The valuations of the near octagon $\mathbb{G}_4$

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Abstract

In [4] it was shown that the dual polar space $DH(2n−1,4)$, $n \geq 2$, has a sub near-2n-gon $\mathbb{G}_n$ with a large automorphism group. In this paper, we classify the valuations of the near octagon $\mathbb{G}_4$. We show that each such valuation is either classical, the extension of a non-classical valuation of a $\mathbb{G}_3$-hex or is associated with a valuation of Fano-type of an $\mathbb{H}_3$-hex. In order to describe the latter type of valuation we must study the structure of $\mathbb{G}_4$ with respect to an $\mathbb{H}_3$-hex. This study also allows us to construct new hyperplanes of $\mathbb{G}_4$. We also show that each valuation of $\mathbb{G}_4$ is induced by a (classical) valuation of the dual polar space $DH(7,4)$.

Keywords: near polygon, generalized quadrangle, dual polar space, valuation, hyperplane.

MSC2000: 51A50, 51E12, 05B25

1 Introduction

1.1 Basic definitions

Let $\mathcal{S}$ be a dense near 2n-gon, i.e. $\mathcal{S}$ is a partial linear space which satisfies the following properties:

(i) For every point $p$ and every line $L$, there exists a unique point $\pi_L(p)$ on $L$ nearest to $p$. Here, distances $d(\cdot,\cdot)$ are measured in the collinearity graph of $\mathcal{S}$.

(ii) Every line of $\mathcal{S}$ is incident with at least three points.

(iii) Every two points of $\mathcal{S}$ at distance 2 from each other have at least two common neighbours.
(iv) The maximal distance between two points of $S$ is equal to $n$.
A dense near 0-gon is a point, a dense near 2-gon is a line and a dense near quadrange is a generalized quadrange (Payne and Thas [17]).

For every point $y$ of $S$ and every non-empty set $X$ of points, we define $d(y, X) := \min \{d(y, x) \mid x \in X\}$. If $X$ is a non-empty set of points of $S$, then for every $i \in \mathbb{N}$, $\Gamma_i(X)$ denotes the set of points $y$ of $S$ at distance $i$ from $X$. If $X$ is a singleton $\{x\}$, we will also write $\Gamma_i(x)$ instead of $\Gamma_i(X)$.

One of the following two cases occurs for two lines $K$ and $L$ of $S$ (see e.g. [5, Theorem 1.3]): (i) there exist unique points $k^* \in K$ and $l^* \in L$ such that $d(k, l) = d(k, k^*) + d(k^*, l^*) + d(l^*, l)$ for all $k \in K$ and $l \in L$; (ii) the map $K \rightarrow L; x \mapsto \pi_L(x)$ is a bijection and its inverse is equal to the map $L \rightarrow K; y \mapsto \pi_K(y)$. If the latter case occurs, then $K$ and $L$ are called parallel.

By Theorem 4 of Brouwer and Wilbrink [2], every two points $x$ and $y$ of $S$ at distance $\delta \in \{0, \ldots, n\}$ from each other are contained in a unique convex subspace $\langle x, y \rangle$ of diameter $\delta$. These convex subspaces are called quads, respectively hexes, if $\delta = 2$, respectively $\delta = 3$. The lines and quads through a given point $x$ of $S$ define a linear space which is called the local space at $x$. If $X_1, X_2, \ldots, X_k$ are non-empty sets of points, then $\langle X_1, X_2, \ldots, X_k \rangle$ denotes the smallest convex subspace containing $X_1 \cup X_2 \cup \cdots \cup X_k$. Clearly, $\langle X_1, X_2, \ldots, X_k \rangle$ is the intersection of all convex subspaces containing $X_1 \cup X_2 \cup \cdots \cup X_k$.

A point $x$ of $S$ is called classical with respect to a convex subspace $F$ of $S$ if there exists a (necessarily unique) point $\pi_F(x)$ in $F$ such that $d(x, y) = d(x, \pi_F(x)) + d(\pi_F(x), y)$ for every point $y$ of $F$. Every point of $\Gamma_i(F)$ is classical with respect to $F$. A convex subspace $F$ of $S$ is called classical (in $S$) if every point of $S$ is classical with respect to $F$. Every line of $S$ is classical. If every quad of $S$ is classical, then $S$ is a so-called dual polar space (Cameron [3]). The near polygon $S$ is then isomorphic to a geometry $\Delta$ whose points and lines are the maximal and next-to-maximal singular subspaces of a given polar space $\Pi$ (natural incidence). A proper convex subspace $F$ of $S$ is called big (in $S$) if every point of $S$ has distance at most 1 from $F$. If $F$ is big, then $F$ is also classical. If $F$ is big and if every line of $S$ is incident with precisely 3 points, then we can define a reflection $R_F$ about $F$ which is an automorphism of $S$: if $x \in F$, then we define $R_F(x) := x$; if $x \not\in F$, then $R_F(x)$ is the unique point on the line $x\pi_F(x)$ different from $x$ and $\pi_F(x)$.

Near polygons were introduced by Shult and Yamushka [18]. We refer to (Chapter 2 of) De Bruyn [5] for more background information on (dense) near polygons.

A function $f$ from the point-set of $S$ to $\mathbb{N}$ is called a valuation of $S$ if it satisfies the following properties:
(V1) $f^{-1}(0) \neq \emptyset$;

(V2) every line $L$ of $S$ contains a unique point $x_L$ such that $f(x) = f(x_L) + 1$ for every point $x$ of $L$ different from $x_L$;

(V3) every point $x$ of $S$ is contained in a (necessarily unique) convex subspace $F_x$ such that the following properties are satisfied for every $y \in F_x$:

(i) $f(y) \leq f(x)$;

(ii) if $z$ is a point collinear with $y$ such that $f(z) = f(y) - 1$, then $z \in F_x$.

Valuations of dense near polygons were introduced in De Bruyn and Van-decasteele [11]. For many classes of dense near polygons, see [10], it can be shown that property (V3) is a consequence of property (V2).

If $f$ is a valuation of $S$, then we denote by $O_f$ the set of points with value 0. A quad $Q$ of $S$ is called special (with respect to $f$) if it contains two distinct points of $O_f$, or equivalently (see [11]), if it intersects $O_f$ in an ovoid of $Q$. We denote by $G_f$ the partial linear space with points the elements of $O_f$ and with lines the special quads (natural incidence).

**Proposition 1.1 (Proposition 2.12 of [11])** Let $S$ be a dense near polygon and let $F = (P',{\mathcal L}',I')$ be a (not necessarily convex) subpolygon of $S$ for which the following holds: (1) $F$ is a dense near polygon; (2) $F$ is a subspace of $S$; (3) if $x$ and $y$ are two points of $F$, then $d_F(x,y) = d_S(x,y)$. Let $f$ denote a valuation of $S$ and put $m := \min\{f(x) | x \in P'\}$. Then the map $f_F : P' \rightarrow \mathbb{N}; x \mapsto f(x) - m$ is a valuation of $F$.

**Definition.** The valuation $f_F$ of $F$ defined in Proposition 1.1 is called the valuation of $F$ induced by $f$.

**Examples.** Let $S = (P,\mathcal L,1)$ be a dense near $2n$-gon, $n \geq 2$.

(1) For every point $x$ of $S$, the map $P \rightarrow \mathbb{N}; y \mapsto d(x,y)$ is a valuation of $S$ which we call a classical valuation.

(2) Suppose $O$ is an ovoid of $S$, i.e. a set of points meeting each line in a unique point. For every point $x$ of $S$, we define $f_O(x) = 0$ if $x \in O$ and $f_O(x) = 1$ otherwise. Then $f_O$ is a valuation of $S$ which we call an ovoidal valuation.

(3) Let $x$ be a point of $S$ and let $O$ be a set of points at distance $n$ from $x$ having a unique point in common with every line at distance $n - 1$ from $x$. For every point $y$ of $S$, we define $f(y) = d(x,y)$ if $d(x,y) \leq n - 1$,
$f(y) = n - 2$ if $y \in O$ and $f(y) = n - 1$ otherwise. Then $f$ is a valuation of $S$ which we call a semi-classical valuation.

(4) Suppose $F = (P', L', I')$ is a convex subspace of $S$ which is classical in $S$. Suppose that $f' : P' \rightarrow N$ is a valuation of $F$. Then the map $f : P \rightarrow N; x \mapsto f(x) := d(x, \pi_F(x)) + f'(\pi_F(x))$ is a valuation of $S$. We call $f$ the extension of $f'$.

In the literature, valuations have been used for the following important applications: (i) classification of dense near polygons ([9], [16]); (ii) constructions of new hyperplanes of dense near polygons, in particular of dual polar spaces ([8], [12]); (iii) classification of certain hyperplanes of dense near polygons, in particular of dual polar spaces ([6]); (iv) study of isometric full embeddings between dense near polygons, in particular between dual polar spaces ([7], [14], [15]).

We will now define two classes of dense near polygons which will be important throughout this paper.

(I) Let $X$ be a set of size $2n + 2$, $n \geq 2$, and let $H_n = (P, L, I)$ be the following point-line geometry:

(i) $P$ is the set of all partitions of $X$ in $n + 1$ subsets of size 2;

(ii) $L$ is the set of all partitions of $X$ in $n - 1$ subsets of size 2 and one subset of size 4;

(iii) a point $p \in P$ is incident with a line $L \in L$ if and only if the partition determined by the point $p$ is a refinement of the partition determined by $L$.

By Brouwer, Cohen, Hall and Wilbrink [1], see also De Bruyn [5, Section 6.2], $H_n$ is a dense near $2n$-gon with three points per line. The near polygon $H_n$ has $\frac{(2n+2)!}{2^{n+1}(n+1)!}$ points and each point is incident with $\binom{n+1}{2}$ lines. Every quad of $H_n$ is isomorphic to either the $(3 \times 3)$-grid or the generalized quadrangle $W(2)$. Every $W(2)$-quad is classical in $H_n$. By De Bruyn [10, Corollary 1.4], a map $f : P \rightarrow N$ is a valuation of $H_n$ if and only if it satisfies properties (V1) and (V2).

The near hexagon $H_3$ will be of interest in this paper. Every $W(2)$-quad of $H_3$ is big. Every local space of $H_3$ is isomorphic to the Fano-plane in which a point has been removed. Hence, every point of $H_3$ is contained in three grid-quads and these grid-quads partition the set of lines through $x$. If $x$ is a point of $H_3$ at distance 2 from a grid-quad $Q$, then $\Gamma_2(x) \cap Q$ is an ovoid of $Q$. Moreover, the three quads through $x$ which meet $Q$ are grids.

(II) Let $H(2n - 1, 4)$, $n \geq 2$, denote the Hermitian variety $X_0^3 + X_1^3 + \cdots + X_{2n-1}^3 = 0$ of $PG(2n - 1, 4)$ (with respect to a given reference system). The
number of nonzero coordinates (with respect to the same reference system) of a point \( p \) of \( \text{PG}(2n - 1, 4) \) is called the \textit{weight} of \( p \). With the Hermitian variety \( H(2n - 1, 4) \), there is associated a dual polar space which is denoted by \( DH(2n - 1, 4) \). The points and lines of \( DH(2n - 1, 4) \) are the maximal and next-to-maximal subspaces of \( H(2n - 1, 4) \) (natural incidence). Let \( \mathbb{G}_n = (\mathcal{P}, \mathcal{L}, I) \) be the following subgeometry of \( DH(2n - 1, 4) \):

(i) \( \mathcal{P} \) is the set of all maximal subspaces of \( H(2n - 1, 4) \) containing \( n \) points with weight 2;
(ii) \( \mathcal{L} \) is the set of all \((n - 2)\)-dimensional subspaces of \( H(2n - 1, 4) \) containing at least \( n - 2 \) points of weight 2;
(iii) incidence is reverse containment.

By De Bruyn [4], see also De Bruyn [5, 6.3], \( \mathbb{G}_n \) is a dense near \( 2n \)-gon with three points on each line and its above-defined embedding in \( DH(2n - 1, 4) \) is isometric, i.e. preserves distances. The near polygon \( \mathbb{G}_n \) has \( \frac{3^n(2n)!}{2^n n!} \) points and each point of \( \mathbb{G}_n \) is contained in precisely \( n(3n - 1) \) lines. Every quad of \( \mathbb{G}_n \) is isomorphic to either the \((3 \times 3)\)-grid, \( W(2) \) or the generalized quadrangle \( Q(5, 2) \). Every \( Q(5, 2) \)-quad is classical in \( \mathbb{G}_n \). By De Bruyn [10, Corollary 1.4], a map \( f : \mathcal{P} \rightarrow \mathbb{N} \) is a valuation of \( \mathbb{G}_n \) if and only if it satisfies properties (V1) and (V2).

1.2 The main result

The near octagon \( \mathbb{G}_4 \) has hexes isomorphic to \( \mathbb{G}_3 \) and \( \mathbb{H}_3 \). Every \( \mathbb{G}_3 \)-hex \( F \) is big in \( \mathbb{G}_4 \) and hence every valuation \( f \) of \( F \) will give rise to a valuation of \( \mathbb{G}_4 \), namely the extension of \( f \). No \( \mathbb{H}_3 \)-hex is big in \( F \). We will later show (Propositions 5.1 and 6.10) that if \( f \) is a valuation of an \( \mathbb{H}_3 \)-hex \( F \) such that \( G_f \) is a Fano-plane, then there exists a unique valuation \( \overline{f} \) of \( \mathbb{G}_4 \) such that \( O_f = O_G \). We will call \( \overline{f} \) a valuation of \textit{Fano-type} of \( \mathbb{G}_4 \). In this paper, we classify all valuations of \( \mathbb{G}_4 \). We will show the following.

\textbf{Theorem 1.2 (Section 6)} If \( f \) is a valuation of \( \mathbb{G}_4 \), then \( f \) is one of the following:

(1) \( f \) is a classical valuation of \( \mathbb{G}_4 \);
(2) \( f \) is the extension of a non-classical valuation of a \( \mathbb{G}_3 \)-hex of \( \mathbb{G}_4 \);
(3) \( f \) is a valuation of Fano-type of \( \mathbb{G}_4 \).

Each of these valuations is induced by a unique (classical) valuation of \( DH(7, 4) \).

Notice that all valuations of \( DH(7, 4) \) are classical by Theorem 6.8 of De Bruyn [5]. In order to describe the valuations of Fano-type of \( \mathbb{G}_4 \) (see Section 5), we must study the structure of \( \mathbb{G}_4 \) with respect to an \( \mathbb{H}_3 \)-hex (Section 4).
This study allows us to construct a class of hyperplanes of $G_4$ (Proposition 4.14).

2 The valuations of the near hexagons $G_3$, $H_3$, $Q(5,2) \times L_3$ and $W(2) \times L_3$

The valuations of the near hexagons $G_3$, $H_3$, $Q(5,2) \times L_3$ and $W(2) \times L_3$ were determined in De Bruyn and Vandecasteele [13].

There are two types of valuations in $G_3$: the classical valuations and the non-classical valuations. In the following lemma, we collect some known facts about non-classical valuations of $G_3$.

Lemma 2.1 ([13]) Suppose $f$ is a non-classical valuation of $G_3$. Then:

(i) $G_f$ is isomorphic to $W(2)$, the linear space obtained from the generalized quadrangle $W(2)$ by adding its ovoids as extra lines.

(ii) $|O_f| = 15$ and every two distinct points of $O_f$ lie at distance 2 from each other.

(iii) Every point with value 1 is contained in a unique special quad.

(iv) Every $Q(5,2)$-quad $Q$ of $G_3$ contains a unique point with value 0. Moreover, $f(y) = d(y, Q \cap O_f)$ for every point $y$ of $Q$.

(v) Every point $x$ of $O_f$ is contained in three special grid-quads and two special $W(2)$-quads. These five quads determine a partition of the set of lines through $x$.

If $f$ is a valuation of $H_3$, then any two distinct points of $O_f$ lie at distance 2 from each other. There are four types of valuations in the near hexagon $H_3$: the classical valuations, the extensions of the ovoidal valuations of the $W(2)$-quads (valuations of extended type), the valuations $f$ for which $G_f$ is a line of size 3 (valuations of grid-type) and the valuations $f$ for which $G_f$ is a Fano-plane (valuations of Fano-type). In the following two lemmas, we collect some known facts about valuations of grid-type and Fano-type.

Lemma 2.2 ([13]) Let $f$ be a valuation of grid-type of $H_3$. Then $O_f$ is an ovoid of a grid-quad $Q$ of $H_3$. If $d(x, O_f) \leq 2$, then $f(x) = d(x, O_f)$. If $d(x, O_f) = 3$, then $f(x) = 1$.

Lemma 2.3 ([13]) Let $f$ be a valuation of Fano-type of $H_3$. Then:

(i) Every $W(2)$-quad $R$ contains a unique point of $O_f$ and $f(y) = d(y, O_f \cap R)$ for every $y \in R$.  

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(ii) Every grid-quad intersects \(O_f\) in either the empty set or an ovoid of the grid-quad. If a grid-quad \(Q\) is disjoint from \(O_f\), then \(Q\) intersects the set of points with value 1 in an ovoid of \(Q\).

(iii) For every \(x \in O_f\), the three grid-quads through \(x\) are special.

(iv) Every point with value 1 is contained in a unique special quad.

Lemma 2.4 Let \(f\) be a valuation of Fano-type of \(\mathbb{H}_3\). Let \(Q\) be a \(W(2)\)-quad of \(\mathbb{H}_3\) and let \(G_2\) and \(G_3\) be two grid-quads of \(\mathbb{H}_3\) such that (i) \(Q\), \(G_2\) and \(G_3\) are mutually disjoint, and (ii) \(\mathcal{R}_Q(G_2) = G_3\). Put \(G_1 := \pi_Q(G_2) = \pi_Q(G_3)\). Then one of the following cases occurs:

(1) There exists precisely one \(i \in \{2, 3\}\) such that \(|G_i \cap O_f| = 3\) and \(|G_{5-i} \cap O_f| = 0\). Moreover, the unique point in \(O_f \cap Q\) is not contained in \(G_1\).

(2) \(|G_2 \cap O_f| = |G_3 \cap O_f| = 0\) and the unique point in \(O_f \cap Q\) is contained in \(G_1\).

Proof. Let \(x^*\) denote the unique point of \(O_f \cap Q\). Recall that \(f(y) = d(y, x^*)\) for every \(y \in Q\). We distinguish two cases.

(1) Suppose \(x^*\) is not contained in \(G_1\). Put \(\Gamma_1(x^*) \cap G_1 = \{x_1, x_2, x_3\}\) and let \(L_i, i \in \{1, 2, 3\}\), denote the unique line through \(x_i\) meeting \(G_2\) and \(G_3\). Since \(x^* \notin G_1\), we have \(d(x^*, G_2) = d(x^*, G_3) = 2\). Hence, each of the three quads through \(x^*\) meeting \(G_2\) \((G_3)\) is a grid. So, \(\langle x^* x_1, L_1 \rangle, \langle x^* x_2, L_2 \rangle\) and \(\langle x^* x_3, L_3 \rangle\) are the three grid-quads through \(x^*\) meeting \(G_2\) \((G_3)\) in a point. By Lemma 2.3(iii) these three grid-quads are special with respect to the valuation \(f\) (recall \(x^* \in O_f\)). Hence, \(|L_1 \cap O_f| = 1\). Choose \(i \in \{2, 3\}\) such that \(G_i \cap L_1 \cap O_f \neq \emptyset\). Then again by Lemma 2.3(iii), \(|G_i \cap O_f| = 3\). Since every point of \(G_1 \setminus \{x_1, x_2, x_3\}\) has value 2, no point of \((G_2 \cup G_3) \setminus (L_1 \cup L_2 \cup L_3)\) belongs to \(O_f\) by property (V2) in the definition of valuation. It follows that \(G_i \cap O_f = (G_i \cap L_1) \cup (G_i \cap L_2) \cup (G_i \cap L_3)\). For every \(j \in \{1, 2, 3\}\), \(L_j \cap G_i\) has value 0 and \(L_j \cap Q\) has value 1. Hence, \(L_j \cap G_{5-i}\) has value 1 by property (V2). Together with \((G_{5-i} \setminus (L_1 \cup L_2 \cup L_3)) \cap O_f = \emptyset\), this implies that \(G_{5-i} \cap O_f = \emptyset\).

(2) Suppose that the unique point \(x^*\) in \(O_f \cap Q\) is contained in \(G_1\). Suppose \(y^*\) is a point of \(O_f \cap G_2\). Then \(d(x^*, y^*) = 2\). Hence, the unique point \(z^*\) of \(G_2\) collinear with \(x^*\) is also collinear with \(y^*\). It follows that \(\langle x^*, y^* \rangle\) and \(G_2\) are two special grid-quads meeting in the line \(y^* z^*\), a contradiction. Hence, \(G_2 \cap O_f = \emptyset\). In a similar way, one shows that \(G_3 \cap O_f = \emptyset\).

The near hexagon \(Q(5, 2) \times \mathbb{L}_3\) is obtained by taking three isomorphic copies of the generalized quadrangle \(Q(5, 2)\) and joining the corresponding points to form lines of size 3. There are two types of valuations in \(Q(5, 2) \times \mathbb{L}_3\)
the classical valuations and the extensions of the ovoidal valuations of the grid-quads.

The near hexagon $W(2) \times \mathbb{L}_3$ is obtained by taking three isomorphic copies of the generalized quadrangle $W(2)$ and joining the corresponding points to form lines of size 3. There are four types of valuations in $W(2) \times \mathbb{L}_3$: the classical valuations, the extensions of the ovoidal valuations of the grid-quads, the extensions of the ovoidal valuations of the $W(2)$-quads and the semi-classical valuations.

3 Properties of the near octagon $G_4$

We start with some properties of the near $2n$-gon $G_n$, $n \geq 3$, whose proofs can be found in the book [5]. Let $U$ denote the set of points of weight 1 and 2 of $\mathrm{PG}(n-1,4)$ (with respect to a certain reference system) and let $L_U$ denote the linear space induced on the set $U$ by the lines of $\mathrm{PG}(n-1,4)$. Then every local space of $G_n$ is isomorphic to $L_U$. Every quad of $G_n$, $n \geq 3$, is isomorphic to either the $(3 \times 3)$-grid, $W(2)$ or $Q(5,2)$. The near polygon $G_n$, $n \geq 3$, has two types of lines:

(i) **SPECIAL LINES**: these are lines which are not contained in a $W(2)$-quad.

(ii) **ORDINARY LINES**: these are lines which are contained in at least one $W(2)$-quad.

There are two possible grid-quads in $G_n$, $n \geq 3$.

(i) **GRID-QUADS OF TYPE I**: these grid-quads contain three ordinary and three special lines; the lines of each type partition the point set of the grid.

(ii) **GRID-QUADS OF TYPE II**: these grid-quads contain six ordinary lines. If $n = 3$, then every grid-quad is of type I. If $n \geq 4$, then both types of grid-quads occur.

The automorphism group of $G_n$, $n \geq 3$, acts transitively on the set of special lines, the set of ordinary lines, the set of $Q(5,2)$-quads, the set of $W(2)$-quads, the set of grid-quads of type I and the set of grid-quads of type II.

In the following lemma, we collect some properties of the near octagon $G_4$.

**Lemma 3.1** (1) The near octagon $G_4$ has 8505 points, each line of $G_4$ contains 3 points and each point of $G_4$ is contained in 22 lines.

(2) Every quad of $G_4$ is isomorphic to either the $(3 \times 3)$-grid, $W(2)$ or $Q(5,2)$. Every $Q(5,2)$-quad is classical in $G_4$. 8
(3) Every hex of $G_4$ is isomorphic to either $G_{3}$, $H_{3}$, $W(2) \times L_{3}$ or $Q(5, 2) \times L_{4}$. Every $G_{3}$-hex is big in $G_{4}$.

(4) If $x$ is a point of $G_{4}$, then every $Q(5, 2)$-quad through $x$ contains precisely two special lines through $x$. Conversely, every two distinct special lines through $x$ are contained in a unique $Q(5, 2)$-quad.

(5) If $x$ is a point of $G_{4}$, then every $G_{3}$-hex through $x$ contains precisely three special lines through $x$. Conversely, every three distinct special lines through $x$ are contained in a unique $G_{3}$-hex.

(6) Every point is contained in 4 special lines, 18 ordinary lines, 36 grid-quads of type I, 27 grid-quads of type II, 36 $W(2)$-quads, 6 $Q(5, 2)$-quads, 4 $G_{3}$-hexes, 18 $Q(5, 2) \times L_{3}$-hexes, 36 $W(2) \times L_{3}$-hexes and 27 $H_{3}$-hexes.

(7) Every special line is contained in 9 grid-quads of type I, 0 grid-quads of type II, 0 $W(2)$-quads, 3 $Q(5, 2)$-quads, 0 $H_{3}$-hexes, 3 $G_{3}$-hexes, 9 $Q(5, 2) \times L_{3}$-hexes and 9 $W(2) \times L_{3}$-hexes.

(8) Every ordinary line is contained in 2 grid-quads of type I, 3 grid-quads of type II, 6 $W(2)$-quads, 1 $Q(5, 2)$-quad, 9 $H_{3}$-hexes, 2 $G_{3}$-hexes, 4 $Q(5, 2) \times L_{3}$-hexes and 6 $W(2) \times L_{3}$-hexes.

(9) Every $W(2)$-quad is contained in precisely 1 $G_{3}$-hex, 1 $W(2) \times L_{3}$-hex, 0 $Q(5, 2) \times L_{3}$-hexes and 3 $H_{3}$-hexes.

(10) Every $Q(5, 2)$-quad is contained in precisely 2 $G_{3}$-hexes, 3 $Q(5, 2) \times L_{3}$-hexes, 0 $W(2) \times L_{3}$-hexes and 0 $H_{3}$-hexes.

(11) Every grid-quad of type I is contained in 1 $G_{3}$-hex, 0 $H_{3}$-hexes, 1 $Q(5, 2) \times L_{3}$-hex and 3 $W(2) \times L_{3}$-hexes.

(12) Every grid-quad of type II is contained in 0 $G_{3}$-hexes, 3 $H_{3}$-hexes, 2 $Q(5, 2) \times L_{3}$-hexes and 0 $W(2) \times L_{3}$-hexes.

(13) Suppose the point $x$ of $G_{4}$ is contained in a $Q(5, 2)$-quad $Q$ and a hex $H$, then $Q \cap H$ is either $Q$ or a line of $Q$.

(14) Suppose the point $x$ of $G_{4}$ is contained in a $G_{3}$-hex $H$ and an $H_{3}$-hex $H'$. Then $H \cap H'$ is a $W(2)$-quad.

Proof. Claims (1), (2), (3) (as well as parts of Claims (4), (5), (6), (7) and (8)) were proved in De Bruyn [5, Section 6.3] in a more general context, namely that of the near 2n-gon $G_{n}$, $n \geq 3$. Claims (4)–(14) readily follow from information on the local spaces which we will now provide.

Let $x$ be an arbitrary point of $G_{4}$. Then the local space of $G_{4}$ at the point $x$ is isomorphic to $L_{U}$ where $U$ is the set of all points of weight 1 or 2 of PG(3, 4) with respect to a certain reference system $(\bar{e}_{1}, \bar{e}_{2}, \bar{e}_{3}, \bar{e}_{4})$ of $V(4, 4)$. A convex subspace $F$ through $x$ corresponds to a certain subspace of $L_{U}$ and hence to a certain set $X_{F}$ of points of PG(3, 4). If $F_{1}$ and $F_{2}$ are two convex subspaces through $x$, then $F_{1} \subset F_{2}$ if and only if $X_{F_{1}} \subset X_{F_{2}}$. We discuss all the possibilities for the lines, quads and hexes.
(i) If $F$ is a special line, then $X_F = \{⟨\bar{e}_i⟩\}$ for some $i \in \{1, 2, 3, 4\}$.
(ii) If $F$ is an ordinary line, then $X_F = \{⟨\bar{e}_i + \lambda \bar{e}_j⟩\}$ for two distinct $i, j \in \{1, 2, 3, 4\}$ and a $\lambda \in \mathbb{F}_4^* := \mathbb{F}_4 \setminus \{0\}$.
(iii) If $F$ is a $Q(5, 2)$-quad, then $X_F = \{⟨\bar{e}_j⟩, ⟨\bar{e}_j + \lambda \bar{e}_j⟩| \lambda \in \mathbb{F}_4\}$ for two distinct $i, j \in \{1, 2, 3, 4\}$.
(iv) If $F$ is a $W(2)$-quad, then $X_F = \{⟨\bar{e}_i + \lambda \bar{e}_j⟩, ⟨\bar{e}_i + \mu \bar{e}_k⟩, ⟨\lambda \bar{e}_j + \mu \bar{e}_k⟩\}$ for three mutually distinct $i, j, k \in \{1, 2, 3, 4\}$ and some $\lambda, \mu \in \mathbb{F}_4^*$.
(v) If $F$ is a grid-quad of type I, then $X_F = \{⟨\bar{e}_i⟩, ⟨\bar{e}_j + \lambda \bar{e}_k⟩\}$ for three mutually distinct $i, j, k \in \{1, 2, 3, 4\}$ and some $\lambda \in \mathbb{F}_4^*$.
(vi) If $F$ is a grid-quad of type II, then $X_F = \{⟨\bar{e}_i + \lambda \bar{e}_j⟩, ⟨\bar{e}_k + \mu \bar{e}_l⟩\}$ for some $\lambda, \mu \in \mathbb{F}_4^*$ and some $i, j, k, l$ satisfying $\{i, j, k, l\} = \{1, 2, 3, 4\}$.
(vii) If $F$ is a $G_3$-hex, then $X_F = ⟨\bar{e}_i, \bar{e}_j, \bar{e}_k⟩ \setminus U$ for three mutually distinct $i, j, k \in \{1, 2, 3, 4\}$.
(viii) If $F$ is an $H_3$-hex, then $X_F = \alpha \cap U$ where $\alpha$ is a plane of PG(3, 4) disjoint from $\{⟨\bar{e}_1⟩, ⟨\bar{e}_2⟩, ⟨\bar{e}_3⟩, ⟨\bar{e}_4⟩\}$. So, $|X_F| = 6$ and $X_F$ contains a unique point of each of the lines $⟨\bar{e}_i, \bar{e}_j⟩$, $i, j \in \{1, 2, 3, 4\}$ with $i \neq j$.
(ix) If $F \cong Q(5, 2) \times \mathbb{L}_3$, then $X_F = \{⟨\bar{e}_i + \lambda \bar{e}_j⟩\} \cup \{⟨\bar{e}_i⟩, ⟨\bar{e}_k + \mu \bar{e}_l⟩| \lambda \in \mathbb{F}_4\}$ for some $\lambda \in \mathbb{F}_4^*$ and some $i, j, k, l$ satisfying $\{i, j, k, l\} = \{1, 2, 3, 4\}$.
(x) If $F \cong W(2) \times \mathbb{L}_3$, then $X_F = \{⟨\bar{e}_i + \lambda \bar{e}_j⟩, ⟨\bar{e}_i + \mu \bar{e}_k⟩, ⟨\lambda \bar{e}_j + \mu \bar{e}_k⟩, ⟨\bar{e}_i⟩\}$ for some $\lambda, \mu \in \mathbb{F}_4^*$ and some $i, j, k, l$ satisfying $\{i, j, k, l\} = \{1, 2, 3, 4\}$. 

4 Structure of $G_4$ with respect to an $H_3$-hex

In this section, $H$ denotes a given $H_3$-hex of $G_4$.

Lemma 4.1 Let $x \in \Gamma_2(H)$ and $Q$ a quad of $H$ such that $\Gamma_2(x) \cap Q$ is an ovoid of $Q$. Then:

(1) $⟨x, Q⟩$ is a hex of $G_4$;
(2) if $Q$ is a $W(2)$-quad, then $⟨x, Q⟩ \cong G_3$;
(3) if $Q$ is a grid-quad, then $Q$ is a grid-quad of type II and $⟨x, Q⟩ \cong H_3$.

Proof. (1) Let $x_1$ and $x_2$ be two distinct points of $\Gamma_2(x) \cap Q$ and let $x_3$ be a common neighbour of $x_1$ and $x_2$. Then $x_3 \in Q \setminus \Gamma_2(x)$ has distance 3 from $x$ and $⟨x, x_3⟩$ is a hex. Now, $x_1$ and $x_2$ are two points on a geodesic path from $x_3$ to $x$. Hence, $⟨x, x_1, x_2⟩ \subseteq ⟨x, x_3⟩$. On the other hand, since $x_3$ is a common neighbour of $x_1$ and $x_2$, we also have $⟨x, x_3⟩ \subseteq ⟨x, x_1, x_2⟩$. Hence, $⟨x, x_1, x_2⟩ = ⟨x, x_3⟩$. Since $x_1$ and $x_2$ are two points of $Q$ at distance 2 from each other, $Q = ⟨x, x_2⟩$. It follows that $⟨x, Q⟩ = ⟨x, x_1, x_2⟩ = ⟨x, x_3⟩$ is a hex.
Among the near hexagons which can occur as hex in $G_3$, only $G_3$ has non-big $W(2)$-quads (recall Lemma 3.1(11)). It follows that $\langle x, Q \rangle \cong G_3$.

(3) The grid-quad $Q$ is contained in the $H_3$-hex $H$. Hence, by Lemma 3.1(11), $Q$ is a grid-quad of type II. Since $x \in \Gamma_2(Q)$, the grid-quad $Q$ of type II is not big in the hex $\langle x, Q \rangle$. Among the near hexagons which can occur as hex in $G_4$, only $G_3$ and $H_3$ have non-big grid-quads. By Lemma 3.1(12), a $G_3$-hex cannot contain grid-quads of type II. Hence, $\langle x, Q \rangle \cong H_3$. □

Remark. If $(x, Q)$ is a point-quad pair of a dense near hexagon such that $d(x, Q) = 2$, then $\Gamma_2(x) \cap Q$ is an ovoid of $Q$ since every line of $Q$ contains a unique point nearest to (and hence at distance 2 from) $x$.

**Proposition 4.2** It holds that $|H| = 105$, $|\Gamma_1(H)| = 3360$, $|\Gamma_2(H)| = 5040$ and $|\Gamma_i(H)| = 0$ for every $i \geq 3$. If $x \in \Gamma_2(H)$, then there are two possibilities:

(a) $\Gamma_2(x) \cap H$ is an ovoid of a $W(2)$-quad $Q$ of $H$ and $\langle x, Q \rangle \cong G_3$;

(b) $\Gamma_2(x) \cap H$ is an ovoid of a grid-quad of type II of $H$ and $\langle x, Q \rangle \cong H_3$.

**Proof.** Suppose $y \in \Gamma_i(H)$ with $i \geq 3$. For every line $L$ of $H$, we have $d(y, L) \leq 3$ since $L$ contains a unique point nearest to $y$. Hence $i = 3$ and $|\Gamma_3(y) \cap L| = 1$ for every line $L$ of $H$. It follows that $\Gamma_3(y) \cap H$ is an ovoid of $H$. But this is impossible since $H$ has no ovoids by [13, Lemma 5.5]. Hence, $|\Gamma_i(H)| = 0$ for every $i \geq 3$. Clearly, $|H| = 105$, $|\Gamma_1(H)| = |H| \cdot (22 - 6) \cdot 2 = 3360$ and $|\Gamma_2(H)| = 8505 - |H| - |\Gamma_1(H)| = 5040$.

Suppose $x \in \Gamma_2(H)$. Applying Proposition 1.1 to the classical valuation $f$ of $G_4$ with $O_f = \{x\}$, we find that the map $g : H \rightarrow \mathbb{N}; y \mapsto d(x, y) - 2$ is a valuation of $H$. The valuation $g$ is not classical since each of its values is at most 2. (A classical valuation of a dense near hexagon has maximal value equal to 3.) By Section 2, there are three possibilities:

(a) $O_g = \Gamma_2(x) \cap H$ is an ovoid in a $W(2)$-quad $Q$ of $H$;

(b) $O_g = \Gamma_2(x) \cap H$ is an ovoid in a grid-quad $Q$ of $H$;

(c) $O_g = \Gamma_2(x) \cap H$ is a set of 7 points and $G_g$ is a Fano-plane.

If case (a) occurs, then $\langle x, Q \rangle \cong G_3$ by Lemma 4.1(2). If case (b) occurs, then $Q$ is a grid-quad of type II and $\langle x, Q \rangle \cong H_3$ by Lemma 4.1(3).

We will now prove that case (c) cannot occur. Suppose the contrary. Let $u$ denote an arbitrary point of $O_g$ and let $Q_1$, $Q_2$ and $Q_3$ denote the
three grid-quads of $H$ through $u$. These grid-quads are special with respect to $g$ by Lemma 2.3(iii). Hence, $\Gamma_2(x) \cap Q_1$ is an ovoid of $Q_1$ for every $i \in \{1, 2, 3\}$. By Lemma 4.1(3), the grid-quads $Q_1, Q_2$ and $Q_3$ have type II and $\langle x, Q_1 \rangle \cong \langle x, Q_2 \rangle \cong \langle x, Q_3 \rangle \cong H$. In the near hexagon $\langle x, Q_1 \rangle \cong H$, the quad $\langle x, u \rangle$ is one of the three quads through $x$ which meet $Q_1$. It follows that $\langle x, Q \rangle$ is a grid-quad. By Lemma 3.1(11), $\langle x, u \rangle$ is a grid-quad of type II. By Lemma 3.1(7), every line of $\langle x, u \rangle$ is an ordinary line. Let $L$ be one of the two (ordinary) lines of $\langle x, u \rangle$ through $u$. By Lemma 3.1(8), $L$ is contained in a unique $Q(5, 2)$-quad $Q$. By Lemma 3.1(13), $Q \cap H$ is a line $L'$. Since $Q_1, Q_2$ and $Q_3$ determine a partition of the lines of $H$ through $u$, we have $L' \subseteq Q_i$ for precisely one $i \in \{1, 2, 3\}$. Now, the $H_3$-hex $\langle x, Q_i \rangle$ contains $L'$ and $L \subseteq \langle x, u \rangle$. So, the $Q(5, 2)$-quad $Q = \langle L, L' \rangle$ would be contained in the $H_3$-hex $\langle x, Q_i \rangle$, clearly a contradiction, since $H_3$ has only grid-quads and $W(2)$-quads.\[\Box\]

**Definition.** A point $x$ of $\Gamma_2(H)$ is said to be of type (a), respectively (b), if case (a), respectively case (b), of Proposition 4.2 occurs.

**Lemma 4.3** Let $H'$ be a hex meeting $H$ in a quad $Q$. Then $\Gamma_2(H) \cap H' = \Gamma_2(Q) \cap H'$.

**Proof.** Suppose $x \in \Gamma_2(H) \cap H'$. Then $x$ has distance at least 2 from $Q$. Since $x$ and $Q$ are contained in $H'$, every point of $Q$ has distance at most 3 from $x$. Hence, for every line $L$ of $Q$, $d(x, L) \leq 2$ since $L$ contains a unique point nearest to $x$. It follows that $x \in \Gamma_2(Q) \cap H'$.

Conversely, suppose that $x \in \Gamma_2(Q) \cap H'$. Then $x \not\in H$ since $H \cap H' = Q$. Suppose $x \in \Gamma_1(H)$. Then $x$ is classical with respect to $H$ and $d(x, y) = 1 + d(\pi_H(x), y)$ for every point $y \in H$. It follows that the point $\pi_H(x)$ is collinear with every point of the ovoid $\Gamma_2(x) \cap Q$ of $Q$. This implies that $\pi_H(x) \in Q$. But this is in contradiction with $\pi_H(x) \sim x \in \Gamma_2(Q)$. It follows that $x \in \Gamma_2(H) \cap H'$.\[\Box\]

**Lemma 4.4** In $\Gamma_2(H)$, there are 3360 points of type (a) and 1680 points of type (b).

**Proof.** In a given $G_3$-hex, there are 120 points at distance 2 from any of its $W(2)$-quads. There are 28 $W(2)$-quads in $H$ and each such quad is contained in a unique $G_3$-hex by Lemma 3.1(9). Lemma 4.3 now implies that the total number of points of type (a) in $\Gamma_2(H)$ is equal to $28 \cdot 1 \cdot 120 = 3360$.

In a given $H_3$-hex, there are 24 points at distance 2 from any of its grid-quads. Now, there are 35 grid-quads (of type II) in $H$ and each of these grid-quads is contained in precisely 2 $H_3$-hexes distinct from $H$ (see Lemma
Lemma 4.3 now implies that the number of points of type (b) in $\Gamma_2(H)$ is equal to $35 \cdot 2 \cdot 24 = 1680$.

(Check: The total number of points of $\Gamma_2(H)$ is indeed equal to $3360 + 1680 = 5040$ as shown in Proposition 4.2). $
$
Lemma 4.5 (Chapter 7 of [5]) Suppose one of the following cases occurs:

(i) $Q$ is a grid-quad of $S \cong \mathbb{H}_3$; (ii) $Q$ is a $W(2)$-quad of $S \cong \mathbb{G}_3$. Let $x$ be a point of $S$ at distance 2 from $Q$. Then every line of $S$ through $x$ has a unique point in common with $\Gamma_1(Q)$.

Let $S$ denote the set of lines of $\mathbb{G}_4$ contained in $\Gamma_2(H)$.

Lemma 4.6 Let $x$ be a point of $\Gamma_2(H)$ and let $Q$ be the quad $\langle \Gamma_2(x) \cap H \rangle$. Then the lines through $x$ contained in $S$ are precisely the lines through $x$ not contained in the hex $\langle x, Q \rangle$. If $x$ has type (a), then precisely 10 lines through $x$ are contained in $S$. If $x$ has type (b), then precisely 16 lines through $x$ are contained in $S$.

Proof. If $x$ is a point of type (a), then $Q \cong W(2)$ and $\langle x, Q \rangle \cong \mathbb{G}_3$. If $x$ is a point of type (b), then $Q$ is a grid-quad and $\langle x, Q \rangle \cong \mathbb{H}_3$. By Lemmas 4.3 and 4.5, every line through $x$ contained in $\langle x, Q \rangle$ contains a point of $\Gamma_1(H)$. Conversely, suppose that $L$ is a line through $x$ containing a point $y \in \Gamma_1(H)$. Then $y$ is classical with respect to $H$ and the point $\pi_H(y)$ lies at distance 2 from $x$. Hence, $\pi_H(y) \in Q$ and $L \subseteq \langle x, \pi_H(y) \rangle \subseteq \langle x, Q \rangle$.

So, the number of lines through $x$ contained in $S$ is equal to the number of lines through $x$ not contained in the hex $\langle x, Q \rangle$. If $x$ is a point of type (a), then $x$ is contained in $22 - 12 = 10$ lines of $S$. If $x$ is a point of type (b), then $x$ is contained in $22 - 6 = 16$ lines of $S$. $
$
From Lemmas 4.4 and 4.6, we readily obtain:

Corollary 4.7 $|S| = \frac{1}{6}[3360 \cdot 10 + 1680 \cdot 16] = 20160$.

Lemma 4.8 Let $L = \{x_1, x_2, x_3\}$ be a line of $S$. For every $i \in \{1, 2, 3\}$, put $Q_i := \langle \Gamma_2(x_i) \cap H \rangle$ and $H_i := \langle x_i, Q_i \rangle$. Then $H_1, H_2$ and $H_3$ are mutually disjoint hexes.

Proof. By symmetry, it suffices to show that $H_1 \cap H_2 = \emptyset$. Suppose to the contrary that $u$ is a point of $H_1 \cap H_2$. Every point on a shortest path between $u \in H_1 \cap H_2$ and $x_1 \in H_1$ belongs to $H_1$. If $x_1 \notin H_2$, then since $x_1$ is classical with respect to $H_2$, the point $x_2 = \pi_{H_2}(x_1)$ lies on such a shortest path. Hence, $x_1 \in H_2$ or $x_2 \in H_1$. So, the line $x_1x_2$ is contained in $H_1$ or $H_2$. 

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Lemma 4.5 then implies that $L$ contains a point of $\Gamma_1(H)$. This contradicts the fact that $L \in S$. \hfill \blacksquare

**Lemma 4.9** Let $L = \{x_1, x_2, x_3\}$ be a line of $S$, put $Q_i = \langle \Gamma_2(x_i) \cap H \rangle$ and $H_i = \langle x_i, Q_i \rangle$. If $x_1$ is of type (a), then $x_2$ and $x_3$ have the same type and $\mathcal{R}_{H_1}(H_2) = H_3$.

**Proof.** By Proposition 4.2, $Q_1 \cong W(2)$ and $H_1 \cong \mathcal{G}_3$. So, $H_1$ is big in $\mathcal{G}_4$. By Lemma 4.8, $H_1$ and $H_2$ are mutually disjoint. Let $H'_2$ be the reflection of $H_2$ about $H_1$ (in the near octagon $\mathcal{G}_4$) and let $Q'_2$ denote the reflection of $Q_2$ about $Q_1$ (in the near hexagon $H$). Then $Q'_3 \cong Q_2$, $H'_3 \cong H_2$ and $Q'_3 \subset H_3$. Since $x_2$ is a point of $H_2$ at distance 2 from the quad $Q_2$ of $H_2$, $x_3 = \mathcal{R}_{H_1}(x_2)$ is a point of $H'_3 = \mathcal{R}_{H_1}(H_2)$ at distance 2 from $Q'_3 = \mathcal{R}_{H_1}(Q_2)$. So, $\Gamma_2(x_3) \cap Q'_3$ is an ovoid of $Q'_3$. This implies that $Q_3 = Q'_3$ and $H_3 = H'_3$. Since $H'_3 \cong H_2$, $x_3$ is of the same type as $x_2$. \hfill \blacksquare

**Lemma 4.10** Every point $x$ of type (a) of $\Gamma_2(H)$ is contained in precisely 6 lines of $S$ which only contain points of type (a).

**Proof.** Put $Q := \langle \Gamma_2(x) \cap H \rangle$.

Let $\{x, x_1, x_2\}$ be a line of $S$ through $x$ which only contains points of type (a) and let $Q_i = \langle \Gamma_2(x_i) \cap H \rangle$, $i \in \{1, 2\}$. Then by Lemmas 4.8 and 4.9, the $W(2)$-quads $Q$, $Q_1$ and $Q_2$ are mutually disjoint and $Q_2$ is the reflection of $Q_1$ about $Q$ (in the near hexagon $H$).

Let $Q'$ be a $W(2)$-quad of $H$ disjoint from $Q$ and let $H'$ denote the unique $\mathcal{G}_3$-hex through $Q'$ (recall Lemma 3.1(9)). We prove that $\langle x, Q \rangle \cap H' = \emptyset$. Suppose to the contrary that $\langle x, Q \rangle \cap H'$ contains a point $u$. If $u \in H$, then $u \in Q = \langle x, Q \rangle \cap H$ and $u \in Q' = H' \cap H$, a contradiction. If $u \in \Gamma_1(H)$, then $u \notin \Gamma_2(Q) \cup \Gamma_2(Q')$ by Lemma 4.3 and hence $\pi_H(u) \in Q \cap Q'$, a contradiction. If $u \in \Gamma_2(H)$, then $u \in \Gamma_2(Q) \cap \Gamma_2(Q')$ and hence $\Gamma_2(u) \cap H \subseteq Q \cap Q'$, again a contradiction. So, the big $\mathcal{G}_3$-hexes $\langle x, Q \rangle$ and $H'$ are disjoint. Hence, the line $x\pi_{H'}(x)$ belongs to $S$ by Lemma 4.6. Since $x$ and $\pi_{H'}(x)$ are points of type (a), also the third point of $x\pi_{H'}(x)$ has type (a) by Lemma 4.9. So, the $W(2)$-quad $Q'$ determines a line of $S$ through $x$ which only consists of points of type (a). If we denote by $Q'' \cong W(2)$ the reflection of $Q'$ about $Q$ (in $H$) and by $H''$ the unique $\mathcal{G}_3$-hex through $Q''$, then $H'' = \mathcal{R}_{H'}(\langle x, Q \rangle)$ and $x\pi_{H'}(x) = x\pi_{H''}(x)$. So, the $W(2)$-quads $Q'$ and $Q''$ determine the same line of $S$ through $x$.

Since there are 12 $W(2)$-quads in $H$ disjoint with $Q$, it follows from the above discussion that there are $\frac{12 \times 6}{2} = 18$ lines of $S$ through $x$ containing only points of type (a). \hfill \blacksquare

From Lemmas 4.4 and 4.10, we readily obtain:
Corollary 4.11 There are \( \frac{3360 \cdot 6}{3} = 6720 \) lines of \( S \) containing precisely three points of type \((a)\).

Lemma 4.12 There are 13440 lines of \( S \) containing one point of type \((a)\) and two points of type \((b)\).

Proof. Let \( x \) be one of the 3360 points of type \((a)\). By Lemmas 4.6, 4.9 and 4.10, \( x \) is contained in \( 10 - 6 = 4 \) lines of \( S \) which contain a unique point of type \((a)\). Hence, the required number is equal to \( 3360 \cdot 4 = 13440 \). \( \blacksquare \)

By Corollary 4.7, Corollary 4.11 and Lemma 4.12, we obtain:

Corollary 4.13 There are two types of lines in \( S \):

(1) Lines of \( S \) only containing points of type \((a)\).

(2) Lines of \( S \) containing a unique point of type \((a)\) and two points of type \((b)\).

Recall that a hyperplane of a point-line geometry is a proper subspace meeting each line.

Proposition 4.14 Let \( X \) denote the set of points of \( G_4 \) consisting of the points of \( H \), the points of \( \Gamma_1(H) \) and the points of type \((a)\) of \( \Gamma_2(H) \). Then \( X \) is a hyperplane of \( G_4 \).

Proof. We need to prove that every line \( L \) containing a point \( x \) of type \((b)\) of \( \Gamma_2(H) \) intersects \( X \) in a unique point. Put \( Q := \langle \Gamma_2(x) \cap H \rangle \). Then \( Q \) is a grid-quad of type II and \( \langle x, Q \rangle \cong \mathbb{H}_3 \).

If \( L \) is not contained in \( \langle x, Q \rangle \), then \( L \in S \) by Lemma 4.6. Corollary 4.13 then implies that \( |L \cap X| = 1 \).

If \( L \) is contained in \( \langle x, Q \rangle \), then \( L \) contains a unique point \( y \) of \( \Gamma_1(Q) \) by Lemma 4.5. Let \( z \in \Gamma_2(Q) \) denote the third point on the line \( L \). By Lemma 4.3 applied to the hex \( H' = \langle x, Q \rangle \), \( z \in \Gamma_2(H) \). Since \( z \in \Gamma_2(Q) \) and \( Q \) are contained in the hex \( \langle x, Q \rangle \), \( \Gamma_2(z) \cap Q \) is an ovoid of \( Q \). It follows that \( \Gamma_2(z) \cap H = \Gamma_2(z) \cap Q \). Since \( Q \) is a grid, \( z \) is of point of type \((b)\) and \( y \) is the unique point of \( X \) contained in \( L \). \( \blacksquare \)

5 A new class of valuations of \( G_4 \)

Let \( H \) denote a hex of \( G_4 \) isomorphic to \( \mathbb{H}_3 \) and let \( f \) denote a valuation of Fano-type of \( H \). Recall that every point of \( \Gamma_1(H) \) is classical with respect to \( H \). Lemma 2.3(i)+(ii) allows us to define the following function \( \overline{f} \) from the point-set of \( G_4 \) to \( \mathbb{N} \):

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(i) If \( x \in H \), then we define \( \overline{f}(x) := f(x) \).
(ii) If \( x \in \Gamma_1(H) \), then we define \( \overline{f}(x) := 1 + f(\pi_H(x)) \).
(iii) If \( x \) is a point of type (a) of \( \Gamma_2(H) \), then \( \overline{f}(x) := d(x,x^*) \), where \( x^* \) is the unique point of \( O_f \) contained in the \( W(2) \)-quad \( \langle \Gamma_2(x) \rangle \cap H \).
(iv) Let \( x \) be a point of type (b) of \( \Gamma_2(H) \) such that \( |O_f \cap Q| = 3 \), where \( Q \) is the unique grid-quad of \( H \) containing \( \Gamma_2(x) \cap H \). Then \( \overline{f}(x) := 2 \) if \( \Gamma_2(x) \cap (O_f \cap Q) \neq \emptyset \) and \( \overline{f}(x) := 1 \) otherwise.
(v) Let \( x \) be a point of type (b) of \( \Gamma_2(H) \) such that \( |O_f \cap Q| = 0 \) where \( Q \) is the unique grid-quad of \( H \) containing \( \Gamma_2(x) \cap H \). Let \( X \) denote the ovoid of \( Q \) consisting of all points with \( f \)-value 1. We define \( \overline{f}(x) := 3 \) if \( \Gamma_2(x) \cap X \neq \emptyset \) and \( \overline{f}(x) := 2 \) otherwise.

**Proposition 5.1** The map \( \overline{f} \) is a valuation of \( \mathbb{G}_4 \).

**Proof.** Recall that a function from the point-set of \( \mathbb{G}_4 \) to \( \mathbb{N} \) is a valuation of \( \mathbb{G}_4 \) if and only if it satisfies properties (V1) and (V2). Clearly, \( \overline{f} \) satisfies property (V1). It remains to show that \( \overline{f} \) also satisfies property (V2). Let \( L \) be an arbitrary line of \( \mathbb{G}_4 \). We can distinguish 6 possibilities by corollary 4.13:

1. \( L \) is contained in \( H \). Then \( L \) satisfies property (V2) with respect to \( \overline{f} \) since \( L \) satisfies property (V2) with respect to \( f \).
2. \( L \) intersects \( H \) in a unique point \( x_L \). Then \( \overline{f}(x) = f(x_L) + 1 = \overline{f}(x_L) + 1 \) for every point \( x \) of \( L \setminus \{x_L\} \). So, \( L \) satisfies property (V2).
3. \( L \subseteq \Gamma_1(H) \). Then \( \pi_H(L) := \{\pi_H(x) \mid x \in L\} \) is a line of \( H \) parallel with \( L \). For every point \( x \) of \( L \), \( \overline{f}(x) = f(\pi_H(x)) + 1 \). Since \( \pi_H(L) \) satisfies property (V2) with respect to \( f \), \( L \) satisfies property (V2) with respect to \( \overline{f} \).
4. \( L \cap \Gamma_1(H) \neq \emptyset \) and \( L \cap \Gamma_2(H) \neq \emptyset \). Let \( x \) denote an arbitrary point of \( L \cap \Gamma_2(H) \) and let \( Q \) denote the unique quad of \( H \) containing \( \Gamma_2(x) \cap H \). Then \( \langle x, Q \rangle \) is a hex containing \( L \).

From the definition of \( \overline{f} \), we see that there exists a constant \( \epsilon \in \{-1,0\} \) such that the map \( u \mapsto \overline{f}(u) + \epsilon \) defines a valuation \( f' \) of \( \langle x, Q \rangle \). If \( x \) is a point of type (a), then \( \epsilon = 0 \) and \( f' \) is a classical valuation of \( \langle x, Q \rangle \cong \mathbb{G}_3 \) by Lemma 2.3(i). If \( x \) is a point of type (b), then \( \epsilon = 0 \) if \( |O_f \cap Q| = 3 \) and \( \epsilon = -1 \) if \( |O_f \cap Q| = 0 \). Moreover, by Lemma 2.2, \( f' \) is a valuation of grid-type of \( \langle x, Q \rangle \cong \mathbb{H}_3 \).

By the previous paragraph, the line \( L \subseteq \langle x, Q \rangle \) satisfies property (V2) with respect to \( \overline{f} \).
5. \( L \subseteq \Gamma_2(H) \) and every point of \( L \) is of type (a). Put \( L = \{x_1, x_2, x_3\} \) and let \( Q_i \), \( i \in \{1,2,3\} \), denote the unique \( W(2) \)-quad of \( H \) containing
$O_i = \Gamma_2(x_i) \cap H$. The set $O_i$ is an ovoid of $Q_i$. Put $H_i := \langle x_i, Q_i \rangle$, $i \in \{1, 2, 3\}$. By Lemmas 4.8 and 4.9, $H_1$, $H_2$ and $H_3$ are three mutually disjoint $G_3$-hexes, $\mathcal{R}_{H_1}(H_2) = H_3$ and $\mathcal{R}_{Q_1}(Q_2) = Q_3$ (reflection about the big $W(2)$-quad $Q_1$ in the $H_3$-hex $H$). So, every line meeting $Q_1$ and $Q_2$ also meets $Q_3$. We have $\mathcal{R}_{H_1}(O_2) = \mathcal{R}_{H_1}(\Gamma_2(x_2) \cap Q_2) = \Gamma_2(x_3) \cap Q_3 = O_3$. In a similar way, one can prove that $O_3 = \mathcal{R}_{H_2}(O_1)$. It follows that $O_1 \cup O_2 \cup O_3$ is the union of 5 lines which meet $Q_1$, $Q_2$ and $Q_3$. Let $u_i^*; i \in \{1, 2, 3\}$, denote the unique point of $Q_i$ with $f$-value 0 (recall Lemma 2.3(i)). Since every two points of $O_f$ lie at distance 2 from each other, $d(u_i^*, u_2^*) = 2$. Since $u_1^*$ is classical with respect to $Q_2$, the unique point $v$ of $Q_2$ collinear with $u_i^*$ is collinear with $u_2^*$. Let $w$ denote the point of the line $u_i^*v$ distinct from $u_i^*$ and $v$. The quad $\langle u_1^*, u_2^*, u_3^* \rangle$ contains the line $u_1^*v$ and hence contains the point $w \in Q_3$. Since the local space of $H$ at the point $w$ is a Fano plane minus a point, the quads $\langle u_1^*, u_2^* \rangle$ and $Q_3$ meet in a line. Since $u_1^*, u_2^* \in O_f$, the quad $\langle u_1^*, u_2^* \rangle$ of $H$ is special with respect to $f$. So, $\langle u_1^*, u_2^* \rangle$ is a grid and the line $\langle u_1^*, u_2^* \rangle \cap Q_3$ contains a unique point of $O_f$ which necessarily coincides with $u_3^*$. The points $u_1^*$, $\pi_{Q_1}(u_2^*)$ and $\pi_{Q_1}(u_3^*)$ of $Q_1$ form a line of $Q_1$ which intersects $O_1$ in a unique point. It follows that $O_1 \cup O_2 \cup O_3$ has a unique point $u^*$ in common with $\{u_1^*, u_2^*, u_3^*\}$. If $i \in \{1, 2, 3\}$ such that $u^* = u_i^*$, then $\overline{f}(x_i) = 2$ and $\overline{f}(x_j) = 3$ for all $j \in \{1, 2, 3\} \setminus \{i\}$. This proves that $L$ satisfies property (V2).

(6) $L \subseteq \Gamma_2(H)$, $L$ contains a unique point $x_1$ of type (a) and two points $x_2$ and $x_3$ of type (b). Let $Q_1$ denote the unique $W(2)$-quad of $H$ containing all points of $\Gamma_2(x_1) \cap H$ and put $H_i := \langle x_i, Q_i \rangle$. Let $G_i$, $i \in \{2, 3\}$, denote the grid-quad of $H$ containing all points of $\Gamma_2(x_i) \cap H$ and put $H_i := \langle x_i, G_i \rangle$. Then $H_1 \cong G_3$ and $H_2 \cong H_3 \cong H_3$. Moreover, by Lemmas 4.8 and 4.9, $H_1$, $H_2$ and $H_3$ are mutually disjoint, $\mathcal{R}_{H_1}(H_2) = H_3$ and $\mathcal{R}_{Q_1}(G_2) = G_3$. Put $G_1 := \pi_{Q_1}(G_2) = \pi_{Q_1}(G_3)$. We have $\mathcal{R}_{H_1}(\Gamma_2(x_2) \cap G_2) = \Gamma_2(x_3) \cap G_3$. Moreover, $\pi_{Q_1}(\Gamma_2(x_2) \cap G_2) = \pi_{Q_1}(\Gamma_2(x_3) \cap G_3) = \Gamma_2(x_1) \cap G_1$ since every line connecting a point of $\Gamma_2(x_2) \cap G_2 \subseteq \Gamma_3(x_1)$ and $\Gamma_2(x_3) \cap G_3 \subseteq \Gamma_3(x_1)$ contains a unique point nearest to $x_1$. We distinguish four possibilities (cf. Lemma 2.4):

(i) $|G_2 \cap O_f| = |G_3 \cap O_f| = 0$, the unique point $x^*$ in $O_f \cap Q_1$ is contained in $G_1$ and $d(x^*, x_1) = 2$. Then the unique line through $x^*$ meeting $G_2$ and $G_3$ intersects $G_2$ and $G_3$ in points with $f$-value 1 belonging respectively to $\Gamma_2(x_2)$ and $\Gamma_2(x_3)$. It follows that $\overline{f}(x_1) = 2$ and $\overline{f}(x_2) = \overline{f}(x_3) = 3$. So, $L$ satisfies property (V2).

(ii) $|G_2 \cap O_f| = |G_3 \cap O_f| = 0$, the unique point $x^*$ in $O_f \cap Q_1$ is contained in $G_1$ and $d(x^*, x_1) = 3$. Hence, the ovoid $\Gamma_2(x_1) \cap G_1$ of $G_1$ contains two points with $f$-value 1 and one point with $f$-value 2 (recall Lemma 2.3(i)). Since each of the three lines meeting $\Gamma_2(x_1) \cap G_1$, $\Gamma_2(x_2) \cap G_2$ and $\Gamma_2(x_3) \cap G_3$
contains a unique point with smallest \( f \)-value, there exists an \( i \in \{2, 3\} \) such that (a) the ovoid \( \Gamma_2(x_i) \cap G_i \) contains two points with \( f \)-value 1 and 1 point with \( f \)-value 1, and (b) the ovoid \( \Gamma_2(x_{5-i}) \cap G_{5-i} \) contains three points with \( f \)-value 2. It follows that \( \overline{f}(x_i) = 3 \), \( \overline{f}(x_i) = 3 \) and \( \overline{f}(x_{5-i}) = 2 \). So, \( L \) satisfies property (V2).

(iii) There exists an \( i \in \{2, 3\} \) such that \( |G_i \cap O_f| = 3 \) and \( |G_{5-i} \cap O_f| = 0 \). Moreover, we assume that \( d(x_1, x^*) = 2 \), where \( x^* \) is the unique point in \( O_f \cap Q_1 \). (Recall \( x^* \not\in G_1 \).) Since \( \{x^*\} \cup \{\Gamma_2(x_1) \cap G_1\} \) is contained in the ovoid \( \Gamma_2(x_1) \cap Q_1 \) of \( Q_1 \), no point of \( \Gamma_2(x_1) \cap G_1 \) is collinear with \( x^* \). So, \( \Gamma_2(x_1) \cap G_1 \) only contains points with \( f \)-value 2 (recall Lemma 2.3(i)). Since every line meeting \( \Gamma_2(x_1) \cap G_1, \Gamma_2(x_2) \cap G_2 \) and \( \Gamma_2(x_3) \cap G_3 \) has a unique point with smallest \( f \)-value and \( G_i \) contains only points with \( f \)-value 0 or 1, \( \Gamma_2(x_1) \cap G_i \) only contains points with \( f \)-value 1 and \( \Gamma_2(x_{5-i}) \cap G_{5-i} \) only contains points with \( f \)-value 2. It follows that \( \overline{f}(x_1) = 2 \), \( \overline{f}(x_i) = 1 \) and \( \overline{f}(x_{5-i}) = 2 \). This proves that \( L \) satisfies property (V2) with respect to \( \overline{f} \).

(iv) There exists an \( i \in \{2, 3\} \) such that \( |G_i \cap O_f| = 3 \) and \( |G_{5-i} \cap O_f| = 0 \). Moreover, we assume that \( d(x_1, x^*) = 3 \) where \( x^* \) is the unique point in \( O_f \cap Q_1 \). (Recall \( x^* \not\in G_1 \).) Then \( \Gamma_2(x_1) \cap G_1 \subseteq \Gamma_2(x_1) \cap Q_1 \) contains at least one point with \( f \)-value 1 (collinear with \( x^* \)). The unique line through each such point meeting \( G_2 \) and \( G_3 \) contains a unique point with smallest \( f \)-value. Hence, \( \Gamma_2(x_1) \cap G_i \) contains at least one point with \( f \)-value 0 and \( \Gamma_2(x_{5-i}) \cap G_{5-i} \) contains at least one point with \( f \)-value 1 (recall that every point of \( G_i \) has \( f \)-value 0 or 1). It follows that \( \overline{f}(x_1) = 3 \), \( \overline{f}(x_i) = 2 \) and \( \overline{f}(x_{5-i}) = 3 \). This proves that \( L \) satisfies property (V2).

\[ \Box \]

The valuation \( \overline{f} \) of \( G_4 \) defined above is called a valuation of Fano-type of \( G_4 \).

6 The classification of the valuations of \( G_4 \)

6.1 Some lemmas

During the classification of the valuations of \( G_4 \), we will need the following three properties which hold for valuations of general near polygons:

\textbf{Lemma 6.1 ([11])} Let \( f \) be a valuation of a dense near 2\( n \)-gon \( S \).

(i) \( f \) is a classical valuation if and only if there exists a point with value \( n \).

(ii) If \( d(x, O_f) \leq 2 \), then \( f(x) = d(x, O_f) \).

(iii) No two distinct special quads intersect in a line.

Now, suppose that \( f \) is a valuation of \( G_4 \).
Lemma 6.2 If $x, y \in O_f$, then $d(x, y)$ is even.

Proof. By Property (V2), $d(x, y) \neq 1$. Suppose $d(x, y) = 3$. Let $H$ denote the unique hex through $x$ and $y$. If $f'$ denotes the valuation of $H$ induced by $f$ (recall Proposition 1.1), then $O_{f'}$ contains two points at distance 3 from each other. This is impossible since none of the near hexagons $G_3$, $W(2) \times L_3$, $Q(5, 2) \times L_3$, $H_3$ has such valuations.

Lemma 6.3 If there exists a $G_3$-hex $H$ such that $|H \cap O_f| = 15$, then $O_f = H \cap O_f$.

Proof. Since $|H \cap O_f| = 15$, the valuation $f'$ of $H \cong G_3$ induced by $f$ is non-classical. Suppose $x \in O_f \setminus H$. Since $H$ is big in $G_4$, $x$ is classical with respect to $H$. The point $\pi_H(x)$ has value 1 and hence is contained in a unique quad $Q$ of $H$ which is special with respect to $f'$ (recall Lemma 2.1(iii)). If $y$ is a point of $Q \cap O_f$ at distance 2 from $\pi_H(x)$, then $d(x, y) = 3$, contradicting Lemma 6.2.

Lemma 6.4 If $x$ and $y$ are two different points of $O_f$, then $d(x, y) = 2$.

Proof. Suppose the contrary. Then $d(x, y) = 4$ by Lemma 6.2. Let $H$ denote an arbitrary $G_3$-hex through $x$. Since $y \in O_f \setminus H$, the valuation induced in $H$ is classical by Lemma 6.3 (recall that $|O_g| = 15$ for every non-classical valuation $g$ of $G_3$). Hence, $f(\pi_H(y)) = d(x, \pi_H(y)) = 3$. On the other hand, since $d(\pi_H(y), y) = 1$ and $f(y) = 0$, it holds that $f(\pi_H(y)) = 1$, a contradiction.

Lemma 6.5 One of the following cases occurs:

(A) $|O_f| = 1$;
(B) There exists a unique $G_3$-hex $H$ such that $O_f \subseteq H$ and $|H \cap O_f| = 15$;
(C) $|O_f| \geq 2$ and every special quad is a grid-quad of type II.

Proof. Suppose $|O_f| \geq 2$ and let $x_1$ and $x_2$ denote two distinct points of $O_f$. Then $d(x_1, x_2) = 2$ by Lemma 6.4. Let $Q$ denote the unique special quad through $x_1$ and $x_2$. Then $Q$ is not isomorphic to $Q(5, 2)$ since this generalized quadrangle has no ovoids (Payne and Thas [17]). If $Q$ is a $W(2)$-quad or a grid-quad of type I, then $Q$ is contained in a unique $G_3$-hex $H$, see Lemma 3.1(9)+(11). Since $Q \cap O_f \subseteq H \cap O_f$, the valuation of $H$ induced by $f$ is non-classical and hence $|H \cap O_f| = 15$ by Lemma 2.1. By Lemma 6.3, it then follows that $O_f = H \cap O_f$. The lemma is now clear.
6.2 Treatment of case (A) of Lemma 6.5

**Proposition 6.6** If $f$ is a valuation of $\mathbb{G}_4$ such that $|O_f| = 1$, then $f$ is a classical valuation.

**Proof.** Put $O_f = \{x\}$ and let $H$ denote an arbitrary $\mathbb{G}_3$-hex through $x$. By Lemma 3.1(5)+(6), there exists a unique special line $L$ through $x$ not contained in $H$. Let $x'$ denote an arbitrary point of $L \setminus \{x\}$. By Lemmas 3.1(5)+(6), there exists a unique $\mathbb{G}_3$-hex $H'$ through $x'$ not containing the special line $L$. We will show that the valuation $f'$ of $H'$ induced by $f$ is classical. Suppose the contrary and let $Q$ denote a grid-quad of $H'$ which is special with respect to $f'$. (Such a grid-quad exists by Lemma 2.1(v).) By Lemma 3.1(12), $Q$ is a grid-quad of type I. By Lemma 3.1(11), $Q$ is contained in a unique $Q(5,2) \times \mathbb{L}_3$-hex $H''$. By Lemma 3.1(4)+(5), $H''$ has two special lines through $x'$. One of these lines is contained in the grid-quad $Q$ of type I. The other special line $L'$ cannot be contained in $H'$ since otherwise $H'' = \langle Q, L \rangle = H'$, which is clearly absurd. Since there is only 1 special line through $x'$ not contained in the $\mathbb{G}_3$-hex $H'$ (Lemmas 3.1(5)+(6)), we must have $L' = L$. Now, the valuation of $H'' = \langle L, Q \rangle$ induced by $f$ contains a unique point with value 0 (namely $x$) and a point with value 1 at distance 3 from it (which is contained in $\Gamma_2(x') \cap Q$). But $Q(5,2) \times \mathbb{L}_3$ does not have valuations of this type. So, we have a contradiction. It follows that the valuation induced in $H'$ is classical. This implies that every point of $H'$ at distance 3 from $x'$ has value 4. By Lemma 6.1(i), it then follows that $f$ is classical. ■

6.3 Treatment of case (B) of Lemma 6.5

**Proposition 6.7** If $f$ is a valuation of $\mathbb{G}_4$ such that $O_f$ is a set of 15 points in a $\mathbb{G}_3$-hex $H$ of $\mathbb{G}_4$, then $f$ is the extension of a non-classical valuation of $\mathbb{G}_3$.

**Proof.** Let $f'$ denote the valuation of $H$ induced by $f$. Then $f'$ is a non-classical valuation of $H$ with $O_{f'} = O_f$. Hence, $f(x) = f'(x)$ for every point $x \in H$. Now, let $x$ be an arbitrary point of $\mathbb{G}_4$ not contained in $H$. Recall that the $\mathbb{G}_3$-hex $H$ is big in $\mathbb{H}_4$. So, $x$ is collinear with the point $\pi_H(x)$ of $H$. Let $Q$ denote an arbitrary $Q(5,2)$-quad of $H$ through $\pi_H(x)$. Among the near hexagons which can occur as hex in $\mathbb{G}_4$, only $\mathbb{G}_3$ and $Q(5,2) \times \mathbb{L}_3$ have $Q(5,2)$-quads. It follows that the hex $\langle x, Q \rangle = \langle x \pi_H(x), Q \rangle$ is isomorphic to $\mathbb{G}_3$ or $Q(5,2) \times \mathbb{L}_3$. The hex $\langle x, Q \rangle$ contains a unique point of $O_f$, namely the unique point of $O_f$ in $Q$ (recall Lemma 2.1(iv)). Now, all valuations of the near hexagons $\mathbb{G}_3$ and $Q(5,2) \times \mathbb{L}_3$ which contain a unique point with value
0 are classical. In particular, the valuation induced in \( \langle x, Q \rangle \) by \( f \) is classical. Hence, \( f(x) = d(x, O_f \cap Q) = 1 + d(\pi_H(x), O_f \cap Q) = 1 + f'(\pi_H(x)) \), where the latter equality follows from Lemma 2.1(iv). This proves that \( f \) is the extension of \( f' \).

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### 6.4 Treatment of case (C) of Lemma 6.5

In this subsection, we suppose that \( f \) is a valuation of \( \mathbb{G}_4 \) such that \( |O_f| \geq 2 \) and such that every special quad is a grid-quad of type II. By Lemma 6.4, every two distinct points of \( O_f \) are contained in a unique special quad. Since a special grid-quad contains three points of \( O_f \), we have \( |O_f| \geq 3 \).

**Lemma 6.8** It holds that \( |O_f| > 3 \).

**Proof.** Suppose to the contrary that \( |O_f| = 3 \). Let \( Q \) denote the unique special grid-quad of type II and put \( \{x_1, x_2, x_3\} = Q \cap O_f \). By Lemma 3.1(12), there exists a \( Q(5,2) \times \mathbb{L}_3 \)-hex \( F \) through \( Q \). This hex contains precisely 2 special lines through \( x_1 \) by Lemmas 3.1(4)+(5). So, \( F \) has an ordinary line \( L \) through \( x_1 \) not contained in \( Q \). Let \( y \in L \setminus \{x_1\} \). By Lemmas 3.1(6)+(8), there exists a \( \mathbb{G}_3 \)-hex \( H' \) through \( y \) not containing the line \( L \). Let \( f' \) denote the valuation of \( H' \) induced by \( f \). Since \( \pi_{H'}(\{x_1, x_2, x_3\}) \subseteq O_f \), \( f' \) is non-classical. By Lemma 2.1(v), there exists a \( W(2) \)-quad \( Q' \) of \( H' \) through \( y \) which is special with respect to \( f' \). Now, by Lemma 3.1(9), \( Q' \) is contained in 1 \( \mathbb{G}_3 \)-hex (namely \( H' \)), 1 \( W(2) \times \mathbb{L}_3 \)-hex (namely the hex \( \langle Q', M \rangle \) where \( M \) is the unique special line through \( y \) not contained in \( H' \) and three \( \mathbb{H}_3 \)-hexes. Hence, \( \langle L, Q' \rangle \) is isomorphic to \( \mathbb{H}_3 \). This implies that \( \langle L, Q' \rangle \) does not contain \( Q \) since \( (L, Q) \cong Q(5,2) \times \mathbb{L}_3 \). It follows that the valuation of \( \langle L, Q' \rangle \cong \mathbb{H}_3 \) induced by \( f \) contains a unique point with value 0 (namely \( x_1 \)) and a point with value 1 at distance 3 from it (which is contained in \( \Gamma_2(y) \cap Q' \)). This is impossible, since \( \mathbb{H}_3 \) does not have such valuations.

**Lemma 6.9** \( O_f \) is a set of 7 points in an \( \mathbb{H}_3 \)-hex of \( \mathbb{G}_4 \).

**Proof.** Let \( x \) denote an arbitrary point of \( O_f \). By Lemmas 6.4 and 6.8, there are two distinct special grid-quads \( G_1 \) and \( G_2 \) (of type II) through \( x \). By Lemma 6.1(iii), \( G_1 \cap G_2 = \{x\} \). Let \( u_1 \) be an arbitrary point of \( (O_f \cap G_1) \setminus \{x\} \). By Lemma 6.4, \( u_1 \) has distance 2 from every point of \( O_f \cap G_2 \). If \( d(u_1, G_2) = 1 \), then \( u_1 \) is classical with respect to \( G_2 \) and all points of \( O_f \cap G_2 \) would be collinear with \( \pi_{G_2}(u_1) \), clearly a contradiction. Hence, \( d(u_1, G_2) = 2 \). Since every line of \( G_2 \) contains a unique point nearest to \( u_1 \), we have \( G_2 \setminus O_f \subseteq \Gamma_3(u_1) \). Now, let \( u_2 \) be an arbitrary point of \( G_2 \setminus O_f \). Then \( \langle u_1, u_2 \rangle \) is a hex. Since \( O_f \cap G_2 \subseteq \Gamma_2(u_1) \), there are two distinct points
v_1 and v_2 of O_f \cap G_2 collinear with u_2 which are on a geodesic path from u_2 to u_1. Hence, G_2 = \langle v_1, v_2 \rangle \subseteq \langle u_1, u_2 \rangle. In particular, x \in \langle u_1, u_2 \rangle. Since x, u_1 \in \langle u_1, u_2 \rangle, we have G_1 = \langle x, u_1 \rangle \subseteq \langle u_1, u_2 \rangle. So, H := \langle G_1, G_2 \rangle is a hex. By Lemma 3.1(12), H is isomorphic to either \mathbb{H}_3 or Q(5, 2) \times \mathbb{L}_3. (Recall that G_1 and G_2 are grids of type II). Now, in the near hexagon Q(5, 2) \times \mathbb{L}_3 any two distinct grid-quads through the same point meet each other in a line. Since G_1 \cap G_2 = \{x\}, we necessarily have H \cong \mathbb{H}_3. Since |G_1 \cap O_f| = |G_2 \cap O_f| = 3, the valuation f_H of H induced by f must be of Fano-type. Hence, |O_f \cap H| = 7.

We show that \Gamma_1(H) \cap O_f = \emptyset. Suppose to the contrary that y is a point of \Gamma_1(H) \cap O_f. Then y is classical with respect to H. Since f(y) = 0, f(\pi_H(y)) = 1 and hence by Lemma 2.3(iv) \pi_H(y) is contained in a unique quad Q of H which is special with respect to f_H. Any point of Q \cap O_{f_H} = Q \cap O_f at distance 2 from \pi_H(y) lies at distance 3 from y, contradicting Lemma 6.4. Hence, \Gamma_1(H) \cap O_f = \emptyset.

We show that f(y) \geq 2 for every point y of type (a) of \Gamma_2(H). Let Q denote the W(2)-quad of H containing all points of \Gamma_2(y) \cap H and let H' be the \mathbb{G}_3-hex \langle y, Q \rangle. Let u denote the unique point of O_f \cap Q (recall Lemma 2.3(i)) and let L be a line of Q through u. If the valuation f_{H'} of H' induced by f is not classical, then by Lemma 2.1(v) there exists a quad of H' through L which is special with respect to f_{H'}. This implies that there is a point of O_{f_{H'}} \subseteq O_f contained in \Gamma_1(H), a contradiction. Hence, f_{H'} is a classical valuation of H'. It follows that f(y) = f_{H'}(y) = d(y, u) \geq 2.

We show that f(y) \geq 1 for every point y of type (b) of \Gamma_2(H). By Lemma 4.6 there exists a line L \in \mathbb{S} through y and this line contains a unique point u of type (a) by Corollary 4.13. Since f(u) \geq 2, we have f(y) \geq 1.

Let H denote the unique \mathbb{H}_3-hex of \mathbb{G}_4 containing all points of O_f and let f' be the valuation of H induced by f. By Lemma 6.9, f' is a valuation of Fano-type of H.

**Proposition 6.10** The valuation f is obtained from f' in the way as described in Section 5.

**Proof.** Let x denote an arbitrary point of \mathbb{G}_4.

If x \in H, then d(x, O_f) \leq 2 and hence f(x) = d(x, O_f) = d(x, O_{f'}) = f'(x) by Lemma 6.1(ii).

If x \in \Gamma_1(H) such that d(\pi_H(x), O_f) \leq 1, then d(x, O_f) \leq 2 and hence f(x) = d(x, O_f) = 1 + d(\pi_H(x), O_f) = 1 + f'(\pi_H(x)) by Lemma 6.1(ii).

Let x \in \Gamma_1(H) such that d(\pi_H(x), O_f) = 2, or equivalently, such that f'(\pi_H(x)) = 2. Let H' denote an arbitrary \mathbb{G}_3-hex through the line x\pi_H(x).
Then $H' \cap H$ is a $W(2)$-quad $Q$ by Lemma 3.1(14). The hex $H'$ contains a unique point $y$ with $f$-value 0, namely the unique point of $O_f$ in $Q$ (recall Lemma 2.3(i)). Hence, the valuation induced in $H'$ is classical. Since $d(\pi_H(x), O_f) = 2$, we have $d(\pi_H(x), y) = 2$. It follows that $f(x) = d(x, y) = 1 + d(\pi_H(x), y) = 3 = 1 + f'(\pi_H(x))$.

Let $x$ denote a point of type (a) of $\Gamma_2(H)$. Let $Q$ denote the $W(2)$-quad of $H$ containing all points of $\Gamma_2(x) \cap H$ and let $x^*$ denote the unique point of $O_f$ in $Q$. The hex $\langle x, Q \rangle$ is isomorphic to $\mathbb{G}_3$ and contains a unique point of $O_f$, namely $x^*$. Hence, the valuation induced in $\langle x, Q \rangle$ is classical. It follows that $f(x) = d(x, x^*)$.

Let $x$ denote a point of type (b) of $\Gamma_2(H)$ such that $|O_f \cap Q| = 3$, where $Q$ is the unique grid-quad of $H$ containing $\Gamma_2(x) \cap H$. The hex $\langle x, Q \rangle$ is isomorphic to $\mathbb{H}_3$ and the valuation of $\langle x, Q \rangle$ induced by $f$ is of grid-type. It follows from Lemma 2.2 that $f(x) = 2$ if $\Gamma_2(x) \cap O_f \cap Q \neq \emptyset$ and $f(x) = 1$ otherwise.

Let $x$ denote a point of type (b) of $\Gamma_2(H)$ such that $|O_f \cap Q| = 0$, where $Q$ is the unique grid-quad of $H$ containing $\Gamma_2(x) \cap H$. By Lemma 2.3(ii), the points with $f$-value 1 determine an ovoid of $Q$. So, the grid-quad $Q$ is special with respect to the valuation $f'$ of $\langle x, Q \rangle \cong \mathbb{H}_3$ induced by $f$. This implies that the valuation $f'$ is either of grid-type or of Fano-type. We will show that the latter possibility cannot occur.

Suppose that $f'$ is a valuation of Fano-type. Let $u$ denote one of the three points of $Q$ with $f$-value 1. By Lemma 2.3(iv), there exists a point $v \notin Q$ of $O_f$ collinear with $u$. Let $G \neq Q$ denote a grid-quad of $\langle x, Q \rangle$ through $u$ (which is special with respect to $f'$). Then $G \cap Q = \{u\}$. Let $w$ be a point of $G \cap \Gamma_2(u)$. If $w \in \Gamma_1(Q)$, then $w$ is classical with respect to $Q$, $\pi_Q(w)$ would be a common neighbour of $u$ and $w$, and the quad $G = \langle u, w \rangle$ would contain the line $u\pi_Q(w)$ of $Q$, a contradiction. So, $w \in \Gamma_2(Q)$. By Lemma 4.3 applied to the hexes $\langle x, Q \rangle$ and $H$, we see that $w \in \Gamma_2(H)$. So, there exists a unique hex through $w$ meeting $H$ in a quad and this hex coincides with $\langle x, Q \rangle$. This implies that the hex $\langle vu, G \rangle \neq \langle x, Q \rangle$ intersects $H$ in the line $uv$. It follows that the valuation induced in $\langle vu, G \rangle$ contains a unique point with value 0 (namely $v$) and a point with value 1 at distance 3 from it (which is contained in $\Gamma_2(u) \cap G$). Among the near hexagons which can occur as hex in $\mathbb{G}_3$, only $W(2) \times \mathbb{L}_3$ has such valuations. So, $\langle vu, G \rangle \cong W(2) \times \mathbb{L}_3$ and the valuation induced in $\langle vu, G \rangle$ is semi-classical. But in a $W(2) \times \mathbb{L}_3$-hex, every grid-quad is of type I, while the grid-quad $G$ has type II since it is contained in the $\mathbb{H}_3$-hex $\langle x, Q \rangle$ (recall Lemma 3.1(11)+(12)). So, we have a contradiction and the valuation $f'$ must be of grid-type.
By Lemma 2.2 it now follows \( f(x) = 3 \) if \( \Gamma_2(x) \cap Q \) has a point with \( f' \)-value 1 and \( f(x) = 2 \) otherwise.

This proves the proposition.  

\section{6.5 A lemma}

Recall that by Section 1.1, the near polygon \( \mathbb{G}_n \) can be isometrically embedded into the dual polar space \( DH(2n - 1, 4) \).

\textbf{Lemma 6.11} Let \( F \) be a hex of \( \mathbb{G}_4 \) and let \( f \) be a valuation of \( F \). Suppose that one of the following cases occurs: (i) \( F \cong \mathbb{H}_3 \) and \( f \) is a valuation of Fano-type of \( F \); (ii) \( F \cong \mathbb{G}_3 \) and \( f \) is a non-classical valuation of \( F \). Suppose also that \( \mathbb{G}_4 \) is isometrically embedded into the dual polar space \( DH(7, 4) \). Then there exists a unique point \( x \in DH(7, 4) \setminus \mathbb{G}_4 \) such that \( O_f \subseteq \Gamma_1(x) \).

\textbf{Proof.} For every convex subspace \( A \) of \( \mathbb{G}_4 \), there exists a unique convex subspace \( \mathcal{A} \) of \( DH(7, 4) \) containing \( A \) and having the same diameter as \( A \). If \( A \) has diameter \( \delta \) and if \( x_1 \) and \( x_2 \) are two points of \( A \) at distance \( \delta \) from each other, then \( \mathcal{A} \) is the unique convex subspace of \( DH(7, 4) \) containing \( x_1 \) and \( x_2 \).

Let \( Q \) be a quad of \( F \) which is special with respect to \( f \). We moreover assume that \( Q \) is a \( W(2) \)-quad if we are in case (ii) of the lemma. Put \( Q \cap O_f = \{x_1, x_2, \ldots, x_k\} \), where \( k = 3 \) (case (i)) or \( k = 5 \) (case (ii)). Let \( y \) be an arbitrary point of \( O_f \setminus Q \). Then \( d(y, x_i) = 2 \) for every \( i \in \{1, \ldots, k\} \). If \( d(y, Q) = 1 \), then \( y \) is classical with respect to \( Q \) and all points of the ovoid \( Q \cap O_f = \{x_1, \ldots, x_k\} \) of \( Q \) would be collinear with \( \pi_Q(y) \), clearly a contradiction. Hence, \( d(y, Q) = 2 \). Since every point of \( Q \cap O_f \) is collinear with \( y \), we have \( y \notin Q \). Since the quad \( Q \) of \( DH(7, 4) \) is big in the hex \( F \) of \( DH(7, 4) \), this implies that \( d(y, Q) = 1 \). Since \( d(y, x_i) = 2 \) and \( y \) is classical with respect to \( Q \), we have \( d(\pi_Q(y), x_i) = 1 \) for every \( i \in \{1, \ldots, k\} \). For every \( i \in \{1, \ldots, k\} \), let \( Q_i \) denote the unique quad of \( \mathbb{G}_4 \) through \( y \) and \( x_i \). Since \( y, x_i \in Q_i \cap O_f \), \( Q_i \) is special with respect to \( f \). So, \( Q_i \) is either a \((3 \times 3)\)-grid or a \( W(2) \)-quad and there exists a unique ovoid \( O_f \) of \( Q_i \) containing \( y \) and \( x_i \). Now, the \( k \) quads \( Q_1, \ldots, Q_k \) are all the quads through \( y \) which are special with respect to \( f \). Since any two distinct points of \( O_f \) lie at distance 2 from each other, we necessarily have \( O_f = O_1 \cup O_2 \cup \cdots \cup O_k \).

We prove that \( \pi_Q(y) \notin \mathbb{G}_4 \). Suppose to the contrary that \( \pi_Q(y) \in \mathbb{G}_4 \). Since \( \pi_Q(y) \) is collinear with the points \( x_1, \ldots, x_k \), we would then have that \( \pi_Q(y) \in Q \). This is impossible since \( d(y, Q) = 2 \). Hence, \( \pi_Q(y) \notin \mathbb{G}_4 \).
Since $\pi_\Gamma(y)$ is collinear with the points $y$ and $x_i$, $i \in \{1, \ldots, k\}$, $\pi_\Gamma(y)$ is contained in $Q_i$. So, $\Gamma_1(\pi_\Gamma(y)) \cap Q_i$ is an ovoid of $Q_i$ containing $y$ and $x_i$. It follows that $\Gamma_1(\pi_\Gamma(y)) \cap Q_i = O_i$. Hence, $O_f = O_1 \cup O_2 \cup \cdots \cup O_k \subseteq \Gamma_1(\pi_\Gamma(y))$.

Conversely, suppose $z$ is a point of $DH(7, 4) \setminus G_4$ such that $O_f \subseteq \Gamma_1(z)$. Since $z$ is collinear with the points $x_1, \ldots, x_k$, we have $z \in Q_i$. Since $z$ is collinear with $y$. We necessarily have $z = \pi_\Gamma(y)$.

\section{6.6 The valuations of $G_4$ are induced by valuations of $DH(7, 4)$}

Let the near octagon $G_4$ be isometrically embedded in $DH(7, 4)$. For every point $x$ of $DH(7, 4)$, the classical valuation $g_x$ of $DH(7, 4)$ with $O_{g_x} = \{x\}$ induces a valuation $f_x$ of $G_4$. It holds that $\max\{f_x(u) \mid u \in G_4\} = 4 - d(x, G_4)$ in view of the following result which holds for general dense near polygons.

\textbf{Lemma 6.12 (Proposition 2.2 of [14])} Let $S$ be a dense near $2n$-gon and let $F = (\mathcal{P}', \mathcal{L}', \Gamma')$ be a dense near $2n$-gon which is fully and isometrically embedded in $S$. Let $x$ be a point of $S$ and let $f_x$ denote the valuation of $F$ induced by the classical valuation $g_x$ of $S$ with $O_{g_x} = \{x\}$, then $d(x, F) = n - M$, where $M$ is the maximal value attained by $f_x$.

If $x \in G_4$, then $f_x$ is a classical valuation of $G_4$ and $O_{f_x} = \{x\}$. If $x \notin G_4$, then $f_x$ is not classical and hence is either the extension of a non-classical valuation of a $G_3$-hex or is a valuation of Fano-type.

\textbf{Proposition 6.13} Let $f$ be a valuation of $G_4$. Then there exists a unique point $x$ of $DH(7, 4)$ such that $f = f_x$.

\textbf{Proof.} Obviously, the proposition holds if $f$ is classical. The required point $x$ is then the unique point contained in $O_f$.

Suppose now that $f$ is non-classical. By the classification of the valuations of $G_4$, we then know that $F := \langle O_f \rangle$ is either $H_3$-hex or a $G_3$-hex of $G_4$. Moreover, if $f'$ denotes the valuation of $F$ induced by $f$, then $O_{f'} = O_f$, $f'$ is a valuation of Fano-type of $F$ if $F \cong H_3$ and $f'$ is a non-classical valuation of $F$ if $F \cong G_3$. By Lemma 6.11, there exists a unique point $x* \in DH(7, 4) \setminus G_4$ such that $O_{f'} \subseteq \Gamma_1(x*)$. Then $O_f \subseteq O_{f'}$. Hence, $O_f = O_{f'}$ and $f = f_x$ by the classification of the valuations of $G_4$.

Conversely, suppose that $f = f_x$ for some point $x$ of $DH(7, 4)$. Since $f$ is non-classical, its maximal value is equal to 3. Lemma 6.12 then implies that $d(x, G_4) = 1$. We have $O_f = \Gamma_1(x) \cap G_4$. Since $O_f \subseteq \Gamma_1(x)$, Lemma 6.11 implies that $x = x*$.
By Proposition 6.13, the number of valuations of $G_4$ is equal to the number of points of $DH(7,4)$. The number of classical valuations of $G_4$ is equal to the number of points of $G_4$, i.e., equal to 8505. The number of valuations of $G_4$ which are extensions of non-classical valuations in $G_3$-hexes is equal to $(\# G_3$-hexes $) \times (\# \text{non-classical valuations in a } G_3$-hex $) = 84 \times 486 = 40824$. The number of valuations of Fano-type of $G_4$ is equal to $(\# H_3$-hexes $) \times (\# \text{valuations of Fano-type in an } H_3$-hex $) = 2178 \times 30 = 65610$. The number $8505 + 40824 + 65610 = 114939$ is indeed equal to the total number of points of $DH(7,4)$.

References


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