, *and* $\mathbf{m}^+\rightharpoonup^*\mathbf{m}$ *in* $L^\infty((0,T),W^{1,2}(\Omega))$. *In particular, there holds* $|\mathbf{m}|=1$ *almost everywhere in* Ω_T, $\mathbf{m}$ *satisfies (12) and there holds* $\mathbf{m}(0,\cdot)=\mathbf{m}_0$ *in the sense of traces.*

Proof. Using the results of Lemma 1, we can prove the statement of this lemma in the identical way as was done in the proof of Lemma 3.2. in [7]. Therefore, we skip the details. □

Lemma 3 *For* $\mathbf{m}$ *as in Lemma 2 and for all* $\phi\in C^\infty(\overline{\Omega}_T)$ *there holds*

$$\int_0^T\Big(\mathbf{m}_t,\phi\Big)\mathrm{d}t-\alpha_G\int_0^T\Big(\mathbf{m}\times\mathbf{m}_t,\phi\Big)\mathrm{d}t=\beta_G\int_0^T\Big(\mathbf{m}\times\nabla\mathbf{m},\nabla\phi\Big)\mathrm{d}t.$$

Proof. For $t\in(0,T)$ let $\phi_h(t,\cdot)=\mathcal{I}_h\phi(t,\cdot)$. Let us define $\mathcal{A}_1,\dots,\mathcal{A}_6$ by

$$\mathcal{A}_1:=\int_0^T\Big(\hat{\mathbf{m}},\phi_h\Big)_h-\Big(\mathbf{m}_t,\phi\Big)\mathrm{d}t,$$

$$\mathcal{A}_2 := \alpha_G \int_0^T \Big(\overline{\mathbf{m}} \times \hat{\mathbf{m}}_t, \phi_h\Big)_h - \Big(\mathbf{m} \times \mathbf{m}_t, \phi\Big) \mathrm{d}t,$$

$$\mathcal{A}_3 := \beta_G \int_0^T \Big(\overline{\mathbf{m}} \times \tilde{\Delta}_h \overline{\mathbf{m}}, \phi_h\Big)_h + \Big(\mathbf{m} \times \nabla \mathbf{m}, \nabla \phi\Big) \mathrm{d}t,$$

$$\mathcal{A}_4 := \alpha_G \int_0^T \Big(\Big[1 - |\overline{\mathbf{m}}|^2\Big] \overline{\mathbf{m}} \times \tilde{\Delta}_h \overline{\mathbf{m}}, \phi_h\Big)_h \mathrm{d}t.$$

$$\mathcal{A}_5 := \int_0^T \Big(\overline{\mathbf{m}} \times \mathbf{r}^+, \phi_h\Big)_h \mathrm{d}t,$$

$$\mathcal{A}_6 := \alpha_G \int_0^T \Big(\overline{\mathbf{m}} \times \mathbf{r}^+, \overline{\mathbf{m}} \times \phi_h\Big)_h \mathrm{d}t.$$

We have

$$\begin{aligned} &\Big(\hat{\mathbf{m}}_t, \phi_h\Big)_h - \Big(\mathbf{m}_t, \phi\Big) \\ &\quad = \Big(\hat{\mathbf{m}}_t, \phi_h\Big)_h - \Big(\hat{\mathbf{m}}_t, \phi_h\Big) + \Big(\hat{\mathbf{m}}_t - \mathbf{m}_t, \phi_h\Big) + \Big(\mathbf{m}_t, \phi_h - \phi\Big). \end{aligned} \tag{20}$$

Using (7) and the approximation properties of $\mathcal{I}_h$ leads to

$$|\mathcal{A}_1| \leq Ch\|\hat{\mathbf{m}}_t\|_2 \|\phi\|_{W^{2,2}} + Ch\|\mathbf{m}_t\|_2 \|\nabla \phi\|_2 + \Big|\int_0^t \Big(\mathbf{m}_t(\zeta), \phi_h(\zeta) - \phi(\zeta)\Big) \mathrm{d}\zeta\Big|,$$

for all $t \in (0,T)$. Consequently, from the weak convergence of $\hat{\mathbf{m}}_t$ to $\mathbf{m}_t$, we obtain for all $t \in (0,T)$ that $|\mathcal{A}_1| \to 0$ for $(h, \tau, \varepsilon) \to 0$. For term $\mathcal{A}_2$ we can write for all $t \in (0,T)$

$$\begin{aligned} &\Big(\overline{\mathbf{m}} \times \hat{\mathbf{m}}_t, \phi_h\Big)_h - \Big(\mathbf{m} \times \mathbf{m}_t, \phi\Big) = \Big(\phi_h \times \overline{\mathbf{m}}, \hat{\mathbf{m}}_t\Big)_h - \Big(\phi \times \mathbf{m}, \mathbf{m}_t\Big) \\ &\quad = \Big(\mathcal{I}_h(\phi_h \times \overline{\mathbf{m}}), \hat{\mathbf{m}}_t\Big)_h - \Big(\mathcal{I}_h(\phi_h \times \overline{\mathbf{m}}), \hat{\mathbf{m}}_t\Big) \\ &\qquad + \Big((\mathcal{I}_h - \mathrm{Id})(\phi_h \times \overline{\mathbf{m}}), \hat{\mathbf{m}}_t\Big) + \Big(\phi_h \times (\overline{\mathbf{m}} - \mathbf{m}), \hat{\mathbf{m}}_t\Big) \\ &\qquad + \Big((\phi_h - \phi) \times \mathbf{m}, \hat{\mathbf{m}}_t\Big) + \Big(\phi \times \mathbf{m}, \hat{\mathbf{m}}_t - \mathbf{m}_t\Big). \end{aligned}$$

For the terms on the right-hand side we can apply (7), and approximation estimates for $\mathcal{I}_h$, to obtain

$$\Big|\Big(\overline{\mathbf{m}} \times \hat{\mathbf{m}}_t, \phi_h\Big)_h - \Big(\mathbf{m} \times \mathbf{m}_t, \phi\Big)\Big|$$

$$\leq Ch\Big[\|\nabla\phi_h\|_2\|\overline{\mathbf{m}}\|_\infty\|\hat{\mathbf{m}}_t\|_2 + \|\phi_h\|_\infty\|\nabla\overline{\mathbf{m}}\|_2\|\hat{\mathbf{m}}_t\|_2\Big]$$
$$+\|\phi_h\|_\infty\|\overline{\mathbf{m}} - \mathbf{m}\|_2\|\hat{\mathbf{m}}_t\|_2 + \|\phi_h - \phi\|_2\|\mathbf{m}\|_\infty\|\hat{\mathbf{m}}_t\|_2 + \Big|\Big(\phi \times \mathbf{m}, \hat{\mathbf{m}}_t - \mathbf{m}_t\Big)\Big|.$$

Next, we use a strong convergence of $\hat{\mathbf{m}}$ to $\mathbf{m}$ and ϕ_h to ϕ in $L^2(\Omega_T)$, and a weak convergence of $\hat{\mathbf{m}}_t$ to $\mathbf{m}_t$ in $L^2(\Omega_T)$, to conclude for all $t \in (0,T)$ that $|\mathcal{A}_2| \to 0$ for $(h,\tau,\varepsilon) \to 0$.

Again, some manipulation gives

$$\begin{aligned}
&\Big(\overline{\mathbf{m}} \times \tilde{\Delta}_h\overline{\mathbf{m}}, \phi_h\Big)_h + \Big(\mathbf{m} \times \nabla\mathbf{m}, \nabla\phi\Big) = \Big(\phi_h \times \overline{\mathbf{m}}, \tilde{\Delta}_h\overline{\mathbf{m}}\Big)_h + \Big(\mathbf{m} \times \nabla\mathbf{m}, \nabla\phi\Big)\\
&\quad = \Big((\mathrm{Id} - \mathcal{I}_h)(\phi_h \times \overline{\mathbf{m}}), \tilde{\Delta}_h\overline{\mathbf{m}}\Big)_h\\
&\qquad -\Big(\nabla(\mathcal{I}_h - \mathrm{Id})(\phi_h \times \overline{\mathbf{m}}), \nabla\overline{\mathbf{m}}\Big) - \Big((\overline{\mathbf{m}} - \mathbf{m}) \times \nabla\overline{\mathbf{m}}, \nabla\phi_h\Big)\\
&\qquad -\Big(\mathbf{m} \times \nabla\overline{\mathbf{m}}, \nabla(\phi_h - \phi)\Big) - \Big(\mathbf{m} \times (\nabla\overline{\mathbf{m}} - \nabla\mathbf{m}), \nabla\phi\Big).
\end{aligned} \tag{21}$$

For the first term on the right-hand side we use (9) and estimates for nodal approximation to get

$$\begin{aligned}
\Big|\Big((\mathrm{Id} - \mathcal{I}_h)(\phi_h \times \overline{\mathbf{m}}), \tilde{\Delta}_h\overline{\mathbf{m}}\Big)_h\Big| &\leq Ch^2h^{-1}\|D^2(\overline{\mathbf{m}} \times \phi_h)\|_2\|\nabla\overline{\mathbf{m}}\|_2\\
&\leq Ch\|\nabla\overline{\mathbf{m}}\|_2\|\nabla\phi_h\|_\infty\|\nabla\overline{\mathbf{m}}\|_2.
\end{aligned} \tag{22}$$

For the other terms on the right-hand side of (21), using a similar argumentation as before, we arrive at

$$\begin{aligned}
&\Big|\Big(\overline{\mathbf{m}} \times \tilde{\Delta}_h\overline{\mathbf{m}}, \phi_h\Big)_h + \Big(\mathbf{m} \times \nabla\mathbf{m}, \nabla\phi\Big)\Big|\\
&\quad = Ch\Big[\|\nabla\overline{\mathbf{m}}\|_2\|\nabla\phi_h\|_\infty\|\nabla\overline{\mathbf{m}}\|_2 + \|\nabla\phi_h\|_2\|\overline{\mathbf{m}}\|_\infty\|\nabla\overline{\mathbf{m}}\|_2\Big]\\
&\qquad +\|\mathbf{m}\|_\infty\|\nabla\overline{\mathbf{m}}\|_2\|\nabla(\phi_h - \phi)\|_2 + \|\overline{\mathbf{m}} - \mathbf{m}\|_2\|\nabla\overline{\mathbf{m}}\|_2\|\nabla\phi_h\|_\infty\\
&\qquad +\Big|\Big(\mathbf{m} \times (\nabla\overline{\mathbf{m}} - \nabla\mathbf{m}), \nabla\phi\Big)\Big|.
\end{aligned}$$

Since $\overline{\mathbf{m}}$ converges strongly to $\mathbf{m}$ in $L^2(\Omega_T)$, and $\nabla\overline{\mathbf{m}}$ converges weakly to $\nabla\mathbf{m}$ in $L^2(\Omega_T)$, we have that the last two terms on the right-hand side converge to 0. So we conclude that $|\mathcal{A}_3| \to 0$ for $(h,\tau,\varepsilon) \to 0$.

Using that $\Big|1 - |\overline{\mathbf{m}}|^2\Big| = \Big|\langle\mathbf{m} - \overline{\mathbf{m}}, \mathbf{m} + \overline{\mathbf{m}}\rangle\Big|$ we get

$$\Big(\Big[1 - |\overline{\mathbf{m}}|^2\Big]\overline{\mathbf{m}} \times \tilde{\Delta}_h\overline{\mathbf{m}}, \phi_h\Big)_h \leq C\|\overline{\mathbf{m}} \times \tilde{\Delta}_h\overline{\mathbf{m}}\|_2\|\phi\|_\infty\|\mathbf{m} - \overline{\mathbf{m}}\|_2.$$

Using the bounds from Lemma 1 and strong convergence of $\overline{\mathbf{m}}$ to $\mathbf{m}$ in $L^2(\Omega_T)$, we conclude that $|\mathcal{A}_4| \to 0$ for $(h, \tau, \varepsilon) \to 0$.

Finally, terms $\mathcal{A}_5$ and $\mathcal{A}_6$ can be estimated by $C\varepsilon\|\phi_h\|_2$.

To finish the proof of the lemma we rewrite (11) as

$$\begin{aligned}&\Big(\hat{\mathbf{m}}_t, \phi_h\Big)_h + \alpha_L\Big(\overline{\mathbf{m}} \times (\overline{\mathbf{m}} \times \tilde{\Delta}_h\overline{\mathbf{m}}), \phi_h\Big)_h + \beta_L\Big(\overline{\mathbf{m}} \times \tilde{\Delta}_h\overline{\mathbf{m}}, \phi_h\Big)_h\\ &\quad = \Big(\overline{\mathbf{m}} \times \mathbf{r}^+, \phi_h\Big)_h,\end{aligned}$$

for $\phi \in V_h$ and all $t \in (0, T)$. Taking $\phi_h(t, \cdot) = \mathcal{I}_h\Big[(\overline{\mathbf{m}} \times \psi_h)(t, \cdot)\Big]$ and using (19) we end up with

$$\begin{aligned}&-\Big(\overline{\mathbf{m}} \times \hat{\mathbf{m}}_t, \psi_h\Big)_h + \alpha_L\Big(\overline{\mathbf{m}} \times \tilde{\Delta}_h\overline{\mathbf{m}}, \psi_h\Big)_h - \beta_L\Big(\overline{\mathbf{m}} \times (\overline{\mathbf{m}} \times \tilde{\Delta}_h\overline{\mathbf{m}}), \psi_h\Big)_h\\ &\quad = \Big(\overline{\mathbf{m}} \times \mathbf{r}^+, \overline{\mathbf{m}} \times \psi_h\Big)_h + \Big(\Big[1 - |\overline{\mathbf{m}}|^2\Big]\overline{\mathbf{m}} \times \tilde{\Delta}_h\overline{\mathbf{m}}, \psi_h\Big)_h,\end{aligned}$$

for all $\psi \in C^\infty(\overline{\Omega}_T)$. We get rid of terms involving $\overline{\mathbf{m}} \times (\overline{\mathbf{m}} \times \tilde{\Delta}_h\overline{\mathbf{m}})$ by a combination of previous two equalities

$$\begin{aligned}&\Big(\hat{\mathbf{m}}_t, \phi_h\Big)_h - \alpha_G\Big(\overline{\mathbf{m}} \times \hat{\mathbf{m}}_t, \psi_h\Big)_h + \beta_G\Big(\overline{\mathbf{m}} \times \tilde{\Delta}_h\overline{\mathbf{m}}, \psi_h\Big)_h\\ &\quad = \Big(\overline{\mathbf{m}} \times \mathbf{r}^+, \phi_h\Big)_h + \alpha_G\Big[\Big(\overline{\mathbf{m}} \times \mathbf{r}^+, \overline{\mathbf{m}} \times \psi_h\Big)_h + \Big(\Big[1 - |\overline{\mathbf{m}}|^2\Big]\overline{\mathbf{m}} \times \tilde{\Delta}_h\overline{\mathbf{m}}, \psi_h\Big)_h\Big].\end{aligned}$$

Using the bounds for $\mathcal{A}_1, \ldots, \mathcal{A}_6$, we verify that

$$\Big|\int_0^T \Big(\mathbf{m}_t, \phi\Big)\mathrm{d}t - \alpha_G\int_0^T \Big(\mathbf{m} \times \mathbf{m}_t, \phi\Big)\mathrm{d}t - \beta_G\int_0^T\Big(\mathbf{m} \times \nabla\mathbf{m}, \nabla\phi\Big)\mathrm{d}t\Big| \le \sum_{i=1}^{6}\Big|\mathcal{A}_i\Big| \to 0,$$

for $(h, \tau, \varepsilon) \to 0$. □

For completeness we mention the algorithm of Bartels from [6] derived from (3).

Algorithm 2 *Given* $\mathbf{m}^j \in V_h$ *and* $\mathbf{r}^j \in V_h$ *satisfying* $\|\mathbf{r}^j\|_h \le \varepsilon$ *find* $\mathbf{m}^{j+1} \in V_h$ *such that for all* $\phi_h \in V_h$ *there holds*

$$\Big(\delta\mathbf{m}^{i+1}, \phi_h\Big)_h - \alpha_G\Big(\mathbf{m}^i \times \delta\mathbf{m}^{i+1}, \phi_h\Big)_h + \beta_G\Big(\mathbf{m}^{i+1/2} \times \tilde{\Delta}_h\mathbf{m}^{i+1/2}, \phi_h\Big)_h$$

$$= \left(\mathbf{m}^{i+1/2} \times \mathbf{r}^{j+1}, \phi_h\right)_h. \tag{23}$$

4 Solution on one time level

In Theorem 1 we proved the conservation of the modulus $|\mathbf{m}^i|$ for (11). Since the scheme (11) is nonlinear, we have to choose the solver to solve the nonlinear system. General solvers such as Newton method, quasi-Newton methods or fixed point iterations, can be successfully used, however by applying those methods we loose the main advantage of the proposed scheme - the conservation of the length of the magnetization in the nodes of the mesh. Therefore we design a fixed point iteration scheme in such a way that this property will not be lost.

We adapt the algorithm for the harmonic map heat flow problem from [7].

Algorithm 3 Input: *parameters h, τ, ε, J as from Theorem 1, $\mathbf{m}^0 \in V_h$ such that $|\mathbf{m}^0(z)| = 1$ for all nodes $z \in \mathcal{N}_h$.*

(a) *Set $i = 0$, $\mathbf{r}^0 = 0$.*
(b) *Set $\mathbf{w}^{i+1,0} = \mathbf{m}^i$.*
  (b1) *Set $l = 0$.*
  (b2) *Compute $\mathbf{w}^{i+1,l+1} \in V_h$ such that*

$$\frac{2}{\tau}\left(\mathbf{w}^{i+1,l+1}, \phi_h\right)_h + \alpha_L\left(\mathbf{w}^{i+1,l+1} \times (\mathbf{w}^{i+1,l} \times \tilde{\Delta}_h \mathbf{w}^{i+1,l}), \phi_h\right)_h$$
$$+\beta_L\left(\mathbf{w}^{i+1,l+1} \times \tilde{\Delta}_h \mathbf{w}^{i+1,l}, \phi_h\right)_h = \frac{2}{\tau}\left(\mathbf{m}^i, \phi_h\right)_h,$$

for all $\phi_h \in V_h$. Set $\mathbf{e}^{i+1,l+1} = \mathbf{w}^{i+1,l+1} - \mathbf{w}^{i+1,l}$ and

$$\mathbf{r}^{i+1} = \alpha_L(\mathbf{w}^{i+1,l+1} \times \tilde{\Delta}_h \mathbf{e}^{i+1,l+1} + \mathbf{e}^{i+1,l+1} \times \tilde{\Delta}_h \mathbf{w}^{i+1,l}) + \beta_L \tilde{\Delta}_h \mathbf{e}^{i+1,l+1}.$$

  (b3) *Go to (c) if $\|\mathbf{r}^{i+1}\|_h \leq \varepsilon$; set $l = l+1$ and continue with (b2) otherwise.*
(c) *Set $\mathbf{m}^{i+1} = 2\mathbf{w}^{i+1,l+1} - \mathbf{m}^i$.*
(d) *Stop if $i + 1 = J$; set $i = i + 1$ and go to (b) otherwise.*

Output: *Sequences $(\mathbf{m}^i)_{i=0,\dots,J}$ and $(\mathbf{r}^i)_{i=0,\dots,J}$.*

In the following theorem we show that all steps in Algorithm 3 are well–defined. Further we show that the sequences generated by the algorithm satisfy (11), and that the algorithm terminates if $\tau = \mathcal{O}(h^2)$.

Theorem 2 *Let $0 \leq i \leq J - 1$ and $\mathbf{m}^i \in V_h$ such that $|\mathbf{m}^i(z)| = 1$ for all nodes $z \in \mathcal{N}_h$. Then, for all $l \geq 0$ the system in (b2) admits a unique solution*

$\mathbf{w}^{i+1,l+1} \in V_h$ *such that* $|\mathbf{w}^{i+1,l+1}(\mathbf{x}_m)| \leq 1$ *and* $|(2\mathbf{w}^{i+1,l+1} - \mathbf{m}^i)(z)| = 1$ *for all nodes* $z \in \mathcal{N}_h$. *Moreover, there holds*

$$\|\mathbf{e}^{i+1,l+1}\|_h \leq c_1^2 \tau h^{-2} \frac{|\alpha_L| + |\beta_L|}{2} \|\mathbf{e}^{i+1,l}\|_h. \tag{24}$$

If for all $0 \leq i \leq J-1$ *the iteration (b1)–(b3) converges then there holds (11).*

We do not provide the proof of this theorem since it can be done in the same way as the proof of Theorem 4.1 from [7]. The only difference lies in the terms with coefficient β_L in (b2). This discrepancy leads to an extra term in the definition of $\mathbf{r}^i$ in (b2). The analysis, however, will be the same.

Remark 3 *As already pointed out at the beginning of Section 2, the assumption on quasi-uniformity of the triangulation can be relaxed. Consider a regular triangulation* $\mathcal{T}_h$ *which is not necessarily quasi-uniform. As before, denote by* h *the maximal element size. Further denote by* h_{min} *the minimal element size. For quasi-uniform families of meshes,* h_{min} *can be expressed as a multiple of* h.

In this general setting, estimates (8)–(10) must be reformulated as

$$\begin{aligned}
&\|\nabla \phi_h\|_2 \leq c_1 h_{min}^{-1} \|\phi_h\|_2, \\
&\|\tilde{\Delta}_h \phi_h\|_h \leq c_1 h_{min}^{-1} \|\nabla \phi_h\|_2, \\
&|\tilde{\Delta}_h \phi_h(z)| \leq c_2 h_{min}^{-2} \|\phi_h\|_\infty.
\end{aligned}$$

Therefore, all subsequent estimates using (8)–(10) must be adapted. For example the upper bound in estimate (22) becomes

$$C h^2 h_{min}^{-1} \|\nabla \overline{\mathbf{m}}\|_2 \|\nabla \phi_h\|_\infty \|\nabla \overline{\mathbf{m}}\|_2.$$

So the assumption $(h, \tau, \varepsilon) \to 0$ *in Lemma 2 changes to* $(h^2 h_{min}^{-1}, \tau, \varepsilon) \to 0$. *The estimate (24) is also changed by replacing* h *with* h_{min}.

5 Conclusions

To compare the behavior of Algorithm 3 with that of Algorithm 2, we analyze the threshold value τ_{max} of the time-step for which we have proved the convergence. In the case of the newly proposed Algorithm 3, from (24) we have that $c_1^2 \tau h^{-2}(|\alpha_L| + |\beta_L|)/2$ must be less than one so for the threshold value τ_{max1} must hold

$$\tau_{max1} = c_1^{-2} \frac{2}{|\alpha_L| + |\beta_L|} h^2.$$

From [6, Theorem 3.1] we have that the threshold value for Algorithm 2 is equal to

$$\tau_{max2} = c_1^{-2} \frac{1}{|\beta_G|} h^2.$$

Using the transformation relations between pairs (α_L, β_L) and (α_G, β_G) mentioned in the introduction, we arrive at the following relation for τ_{max2}

$$\tau_{max2} = c_1^{-2} \frac{|\beta_L|}{\alpha_L^2 + \beta_L^2} h^2.$$

Now we can clearly see how τ_{max} changes with variable β_L. As we already mentioned, our main aim is to show how both algorithms perform when β_L goes to zero. It is evident that for $\beta_L \to 0$, the threshold value τ_{max1} practically does not change, while the threshold value τ_{max2} degenerates to zero.

Acknowledgements

I would like to thank Prof. Roger Van Keer for his support and interest.

References

[1] F. Alouges, A. Soyeur, On global weak solutions for Landau-Lifshitz equations: Existence and nonuniqueness, Nonlinear Anal. 18 (11) (1992) 1071–1094.

[2] F. Alouges, A new finite element scheme for Landau-Lifshitz equations, Discrete and Continuous Dynamical Systems - Series S 18 (2) (2008) 187–196.

[3] F. Alouges, P. Jaisson, Convergence of a finite element discretization for the Landau-Lifshitz equations in micromagnetism, Math. Models Methods Appl. Sci. 16 (2) (2006) 299–316.

[4] L'. Baňas, S. Bartels, A. Prohl, A convergent implicit discretization of the Maxwell-Landau-Lifshitz-Gilbert equation, SIAM J. Numer. Anal. to appear.

[5] L'. Baňas, A. Prohl, M. Slodička, Modeling of thermally assisted magnetodynamics, SIAM J. Numer. Anal. to appear.

[6] S. Bartels, Constraint preserving, inexact solution of implicit discretizations of Landau-Lifshitz-Gilbert equations and consequences for convergence, PAMM 6 (1) (2006) 19–22.

[7] S. Bartels, A. Prohl, Constraint preserving implicit finite element discretization of harmonic map heat flow into spheres, Math. Comp. 76 (260) (2007) 1847–1859.

[8] S. Bartels, A. Prohl, Convergence of an implicit finite element method for the Landau-Lifshitz-Gilbert equation, SIAM J. Numer. Anal. 44 (4) (2006) 1405–1419.

[9] I. Cimrák, A Survey on the Numerics and Computations for the Landau-Lifshitz Equation of Micromagnetism, *Arch. Comp. Meth. Eng.* 15 (3) (2008) 277–309.

[10] I. Cimrák, M. Slodička, Optimal convergence rate for Maxwell-Landau-Lifshitz system, Phys. B 343 (1–4) (2004) 236–240.

[11] M. d'Aquino, Nonlinear Magnetization Dynamics in Thin-films and Nanoparticles, PhD thesis (2004).

[12] M. d'Aquino, C. Serpico, G. Miano, Geometrical integration of Landau-Lifshitz-Gilbert equation based on the mid-point rule, J. Comp. Phys. 209 (2005) 730–753.

[13] W. E, X.-P. Wang, Numerical methods for the Landau-Lifshitz equation, SIAM J. Numer. Anal. 38 (5) (2000) 1647–1665.

[14] B. Guo, M. Hong, The Landau-Lifshitz equation of the ferromagnetic spin chain and harmonic maps, Calc. Var. 1 (3) (1993) 311–334.

[15] A. Prohl, *Computational Micromagnetism,* Advances in Numerical Mathematics, B.G. Teubner, Stuttgart, 2001.

[16] C. Serpico, I.D. Mayergoyz, G. Bertotti, Numerical technique for integration of the Landau-Lifshitz equation, J. Appl. Phys. 89 (11) (2001) 6991–6993.

[17] A. Visintin, On Landau-Lifshitz' equations for ferromagnetism, Japan J. Appl. Math. 2 (1985) 69–84.

Author: Ivan Cimrák
Affiliation: *NfaM*^{2} Research Group, Department of Mathematical Analysis, Ghent University, Belgium
Address: Department of Mathematical analysis, Galglaan 2, B-9000 Ghent, Belgium
E-mail: ivan.cimrak@ugent.be