On the nucleus of the Grassmann embedding of the symplectic dual polar space
\(DSp(2n, \mathbb{F}), \text{char}(\mathbb{F}) = 2\)

Rieuwert J. Blok, Ilaria Cardinali and Bart De Bruyn

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Abstract

Let \(n \geq 3\) and let \(\mathbb{F}\) be a field of characteristic 2. Let \(DSp(2n, \mathbb{F})\) denote the dual polar space associated with the building of Type \(C_n\) over \(\mathbb{F}\) and let \(G_{n-2}\) denote the \((n - 2)\)-Grassmannian of type \(C_n\). Using the bijective correspondence between the points of \(G_{n-2}\) and the quads of \(DSp(2n, \mathbb{F})\), we construct a full projective embedding of \(G_{n-2}\) into the nucleus of the Grassmann embedding of \(DSp(2n, \mathbb{F})\). This generalizes a result of the paper [9] which contains an alternative proof of this fact in the case when \(n = 3\) and \(\mathbb{F}\) is finite.

1 Introduction and preliminaries

We assume the reader is familiar with the concept of a partial linear rank two incidence geometry \(\Gamma = (\mathcal{P}, \mathcal{L})\), also called a point-line geometry (See e.g. [5, 18]). The distance \(\text{dist}(x, y)\) between two points \(x, y \in \mathcal{P}\) of \(\Gamma\) will be measured in the collinearity graph of \(\Gamma\), that is the graph \((\mathcal{P}, E)\) whose set of edges consists of all unordered pairs of points belonging to a line of \(\Gamma\). By a subspace of \(\Gamma\) we mean a subset \(S\) of \(\mathcal{P}\) such that if \(l \in \mathcal{L}\) and \(|l \cap S| \geq 2\), then \(l \subseteq S\). If \(S\) is a subspace of \(\Gamma\), then we denote by \(\bar{S}\) the point-line geometry \((S, \mathcal{L}_S)\) where \(\mathcal{L}_S := \{l \in \mathcal{L} \mid l \subseteq S\}\). A subspace \(S\) is called convex if for any three points \(x, y, z \in \mathcal{P}\), \(\text{dist}(x, y) + \text{dist}(y, z) = \text{dist}(x, z)\) and \(x, z \in S\) imply that also \(y \in S\). The maximal distance between two points of a convex subspace \(S\) is called the diameter of \(S\).

Let \(\Gamma_1 = (\mathcal{P}_1, \mathcal{L}_1)\) and \(\Gamma_2 = (\mathcal{P}_2, \mathcal{L}_2)\) be two point-line geometries with respective distance functions \(\text{dist}_1(\cdot, \cdot)\) and \(\text{dist}_2(\cdot, \cdot)\). A full embedding of \(\Gamma_1\) into \(\Gamma_2\) is an injective mapping \(e\) from \(\mathcal{P}_1\) to \(\mathcal{P}_2\) such that \(e(L) := \{e(x) \mid x \in L\}\)
is a line of $\Gamma_2$ for every line $L$ of $\Gamma_1$. A full embedding is called isometric if $\text{dist}_2(e(x), e(y)) = \text{dist}_1(x, y)$ for all $x, y \in \mathcal{P}_1$. If $\Gamma_2$ is a projective space and if $e(\mathcal{P}_1)$ generates the whole of $\Gamma_2$, then $e$ is called a full projective embedding. In this case, the dimension of the projective space $\Gamma_2$ is called the projective dimension of $e$. Isomorphisms between full (projective) embeddings, which we will denote by the symbol $\cong$, are defined in the usual way.

Suppose $e : \Gamma \to \Sigma$ is a full embedding of the point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ into the projective space $\Sigma$, and $\alpha$ is a subspace of $\Sigma$ satisfying:

(C1) $e(x) \not\in \alpha$ for every point $x \in \mathcal{P}$;

(C2) $\langle \alpha, e(x) \rangle \neq \langle \alpha, e(y) \rangle$ for every two distinct points $x$ and $y$ of $\Gamma$.

Then the mapping $e/\alpha : \Gamma \to \Sigma/\alpha; x \mapsto \langle \alpha, e(x) \rangle$ is a full embedding of $\Gamma$ into the quotient projective space $\Sigma/\alpha$. If $e_1 : \Gamma \to \Sigma_1$ and $e_2 : \Gamma \to \Sigma_2$ are two full projective embeddings of $\Gamma$, then we say that $e_1 \geq e_2$ if there exists a subspace $\alpha$ of $\Sigma_1$ satisfying (C1), (C2) such that $e_1/\alpha$ is isomorphic to $e_2$.

An important class of point-line geometries are the dual polar spaces. With every non-degenerate polar space $\Pi$ of rank $n$, there is associated a dual polar space $\Delta$ of rank $n$. The points of $\Delta$ are the maximal singular subspaces of $\Pi$ (i.e. the singular subspaces of projective dimension $n - 1$) and the lines of $\Delta$ are the sets $L_\alpha$ of maximal singular subspaces containing a given singular subspace $\alpha$ of projective dimension $n - 2$. There is a bijective correspondence between the possibly empty singular subspaces of $\Pi$ and the convex subspaces of $\Delta$. If $\beta$ is an $(n - 1 - k)$-dimensional, $k \in \{0, \ldots, n\}$, singular subspace of $\Pi$, then the set of all singular subspaces containing $\beta$ is a convex subspace of $\Delta$ of diameter $k$. These convex subspaces are called quads if $k = 2$, hexes if $k = 3$ and maxes if $k = n - 1$. If $F$ is a convex subspace of $\Delta$ of diameter $k \geq 2$, then $\tilde{F}$ is a dual polar space of rank $k$.

Suppose $\Delta = (\mathcal{P}, \mathcal{L})$ is a thick dual polar space of rank $n$. For every point $x$ of $\Delta$, let $H_x$ denote the set of points at non-maximal distance (i.e. distance at most $n - 1$) from $x$. If $e : \Delta \to \Sigma$ is a full projective embedding of $\Delta$ and if $F$ is a convex subspace of $\Delta$, then $e$ induces a full embedding $e_F$ of $\tilde{F}$ into a subspace $\Sigma_F$ of $\Sigma$. A full embedding $e$ of $\Delta$ into a projective space $\Sigma$ is called polarized if $\langle e(H_x) \rangle$ is a hyperplane of $\Sigma$ for every point $x$ of $\Delta$. If $e$ is a full polarized embedding of $\Delta$, then $\mathcal{N}_e := \bigcap_{x \in \mathcal{P}} \langle e(H_x) \rangle$ is called the nucleus of the embedding $e$. The nucleus $\mathcal{N}_e$ satisfies the properties (C1) and (C2) above and hence there exists a full embedding $\tilde{e} := e/\mathcal{N}_e$ of $\Delta$ into the projective space $\Sigma/\mathcal{N}_e$. If $e_1$ is an arbitrary full polarized embedding of $\Delta$, then by Cardinali, De Bruyn and Pasini [8], $e_1 \geq \tilde{e}$ and $\tilde{e}_1 \cong \tilde{e}$. The embedding $\tilde{e}$ is called the minimal full polarized embedding of $\Delta$. The following is also proved in [8].
Lemma 1.1 If $F$ is a convex subspace of diameter at least 2 of $\Delta$, then $(\bar{e})_F$ is isomorphic to the minimal full polarized embedding of $\tilde{F}$.

Now, let $n \geq 3$ and let $F$ be a field of characteristic 2. Consider the dual polar space $DSp(2n, F)$ associated with the building of type $C_n$ over $F$ (see Section 2). This dual polar space admits a full embedding $e_{gr}$ into a projective space of dimension $\binom{2n}{n} - \binom{2n}{n-2} - 2n - 1$, called the Grassmann embedding of $DSp(2n, F)$. If $Q$ is a quad of $DSp(2n, F)$, then $e_Q := (e_{gr})_Q$ is isomorphic to the Grassmann embedding of $DSp(4, F) \cong O(5, F)$ into $PG(4, F)$ and hence $\mathcal{N}_{e_Q}$ is a singleton, see e.g. Section 2. Let $G_{n-2}$ denote the following point-line geometry: the points of $G_{n-2}$ are the quads of $DSp(2n, F)$ and the lines of $G_{n-2}$ are the sets of quads of $DSp(2n, F)$ which contain a given line of $DSp(2n, F)$ and which are contained in a given hex of $DSp(2n, F)$. The following is the main result of this paper:

Main Theorem (1) The dimension of $\mathcal{N}_{e_{gr}}$ is equal to $\binom{2n}{n} - \binom{2n}{n-2} - 2n - 1$. (2) For every quad $Q$ of $DSp(2n, F)$, the singleton $\mathcal{N}_{e_Q}$ is contained in $\mathcal{N}_{e_{gr}}$. (3) The map $Q \mapsto \mathcal{N}_{e_Q}$ defines a full projective embedding of $G_{n-2}$ into $\mathcal{N}_{e_{gr}}$.

The geometry $G_{n-2}$ is isomorphic to the $(n-2)$-Grassmannian of type $C_n$, that is the point-line geometry with points the objects of rank $n-2$ of $C_n$ (i.e. the spaces of vector dimension $n-2$) and with lines the sets $l_{[A,B]} := \{x \mid \text{rank}(x) = n-2, A \subset x \subset B\}$, where $A$ and $B$ are objects of rank $n-3$ and $n-1$, respectively.

Grassmannians of polar spaces have attracted some attention recently. Embeddings, generating ranks, special subspaces, and hyperplane complements have recently been under investigation in the literature, see e.g. [1, 13, 11, 2, 12, 3, 4].

We will prove the main theorem in Section 4. This main theorem generalizes Theorem 1.3 of the paper [9] which contains an alternative proof of Main Theorem for the case when $n = 3$ and $F$ is a finite field of even characteristic.

2 Notation and the dimension of $\mathcal{N}_{e_{gr}}$

Let $n \in \mathbb{N} \setminus \{0, 1\}$ and let $F$ be a field of characteristic 2. Let $V$ be a 2n-dimensional vector space over $F$ equipped with a non-degenerate alternating form $(\cdot, \cdot)$. An ordered basis $(\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n, \bar{f}_1, \bar{f}_2, \ldots, \bar{f}_n)$ of $V$ is called a hyperbolic basis of $V$ (with respect to $(\cdot, \cdot)$) if $(\bar{e}_i, \bar{e}_j) = (\bar{f}_i, \bar{f}_j) = 0$ and $(\bar{e}_i, \bar{f}_j) = \delta_{ij}$ for all $i, j \in \{1, \ldots, n\}$. Let $\wedge^n V$ denote the $n$-th exterior power of $V$ and let $W$ denote the subspace of $\wedge^n V$ generated by all vectors of the
form \( \tilde{v}_1 \land \tilde{v}_2 \land \cdots \land \tilde{v}_n \), where \( \tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_n \) are vectors of \( V \) satisfying \( (\tilde{v}_i, \tilde{v}_j) = 0 \) for all \( i, j \in \{1, \ldots, n\} \). The dimension of \( W \) is equal to \( \binom{2n}{n} - \binom{2n}{n-2} \), see e.g. De Bruyn [14].

The subspaces of \( V \) which are totally isotropic with respect to \((\cdot, \cdot)\) define a building of type \( C_n \). We denote the associated dual polar space by \( DSp(2n, F) \). We denote by \( DO(2n + 1, F) \) the dual polar space associated with the building of type \( B_n \) which arises from a vector space of dimension \( 2n + 1 \) over \( F \) equipped with a non-degenerate quadratic form of Witt-index \( n \). The dual polar spaces \( DSp(2n, F) \) and \( DO(2n + 1, F) \) are isomorphic if and only if the field \( F \) is perfect (see e.g. De Bruyn and Pasini [17]). In [17] it is also shown that for any field \( F \) (of characteristic \( 2 \)), there exists an isometric full embedding of \( DSp(2n, F) \) into \( DO(2n + 1, F) \).

For every point \( p = \langle \tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_n \rangle \) of \( DSp(2n, F) \), let \( e_{gr}(p) \) denote the point \( \langle \tilde{v}_1 \land \tilde{v}_2 \land \cdots \land \tilde{v}_n \rangle \) of \( PG(W) \). Notice that \( e_{gr}(p) \) is independent of the generating set \( \{\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_n\} \) of \( p \). The map \( e_{gr} \) defines a full embedding of \( DSp(2n, F) \) into \( PG(W) \), called the Grassmann embedding of \( DSp(2n, F) \). Let \( \mathcal{N} := N_{e_{gr}} \) denote the nucleus of the embedding \( e_{gr} \). The dual polar space \( DO(2n + 1, F) \) admits a full polarized embedding into the projective space \( PG(2^n - 1, F) \) called the spin embedding of \( DO(2n + 1, F) \) (Chevalley [10], Buekenhout and Cameron [6], Cameron [7]). In view of the existence of an isometric full embedding of \( DSp(2n, F) \) into \( DO(2n + 1, F) \), the dual polar space \( DSp(2n, F) \) admits a full polarized embedding \( e_{sp} \) into a subspace \( \Sigma \) of \( PG(2^n - 1, F) \). Now, by De Bruyn and Pasini [16, Corollary 1.5], any full polarized embedding of a thick dual polar space of rank \( n \) has projective dimension at least \( 2^n - 1 \). Hence \( \Sigma = PG(2^n - 1, F) \) and \( e_{sp} \) is the minimal full polarized embedding of \( DSp(2n, F) \). This implies that the projective dimension of \( \mathcal{N} \) equals \( \binom{2n}{n} - \binom{2n}{n-2} - 2^n - 1 \).

3 The nucleus in the case \( n = 2 \)

In this section, we suppose that \( n = 2 \). Then \( \mathcal{N} \) is a singleton and the vector space \( W \) has dimension 5. Since \( p_1 = \langle \tilde{e}_1, \tilde{e}_2 \rangle \), \( p_2 = \langle \tilde{f}_1, \tilde{f}_2 \rangle \), \( p_3 = \langle \tilde{e}_1, \tilde{f}_2 \rangle \), \( p_4 = \langle \tilde{e}_2, \tilde{f}_1 \rangle \) and \( p_5 = \langle \tilde{e}_1 + \tilde{e}_2, \tilde{f}_1 + \tilde{f}_2 \rangle \) are points of \( DSp(4, F) \) and \( \tilde{e}_1 \land \tilde{e}_2, \tilde{f}_1 \land \tilde{f}_2, \tilde{e}_1 \land \tilde{f}_2, \tilde{e}_2 \land \tilde{f}_1, (\tilde{e}_1 + \tilde{e}_2) \land (\tilde{f}_1 + \tilde{f}_2) \) are linearly independent vectors of \( W \), we have

\[
W = \langle \tilde{e}_1 \land \tilde{e}_2, \tilde{f}_1 \land \tilde{f}_2, \tilde{e}_1 \land \tilde{f}_2, \tilde{e}_2 \land \tilde{f}_1, (\tilde{e}_1 + \tilde{e}_2) \land (\tilde{f}_1 + \tilde{f}_2) \rangle \\
= \langle \tilde{e}_1 \land \tilde{e}_2, \tilde{f}_1 \land \tilde{f}_2, \tilde{e}_1 \land \tilde{f}_2, \tilde{e}_2 \land \tilde{f}_1, \tilde{e}_1 \land \tilde{f}_1 + \tilde{e}_2 \land \tilde{f}_2 \rangle.
\]

Now, the image of \( e_{gr} \) is a quadric \( Q \cong O(5, F) \) of \( PG(W) \). The tangent hyperplane \( T(p_1) \) at the point \( e_{gr}(p_1) \) of \( Q \) is equal to \( \langle \tilde{e}_1 \land \tilde{e}_2, \tilde{e}_1 \land \tilde{f}_2, \tilde{e}_2 \land \tilde{f}_1 \rangle \).
Lemma 4.1. The following lemma, e.g. in [8, Proposition 4.10]. So, for every quad to the Grassmann embedding of $\widetilde{\mathcal{G}}$ in Section 1 with respect to the embedding $\mathcal{N}$. Suppose $\mathcal{F}$ is a quad of $\mathcal{N}$ that coincides with the subspace of $\mathcal{F}$, see e.g. Cardinali, De Bruyn and Pasini [8, Proposition 4.10]. So, for every quad $Q$ of $\mathcal{N}$, the nucleus of $e_Q$ consists of a single point. We will denote this point by $e_{\mathcal{N}}(Q)$. By the following lemma, $e_{\mathcal{N}}$ can be regarded as a map between the set of points of $\mathcal{G}_{n-2}$ and the set of points of $\mathcal{N}$.

Lemma 4.1 (i) For every quad $Q$ of $\mathcal{N}$, $e_{\mathcal{N}}(Q) \in \mathcal{N}$.

(ii) $\mathcal{N}$ coincides with the subspace of $\mathcal{F}$ generated by the points $e_{\mathcal{N}}(Q)$, where $Q$ is a quad of $\mathcal{N}$.

Proof Suppose $\mathcal{N}'$ is a subspace satisfying properties (C1) and (C2) of Section 1 with respect to the embedding $e_{gr}$. Then for every quad $Q$ of $\mathcal{N}$, $\mathcal{N}' \cap \Sigma_Q$ satisfies properties (C1) and (C2) with respect to the embedding $e_Q$. Moreover,

$$e_Q/(\mathcal{N}' \cap \Sigma_Q) \cong (e_{gr}/\mathcal{N})_Q.$$  \hspace{1cm} (1)

(i) Since $e_{gr}/\mathcal{N}$ is the minimal full polarized embedding of $\mathcal{N}$, $\mathcal{N}' \cap \Sigma_Q$ is isomorphic to the minimal full polarized embedding of $Q$ for every quad $Q$ of $\mathcal{N}$ (see Lemma 1.1). From (1), it then follows that $\mathcal{N}' \cap \Sigma_Q = \mathcal{N}_{e_Q} = \{e_{\mathcal{N}}(Q)\}$. Hence, $e_{\mathcal{N}}(Q) \in \mathcal{N}$.

(ii) Suppose $\mathcal{N}'$ is the subspace of $\mathcal{N}$ generated by all points $e_{\mathcal{N}}(Q)$ where $Q$ is a quad of $\mathcal{N}$. Then for every quad $Q$ of $\mathcal{N}$, $\mathcal{N}' \cap \Sigma_Q \subseteq \mathcal{N} \cap \Sigma_Q = \{e_{\mathcal{N}}(Q)\}$. Hence, $\mathcal{N}' \cap \Sigma_Q = \{e_{\mathcal{N}}(Q)\}$. By (1), the embedding $e_{gr}/\mathcal{N}'_Q$ has projective dimension 3. Now, by De Bruyn [15, Theorem 1.6], if $e'$ is a full polarized embedding of a dual polar space of rank $n$ such that every induced quad embedding has projective dimension 3, then $e'$ has projective dimension $2^n - 1$. Applying this here, we see that the full
polarized embedding $e_{gr}/\mathcal{N}'$ has projective dimension $2^n - 1$. This implies that $\mathcal{N}' = \mathcal{N}$. 

**Lemma 4.2** $e_{\mathcal{N}}$ maps in a bijective way any line of $\mathbb{G}_{n-2}$ to some line of $\mathcal{N}$.

**Proof** If $H$ is a hex of $DSp(2n, \mathbb{F})$, then the full embedding $e_H$ of $\tilde{H}$ induced by $e_{gr}$ is isomorphic to the Grassmann embedding of $\tilde{H}$. So, it suffices to prove the lemma in the case $n = 3$. Consider the line $L^*$ of $\mathbb{G}_{n-2}$ which consists of all quads of $DSp(6, \mathbb{F})$ which contain a given line $L$ of $DSp(6, \mathbb{F})$.

We can choose a hyperbolic basis $(\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{f}_1, \bar{f}_2, \bar{f}_3)$ of $V$ in such a way that $L = \langle \bar{e}_1, \bar{e}_2 \rangle$. Let $Q$ denote the quad of $DSp(6, \mathbb{F})$ corresponding to $\langle \bar{e}_1 \rangle$ and for every $t \in \mathbb{F}$, let $Q_t$ denote the quad of $DSp(6, \mathbb{F})$ corresponding to $\langle \bar{e}_2 + t \bar{e}_1 \rangle$. Then by Section 3, $e_{\mathcal{N}}(Q) = \langle \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_2 + \bar{e}_1 \wedge \bar{e}_3 \wedge \bar{f}_3 \rangle$.

Since $(\bar{e}_2 + t \bar{e}_1, \bar{e}_1, \bar{e}_3, \bar{f}_2, \bar{f}_1 + t \bar{f}_2, \bar{f}_3)$ is a hyperbolic basis of $V$, we have $e_{\mathcal{N}}(Q_t) = \langle (\bar{e}_2 + t \bar{e}_1) \wedge \bar{e}_1 \wedge (\bar{f}_1 + t \bar{f}_2) + (\bar{e}_2 + t \bar{e}_1) \wedge \bar{e}_3 \wedge \bar{f}_3 \rangle = \langle (\bar{e}_2 \wedge \bar{e}_1 \wedge \bar{f}_1 + \bar{e}_2 \wedge \bar{e}_3 \wedge \bar{f}_3) + t(\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_2 + \bar{e}_1 \wedge \bar{e}_3 \wedge \bar{f}_3) \rangle$. Hence, $e_{\mathcal{N}}$ defines a bijection between the line $L^*$ of $\mathbb{G}_{n-2}$ and a line of $\mathcal{N}$.

**Lemma 4.3** The map $e_{\mathcal{N}}$ is injective.

**Proof** Let $Q_1$ and $Q_2$ be two distinct quads of $DSp(2n, \mathbb{F})$. We need to show that $e_{\mathcal{N}}(Q_1) \neq e_{\mathcal{N}}(Q_2)$.

(i) If $Q_1 \cap Q_2$ is a line, then Lemma 4.2 implies that $e_{\mathcal{N}}(Q_1) \neq e_{\mathcal{N}}(Q_2)$.

(ii) Suppose that $Q_1 \cap Q_2$ is a singleton $\{x\}$. Let $F$ denote the convex subspace of diameter 4 containing $Q_1$ and $Q_2$. Since the embedding $e_F$ of $\tilde{F}$ induced by $e$ is isomorphic to the Grassmann embedding of $\tilde{F}$, we may suppose that $n = 4$. We can choose a hyperbolic basis $(\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4, \bar{f}_1, \bar{f}_2, \bar{f}_3, \bar{f}_4)$ of $V$ in such a way that the point $x$ corresponds to $(\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4)$ and that the quads $Q_1$ and $Q_2$ correspond to $(\bar{e}_1, \bar{e}_2)$ and $(\bar{e}_3, \bar{e}_4)$, respectively. Then by Section 3, $e_{\mathcal{N}}(Q_1) = \langle \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 \wedge \bar{f}_3 + \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_4 \wedge \bar{f}_4 \rangle \neq \langle \bar{e}_3 \wedge \bar{e}_4 \wedge \bar{e}_1 \wedge \bar{f}_1 + \bar{e}_3 \wedge \bar{e}_4 \wedge \bar{e}_2 \wedge \bar{f}_2 \rangle = e_{\mathcal{N}}(Q_2)$.

(iii) Suppose $Q_1$ and $Q_2$ are disjoint. Then there exist maxes $M_1$ and $M_2$ such that $Q_1 \subseteq M_1$, $Q_2 \subseteq M_2$ and $M_1 \cap M_2 = \emptyset$. We can choose a hyperbolic basis $(\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n, \bar{f}_1, \bar{f}_2, \ldots, \bar{f}_n)$ of $V$ in such a way that $M_1$ corresponds to $\langle \bar{e}_1 \rangle$ and $M_2$ corresponds to $\langle \bar{f}_1 \rangle$. Then $e_{\mathcal{N}}(M_1)$ is the subspace of $PG(W)$ generated by all points of the form $\langle \bar{e}_1 \wedge \bar{g}_2 \wedge \bar{g}_3 \wedge \cdots \wedge \bar{g}_n \rangle$, where $\bar{g}_2, \bar{g}_3, \ldots, \bar{g}_n$ are linearly independent vectors of $\langle \bar{e}_2, \ldots, \bar{e}_n, \bar{f}_1, \ldots, \bar{f}_n \rangle$ satisfying $(\bar{g}_i, \bar{g}_j) = 0$ for all $i, j \in \{2, \ldots, n\}$. Similarly, $e_{\mathcal{N}}(M_2)$ is the subspace of $PG(W)$ generated by all points of the form $\langle \bar{f}_1 \wedge \bar{g}_2 \wedge \bar{g}_3 \wedge \cdots \wedge \bar{g}_n \rangle$, where $\bar{g}_2, \bar{g}_3, \ldots, \bar{g}_n$ are linearly independent vectors of $\langle \bar{e}_2, \ldots, \bar{e}_n, \bar{f}_2, \ldots, \bar{f}_n \rangle$ satisfying $(\bar{g}_i, \bar{g}_j) = 0$ for all $i, j \in \{2, \ldots, n\}$. Clearly, $e_{\mathcal{N}}(M_1)$ and $e_{\mathcal{N}}(M_2)$ are disjoint. This implies that $e_{\mathcal{N}}(Q_1) \neq e_{\mathcal{N}}(Q_2)$. 


References


