An alternative definition of the notion valuation in the theory of near polygons

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Abstract

Valuations of dense near polygons were introduced in [9]. A valuation of a dense near polygon $S = (P, L, I)$ is a map $f$ from the point-set $P$ of $S$ to the set $\mathbb{N}$ of nonnegative integers satisfying very nice properties with respect to the set of convex subspaces of $S$. In the present paper, we give an alternative definition of the notion valuation and prove that both definitions are equivalent. In the case of dual polar spaces and many other known dense near polygons, this alternative definition can be significantly simplified.

1 Introduction

1.1 Basic definitions

A near polygon is a partial linear space $S = (P, L, I)$, $I \subseteq P \times L$, with the property that for every point $p \in P$ and every line $L \in L$, there exists a unique point on $L$ nearest to $p$. Here distances $d(\cdot, \cdot)$ are measured in the collinearity graph $\Gamma$ of $S$. If $d$ is the diameter of $\Gamma$, then the near polygon is called a near $2d$-gon. A near 0-gon is a point and a near 2-gon is a line. Near quadrangles are usually called generalized quadrangles (Payne and Thas [11]).

If $X_1$ and $X_2$ are two nonempty sets of points of a near polygon $S$, then $d(X_1, X_2)$ denotes the minimal distance between a point of $X_1$ and a point of $X_2$. If $X_1 = \{x_1\}$, we will also write $d(x_1, X_2)$ instead of $d(\{x_1\}, X_2)$. For every nonempty set $X$ of points of $S$ and every $i \in \mathbb{N}$, $\Gamma_i(X)$ denotes the set of all points $y$ of $S$ for which $d(y, X) = i$. If $X$ is a singleton $\{x\}$, then we will also write $\Gamma_i(x)$ instead of $\Gamma_i(\{x\})$.

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A nonempty set $X$ of points of a near polygon $S$ is called a \textit{subspace} if every line meeting $X$ in at least two points has all its points in $X$. A subspace $X$ is called \textit{convex} if every point on a shortest path between two points of $X$ is also contained in $X$. Having a subspace $X$, we can define a subgeometry $S_X$ of $S$ by considering only those points and lines of $S$ which are contained in $X$. If $X$ is a convex subspace, then $S_X$ is a sub-near-polygon of $S$. If $X_1, X_2, \ldots, X_k$ are objects of $S$ (like points, and nonempty sets of points), then $\langle X_1, X_2, \ldots, X_k \rangle$ denotes the smallest convex subspace of $S$ containing $X_1, X_2, \ldots, X_k$. The maximal distance between two points of a convex subspace $F$ of $S$ is called the \textit{diameter} of $F$.

A near polygon $S$ is called \textit{dense} if every line of $S$ is incident with at least three points and if every two points of $S$ at distance 2 have at least two common neighbours. By Theorem 4 of Brouwer and Wilbrink [3], every two points of a dense near polygon at distance $\delta$ from each other are contained in a unique convex sub-$2\delta$-gon. These convex sub-$2\delta$-gons are called \textit{quads} if $\delta = 2$ and \textit{hexes} if $\delta = 3$. The existence of quads in dense near polygons was already shown in Shult and Yanushka [12, Proposition 2.5]. With every point $x$ of a dense near polygon $S$, there is associated a linear space $L(S,x)$ which is called the \textit{local space} at $x$. The points, respectively lines, of $L(S,x)$ are the lines, respectively quads, through $x$, and incidence is containment.

1.2 The main results

Let $S = (P, L, I)$ be a dense near polygon. A function $f$ from $P$ to $\mathbb{N}$ is called a \textit{valuation} of $S$ if it satisfies the following properties (we call $f(x)$ the \textit{value} of $x$):

(V1) there exists at least one point with value 0;

(V2) every line $L$ of $S$ contains a unique point $x_L$ with smallest value and $f(x) = f(x_L) + 1$ for every point $x$ of $L$ different from $x_L$;

(V3) every point $x$ of $S$ is contained in a (necessarily unique) convex subspace $F_x$ which satisfies the following properties:

(i) $f(y) \leq f(x)$ for every point $y$ of $F_x$.

(ii) every point $z$ of $S$ which is collinear with a point $y$ of $F_x$ and which satisfies $f(z) = f(y) - 1$ also belongs to $F_x$.

Examples. (1) Let $x$ be a given point of $S$ and define $f(y) := d(x,y)$ for every $y \in P$. Then $f$ is a valuation of $S$. We call $f$ a \textit{classical valuation} of $S$.

(2) Let $O$ be an ovoid of $S$, i.e. a set of points of $S$ meeting each line in a unique point. Then define $f(x) := 0$ if $x \in O$ and $f(x) := 1$ if $x \in P \setminus O$. Then $f$ is a valuation of $S$. We call $f$ an \textit{ovoidal valuation} of $S$.

Consider the following property for a function $f : P \to \mathbb{N}$:
(V3’) Through every point $x$ of $S$, there exists a convex subspace $F_x$ of $S$ such that the lines through $x$ contained in $F_x$ are precisely the lines through $x$ containing a point with value $f(x) - 1$.

If Property (V3’) is satisfied, then the convex subspace $F_x$ through $x$ is uniquely determined: if $L_x$ denotes the set of lines through $x$ containing a point with value $f(x) - 1$, then $F_x$ coincides with the smallest convex subspace of $S$ containing all lines of $L_x$. (By Brouwer and Wilbrink [3], see also [7, Theorem 2.14], a convex subspace $F$ of $S$ is completely determined by the set of lines of $F$ through one of its points.)

The following theorem provides an alternative definition of the notion valuation.

**Theorem 1.1 (Section 2)** Let $S = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a dense near polygon and let $f$ be a map from $\mathcal{P}$ to $\mathbb{N}$. Then $f$ is a valuation of $S$ if and only if $f$ satisfies Properties (V1), (V2) and (V3’).

It will turn out that in many dense near polygons, Property (V3’) is a consequence of Property (V2). We first observe the following.

**Theorem 1.2 (Section 2)** Let $S = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a dense near polygon and let $f$ be a map from $\mathcal{P}$ to $\mathbb{N}$ satisfying Property (V2). Then for every point $x$ of $S$, the set of lines through $x$ containing a point with value $f(x) - 1$ is a subspace of the local space $L(S, x)$.

**Definition.** Let $S = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a dense near polygon and let $x$ be a point of $S$. We say that the local space $L(S, x)$ at $x$ is regular if for every subspace $S$ of $L(S, x)$, there exists a convex subspace $F_S$ through $x$ such that the lines through $x$ contained in $F_S$ are precisely the elements of $S$.

In Sections 3 and 4, we will give a description of the known examples of dense near polygons. Among other examples, we will discuss there the class of the dual polar spaces and two near hexagons which we will denote by $E_1$ and $E_2$. We will prove the following:

**Theorem 1.3 (Section 3)** (a) All local spaces of a thick dual polar space are regular.

(b) Let $S$ be a known dense near polygon not containing hexes isomorphic to $E_1$ or $E_2$. Then every local space of $S$ is regular.

**Remarks.** (1) Every local space of the near hexagon $E_1$ is isomorphic to the complete graph of 12 vertices (regarded as linear space). No such local space is regular: subspaces containing $i \in \{3, 4, \ldots, 11\}$ points do not correspond with convex subspaces.

(2) Every local space of the near hexagon $E_2$ is isomorphic to PG(3, 2) (regarded as linear space). No such local space is regular: subspaces carrying the structure of a PG(2, 2) do not correspond with convex subspaces.

By Theorems 1.1, 1.2 and 1.3, we have
Corollary 1.4 Let $S = (\mathcal{P}, \mathcal{L}, I)$ be a dense near polygon every local space of which is regular. Then a map $f : \mathcal{P} \to \mathbb{N}$ is a valuation of $S$ if and only if it satisfies Properties (V1) and (V2). In particular, this holds if $S$ is a thick dual polar space or a known dense near polygon without hexes isomorphic to $E_1$ or $E_2$.

The following theorem shows that the conclusion of Corollary 1.4 is not necessarily true in case there are hexes isomorphic to $E_1$ or $E_2$.

Theorem 1.5 (Section 4) Let $S$ be a near hexagon isomorphic to either $E_1$ or $E_2$. Then there exists a map $f : \mathcal{P} \to \mathbb{N}$ which satisfies Properties (V1) and (V2) and which is not a valuation of $S$.

2 Proofs of Theorems 1.1 and 1.2

In this section $S = (\mathcal{P}, \mathcal{L}, I)$ is a dense near polygon and $f$ is a map from $\mathcal{P}$ to the set $\mathbb{N}$ of nonnegative integers.

Lemma 2.1 If $f$ is a valuation of $S$, then $f$ satisfies Property (V3').

Proof. Let $x$ be an arbitrary point of $S$ and let $F_x$ denote the necessarily unique convex subspace through $x$ for which Property (V3) is satisfied. Let $L$ be an arbitrary line through $x$.

If $L \subseteq F_x$, then by (V3,i), $f(y) \leq f(x)$ for every $y \in L$. Hence, $L$ contains a unique point with value $f(x) - 1$ by (V2).

Conversely, suppose that $L$ contains a point with value $f(x) - 1$. Then $L \subseteq F_x$ by (V3,ii).

So, $F'_x := F_x$ is the unique convex subspace of $S$ through $x$ such that the lines through $x$ contained in $F'_x$ are precisely the lines through $x$ containing a point with value $f(x) - 1$.

Lemma 2.2 Suppose $f$ satisfies Property (V2) and let $Q$ be a quad of $S$. Then precisely one of the following two cases occurs:

(i) there exists a point $x^* \in Q$ such that $f(x) = f(x^*) + d(x^*, x)$ for every $x \in Q$;
(ii) there exists an ovoid $O$ of $Q$ and an $m^* \in \mathbb{N}$ such that $f(x) = m^* + d(x, O)$ for every $x \in Q$.

Proof. It is well-known that for every point $x$ of $Q$, the set $\{x\} \cup (\Gamma_1(x) \cap Q)$ is a maximal subspace of $Q$. This implies that the set $\Gamma_2(x) \cap Q$ is connected. We can distinguish the following two cases:

(i) There exist points $x^*, y_1 \in Q$ such that $f(y_1) - f(x^*) \geq 2$. By Property (V2), $d(y_1, x^*) \geq 2$. Hence, $d(y_1, x^*) = 2$ and $f(y_1) - f(x^*) = 2$. Every point collinear with $x^*$ and $y_1$ has value $f(x^*) + 1$ by (V2). Hence, also every point of $\Gamma_1(x^*) \cap Q$ has value $f(x^*) + 1$ by (V2).
Now, suppose \( y_2 \) is a point of \( Q \cap \Gamma_2(x^*) \) at distance 1 from \( y_1 \). Then \( f(y_1) = f(x^*) + 2 \) and the unique point on the line \( y_1 y_2 \) collinear with \( x^* \) has value \( f(x^*) + 1 \). By (V2) applied to the line \( y_1 y_2 \), it follows that \( f(y_2) = f(x^*) + 2 \).

Now, invoking the connectedness of \( \Gamma_2(x^*) \cap Q \), we see that every point of \( \Gamma_2(x^*) \cap Q \) has value \( f(x^*) + 2 \).

Summarizing, we have that \( f(x) = f(x^*) + d(x^*, x) \) for every \( x \in Q \).

(ii) \( |f(x_1) - f(x_2)| \leq 1 \) for any two points \( x_1 \) and \( x_2 \) of \( Q \). Put \( m^* := \min \{ f(x) \mid x \in Q \} \). Then \( f(x) \in \{ m^*, m^* + 1 \} \). Since every line of \( Q \) contains a unique point with smallest value, the set of points of \( Q \) with value \( m^* \) is an ovoid of \( Q \). Hence, \( f(x) = m^* + d(x, O) \) for every \( x \in Q \).

The following lemma is precisely Theorem 1.2.

**Lemma 2.3** If \( f \) satisfies Property (V2), then for every point \( x \) of \( S \), the set of lines through \( x \) containing a point with value \( f(x) - 1 \) is a subspace of the local space \( \mathcal{L}(S, x) \).

**Proof.** Let \( L_1 \) and \( L_2 \) be two lines through \( x \) containing a (unique) point with value \( f(x) - 1 \) and let \( Q \) denote the unique quad through \( L_1 \) and \( L_2 \). We need to show that every line of \( Q \) through \( x \) contains a point with value \( f(x) - 1 \). By Lemma 2.2, one of the following two cases occurs:

1. There exists a point \( x^* \in Q \) such that \( f(u) = f(x^*) + d(x^*, u) \) for every \( u \in Q \). Since there are at least two lines of \( Q \) through \( x \) containing a point with value \( f(x) - 1 \), we necessarily have \( d(x^*, x) = 2 \). But then every line of \( Q \) through \( x \) contains a unique point with value \( f(x) - 1 = f(x^*) + 1 \), namely the unique point on that line collinear with \( x^* \).

2. There exists an ovoid \( O \) of \( Q \) and an \( m^* \in \mathbb{N} \) such that \( f(u) = m^* + d(u, O) \) for every \( u \in Q \). Since \( L_1 \) and \( L_2 \) contain points with value \( f(x) - 1 \), \( x \) does not belong to \( O \). Clearly, every line of \( Q \) through \( x \) contains a unique point with value \( f(x) - 1 \), namely the unique point on that line belonging to \( O \).

**Lemma 2.4** Suppose \( f \) satisfies Property (V2). Let \( F \) be a convex subspace of \( S \) and put \( M := \max \{ f(x) \mid x \in F \} \). If \( x \) and \( y \) are two points of \( F \) such that \( f(x) = f(y) = M \), then \( x \) and \( y \) are connected by a path which entirely consists of points of \( F \) with value equal to \( M \).

**Proof.** We will prove the lemma by induction on \( d(x, y) \). The lemma trivially holds if \( d(x, y) \leq 1 \). So, suppose \( d(x, y) \geq 2 \). Let \( L_x \) be an arbitrary line through \( x \) contained in the convex subspace \( \langle x, y \rangle \subseteq F \) and let \( u \) denote the unique point on \( L_x \) at distance \( d(x, y) - 1 \) from \( y \). Let \( L_y \) denote a line of \( \langle x, y \rangle \) through \( y \) not contained in \( \langle u, y \rangle \). Then every point of \( L_x \) has distance \( d(x, y) - 1 \) from a unique point of \( L_y \). Now, since (V2) holds, the lines \( L_x \) and \( L_y \) contain unique points with value \( M - 1 \). So, since \( |L_x|, |L_y| \geq 3 \), there exist points \( x' \in L_x \) and \( y' \in L_y \) such that \( d(x', y') = d(x, y) - 1 \) and \( f(x') = f(y') = M \). By the induction hypothesis, \( x' \) and \( y' \) are connected by a path which entirely consists of points of \( F \) with value equal to \( M \). Hence, also \( x \) and \( y \) are connected by a path which entirely consists of points of \( F \) with value equal to \( M \).
Lemma 2.5 Suppose \( f \) satisfies Property (V2). Let \( Q \) be a quad of \( S \), let \( x \) and \( y \) be two distinct collinear points of \( Q \) such that \( f(x) = f(y) \) and let \( L_x \) and \( L_y \) be two lines of \( Q \) different from \( xy \) such that \( x \in L_x \) and \( y \in L_y \). Then the following holds: if \( L_x \) contains a point with value \( f(x) - 1 \), then \( L_y \) contains a point with value \( f(y) - 1 \).

Proof. By Lemma 2.2, we can distinguish two possibilities:

1. There exists a point \( x^* \in Q \) such that \( f(u) = f(x^*) + d(x^*, u) \) for every point \( u \in Q \). Since \( f(x) = f(y) \), either \( x^*, x, y \) are contained on a line or \( x, y \in Q \cap \Gamma_2(x^*) \).

In the former case, no line of \( Q \) through \( x \) distinct from \( xy \) contains a point with value \( f(x) - 1 \). In the latter case, every line of \( Q \) through \( x \) contains a unique point with value \( f(x) - 1 \). But in this case, also every line of \( Q \) through \( y \) contains a unique point with value \( f(y) - 1 \).

2. There exists an ovoid \( O \) of \( Q \) and an \( m^* \in \mathbb{N} \) such that \( f(u) = m^* + d(u, O) \) for every \( u \in Q \). Then \( x, y \notin O \). In this case, every line of \( Q \) through \( x \) contains a unique point with value \( f(x) - 1 \) and every line of \( Q \) through \( y \) contains a unique point with value \( f(y) - 1 \).

Definition. If \( f \) satisfies Property (V3'), then for every point \( x \) of \( S \), we denote by \( F_x \) the unique convex subspace of \( S \) through \( x \) such that the lines through \( x \) contained in \( F_x \) are precisely the lines through \( x \) containing a point with value \( f(x) - 1 \).

Lemma 2.6 If \( f \) satisfies Properties (V2) and (V3'), then \( f \) also satisfies Property (V3, i) with respect to the convex subspaces \( F_x, x \in \mathcal{P} \).

Proof. Let \( x \) be a point of \( S \). We need to show that \( f(y) \leq f(x) \) for every point \( y \) of \( F_x \). Suppose the contrary holds. Then choose a \( y \in F_x \) such that \( f(y) > f(x) \) with \( d(x, y) \) as small as possible. Let \( y_1 \) be a point of \( F_x \) collinear with \( y \) at distance \( d(x, y) - 1 \) from \( x \). Then since \( d(x, y_1) < d(x, y) \), \( f(y_1) \leq f(x) \). Hence, \( f(y) = f(x) + 1 \) and \( f(y_1) = f(x) \). By Lemma 2.4, there now exists a path \( y_1, y_2, \ldots, y_k = x \) in \( \langle x, y_1 \rangle \) connecting \( y_1 \) with \( x \) such that \( f(y_i) = f(x) \) for every \( i \in \{1, \ldots, k\} \). (Since \( d(u, x) \leq d(x, y_1) < d(x, y) \), we have \( f(u) \leq f(x) \) for every \( u \in \langle x, y_1 \rangle \).) We now inductively define a line \( L_i, i \in \{1, \ldots, k\} \), of \( F_x \) through \( y_i \) and show that this line contains a point with value \( f(x) + 1 \). Put \( L_1 := y_1 y \subset F_x \). As remarked above \( y \) has value \( f(x) + 1 \). Suppose now that for a certain \( i \in \{1, \ldots, k-1\} \), we have defined the line \( L_i \). Since \( L_i \) contains a point with value \( f(x) + 1 \) and \( y_i y_{i+1} \) contains a point with value \( f(x) + 1 \) (recall (V2)), we have \( L_i \neq y_i y_{i+1} \). Now, let \( Q \) denote the unique quad through \( L_i \) and \( y_i y_{i+1} \) and let \( L_{i+1} \) be a line of \( Q \) through \( y_{i+1} \) distinct from \( y_i y_{i+1} \). Since \( f(y_{i+1}) = f(x) \), there are two possibilities by Property (V2). Either \( L_{i+1} \) contains a point with value \( f(x) - 1 \) or a point with value \( f(x) + 1 \). In the former case, it would follow from Lemma 2.5, that also \( L_i \) would contain a point with value \( f(x) - 1 \), a contradiction. Hence, \( L_{i+1} \) contains a point with value \( f(x) + 1 \). Also, since \( L_i \) and \( y_i y_{i+1} \) are contained in \( F_x \), the quad \( Q \) is contained in \( F_x \) and hence \( L_{i+1} \subset F_x \).

A contradiction is now readily obtained. The line \( L_k \) through \( y_k = x \) is contained in \( F_x \) and contains a point with value \( f(x) + 1 \). But by (V3'), we would also have that \( L_k \)
contains a point with value \( f(x) - 1 \). So, our assumption was wrong and \( f(y) \leq f(x) \) for every point \( y \) of \( F_x \).

**Lemma 2.7** Suppose \( f \) satisfies Properties (V2) and (V3'). Then \( F_y = F_x \) for every point \( x \) of \( S \) and every \( y \in F_x \) with \( f(y) = f(x) \).

**Proof.** By Lemmas 2.4 and 2.6, there exists a path \( y = y_1, y_2, \ldots, y_k = x \) which entirely consists of points of \( F_x \) with value \( f(x) \). We show the following by downwards induction on \( i \in \{1, 2, \ldots, k\} \):

- If \( L \) is a line through \( y_i \) containing a point with value \( f(x) - 1 \), then \( L \subseteq F_x \).
- If \( L \) is a line through \( y_i \) containing a point with value \( f(x) + 1 \), then \( L \) is not contained in \( F_x \).

Obviously, this claim holds if \( i = k \). So, suppose \( i < k \) and that the claim holds for the number \( i+1 \).

Let \( L \) be a line through \( y_i \) containing a point with value \( f(x) - 1 \). If \( L = y_i y_{i+1} \), then \( L \subseteq F_x \). So, suppose \( L \neq y_i y_{i+1} \). Let \( L' \) be a line through \( y_{i+1} \) distinct from \( y_i y_{i+1} \) contained in the quad \( \langle L, y_i y_{i+1} \rangle \). By Lemma 2.5, \( L' \) contains a point with value \( f(x) - 1 \). Hence, \( L' \subseteq F_x \) by the induction hypothesis. Since \( L' \) and \( y_i y_{i+1} \) are contained in \( F_x \), the quad \( \langle L', y_i y_{i+1} \rangle = \langle L, y_i y_{i+1} \rangle \) is contained in \( F_x \). Hence, \( L \subseteq F_x \).

Let \( L \) be a line through \( y_i \) containing a point with value \( f(x) + 1 \). Then \( L \) cannot be contained in \( F_x \) by Lemma 2.6.

Hence, the above claim holds for every \( i \in \{1, 2, \ldots, k\} \). The fact that it holds for \( i = 1 \) implies that \( F_y = F_x \). ■

**Lemma 2.8** Suppose \( f \) satisfies Properties (V2) and (V3'). Then \( f \) also satisfies Property (V3,ii) with respect to the convex subspaces \( F_x, x \in P \).

**Proof.** Let \( x \) be an arbitrary point of \( S \), let \( y \) be a point of \( F_x \) and let \( z \) be a point collinear with \( y \) for which \( f(z) = f(y) - 1 \). We need to show that \( z \in F_x \). We will prove this by induction on the number \( f(x) - f(y) \) (which is nonnegative by Lemma 2.6). If \( f(x) = f(y) \), then we have that \( z \in F_y = F_x \) by Lemma 2.7. So, suppose \( f(x) > f(y) \). Then \( x \notin F_y \) by Lemma 2.6. So, \( F_x \nsubseteq F_y \) and there exists a line \( L \) through \( y \) contained in \( F_x \) but not in \( F_y \). Let \( u \) be an arbitrary point of \( L \setminus \{y\} \). Then \( f(u) = f(y) + 1 \) and hence \( d(u, z) = 2 \) by (V2). Let \( v \) denote a common neighbour of \( u \) and \( z \) distinct from \( y \). Then \( f(v) = f(y) \). The line \( uv \) is a line through \( u \in F_x \) containing a point with value \( f(u) - 1 \), namely the point \( v \). By the induction hypothesis, \( uv \subseteq F_x \). Since also \( L \subseteq F_x \), the quad \( \langle uv, L \rangle = \langle u, z \rangle \) is contained in \( F_x \). Hence, \( z \in F_x \). ■

Theorem 1.1 is an immediate corollary of Lemmas 2.1, 2.6 and 2.8.
3 Proof of Theorem 1.3

Every known dense near polygon without hexes isomorphic to \( E_1 \) and \( E_2 \) is up to isomorphism either a line, a thick dual polar space of rank \( n \geq 2 \), a near polygon \( \mathbb{I}_n \) for some \( n \geq 2 \), a near polygon \( \mathbb{H}_n \) for some \( n \geq 2 \), a near 2n-gon \( \mathbb{G}_n \) for some \( n \geq 2 \), the near hexagon \( E_3 \), or is obtained from these near polygons by successive application of the product and gluing constructions. So, in order to prove Theorem 1.3, it suffices to verify the following: (I) all local spaces of a thick dual polar space of rank \( n \geq 2 \) are regular; (II) all local spaces of the near 2n-gon \( \mathbb{I}_n \), \( n \geq 2 \), are regular; (III) all local spaces of the near 2n-gon \( \mathbb{H}_n \), \( n \geq 2 \), are regular; (IV) all local spaces of the near 2n-gon \( \mathbb{G}_n \), \( n \geq 2 \), are regular; (V) all local spaces of the product of near polygons \( A_1 \times A_2 \) is regular; (VII) if \( A_1 \) and \( A_2 \) are two dense near polygons of diameter at least 2 such that every local space of \( A_i \), \( i \in \{1, 2\} \), is regular, then also every local space of any glued near polygon of type \( A_1 \otimes A_2 \) is regular.

(I) Let \( \Pi \) be a nondegenerate thick polar space of rank \( n \geq 2 \). With \( \Pi \) there is associated a dual polar space \( \Delta \) whose points are the maximal (i.e. \((n-1)\)-dimensional) totally singular subspaces of \( \Pi \) and whose lines are the \((n-2)\)-dimensional totally singular subspaces of \( \Pi \) (natural incidence). If \( \gamma \) is an \((n-1-k)\)-dimensional totally singular subspace of \( \Pi \), then the set of all maximal singular subspaces of \( \Pi \) containing \( \gamma \) is a convex subspace \( F_\gamma \) of \( \Delta \). Conversely, every convex subspace of \( \Delta \) is obtained in this way. Now, let \( \alpha \) be an arbitrary point of \( \Delta \). Then \( \alpha \) can be regarded as an \((n-1)\)-dimensional projective space. From this point of view, the local space \( \mathcal{L}(\Delta, \alpha) \) at \( \alpha \) is nothing else than the dual projective space associated with \( \alpha \). If \( S \) is a subspace of \( \mathcal{L}(\Delta, \alpha) \), then \( S \) consists of all hyperplanes of \( \alpha \) which contain a given subspace \( \beta \) of \( \alpha \). Then \( S \), regarded as set of lines of \( \Delta \), consists of all lines of \( \Delta \) through \( \alpha \) contained in \( F_\beta \). This proves that every local space of \( \Delta \) is regular.

(II) Consider a nonsingular parabolic quadric \( Q(2n, 2) \), \( n \geq 2 \), in \( \text{PG}(2n, 2) \) and a hyperplane of \( \text{PG}(2n, 2) \) intersecting \( Q(2n, 2) \) in a nonsingular hyperbolic quadric \( Q^+(2n-1, 2) \). Let \( \mathbb{I}_n = (\mathcal{P}, \mathcal{L}, 1) \) be the following point-line geometry: (i) \( \mathcal{P} \) is the set of all maximal subspaces (of dimension \( n-1 \)) of \( Q(2n, 2) \) not contained in \( Q^+(2n-1, 2) \); (ii) \( \mathcal{L} \) is the set of all \((n-2)\)-dimensional subspaces of \( Q(2n, 2) \) not contained in \( Q^+(2n-1, 2) \); (iii) incidence is reverse containment. Then by Brouwer et al. [2], \( \mathbb{I}_n \) is a dense near 2n-gon. If \( \gamma \) is an \((n-1-k)\)-dimensional subspace of \( Q(2n, 2) \) which is not contained in \( Q^+(2n-1, 2) \) if \( k \in \{0, 1\} \), then the set \( F_\gamma \) of all maximal subspaces of \( Q(2n, 2) \) containing \( \gamma \) is a convex subspace \( F_\gamma \) of \( \mathbb{I}_n \). Conversely, every convex subspace of \( \mathbb{I}_n \) is obtained in this way. It follows that every local space of \( \mathbb{I}_n \) is isomorphic to the projective space \( \text{PG}(n-1, 2) \) (regarded as linear space) in which a point has been removed. Specifically, if \( \alpha \) is a point of \( \mathbb{I}_n \), then \( \mathcal{L}(\mathbb{I}_n, \alpha) \) is the dual projective space associated with \( \alpha \cong \text{PG}(n-1, 2) \) in which the point \( \alpha \cap Q^+(2n-1, 2) \) has been removed. Now, let \( S \) be a subspace of \( \mathcal{L}(\mathbb{I}_n, \alpha) \). Then there exists a subspace \( \beta \) in \( \alpha \) distinct from \( \alpha \cap Q^+(2n-1, 2) \) such that \( \beta \) is contained in
all hyperplanes of $\alpha$ through $\beta$ distinct from $\alpha \cap Q^+(2n - 1, 2)$. The following obviously holds: the lines through $\alpha$ contained in the convex subspace $F_\beta$ are precisely the elements of $S$. This proves that every local space of $\mathbb{I}_n$ is regular. More information on the convex subpolygons of the dense near $2n$-gon $\mathbb{I}_n$ can be found in [7, Section 6.4].

(III) Let $A$ be a set of size $2n + 2$, $n \geq 2$. Let $\mathbb{H}_n = (\mathcal{P}, \mathcal{L}, I)$ be the following point-line geometry: (i) $\mathcal{P}$ is the set of all partitions of $A$ in $n + 1$ subsets of size 2; (ii) $\mathcal{L}$ is the set of all partitions of $A$ in $n - 1$ subsets of size 2 and one subset of size 4; (iii) a point $p \in \mathcal{P}$ is incident with a line $L \in \mathcal{L}$ if and only if the partition defined by $p$ is a refinement of the partition defined by $L$. By De Bruyn [6], $\mathbb{H}_n$ is a dense near $2n$-gon. If $\mathcal{M}_n$ denotes the partial linear space whose points, respectively lines, are the subsets of size 2, respectively size 3, of the set $\{A_1, A_2, \ldots, A_{n+1}\}$ (natural incidence), then every local space of $\mathbb{H}_n$ is isomorphic to the linear space $\mathcal{L}_n$ obtained from $\mathcal{M}_n$ by adding lines of size 2. In fact, for every point $x$ of $\mathbb{H}_n$, we can construct the following explicit isomorphism $\phi_x$ between $\mathcal{L}(\mathbb{H}_n, x)$ and $\mathcal{L}_n$. Recall that the point $x$ is a partition $\{A_1, A_2, \ldots, A_{n+1}\}$ of $A$ in $n + 1$ subsets of size 2. Then for every line $L$ of $\mathbb{H}_n$, put $\phi_x(L) := \{A_i, A_j\}$ where $A_i$ and $A_j$ are the unique elements of $\{A_1, A_2, \ldots, A_{n+1}\}$ such that $A_i \cup A_j$ is contained in the partition defined by $L$.

Now, let $x$ be an arbitrary point of $\mathbb{H}_n$ and let $S$ be an arbitrary subspace of $\mathcal{L}(\mathbb{H}_n, x)$. As before, let $\{A_1, \ldots, A_{n+1}\}$ be the partition of $A$ corresponding with $x$ and let $\phi_x$ be the isomorphism between $\mathcal{L}(\mathbb{H}_n, x)$ and $\mathcal{L}_n$ as defined above. Then $\phi_x(S)$ is a subspace of $\mathcal{L}_n$. So, there exist mutually disjoint subsets $\alpha_1, \ldots, \alpha_k$ ($k \geq 0$) of size at least 2 of $\{A_1, A_2, \ldots, A_{n+1}\}$ such that the points of $\phi_x(S)$ are precisely the pairs of $\{A_1, \ldots, A_n\}$ which are contained in $\alpha_i$ for some $i \in \{1, \ldots, k\}$. Now, for every $i \in \{1, \ldots, k\}$, put $B_k := \bigcup_{C \subset \alpha_i} C$ and let $B_{k+1}, \ldots, B_l$ denote those elements of $\{A_1, A_2, \ldots, A_{n+1}\}$ which are not contained in $\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_k$. Then $\{B_1, B_2, \ldots, B_l\}$ is a partition of $A$ in subsets of even size. By Theorem 6.15 of [7], the set of points of $\mathbb{H}_n$ which regarded as partitions of $A$ are refinements of $\{B_1, B_2, \ldots, B_l\}$ is a convex subspace $F_S$ of $\mathbb{H}_n$. The lines of $\mathbb{H}_n$ through $x$ contained in $F_S$ are precisely the elements of $S$. This proves that all local spaces of $\mathbb{H}_n$ are regular. More information on the convex subspaces of the dense near $2n$-gon $\mathbb{H}_n$ can be found in [7, Chapter 6.2].

(IV) Let $H(2n - 1, 4)$, $n \geq 2$, denote the Hermitian variety $X_0^3 + X_1^3 + \cdots + X_{2n-1}^3 = 0$ of $\text{PG}(2n - 1, 4)$ (with respect to a given reference system). If $p$ is a point of $\text{PG}(2n - 1, 4)$, then the number of nonzero coordinates of $p$ (with respect to the same reference system) is called the weight of $p$. The set of all $i \in \{0, 1, \ldots, 2n - 1\}$ such that the $i$-th coordinate of $p$ is nonzero is called the support of $p$. The Hermitian variety $H(2n - 1, 4)$ consists of all points of $\text{PG}(2n - 1, 4)$ of even weight.

Let $\mathcal{G}_n = (\mathcal{P}, \mathcal{L}, I)$ be the following point-line geometry: (i) $\mathcal{P}$ is the set of all maximal subspaces of $H(2n - 1, 4)$ generated by $n$ points of weight 2 whose supports are mutually disjoint; (ii) $\mathcal{L}$ is the set of all $(n - 2)$-dimensional subspaces of $H(2n - 1, 4)$ which contain $n - 2$ points of weight 2 whose supports are mutually disjoint; (iii) incidence is reverse containment. By De Bruyn [6], $\mathcal{G}_n$ is a dense near $2n$-gon. Every line $L$ of $\mathcal{G}_n$ is generated by a unique set of $n - 1$ points whose supports are mutually disjoint. This
set either consists of \( n - 1 \) points of weight 2 or \( n - 2 \) points of weight 2 and 1 point of weight 4. If \( \gamma \) is a subspace of \( H(2n - 1, 4) \) generated by points (of even weight) whose supports are mutually disjoint, then the set of all generators of \( H(2n - 1, 4) \) containing \( \gamma \) is a convex subspace \( F_{\gamma} \) of \( G_n \) ([7, Theorem 6.27]). Conversely, every convex subspace of \( G_n \) is obtained in this way.

Now, consider a reference system in the projective space \( \text{PG}(n - 1, 4) \) and let \( U_i, i \geq 1 \), be the set of points of \( \text{PG}(n - 1, 4) \) of weight \( i \) (with respect to that reference system). Let \( L_n \) denote the linear space induced on \( U_1 \cup U_2 \) by the lines of \( \text{PG}(n - 1, 4) \). Then every local space of \( G_n \) is isomorphic to \( L_n \). In fact for every point \( x \) of \( G_n \), we can construct the following explicit isomorphism \( \phi_x \) between \( L(G_n, x) \) and \( L_n \). Recall that a point \( x \) of \( G_n \) is generated by \( n \) points \( p_1, p_2, \ldots, p_n \) of weight 2 whose supports are mutually disjoint. Put \( \text{PG}(n - 1, 4) = \langle p_1, p_2, \ldots, p_n \rangle \). If \( L \) is a line through \( x \), then one of the following two cases occurs: (1) there exists a unique \( i \in \{1, \ldots, n\} \) such that \( L = \langle \{p_1, \ldots, p_n\} \setminus \{p_i\} \rangle \); (2) there exists a unique pair \( \{i, j\} \subseteq \{1, \ldots, n\} \) such that \( L = \langle \{p_1, \ldots, p_n\} \setminus \{p_i, p_j\} \cup \{r\} \rangle \), where \( r \) is some point (of weight 4) of \( p_i p_j \setminus \{p_i, p_j\} \).

In the former case, we define \( \phi_x(L) := p_i \) and in the latter case, \( \phi_x(L) := r \).

Now, let \( x \) be an arbitrary point of \( G_n \). Then we know that \( x = \langle p_1, p_2, \ldots, p_n \rangle \) where \( p_1, p_2, \ldots, p_n \) are points of weight 2 whose supports are mutually disjoint. For every \( p \in \langle p_1, p_2, \ldots, p_n \rangle \), let \( X_p \) be the smallest subset of \( \{1, \ldots, n\} \) such that \( p \in \langle p_i \mid i \in X_p \rangle \). For all \( i, j \in X_p \) with \( i \neq j \), let \( p_{i,j} \) denote the unique point in the singleton \( \langle p_i, p_j \rangle \cap \langle p, \{p_k \mid k \in X_p \setminus \{i, j\}\} \rangle \).

Now, let \( S \) be an arbitrary subspace of \( L(G_n, x) \). Since \( \phi_x(S) \) is a subspace of \( L_n \), we can find a subset \( A = \{p_{i_1}, \ldots, p_{i_k}\} \subseteq \{p_1, \ldots, p_n\} \) (\( k \geq 0 \)) and points \( q_1, \ldots, q_l \in \langle p_1, \ldots, p_n \rangle \) (\( l \geq 0 \)) such that: (i) \( |X_{q_1}|, \ldots, |X_{q_l}| \geq 2 \); (ii) the sets \( \{p_{i_1}, \ldots, p_{i_k}\}, \{p_i \mid i \in X_{q_1}\}, \ldots, \{p_i \mid i \in X_{q_l}\} \) are mutually disjoint; (iii) a point of \( L_n \) belongs to \( \phi_x(S) \) if and only if it belongs to \( \langle p_{i_1}, \ldots, p_{i_k}\rangle \) or is of the form \( \langle q_i \rangle_{(j,k)} \) for some \( i \in \{1, \ldots, l\} \) and some \( j, k \in X_{q_i} \) with \( j \neq k \). Let \( q_{l+1}, \ldots, q_m \) denote those points of \( \{p_1, \ldots, p_n\} \) which are not contained in \( \{p_{i_1}, \ldots, p_{i_k}\} \cup \{p_i \mid i \in X_{q_1}\} \cup \cdots \cup \{p_i \mid x \in X_{q_l}\} \). Then the supports of the points \( q_{l+1}, \ldots, q_m \) of \( \text{PG}(2n - 1, 4) \) are mutually disjoint. This means that there is a convex subspace \( F_{\beta} \) of \( G_n \) associated with the subspace \( \beta = \langle q_1, \ldots, q_m \rangle \) of \( H(2n - 1, 4) \). Now, a line \( L \) of \( G_n \) through \( x \) belongs to \( F_{\beta} \) if and only if \( L \in S \). This proves that every local space of \( G_n \) is regular. More information on the convex subspaces of the near \( 2n \)-gon \( G_n \) can be found in [7, Section 6.3].
is due to Aschbacher [1]. Every local space of \( \mathbb{E}_3 \) is isomorphic to the linear space \( \overline{W}(2) \) obtained from the generalized quadrangle \( W(2) \) of order 2 by adding its ovoids as extra lines (see [7, Theorem 6.98]). Since every subspace of \( \overline{W}(2) \) is either the empty set, a singleton, a line or the whole space, every local space of \( \mathbb{E}_3 \) is regular.

(VI) Let \( A_1 \) and \( A_2 \) be two dense near polygons of diameter at least 1. Then a product near polygon \( A_1 \times A_2 \) can be defined, see [7, Section 1.6]. Let \( x \) be an arbitrary point of \( A_1 \times A_2 \) and let \( \mathcal{L} \) be a set of lines through \( x \) forming a subspace of \( \mathcal{L}(A_1 \times A_2, x) \). Through \( x \) there are convex subspaces \( F_1 \) and \( F_2 \) such that: (i) \( F_1 \cong A_1, \ F_2 \cong A_2 \); (ii) \( F_1 \cap F_2 = \{x\} \); (iii) every line through \( x \) is contained in either \( F_1 \) or \( F_2 \). Let \( \mathcal{L}_i, \ i \in \{1,2\} \), denote the set of lines through \( x \) contained in \( F_i \). Then \( \mathcal{L}_i \cap \mathcal{L} \) is a subspace of \( \mathcal{L}(F_i, x) \). Since \( \mathcal{L}(F_i, x) \) is regular, there exists a unique convex subspace \( G_i \) of \( F_i \) through \( x \) such that the lines through \( x \) contained in \( G_i \) are precisely the lines of \( \mathcal{L}_i \cap \mathcal{L} \). Now, by [7, Section 4.6], the convex subspace \( \langle G_1, G_2 \rangle \) intersects \( F_1 \) in \( G_1 \) and \( F_2 \) in \( G_2 \). Hence, the set of lines through \( x \) contained in \( \langle G_1, G_2 \rangle \) coincides with \( \mathcal{L} \). This proves that all local spaces of \( A_1 \times A_2 \) are regular.

(VII) Let \( A_1 \) and \( A_2 \) be two dense near polygons of diameter at least 2. If \( A_1 \) and \( A_2 \) satisfy certain nice conditions (see [7, Theorem 4.11]) then a so-called glued near polygon \( A_1 \otimes A_2 \) can be derived from \( A_1 \) and \( A_2 \). Let \( x \) be an arbitrary point of \( A_1 \otimes A_2 \) and let \( \mathcal{L} \) be a set of lines through \( x \) forming a subspace of \( \mathcal{L}(A_1 \otimes A_2, x) \). Through \( x \) there are convex subspaces \( F_1 \) and \( F_2 \) such that: (i) \( F_1 \cong A_1, \ F_2 \cong A_2 \); (ii) \( F_1 \cap F_2 \) is a line \( L \); (iii) every line through \( x \) distinct from \( L \) is contained in either \( F_1 \) or \( F_2 \). Let \( \mathcal{L}_i, \ i \in \{1,2\} \), denote the set of lines through \( x \) contained in \( F_i \). Then \( \mathcal{L}_i \cap \mathcal{L} \) is a subspace of \( \mathcal{L}(F_i, x) \). Since \( \mathcal{L}(F_i, x) \) is regular, there exists a unique convex subspace \( G_i \) of \( F_i \) through \( x \) such that the lines through \( x \) contained in \( G_i \) are precisely the lines of \( \mathcal{L}_i \cap \mathcal{L} \). Now, by [7, Section 4.6], the convex subspace \( \langle G_1, G_2 \rangle \) intersects \( F_1 \) in \( G_1 \) and \( F_2 \) in \( G_2 \). Hence, the set of lines through \( x \) contained in \( \langle G_1, G_2 \rangle \) coincides with \( \mathcal{L} \). This proves that all local spaces of \( A_1 \otimes A_2 \) are regular. For an extensive discussion of glued near polygons and their properties, we refer to [7, Chapter 4].

### 4 Proof of Theorem 1.5

#### 4.1 The case of the near hexagon \( \mathbb{E}_1 \)

Let \( M \) denote the following matrix:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 & 0 \\
\end{bmatrix}
\]
The 12 columns of the matrix $M$ define a set $\mathcal{K}$ of 12 points in $\text{PG}(5,3)$. This set of 12 points, which was first discovered by Coxeter [4], has several nice properties, see e.g. Lemma 4.1 below. For every point $x$ of $\text{PG}(5,3)$, define the generating index $i_\mathcal{K}(x)$ of $x$ as the minimal number of points of $\mathcal{K}$ which are necessary to generate a subspace containing $x$.

**Lemma 4.1 ([4], [8])** (a) The maximal index of a point of $\text{PG}(5,3)$ is equal to 3.
(b) If $L$ is a line of $\text{PG}(5,3)$ through a point $x$ of $\mathcal{K}$, then $L \setminus \{x\}$ contains a unique point with smallest index.
(c) Every $i \in \{1, 2, 3, 4, 5\}$ distinct points of $\mathcal{K}$ generate an $(i-1)$-dimensional subspace of $\text{PG}(5,3)$. The 4-dimensional subspace generated by 5 distinct points of $\mathcal{K}$ contains precisely 6 points of $\mathcal{K}$.
(d) The group of automorphisms of $\text{PG}(5,3)$ stabilizing $\mathcal{K}$ acts 5-transitively on the set of points of $\mathcal{K}$.

Now, embed $\text{PG}(5,3)$ as a hyperplane in the projective space $\text{PG}(6,3)$. Let $\mathbb{E}_1$ be the following point-line geometry: (i) the points of $\mathbb{E}_1$ are the points of $\text{PG}(6,3) \setminus \text{PG}(5,3)$; (ii) the lines of $\mathbb{E}_1$ are the lines of $\text{PG}(6,3)$, not contained in $\text{PG}(5,3)$, which contain a point of $\mathcal{K}$; (iii) incidence is derived from the one of $\text{PG}(6,3)$. Then by De Bruyn and De Clerck [8], $\mathbb{E}_1$ is a dense near hexagon. The first construction of $\mathbb{E}_1$ (using cosets of the extended ternary Golay code) is due to Shult and Yanushka [12, Section 2.5].

Let $\mathcal{P}$ denote the point-set of $\mathbb{E}_1$. In this subsection, we will prove that there exists a map $f : \mathcal{P} \to \mathbb{N}$ satisfying properties (V1) and (V2) such that $\max\{f(x) | x \in \mathcal{P}\} = 2$. In view of the fact that every valuation of $\mathbb{E}_1$ is either classical or ovoidal (see [10, Theorem 1]), this implies that $f$ is not a valuation of $\mathbb{E}_1$. This proves that Theorem 1.5 holds in the case the near hexagon is isomorphic to $\mathbb{E}_1$.

Let $x$ be an arbitrary point of $\text{PG}(5,3)$ with index 3. For every $y_1 \in \mathcal{K}$, the line $xy_1$ contains a unique point $y'_1$ with index 2 (Lemma 4.1(b)). By Lemma 4.1(c), there exist unique points $y_2$ and $y_3$ of $\mathcal{K}$ such that $y'_1 \in \{y_2, y_3\}$. Define $A_{y_1} := \{y_1, y_2, y_3\}$ and $\alpha_{y_1} := \langle y_1, y_2, y_3 \rangle$. By Lemma 4.1(c), $\alpha_{y_1}$ is a plane which intersects $\mathcal{K}$ in the set $A_{y_1}$. Obviously, $A_{y_1} = A_{y_2} = A_{y_3}$ and $x \in \alpha_{y_1} = \alpha_{y_2} = \alpha_{y_3}$. So, the set $U := \{\alpha_y | y \in \mathcal{K}\}$ has size $|\mathcal{K}| = 4$. By Lemma 4.1(c), any two distinct planes of $U$ intersect in the point $x$. Put $U = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Then for every $i \in \{1, 2, 3, 4\}$, there exists a unique line $L_i \subseteq \alpha_i$ through $x$ disjoint from $\alpha_i \cap \mathcal{K}$. Let $\alpha$ be the subspace of $\text{PG}(5,3)$ generated by the lines $L_1, L_2, L_3$ and $L_4$.

We claim that $\alpha$ is a plane of $\text{PG}(5,3)$, i.e. the lines of $\alpha$ through $x$ are precisely the lines $L_1, L_2, L_3$ and $L_4$. By Lemma 4.1(d), we may without loss of generality suppose that the points $(1, 0, 0, 0, 0)$, $(0, 1, 0, 0, 0)$ and $(0, 0, 1, 0, 0)$ belong to $\alpha_1$ and that $(0, 0, 0, 1, 0)$ and $(0, 0, 0, 0, 1)$ belong to $\alpha_2$. Then the sixth point of $\mathcal{K}$ in the subspace $\langle(1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0), (0, 0, 0, 1, 0), (0, 0, 0, 0, 1) \rangle$ is the unique point of $\alpha_2 \cap \mathcal{K}$ distinct from $(0, 0, 0, 1, 0)$ and $(0, 0, 0, 0, 1)$. This point is equal to $(1, 1, -1, -1, 1)$. It follows that the point $x$ is equal to $(1, 1, -1, 0, 0, 0)$. The remaining
planes of \( U \) are (up to transposition) \( \alpha_3 = \{(0, 0, 0, 0, 0, 1), (1, 0, 1, -1, 1, 1), (0, -1, -1, -1, -1, -1)\} \) and \( \alpha_4 = \{(1, 1, 1, -1, -1), (1, -1, 1, 0, 1)\} \). We can now easily calculate the lines \( L_1, L_2, L_3 \) and \( L_4 \):

\[
L_1 = \{(1, 1, -1, 0, 0, 0), (1, -1, 0, 0, 0, 0)\}, \\
L_2 = \{(1, 1, -1, 0, 0, 0), (0, 0, 0, 1, 1, 0)\}, \\
L_3 = \{(1, 1, -1, 0, 0, 0), (1, -1, 0, 1, 1, 0)\}, \\
L_4 = \{(1, 1, -1, 0, 0, 0), (1, -1, 0, -1, -1, -1)\}.
\]

Hence, \( \alpha = \langle L_1, L_2, L_3, L_4 \rangle \) is a plane of \( \PG(5, 3) \). Now, let \( B \) be an arbitrary 3-space of \( \PG(6, 3) \) through \( \alpha \) not contained in \( \PG(5, 3) \). Let \( \mathcal{A} \) denote the projective plane obtained by taking the quotient space of \( \PG(6, 3) \) over the subspace \( B \). The 13 points \( A_1, A_2, \ldots, A_{13} \) of \( \mathcal{A} \) are the 13 4-spaces of \( \PG(6, 3) \) containing \( B \). Without loss of generality, we may suppose that \( A_i = \langle B, \alpha_i \rangle \) for every \( i \in \{1, 2, 3, 4\} \). By Lemma 4.1(c), \( \{A_1, A_2, A_3, A_4\} \) is a set of 4 points of \( \mathcal{A} \), no three of which are collinear.

Now, for every map \( \mu : \{A_1, \ldots, A_{13}\} \to \mathbb{N} \) satisfying \( \mu(A_1) = \mu(A_2) = \mu(A_3) = \mu(A_4) = 1 \), let \( f_\mu \) be the following map from \( \mathcal{P} \) to \( \mathbb{N} \): if \( y \notin B \setminus \PG(5, 3) \), then \( f_\mu(y) = 0 \); if \( y \in A_i \setminus (\PG(5, 3) \cup B) \) for a certain \( i \in \{1, \ldots, 13\} \), then \( f_\mu(y) = \mu(A_i) \). The function \( f_\mu \) satisfies properties (V1) and (V2) if and only if

\[ (*) \text{ for every line } \chi \text{ of } \mathcal{A} \text{ containing } A_i, \ i \in \{1, 2, 3, 4\}, \text{ there exists a unique point } z_{\chi,i} \in \chi \setminus \{A_i\} \text{ such that } \mu(z) = \mu(z_{\chi,i}) + 1 \text{ for every } z \in \chi \setminus \{A_i, z_{\chi,i}\}. \]

We show that there exists a function \( \mu : \{A_1, \ldots, A_{13}\} \to \mathbb{N} \) satisfying Property \((*)\). Let \( A_5, A_6 \) and \( A_7 \) be those points of \( \mathcal{A} \) such that \( \{A_1, A_2, A_3, A_7\} \) and \( \{A_3, A_4, A_6, A_7\} \) are lines of \( \mathcal{A} \). Then define \( \mu(A_1) = \mu(A_2) = \mu(A_3) = \mu(A_4) = \mu(A_5) = \mu(A_6) = 1, \mu(A_7) = 0 \) and \( \mu(A) = 2 \) for every point \( A \) of \( \mathcal{A} \) not contained in \( \{A_1, \ldots, A_7\} \). Then \( \mu \) satisfies Property \((*)\). Hence, \( f_\mu \) is a map satisfying (V1) and (V2). Since the maximal value attained by \( f_\mu \) is equal to 2, \( f_\mu \) is not a valuation of \( E_1 \). (Recall that every valuation of \( E_1 \) is either classical or ovoidal.)

### 4.2 The case of the near hexagon \( \mathbb{E}_2 \)

Let \( D \) denote the unique Steiner system \( S(5, 8, 24) \). (Recall that there are 24 points in such a Steiner system, each block contains 8 points and every five distinct points are contained in a unique block.) If \( B_1 \) and \( B_2 \) are two distinct blocks of \( S(5, 8, 24) \), then \( |B_1 \cap B_2| \in \{0, 2, 4\} \). Moreover, if \( |B_1 \cap B_2| = 0 \), then the complement of \( B_1 \cup B_2 \) is again a block. From \( S(5, 8, 24) \), we can construct the following incidence structure \( \mathbb{E}_2 \): (i) the points of \( \mathbb{E}_2 \) are the blocks of \( S(5, 8, 24) \); (ii) the lines of \( \mathbb{E}_2 \) are the triples of mutually disjoint blocks; (iii) incidence is containment. Then \( \mathbb{E}_2 \) is a dense near hexagon by Shult and Yanushka [12, p. 40] (see also [7, Section 6.6]).

Now, let \( x \) and \( y \) be two given distinct points of \( D \). If \( B \) is a block of \( D \), then we define \( f(B) := 0 \) if \( x, y \in B \), \( f(B) := 1 \) if \( x, y \notin B \) and \( f(B) := 2 \) if \( |\{x, y\} \cap B| = 1 \). Clearly, \( f \) satisfies Properties (V1) and (V2). The map \( f \) cannot be a classical valuation.
of $\mathbb{E}_2$ (since $f(B) \neq 3$ for any block $B$ of $\mathcal{D}$), nor an ovoidal valuation of $\mathbb{E}_2$ (there exists a block $B$ with $f(B) = 2$). Now, by [10, Theorem 2] (see also [7, Theorem 6.81]), every valuation of $\mathbb{E}_2$ is either classical or ovoidal. Hence, $f$ is not a valuation.

References