Abstract

We investigate the functional code \( C_h(X) \) introduced by G. Lachaud [10] in the special case where \( X \) is a non-singular Hermitian variety in \( \text{PG}(N, q^2) \) and \( h = 2 \). In [4], F. Edoukou solved the conjecture of Sørensen [11] on the minimum distance of this code for a Hermitian variety \( X \) in \( \text{PG}(3, q^2) \). In this paper, we will answer the question about the minimum distance in general dimension \( N \), with \( N < O(q^2) \). We also prove that the small weight codewords correspond to the intersection of \( X \) with the union of 2 hyperplanes.

1 Introduction

We study the functional code \( C_2(X) \) in \( \text{PG}(N, q^2) \), where \( X \) is a non-singular Hermitian variety \( H(N, q^2) \). Let \( X = \{ P_1, \ldots, P_n \} \), where we normalize the coordinates of these points with respect to the leftmost non-zero coordinate. Let \( \mathcal{F} \) be the set of all homogeneous quadratic polynomials \( f(X_0, \ldots, X_N) \) defined by \( N + 1 \) variables with coefficients in \( \mathbb{F}_{q^2} \). The functional code \( C_2(X) \) is the linear code

\[
C_2(X) = \{ (f(P_1), \ldots, f(P_n)) || f \in \mathcal{F} \cup \{0\} \}.
\]

This linear code has length \( n = |X| \) and dimension \( k = \binom{N+2}{2} \) over \( \mathbb{F}_{q^2} \). The third fundamental parameter of this linear code is its minimum distance \( d \). Since the code is linear, this minimum distance corresponds to the minimum weight of the code. The small weight codewords, i.e., the codewords having the minimum weight or a weight close to the minimum weight, arise from the quadrics having the (almost) largest intersections with \( X \).

Sørensen [11] conjectured that the maximum size for the intersection of a quadric \( Q \) with the Hermitian variety \( H(3, q^2) \) in \( \text{PG}(3, q^2) \) is equal to \( 2q^3 + 2q^2 - q + 1 \). The correctness of this conjecture was proven by Edoukou in [3].
More precisely, Edoukou not only proved that the maximum size for the intersection of a quadric $Q$ with the Hermitian variety $H(3, q^2)$ in $PG(3, q^2)$ is equal to $2q^3 + 2q^2 - q + 1$; he also proved that the second largest intersection size of a quadric $Q$ with the Hermitian variety $H(3, q^2)$ in $PG(3, q^2)$ is at most $2q^3 + q^2 + 1$.

Regarding the largest intersection sizes of a quadric $Q$ with the Hermitian variety $H(4, q^2)$ in $PG(4, q^2)$, Edoukou [5] determined the five largest intersection sizes, leading to the 5 smallest weights for the code $C_2(X)$, $X = H(4, q^2)$.

In [5, Conjecture 2, p. 145], he also stated that the five smallest weights for the code $C_2(X)$, $X = H(N, q^2)$, arise from the intersections of $X$ with the quadrics which are the union of two distinct hyperplanes.

We determine the 5 smallest weights of $C_2(X)$, $X = H(N, q^2)$, $N < O(q^2)$, via geometrical arguments, and prove the validity of the conjecture of Edoukou for $N < O(q^2)$. These 5 smallest weights will be the small weights of the code $C_2(X)$, $X = H(N, q^2)$, on which we will concentrate.

First of all, we will investigate the different intersections of quadrics $Q$ in $PG(4, q^2)$ with $H(4, q^2)$; leading to a lower bound on the intersection size guaranteeing that any quadric having more than this number of points in common with $H(4, q^2)$ must be the union of two hyperplanes. We use this result to find a bound on the intersection sizes of absolutely irreducible quadrics with the non-singular Hermitian variety $H(N, q^2)$. Here this lower bound on the intersection size guarantees that $Q$ is the union of 2 hyperplanes. Using this bound, we prove that the small weight codewords correspond to quadrics which are the union of 2 hyperplanes. There are several possibilities for the intersection of such a quadric with a non-singular Hermitian variety $X$. So we can construct tables with the 5 smallest weights of the functional code $C_2(X)$, $X$ a non-singular Hermitian variety in $PG(N, q^2)$, $N < O(q^2)$.

The results of this article continue the research on the small weight codewords of functional codes performed in [6, 7]. In [6], we determined the smallest weights of the non-zero codewords of the functional codes $C_2(Q)$, which are defined by the intersections of all quadrics with a non-singular quadric $Q$ in $PG(N, q)$, and in [7], we determined the smallest weights of the non-zero codewords of the functional codes $C_{herm}(X)$, which are defined by the intersections of all Hermitian varieties with a non-singular Hermitian variety in $PG(N, q^2)$. In these cases, the smallest weight codewords arise in [6] from the intersections of $Q$ with the quadrics which are the union of two hyperplanes, and in [7] from the intersections of $X$ with the Hermitian varieties which are the union of $q + 1$ hyperplanes through a common $(N - 2)$-dimensional space of $PG(N, q^2)$.

In the article [6], the crucial element was the fact that the intersection $V$ of two quadrics $Q$ and $Q'$ lies in all the $q + 1$ quadrics $\lambda Q + \mu Q'$, $(\lambda, \mu) \in \mathbb{F}_q^2 \setminus \{(0, 0)\}$, of the pencil of quadrics defined by $Q$ and $Q'$ and similarly for the second article [7], the crucial element was the fact that the intersection $V$ of two Hermitian varieties $X$ and $X'$ in $PG(N, q^2)$ lies in all the $q + 1$ Hermitian varieties $\lambda X + \mu X'$, $(\lambda, \mu) \in \mathbb{F}_q^2 \setminus \{(0, 0)\}$, of the pencil of Hermitian varieties defined by $X$ and $X'$. This enabled us to obtain results for general dimensions $N$.

We cannot use this fact in this article. A quadric and a Hermitian variety do not define together a pencil of quadrics or of Hermitian varieties. This implied that different
arguments had to be used, which enabled us to obtain results up to dimension \( N < O(q^2) \) for the Hermitian variety \( X \) in \( \text{PG}(N, q^2) \).

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2 Quadrics and Hermitian varieties

By \( \pi_i \), we denote a projective subspace of dimension \( i \) in \( \text{PG}(N, q^2) \). We will often use the term space instead of projective subspace. The space generated by two spaces \( \pi_i \) and \( \pi_{i'} \) is denoted by \( (\pi_i, \pi_{i'}) \).

For the fundamental properties of quadrics and Hermitian varieties, we refer to [9, Chapters 22 and 23]. We repeat the relevant properties for the arguments in this article.

The non-singular quadrics in \( \text{PG}(N, q^2) \) are equal to:

- the non-singular parabolic quadrics \( Q(N, q^2) \) in \( \text{PG}(N = 2N', q^2) \) having standard equation \( X_0^2 + X_1X_2 + \cdots + X_{2N'-1}X_{2N'} = 0 \). These quadrics contain \( q^{4N'-2} + \cdots + q^2 + 1 \) points, and the largest dimensional spaces contained in a non-singular parabolic quadric of \( \text{PG}(2N', q^2) \) have dimension \( N' - 1 \),

- the non-singular hyperbolic quadrics \( Q^+(N, q^2) \) in \( \text{PG}(N = 2N' + 1, q^2) \) having standard equation \( f(X_0, X_1) = X_0X_1 + \cdots + X_{2N'}X_{2N'+1} = 0 \), where \( f(X_0, X_1) \) is an irreducible quadratic polynomial over \( \mathbb{F}_{q^2} \). These quadrics contain \( (q^{2N'} + 1)/(q^2 - 1) = q^{4N'} + q^{4N'-2} + \cdots + q^{2N'+2} + q^{2N'-2} + \cdots + q^2 + 1 \) points, and the largest dimensional spaces contained in a non-singular hyperbolic quadric of \( \text{PG}(N = 2N' + 1, q^2) \) have dimension \( N' \),

- the non-singular elliptic quadrics \( Q^-(N, q^2) \) in \( \text{PG}(N = 2N' + 1, q^2) \) having standard equation \( f(X_0, X_1) = X_0 + \cdots + X_{2N'}X_{2N'+1} = 0 \), where \( f(X_0, X_1) \) is an irreducible quadratic polynomial over \( \mathbb{F}_{q^2} \). These quadrics contain \( (q^{2N'} + 1)/(q^2 - 1) = q^{4N'} + q^{4N'-2} + \cdots + q^{2N'+2} + q^{2N'-2} + \cdots + q^2 + 1 \) points, and the largest dimensional spaces contained in a non-singular elliptic quadric of \( \text{PG}(2N' + 1, q^2) \) have dimension \( N' - 1 \).

The non-singular Hermitian variety \( H(N, q^2) \) in \( \text{PG}(N, q^2) \) has standard equation \( X_0^{q+1} + X_1^{q+1} + \cdots + X_N^{q+1} = 0 \). This variety contains \( \frac{(q^{N+1} - (-1)^N)(q^{N+1} - 1)}{q^2 - 1} \) points, and the largest dimensional spaces contained in a non-singular Hermitian variety of \( \text{PG}(N, q^2) \) have dimension \( \lfloor \frac{N+1}{2} \rfloor \), where \( \lfloor x \rfloor \) denotes the largest integer smaller than or equal to \( x \).

All the quadrics and Hermitian varieties of \( \text{PG}(N, q^2) \), including the non-singular ones, can be described as a quadric/Hermitian variety having an \( s \)-dimensional vertex \( \pi_s \) of singular points, \( s \geq -1 \), and having a non-singular base \( X_{N-s-1} \) in an \((N - s - 1)\)-dimensional space skew to \( \pi_s \). We denote such a quadric or Hermitian variety in \( \text{PG}(N, q^2) \) with vertex \( \pi_s \) and base \( X_{N-s-1} \) by \( \pi_s X_{N-s-1} \). The points \( P \) of the vertex \( \pi_s \) of a quadric or
Hermitian variety $\pi_s X_{N-s-1}$ are called the *singular* points of $\pi_s X_{N-s-1}$, while the points of $\pi_s X_{N-s-1} \setminus \pi_s$ are called *non-singular*. A quadric or Hermitian variety $\pi_s X_{N-s-1}$ is called *singular* when it has a vertex $\pi_s$ of dimension $s \geq 0$.

A line intersecting the quadric or Hermitian variety $X$ in a unique point is called a *tangent line*. A *tangent hyperplane* through a point $P \in X$ is a hyperplane such that all lines through $P$ in this hyperplane are either tangent lines or either contained in $X$. Such a hyperplane is denoted by $T_P(X)$. A non-singular point of a quadric or Hermitian variety $X$ has a unique tangent hyperplane; for a singular point $P$ of $X$, every hyperplane through $P$ is a tangent hyperplane to $X$.

Consider a non-singular quadric or Hermitian variety $X$ in $N$ dimensions, then a non-tangent hyperplane intersects $X$ in a non-singular quadric or non-singular Hermitian variety, and a tangent hyperplane intersects this non-singular quadric or Hermitian variety $X$ in a cone $\pi_0 X'$, with $X'$ a quadric or Hermitian variety in $N - 2$ dimensions of the same type as $X$; see [1, 2] for these properties in the case of Hermitian varieties.

We call the largest dimensional spaces contained in a quadric or Hermitian variety the *generators* of this quadric or Hermitian variety.

The quadrics having the largest size are the union of two distinct hyperplanes of $\text{PG}(N, q^2)$, and have size $2q^{2N-2} + q^{2N-4} + \cdots + q^2 + 1$.

As we mentioned in the introduction, the smallest weight codewords of the code $C_2(X)$ correspond to the quadrics $Q$ having the largest intersections with the Hermitian variety $X$ of $\text{PG}(N, q^2)$. We will show that the largest intersections arise from the quadrics $Q$ that are the union of two distinct hyperplanes of $\text{PG}(N, q^2)$, when $N < O(q^2)$. This proves the conjecture of F.A.B. Edoukou [5] in small dimensions $N$.

Finally, the set of $q + 1$ transversals of three pairwise skew lines in $\text{PG}(3, q)$ is called a *regulus*. Three lines of a regulus define again a regulus, called the *opposite regulus*. A hyperbolic quadric $Q^+(3, q)$ is a pair of complementary reguli.

## 3 Dimension 4

The goal is to look for a bound $W_4$ on the intersection size of an absolutely irreducible quadric $Q$ with the Hermitian variety $X (= H(4, q^2))$, in such a way that if the intersection size of $Q \cap X$ is larger than this bound, then the quadric $Q$ has to be the union of 2 hyperplanes. Therefore we search for the largest intersection size of an absolutely irreducible quadric with $X$. This problem was first investigated by Edoukou [5]. We present here an alternative approach.

### 3.1 The quadric $Q$ is the non-singular quadric $Q(4, q^2)$

**Lemma 3.1** If $Q^+(3, q^2) \cap H(3, q^2)$ contains 3 skew lines, then the intersection consists of $2(q+1)$ lines forming a hyperbolic quadric $Q^+(3, q)$ and $|Q^+(3, q^2) \cap H(3, q^2)| = 2q^3 + q^2 + 1$.

**Proof.** This is [8, Lemma 19.3.1].
This implies that
\[ |Q^+(3, q^2) \cap H(3, q^2)| = (q + 1)(q^2 + 1) + (q^2 - q)(q + 1) = 2q^3 + q^2 + 1. \]

\[ \square \]

**Lemma 3.2** If \( Q^+(3, q^2) \cap H(3, q^2) \) contains at most 2 skew lines, then \(|Q^+(3, q^2) \cap H(3, q^2)| \leq q^3 + 3q^2 - q + 1.\)

**Proof.** (see also [3]) We count according to the lines of one regulus of \( Q^+(3, q^2) \):
\[ |Q^+(3, q^2) \cap H(3, q^2)| \leq 2(q^2 + 1) + (q^2 - 1)(q + 1) \leq q^3 + 3q^2 - q + 1. \]

\[ \square \]

**Lemma 3.3** Let \( L \) be a line of \( Q(4, q^2) \) containing at most \( q \) points of \( Q(4, q^2) \cap H(4, q^2) \), then \(|Q(4, q^2) \cap H(4, q^2)| \leq q^5 + 3q^4 + 2q^2 + q + 1.\)

**Proof.** Let \( P \in L \) with \( P \notin Q(4, q^2) \cap H(4, q^2) \). Take a line \( M \) of \( Q(4, q^2) \) intersecting \( L \) in \( P \). Consider the plane \( \langle L, M \rangle \). Then \( \langle L, M \rangle \) lies in the tangent hyperplane \( T_P(Q(4, q^2)) \) to \( Q(4, q^2) \) and on \( q^2 \) 3-dimensional spaces sharing a hyperbolic quadric \( Q^+(3, q^2) \) with \( Q(4, q^2) \). No \( Q^+(3, q^2) \) can intersect \( H(4, q^2) \) in \( q + 1 \) lines of both reguli, since \( L \) has only \( q \) points of the intersection \( Q(4, q^2) \cap H(4, q^2) \). So \(|Q(4, q^2) \cap H(4, q^2)| \leq q^2(q^3 + 3q^2 - q + 1) + |T_P(Q(4, q^2)) \cap Q(4, q^2) \cap H(4, q^2)|.

If \( P \notin Q(4, q^2) \cap H(4, q^2) \), then \(|T_P(Q(4, q^2)) \cap Q(4, q^2) \cap H(4, q^2)| \leq (q + 1)(q^2 + 1).\) So \(|Q(4, q^2) \cap H(4, q^2)| \leq q^5 + 3q^4 + 2q^2 + q + 1.\)

\[ \square \]

**Remark 3.4** From now on, we assume that every line of \( Q(4, q^2) \) shares at least \( q + 1 \) points with \( H(4, q^2) \). So all lines of \( Q(4, q^2) \) share \( q + 1 \) or \( q^2 + 1 \) points with \( H(4, q^2) \), since a line having more than \( q + 1 \) points of \( H(4, q^2) \) is contained in \( H(4, q^2) \).

**Lemma 3.5** Let \( P \in Q(4, q^2) \cap H(4, q^2) \), then \( T_P(Q(4, q^2)) \neq T_P(H(4, q^2)) \).

**Proof.** Assume that \( T_P(Q(4, q^2)) = T_P(H(4, q^2)) \). Let \( Q(2, q^2) \) be the base of \( T_P(Q(4, q^2)) \cap Q(4, q^2) \) and let \( H(2, q^2) \) be the base of \( T_P(H(4, q^2)) \cap H(4, q^2) \). Take a line \( L \) through \( P \) to a point of \( Q(2, q^2) \setminus H(2, q^2) \). This line \( L \) only shares \( P \) with \( H(4, q^2) \), while it should contain at least \( q + 1 \) points of \( H(4, q^2) \).

\[ \square \]

**Lemma 3.6** Assume that all lines of \( Q(4, q^2) \) share \( q + 1 \) or \( q^2 + 1 \) points with \( H(4, q^2) \), then \(|Q(4, q^2) \cap H(4, q^2)| \leq q^5 + 3q^4 - 4q^2 + 3q + 1.\)
Proof. Let \( P \) be a point of \( Q(4, q^2) \) not lying in the intersection \( Q(4, q^2) \cap H(4, q^2) \), and take 2 lines \( L \) and \( M \) of \( Q(4, q^2) \) through \( P \). All \( q^2 + 1 \) lines of \( Q(4, q^2) \) through \( P \) contain \( q + 1 \) points of \( Q(4, q^2) \cap H(4, q^2) \), so \( |T_P(Q(4, q^2)) \cap Q(4, q^2) \cap H(4, q^2)| = (q + 1)(q^2 + 1) \).

Consider the \( q + 1 \) points \( P_1, \ldots, P_{q+1} \) of \( L \cap Q(4, q^2) \cap H(4, q^2) \). They lie on at most 2 lines contained in \( Q(4, q^2) \cap H(4, q^2) \) (Lemma 3.5). For, such a line through a point \( P_i \) lies in the tangent hyperplanes \( T_P(Q(4, q^2)) \) and \( T_P(H(4, q^2)) \). But these tangent hyperplanes only have a plane in common and this plane has at most two lines through \( P \) contained in \( Q(4, q^2) \cap H(4, q^2) \). So at most 2 of the \( q^2 \) distinct hyperbolic quadrics \( Q^+ (3, q^2) \) of \( Q(4, q^2) \) through \( (L, M) \) can intersect \( H(4, q^2) \) in \( 2(q + 1) \) lines, so we get at most twice \( 2q^3 + q^2 + 1 - 2(q + 1) = 2q^3 + q^2 - 2q - 1 \) extra intersection points. At least \( q^2 - 2 \) times, we get at most \( q^3 + 3q^2 - q + 1 - 2(q + 1) = q^3 + 3q^2 - 3q - 1 \) extra intersection points.

So in total there are at most \( q^3 + 3q^2 - 4q^2 + 3q + 1 \) intersection points. \( \square \)

### 3.2 The quadric cone \( Q = \pi_0 Q^- (3, q^2) \)

**Case I:** \( H(4, q^2) \cap \pi_0 Q^- (3, q^2) \) does not contain a line.

Then the \( q^4 + 1 \) lines through \( \pi_0 \) on \( Q^- (3, q^2) \) have at most \( q + 1 \) points of \( H(4, q^2) \). So

\[
|H(4, q^2) \cap \pi_0 Q^- (3, q^2)| \leq (q + 1)(q^4 + 1) \leq q^5 + q^4 + q + 1. \tag{1}
\]

This upper bound is also determined in [5, Subsection 3.3.1].

**Case II:** \( H(4, q^2) \cap \pi_0 Q^- (3, q^2) \) contains at least one line.

**Lemma 3.7** If \( H(4, q^2) \cap \pi_0 Q^- (3, q^2) \) contains at least one line \( L \), then \( H(4, q^2) \cap \pi_0 Q^- (3, q^2) \) contains at most \( 2(q + 1) \) lines.

**Proof.** Since \( L \subseteq H(4, q^2) \cap \pi_0 Q^- (3, q^2) \), necessarily \( \pi_0 \subseteq H(4, q^2) \cap \pi_0 Q^- (3, q^2) \). Every line \( L' \) of \( H(4, q^2) \cap \pi_0 Q^- (3, q^2) \) passes through \( \pi_0 \), so lies in the tangent hyperplane \( T_{\pi_0}(H(4, q^2)) \). This hyperplane intersects \( \pi_0 Q^- (3, q^2) \) in a cone \( \pi_0 Q(2, q^2) \) if there are at least two lines contained in \( H(4, q^2) \cap \pi_0 Q^- (3, q^2) \). Since \( L \subseteq H(4, q^2) \cap \pi_0 Q^- (3, q^2) \), it defines a point of \( H(2, q^2) \cap Q(2, q^2) \), with \( H(2, q^2) \) and \( Q(2, q^2) \) the basis of the tangent cone \( T_{\pi_0}(H(4, q^2)) \) and of \( \pi_0 Q^- (3, q^2) \cap T_{\pi_0}(H(4, q^2)) \). By Bézout’s theorem, \( |H(2, q^2) \cap Q(2, q^2)| \leq 2(q + 1) \). So at most \( 2(q + 1) \) lines of \( \pi_0 Q^- (3, q^2) \) lie completely on \( H(4, q^2) \). \( \square \)

By the previous lemma, we have:

\[
|H(4, q^2) \cap \pi_0 Q^- (3, q^2)| \leq 2(q + 1)(q^2 + 1) + (q^4 - 2q - 1)(q + 1) \leq q^5 + q^4 + 2q^3 - q + 1. \tag{3}
\]
3.3 The quadric cone $Q = \pi_0 Q^+(3, q^2)$

(see also [5, Section 3.1]) We can describe $\pi_0 Q^+(3, q^2)$ by $q^2 + 1$ planes defined by $\pi_0$ and the lines of one regulus of $Q^+(3, q^2)$. No plane lies completely on $H(4, q^2)$, so every plane shares at most $q^3 + q^2 + 1$ points, of a cone $PH(1, q^2)$, with $H(4, q^2)$. Hence,

$$|H(4, q^2) \cap \pi_0 Q^+(3, q^2)| \leq (q^2 + 1)(q^3 + q^2 + 1) \leq q^5 + q^4 + q^3 + 2q^2 + 1.$$  \hspace{1cm} (5) \hspace{1cm} (6)

3.4 The quadric cone $Q = \pi_1 Q(2, q^2)$

(see also [5, Section 3.1]) Also this quadric can be described by $q^2 + 1$ planes, so as above

$$|H(4, q^2) \cap \pi_1 Q(2, q^2)| \leq q^5 + q^4 + q^3 + 2q^2 + 1.$$

3.5 The quadric cone $Q = \pi_2 Q^-(1, q^2)$

Then we have in fact the intersection of a plane with $H(4, q^2)$. So this intersection size will be smaller than the previous bounds.

3.6 Conclusion

Let $Q$ be a quadric in $\text{PG}(4, q^2)$.

**Theorem 3.8** If $|Q \cap H(4, q^2)| > q^5 + 3q^4 + 2q^2 + q + 1$, then $Q$ is the union of 2 hyperplanes.

**Proof.** From Lemmata 3.3 and 3.6, we know that the intersection size of the non-singular quadric $Q(4, q^2)$ with $H(4, q^2)$ is at most $q^5 + 3q^4 + 2q^2 + q + 1$. For the different intersection sizes of other quadrics with $H(4, q^2)$, (2), (4), and (6) learn us that they are smaller than the previous one. So this proves the theorem. \hfill \Box

From now on, we will denote this bound by $W_4 = q^5 + 3q^4 + 2q^2 + q + 1$.

4 General case

Let $Q$ be a quadric in $\text{PG}(N, q^2)$.

**Theorem 4.1** If $|Q \cap H(N, q^2)| > (q^2 + 2)^{N-4}W_4$, then $Q$ is the union of two hyperplanes, for dimension $N < O(q^2)$.

**Proof.** Part 1. Denote $(q^2 + 2)^{N-4}W_4$ by $W_N$. The bound is valid for $N = 4$ (Theorem 3.8).

Suppose that the lemma holds for dimension $N - 1$. By induction, we show that the bound is true for dimension $N$. 

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Select \((q^2 + 2)^{N-4}W_4\) points \(P\) of \(Q \cap H(N, q^2)\) and count the incidences \((P, H)\), with \(P \in Q \cap H(N, q^2)\) and \(H\) a tangent hyperplane to \(H(N, q^2)\). This gives
\[
((q^2 + 2)^{N-4}W_4)|_{PH(N-2, q^2)}| = |H(N, q^2)|X_N,
\]
with \(X_N\) the average number of those \((q^2 + 2)^{N-4}W_4\) points of \(Q \cap H(N, q^2)\) in a tangent hyperplane to \(H(N, q^2)\).

So some tangent hyperplane \(T_P(H(N, q^2))\), \(P \in H(N, q^2)\), contains at most
\[
X_N \leq \frac{(q^2 + 2)^{N-4}W_4((q^2-1)(-1)^N(q^{N-2} + (-1)^{N-1})q^2 + q^2 - 1)}{q^{N+1}(-1)^N(q^N + (-1)^{N+1})}
\]
of those points.

There remain more than \((q^2 + 2)W_{N-1} - W_{N-1}(1 + \frac{2}{q^2-1})\) points in \(Q \cap H(N, q^2)\), not lying in this tangent hyperplane \(T_P(H(N, q^2))\). Take an arbitrary \(H(N - 3, q^2)\) on the base \(H(N - 2, q^2)\) of \(T_P(H(N, q^2)) \cap H(N, q^2)\). We do not know \(|H(N - 3, q^2) \cap Q \cap H(N, q^2)|\), but we know that the \((q^2 + 1)\) hyperplanes through \(\langle P, H(N - 3, q^2) \rangle\) are \(T_P(H(N, q^2))\), the only tangent hyperplane through \(\langle P, H(N - 3, q^2) \rangle\), and \(q^2\) hyperplanes intersecting \(H(N, q^2)\) in a non-singular Hermitian variety \(H(N - 1, q^2)\).

So one of them, denoted by \(\pi\), contains more than \(\frac{(q^2-1)W_{N-1}}{q^2-1} \geq W_{N-1}\) points of the intersection. Then in this hyperplane \(\pi\), since \(|\pi \cap Q \cap H(N - 1, q^2)| > W_{N-1}\), \(\pi \cap Q\) is the union of two \((N - 2)\)-dimensional spaces.

**Part 2.** The only quadrics containing \((N - 2)\)-dimensional spaces are \(\pi_{N-4}Q^+(3, q^2)\), \(\pi_{N-2}Q^+(1, q^2)\), and \(\pi_{N-3}Q(2, q^2)\).

We want to eliminate the quadrics \(\pi_{N-4}Q^+(3, q^2)\) and \(\pi_{N-3}Q(2, q^2)\); they both can be described as the union of \(q^2 + 1\) \((N - 2)\)-dimensional spaces \(\pi_{N-2}\). The largest intersection of \(\pi_{N-2} \cap H(N, q^2)\) comes from a Hermitian variety which is the union of \(q + 1\) distinct \((N - 3)\)-dimensional spaces sharing an \((N - 4)\)-dimensional space and this has size
\[
(q + 1)q^{2N-6} + q^{2N-8} + \cdots + q^2 + 1 = q^{2N-5} + q^{2N-6} + q^{2N-8} + \cdots + q^2 + 1.
\]

If this would be the case for all these \(q^2 + 1\) distinct \(\pi_{N-2}\), we would get at most an intersection size \((q^2 + 1)(q^{2N-5} + q^{2N-6} + q^{2N-8} + \cdots + q^2 + 1)\) of these quadrics with \(H(N, q^2)\). Since \((q^2 + 2)^{N-4}W_4 > (q^2 + 1)(q^{2N-5} + q^{2N-6} + q^{2N-8} + \cdots + q^2 + 1)\), these quadrics cannot occur.

So \(Q = \pi_{N-2}Q^+(1, q^2)\) which is the union of two hyperplanes. □

**Remark 4.2** The condition \(N < O(q^2)\) arises from the fact that only for \(N < O(q^2)\), the value \((q^2 + 2)^{N-4}W_4\) is smaller than or equal to the intersection size of two hyperplanes with a non-singular Hermitian variety \(H(N, q^2)\). Here, necessarily \(N < q^2/3\).
5 Structure of small weight codewords

We proved in Theorem 4.1 that the small weight codewords of $C_2(X)$, $X$ a non-singular Hermitian variety in $\text{PG}(N,q^2)$, $O(q^2) > N \geq 4$, correspond to the intersections of $X$ with the quadrics consisting of the union of two hyperplanes. We now count the number of codewords obtained via the intersections of $X$ with the union of two hyperplanes.

Consider a quadric $Q$ which is a union of two hyperplanes, then $Q$ defines $q^2 - 1$ codewords of $C_2(X)$, equal to each other up to a non-zero scalar multiple.

It could be that a quadric $Q'$ which also is a union of two hyperplanes, but different from $Q$, defines the same $q^2 - 1$ codewords of $C_2(X)$. However, this can be excluded for $N \geq 4$ in the following way.

If the quadric $Q$, which is the union of the two hyperplanes $\Pi_1$ and $\Pi_2$, and the quadric $Q'$, which is the union of the two hyperplanes $\Pi'_1$ and $\Pi'_2$, define the same codewords of $C_2(X)$, then $(\Pi_1 \cup \Pi_2) \cap X = (\Pi'_1 \cup \Pi'_2) \cap X$. Assume that $\Pi'_1 \neq \Pi_1, \Pi_2$. Then the hyperplane intersection $\Pi'_1 \cap X$ must be contained in the two $(N-2)$-dimensional intersections $\Pi'_1 \cap \Pi_1 \cap X$ and $\Pi'_1 \cap \Pi_2 \cap X$. But the smallest possible intersection size of a hyperplane with $X$ is larger than twice the largest possible intersection size of an $(N-2)$-dimensional space with $X$. So this case does not occur.

Hence, to calculate the number of codewords arising from the union of two hyperplanes, we simply check which unions of two hyperplanes determine codewords of a particular weight (Tables 1, 2 and 3); we then count how many such pairs of hyperplanes there are in $\text{PG}(N,q^2)$, and then we multiply this number by $q^2 - 1$ since a union of two hyperplanes defines $q^2 - 1$ non-zero codewords which are a scalar multiple of each other. For $N \geq 4$, this determines the precise number of codewords of the smallest weights in $C_2(X)$ (Table 3).

We determine the geometrical construction of the smallest weight codewords. They correspond to the intersection of $H(N,q^2)$ with $\pi_{N-2}Q^+(1,q^2)$. The quadric $\pi_{N-2}Q^+(1,q^2)$ consists of two hyperplanes, which we will denote by $\Pi_1$ and $\Pi_2$, through an $(N-2)$-dimensional space $\pi_{N-2}$. We recall that a hyperplane intersects $H(N,q^2)$ either in a non-singular Hermitian variety $H(N-1,q^2)$ or, in case it is a tangent hyperplane, in a cone $\pi_0H(N-2,q^2)$. This $(N-2)$-dimensional space $\pi_{N-2}$ can intersect $H(N,q^2)$ in different ways and this gives us different weight codewords. Starting from the intersection of $\pi_{N-2} \cap H(N,q^2)$, we determine the different intersection sizes and small weights of $C_2(X)$.

For the intersection of $\pi_{N-2}$ with $H(N,q^2)$, there are three possibilities. This intersection is either a non-singular Hermitian variety $H(N-2,q^2)$, a singular Hermitian variety $\pi_0H(N-3,q^2)$ with vertex the point $\pi_0$ and base the non-singular Hermitian variety $H(N-3,q^2)$, or a singular Hermitian variety $LH(N-4,q^2)$ with vertex the line $L$ and base the non-singular Hermitian variety $H(N-4,q^2)$.

In Table 1, we denote the different possibilities for the intersection of $X = H(N,q^2)$ with the union of two hyperplanes $\Pi_1$ and $\Pi_2$. 

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In the second table, we give the intersection sizes: we split the table up into the cases $N$ even and $N$ odd.

### Table 1

|   | $\pi_{N-2} \cap H(N, q^2)$ | $|X \cap (\Pi_1 \cup \Pi_2)|$ |
|---|---------------------------|---------------------------|
| (1) | $H(N - 2, q^2)$ | $2|H(N - 1, q^2)| - |H(N - 2, q^2)|$ |
| (1.2) | $H(N - 2, q^2)$ | $|H(N - 1, q^2)| + |\pi_0 H(N - 2, q^2)| - |H(N - 2, q^2)|$ |
| (1.3) | $H(N - 2, q^2)$ | $2|\pi_0 H(N - 2, q^2)| - |H(N - 2, q^2)|$ |
| (2) | $\pi_0 H(N - 3, q^2)$ | $|H(N - 1, q^2)| + |\pi_0 H(N - 2, q^2)| - |\pi_0 H(N - 3, q^2)|$ |
| (2.2) | $\pi_0 H(N - 3, q^2)$ | $2|\pi_0 H(N - 2, q^2)| - |\pi_0 H(N - 3, q^2)|$ |
| (3) | $LH(N - 4, q^2)$ | $2|\pi_0 H(N - 2, q^2)| - |LH(N - 4, q^2)|$ |

### Table 2 (a): $N$ even

|   | $|X \cap (\Pi_1 \cup \Pi_2)|$ |
|---|---------------------------|
| (1) | $2q^{2N-3} + q^{2N-5} + q^{2N-7} + \ldots + q^{N+1} + q^{N-1} + 2q^{N-2} + q^{N-4} + \ldots + q^2 + 1$ |
| (1.2) | $2q^{2N-3} + q^{2N-5} + q^{2N-7} + \ldots + q^{N+1} + 2q^{N-2} + q^{N-4} + \ldots + q^2 + 1$ |
| (1.3) | $2q^{2N-3} + q^{2N-5} + q^{2N-7} + \ldots + q^{N+1} - q^{N-1} + 2q^{N-2} + q^{N-4} + \ldots + q^2 + 1$ |
| (2) | $2q^{2N-3} + q^{2N-5} + q^{2N-7} + \ldots + q^{N+1} + q^{N-2} + q^{N-4} + \ldots + q^2 + 1$ |
| (2.2) | $2q^{2N-3} + q^{2N-5} + q^{2N-7} + \ldots + q^{N+1} + q^{N-2} + q^{N-4} + \ldots + q^2 + 1$ |
| (3) | $2q^{2N-3} + q^{2N-5} + q^{2N-7} + \ldots + q^{N+1} + q^{N-2} + q^{N-4} + \ldots + q^2 + 1$ |

### Table 2 (b): $N$ odd

|   | $|X \cap (\Pi_1 \cup \Pi_2)|$ |
|---|---------------------------|
| (1) | $2q^{2N-3} + q^{2N-5} + q^{2N-7} + \ldots + q^{N+2} + q^{N} - q^{N-2} + q^{N-4} + \ldots + q^2 + 1$ |
| (1.2) | $2q^{2N-3} + q^{2N-5} + q^{2N-7} + \ldots + q^{N+1} + q^{N-1} - q^{N-2} + q^{N-3} + \ldots + q^2 + 1$ |
| (1.3) | $2q^{2N-3} + q^{2N-5} + q^{2N-7} + \ldots + q^{N} + 2q^{N-1} - q^{N-2} + q^{N-3} + \ldots + q^2 + 1$ |
| (2) | $2q^{2N-3} + q^{2N-5} + q^{2N-7} + \ldots + q^{N} + q^{N+1} + q^{N-1} + q^{N-3} + \ldots + q^2 + 1$ |
| (2.2) | $2q^{2N-3} + q^{2N-5} + q^{2N-7} + \ldots + q^{N} + q^{N} + q^{N-3} + \ldots + q^2 + 1$ |
| (3) | $2q^{2N-3} + q^{2N-5} + q^{2N-7} + \ldots + q^{N} + q^{N-1} + q^{N-3} + \ldots + q^2 + 1$ |

From the intersection sizes listed in Table 2, we now determine the smallest weights for $C_2(X)$ by subtracting the size of the intersection $Q \cap X$ from the length of the code $C_2(X)$. In the same table, we list the number of such codewords. We again split up the table into the cases $N$ even and $N$ odd.
Table 3 (a): \( N \) even, \( N < O(q^2) \)

<table>
<thead>
<tr>
<th>(1.1)</th>
<th>Weight</th>
<th>Number of codewords for ( N \geq 4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( w_1 = q^{N-2}(q^{N+1} - q^{N-1} - q - 1) )</td>
<td>( \frac{(q^{N+1}+1)(q^{N-1})q^{N^2-1} - (q-1)(q^2-q-1)}{2(q+1)} )</td>
</tr>
<tr>
<td>(2.2)</td>
<td>( w_1 + q^{N-2} )</td>
<td>( \frac{(q^{N+1}+1)(q^{N-1})q^{N^2-1} - (q-1)(q^2-q-1)}{2(q^2-q-1)} )</td>
</tr>
<tr>
<td>(1.2)</td>
<td>( w_1 + q^{N-1} )</td>
<td>( \frac{(q^{N+1}+1)(q^{N-1})q^{N^2-1} - (q-1)(q^2-q-1)}{2(q^2-q-1)} )</td>
</tr>
<tr>
<td>(2.1)+(3.1)</td>
<td>( w_1 + q^{N-1} + q^{N-2} )</td>
<td>( \frac{(q^{N+1}+1)(q^{N-1})q^{N^2-1} - (q-1)(q^2-q-1)}{2(q^2-q-1)} )</td>
</tr>
<tr>
<td>(1.3)</td>
<td>( w_1 + 2q^{N-1} )</td>
<td>( \frac{(q^{N+1}+1)(q^{N-1})q^{N^2-1} - (q-1)(q^2-q-1)}{2(q^2-q-1)} )</td>
</tr>
</tbody>
</table>

Table 3 (b): \( N \) odd, \( N < O(q^2) \)

<table>
<thead>
<tr>
<th>(1.3)</th>
<th>Weight</th>
<th>Number of codewords for ( N \geq 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( w_1 = q^{N-2}(q^{N+1} - q^{N-1} - q + 1) )</td>
<td>( \frac{(q^{N+1}+1)(q^{N-1})q^{N^2-1} - (q-1)(q^2-q-1)}{2(q^2-q-1)} )</td>
</tr>
<tr>
<td>(2.1)+(3.1)</td>
<td>( w_1 + q^{N-1} - q^{N-2} )</td>
<td>( \frac{(q^{N+1}+1)(q^{N-1})q^{N^2-1} - (q-1)(q^2-q-1)}{2(q^2-q-1)} )</td>
</tr>
<tr>
<td>(1.2)</td>
<td>( w_1 + q^{N-1} )</td>
<td>( \frac{(q^{N+1}+1)(q^{N-1})q^{N^2-1} - (q-1)(q^2-q-1)}{2(q^2-q-1)} )</td>
</tr>
<tr>
<td>(2.2)</td>
<td>( w_1 + 2q^{N-1} - q^{N-2} )</td>
<td>( \frac{(q^{N+1}+1)(q^{N-1})q^{N^2-1} - (q-1)(q^2-q-1)}{2(q^2-q-1)} )</td>
</tr>
<tr>
<td>(1.1)</td>
<td>( w_1 + 2q^{N-1} )</td>
<td>( \frac{(q^{N+1}+1)(q^{N-1})q^{N^2-1} - (q-1)(q^2-q-1)}{2(q^2-q-1)} )</td>
</tr>
</tbody>
</table>

To conclude this article, we restate the conjecture of Edoukou [5] regarding the smallest weights of the functional codes \( C_2(X) \), \( X \) a non-singular Hermitian variety of \( PG(N,q^2) \); a conjecture which we have proven to be true for small dimensions \( N \).

**Conjecture.** The smallest weights of the functional codes \( C_2(X) \), \( X \) a non-singular Hermitian variety of \( PG(N,q^2) \), arise from the quadrics \( Q \) which are the union of two hyperplanes of \( PG(N,q^2) \).

**References**


Address of the authors:

Department of pure mathematics and computer algebra, Ghent University, Krijgslaan 281-S22, 9000 Ghent, Belgium.
A. Hallez: athallez@cage.ugent.be, L. Storme: ls@cage.ugent.be, http://cage.ugent.be/~ls