The hyperplanes of $DW(5, 2^h)$ which arise from embedding

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Abstract

We show that there are 6 isomorphism classes of hyperplanes of the dual polar space $\Delta = DW(5, 2^h)$ which arise from the Grassmann-embedding. If $h \geq 2$, then these are all the hyperplanes of $\Delta$ arising from an embedding. If $h = 1$, then there are 6 extra classes of hyperplanes as has been shown by Pralle [23] with the aid of a computer. We will give a computer free proof for this fact. The hyperplanes of $DW(5, q)$, $q$ odd, arising from an embedding will be classified in the forthcoming paper [8].

Keywords: symplectic dual polar space, hyperplane, Grassmann-embedding, universal embedding

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1 Introduction

1.1 Dual polar spaces and their hyperplanes

Let $\Pi$ be a nondegenerate polar space of rank $n \geq 2$. With $\Pi$ there is associated a point-line geometry $\Delta$ whose points are the maximal singular subspaces of $\Pi$, whose lines are the next-to-maximal singular subspaces of $\Pi$ and whose incidence relation is reverse containment. The geometry $\Delta$
is called a dual polar space of rank \( n \) (Cameron [4]). \( \Delta \) is a near polygon (Shult and Yanushka [28]; De Bruyn [9]) which means that for every point \( x \) and every line \( L \), there exists a unique point on \( L \) nearest to \( x \). Here, distances \( d(\cdot, \cdot) \) are measured in the point graph of \( \Delta \). For every point \( x \) of \( \Delta \) and every \( i \in \mathbb{N} \), \( \Delta_i(x) \) denotes the set of points of \( \Delta \) at distance \( i \) from \( x \), and \( x^\perp \) denotes the set \( \Delta_0(x) \cup \Delta_1(x) \). For every nonempty set \( X \) of points of \( \Delta \) and every \( i \in \mathbb{N} \), \( \Delta_i(X) \) denotes the set of all points \( x \) of \( \Delta \) for which \( \min\{j \mid \Delta_j(x) \cap X \neq \emptyset \} = i \). The maximal singular subspaces through a given singular \( (n-1-i) \)-dimensional subspace of \( \Pi \) \((0 \leq i \leq n)\) define a convex subspace of \( \Delta \) of diameter \( i \). These convex subspaces are called quads if \( i = 2 \) and maxes if \( i = n-1 \). We will use the notation \( \langle *, * \rangle \) to denote the smallest convex subspace containing the objects \( *, * \). The points and lines contained in a quad define a generalized quadrangle (Payne and Thas [21]). For every point \( x \) and every convex subspace \( F \), there exists a unique point \( \pi_F(x) \) in \( F \) nearest to \( x \) and \( d(x, y) = d(x, \pi_F(x)) + d(\pi_F(x), y) \) for every point \( y \) of \( F \). We call \( \pi_F(x) \) the projection of \( x \) onto \( F \). The convex subspaces through a given point \( x \) of \( \Delta \) define a projective space of dimension \( n-1 \) which we will denote by \( \text{Res}_\Delta(x) \). If \( F \) is a max and if \( A \) is a convex subspace, then either \( A \cap F = \emptyset \) or \( A \subseteq F \) or \( A \cap F \) is a max of \( A \).

We will denote dual polar spaces by putting a “D” in front of the name of the corresponding polar spaces. This paper mainly deals with the dual polar space \( DW(5, q) \) which is the dual of the polar space \( W(5, q) \) of the subspaces of \( \text{PG}(5, q) \) which are totally isotropic with respect to a given symplectic polarity.

A hyperplane of a point-line geometry \( S \) is a proper subspace \( H \) meeting each line. Let \( \Delta \) be a dual polar space of rank \( n \geq 2 \). For every point \( x \) of \( \Delta \), let \( H_x \) denote the set of points of \( \Delta \) at non-maximal distance from \( x \). Since \( \Delta \) is a near polygon, \( H_x \) is a hyperplane of \( \Delta \). We call \( H_x \) the singular hyperplane with deepest point \( x \). Suppose \( A \) is a convex subspace of diameter \( \delta \) of \( \Delta \) and let \( H_A \) be a hyperplane of \( A \). By De Bruyn and Vandecasteele [15, Proposition 1], the set \( H \) of points of \( \Delta \) at distance at most \( n - \delta \) from at least one point of \( H_A \) is a hyperplane \( H \) of \( \Delta \). We call \( H \) the extension of \( H_A \).

If \( H \) is a hyperplane of a thick dual polar space \( \Delta \) and if \( Q \) is a quad of \( \Delta \), then either (i) \( Q \subseteq H \), (ii) there exists a point \( x \) in \( Q \) such that \( x^\perp \cap Q = H \cap Q \), (iii) \( Q \cap H \) is a subquadrangle of \( Q \), or (iv) \( Q \cap H \) is an ovoid of \( Q \), i.e. a set of points of \( Q \) meeting each line in a unique point. If case (i), case (ii), case (iii), respectively case (iv), occurs, then we say that \( Q \)
is deep, singular, subquadrangular, respectively ovoidal, with respect to $H$. A hyperplane $H$ of $\Delta$ is called locally singular, locally ovoidal, respectively locally subquadrangular, if every nondeep quad is singular, ovoidal, respectively subquadrangular, with respect to $H$.

By Lemma 6.1 of [26], every hyperplane of a thick dual polar space is a maximal subspace.

1.2 Projective embeddings

Let $S$ be a point-line geometry and let $V$ be a finite-dimensional vector space. A full embedding of $S$ in $\Sigma := \text{PG}(V)$ is an injective mapping $e$ from the point-set $P$ of $S$ to the point-set of $\Sigma$ satisfying (i) $\langle e(P) \rangle = \Sigma$ and (ii) $e(L) := \{e(x) \mid x \in L\}$ is a line of $\Sigma$ for every line $L$ of $S$. The integer $\dim(\Sigma)$ is called the projective dimension of the embedding. If $e : S \to \Sigma$ is a full embedding, then for every hyperplane $\alpha$ of $\Sigma$, $H(\alpha) := e^{-1}(e(P) \cap \alpha)$ is a hyperplane of $S$. We say that the hyperplane $H(\alpha)$ arises from the embedding $e$.

Two embeddings $e_1 : S \to \Sigma_1$ and $e_2 : S \to \Sigma_2$ are called isomorphic ($e_1 \cong e_2$) if there exists an isomorphism $f : \Sigma_1 \to \Sigma_2$ such that $e_2 = f \circ e_1$. If $e : S \to \Sigma$ is a full embedding of $S$ and if $U$ is a subspace of $\Sigma$ satisfying

(C1) $\langle U, e(p) \rangle \neq U$ for every point $p$ of $S$,
(C2) $\langle U, e(p_1) \rangle \neq \langle U, e(p_2) \rangle$ for any two distinct points $p_1$ and $p_2$ of $S$,

then there exists a full embedding $e/U$ of $S$ in the quotient space $\Sigma/U$, mapping each point $p$ of $S$ to $\langle U, e(p) \rangle$. If $e_1 : S \to \Sigma_1$ and $e_2 : S \to \Sigma_2$ are two embeddings, then we say that $e_1 \succeq e_2$, if there exists a subspace $U$ in $\Sigma_1$ satisfying (C1), (C2) and $e_1/U \cong e_2$. If $e : S \to \Sigma = \text{PG}(V)$ is a full embedding of $S$, then by Ronan [24], there exists a unique (up to isomorphism) full embedding $\tilde{e} : S \to \tilde{\Sigma} = \text{PG}(\tilde{V})$ satisfying the following:
(i) $V$ and $\tilde{V}$ have isomorphic underlying division rings; (ii) $\tilde{e} \succeq e$; (iii) if $e' \succeq e$ for some embedding $e'$ of $S$, then $\tilde{e} \succeq e'$. We say that $\tilde{e}$ is universal relative to $e$. If $\tilde{e}' \cong \tilde{e}$ for any other embedding $e'$ of $S$ with the same underlying division ring, then $\tilde{e}$ is called absolutely universal.

Now, let us turn our attention to thick dual polar spaces. Suppose $e : \Delta \to \Sigma$ is a full embedding of a thick dual polar space $\Delta$ with point-set $P$ in the projective space $\Sigma$. If $H$ is a hyperplane of $\Delta$, then since $H$ is a maximal subspace, $\langle e(H) \rangle$ is either $\Sigma$ or a hyperplane of $\Sigma$. Moreover, if $\langle e(H) \rangle$ is
a hyperplane $\alpha$ of $\Sigma$, then $e(P) \cap \alpha = e(H)$, i.e. $H = e^{-1}(\alpha \cap e(P))$. So, there exists a bijective correspondence between the hyperplanes of $\Sigma$ and the hyperplanes of $\Delta$ which arise from the embedding $e$.

A full embedding $e$ of a thick dual polar space is called polarized if every singular hyperplane arises from $e$.

1.3 Full polarized embeddings of $DW(5, q)$

Suppose $\alpha$ is a two-dimensional subspace of $PG(5, q)$ generated by the points $(x_{i,1}, \ldots, x_{i,6})$, $1 \leq i \leq 3$, of $PG(5, q)$. For every $J = \{i_1, i_2, i_3\} \subseteq I := \{1, 2, \ldots, 6\}$ with $i_1 < i_2 < i_3$, we define

$$\alpha_J := \begin{vmatrix} x_{1,i_1} & x_{1,i_2} & x_{1,i_3} \\ x_{2,i_1} & x_{2,i_2} & x_{2,i_3} \\ x_{3,i_1} & x_{3,i_2} & x_{3,i_3} \end{vmatrix}.$$ 

The elements $\alpha_J$, $J \in \binom{I}{3}$, are the coordinates of a point $\kappa(\alpha)$ of $PG(19, q)$ and this point does not depend on the particular set of three points which we have chosen as generating set for $\alpha$. The map $\kappa$ defines a bijection between the set of planes of $PG(5, q)$ and a Grassmann-variety $G_{5,2,q}$ of $PG(19, q)$. Let $G$ denote the point-line geometry whose points are the planes of $PG(5, q)$ and whose lines are the sets of $q + 1$ planes through a given line and contained in a given 3-space. $G$ is called the Grassmannian of the planes of $PG(5, q)$. The map $\kappa$ defines a full embedding of $G$ in $PG(19, q)$. We refer to Chapter 24 of Hirschfeld and Thas [16] for more background information on Grassmann-varieties. The geometry $G$ has a subgeometry $\Delta \cong DW(5, q)$ whose points are the planes of $PG(5, q)$ which are totally isotropic with respect to a given symplectic polarity $\zeta$. If $\mathcal{I}$ denotes the set of all these totally isotropic planes, then $\langle \kappa(\mathcal{I}) \rangle$ is a 13-dimensional subspace $\Sigma_{gr}$ of $PG(19, q)$ and the map $\kappa$ induces a full polarized embedding $e_{gr}$ of $\Delta$ in $\Sigma_{gr}$, which we call the Grassmann-embedding of $DW(5, q)$. We refer to Cooperstein [7] for more details on the above facts. The following lemma is well-known and straightforward to verify.

**Lemma 1.1** Let $\Delta$ be the dual polar space $DW(5, q)$ and let $e_{gr} : \Delta \rightarrow \Sigma_{gr}$ denote the Grassmann-embedding of $\Delta$. Then for every automorphism $\theta$ of $\Delta$, there exists an automorphism $\phi$ of $\Sigma_{gr}$ such that $e_{gr}(x^\theta) = [e_{gr}(x)]^\phi$ for every point $x$ of $\Delta$. 


By Kasikova and Shult [17, Section 4.6], the dual polar space $DW(5, q)$ admits the absolutely universal embedding. If $q \neq 2$, then the absolutely universal embedding is isomorphic to the Grassmann-embedding, see Cooperstein [7]. This is not true if $q = 2$. The Grassmann-embedding has projective dimension 13, while the universal embedding has projective dimension 14, see Blokhuis and Brouwer [1], Brouwer and Wilbrink [2], Li [18] or Yoshiara [30]. In the sequel, we will denote by $e_{un} : \Delta \to \Sigma_{un}$ the universal embedding of $\Delta$.

If $q = 2^h$, then the dual polar space $DW(5, 2^h)$ is isomorphic to the dual polar space $DQ(6, 2^h)$ and hence admits a full polarized embedding in a 7-dimensional projective space $\Sigma_{sp}$. We call this embedding the spin-embedding of $DW(5, 2^h)$. We refer to Buekenhout and Cameron [3] for the construction of this embedding. With the terminology of Cardani, De Bruyn and Pasini [6], $e_{sp}$ is the minimal full polarized embedding of $DW(5, 2^h)$. So, $e_{sp} \leq e_{gr} \leq e_{un}$.

We denote by $\mathcal{H}_{un}$ the set of all hyperplanes of $DW(5, q)$ which arise from its universal embedding and by $\mathcal{H}_{gr}$ the set of all hyperplanes which arise from its Grassmann-embedding. If $q$ is even, then we denote by $\mathcal{H}_{sp}$ the set of all hyperplanes of $DW(5, q)$ arising from the spin-embedding. If $q$ is odd, then $\mathcal{H}_{gr} = \mathcal{H}_{un}$. If $q = 2$, then $\mathcal{H}_{sp} \subset \mathcal{H}_{gr} \subset \mathcal{H}_{un}$. If $q = 2^h$ with $h \geq 2$, then $\mathcal{H}_{sp} \subset \mathcal{H}_{gr} = \mathcal{H}_{un}$.

If $H_1$ and $H_2$ are two distinct hyperplanes of $\mathcal{H}_{un}$, then we denote by $[H_1, H_2]$ the set of all hyperplanes of $DW(5, q)$ arising from a hyperplane of $\Sigma_{un}$ through $\langle e_{un}(H_1) \rangle \cap \langle e_{un}(H_2) \rangle$. If $q = 2$ and $H_1, H_2 \in \mathcal{H}_{gr}$, then $[H_1, H_2] \subseteq \mathcal{H}_{gr}$. If $q = 2^h$ and $H_1, H_2 \in \mathcal{H}_{sp}$, then $[H_1, H_2] \subseteq \mathcal{H}_{sp}$.

The subsets $[H_1, H_2]$, $H_1, H_2 \in \mathcal{H}_{un}$ and $H_1 \neq H_2$, define a projective space on the set $\mathcal{H}_{un}$, since there exists a natural correspondence between the elements of $\mathcal{H}_{un}$ and the points of the dual of the projective space $\Sigma_{un}$.

1.4 The main results of this paper

In this paper, we determine all hyperplanes of $DW(5, 2^h)$, $h \geq 1$, which arise from the Grassmann-embedding. We will show the following.

**Main Theorem.** The dual polar space $DW(5, 2^h)$, $h \geq 1$, has six isomorphism classes of hyperplanes arising from the Grassmann-embedding.

These six classes are the following:
Type | \# hyperplanes | \# points
--- | --- | ---
I | \((q + 1)(q^4 + 1)(q^4 + 1)\) | \(q^5 + q^4 + 2q^3 + q^2 + q + 1\)
II | \(\frac{1}{2}q^2(q^2 + 1)(q^2 + q + 1)(q^3 + 1)\) | \(q^5 + 2q^4 + 2q^3 + q^2 + q + 1\)
III | \(\frac{1}{2}q^2(q + 1)(q^6 - 1)\) | \(q^5 + 2q^4 + q^3 + q^2 + q + 1\)
IV | \(q^4(q + 1)(q^4 + 1)(q^6 - 1)\) | \(q^5 + q^4 + 2q^3 + q^2 + q + 1\)
V | \(q^3(q^4 - 1)\) | \(q^5 + q^4 + q^3 + q^2 + q + 1\)
VI | \(q^3(q^4 - 1)(q^6 - 1)\) | \(q^5 + q^4 + q^3 + q^2 + q + 1\)

Table 1: The hyperplanes of \(DW(5, q)\), \(q = 2^h\), arising from the Grassmann-embedding

- **TYPE I**: The singular hyperplanes.
- **TYPE II**: The extensions of the \((q + 1) \times (q + 1)\)-subgrids of the quads.
- **TYPE III**: The extensions of the classical ovoids of the quads.
- **TYPE IV**: A hyperplane of \([H_1, H_2] \setminus \{H_1, H_2\}\), where \(H_1\) is the extension of a \((q + 1) \times (q + 1)\)-subgrid of a quad \(Q\) and \(H_2\) is a singular hyperplane whose deepest point belongs to \(H_1 \setminus Q\).
- **TYPE V**: The so-called hexagonal hyperplanes. The points and lines contained in such a hyperplane define a split-Cayley hexagon \(H(2^h)\).
- **TYPE VI**: A hyperplane of \([H_1, H_2] \setminus \{H_1, H_2\}\), where \(H_1\) is the extension of a \((q + 1) \times (q + 1)\)-subgrid of a quad \(Q\) and \(H_2\) is a singular hyperplane whose deepest point does not belong to \(H_1\).

We refer to Van Maldeghem [29] for the definition of the split-Cayley hexagons. Recall that an ovoid of \(Q(4, q)\) is called *classical* if it arises from its natural (i.e. absolutely universal) embedding in \(PG(4, q)\). In Table 1, we list the number of hyperplanes of each type, together with the number of points in each hyperplane.

If \(h \neq 1\), then the Grassmann-embedding is the universal embedding of \(DW(5, 2^h)\). In this case, the above-mentioned hyperplanes are all the hyperplanes which arise from embedding. This is not true for \(h = 1\). The universal embedding has projective dimension 14, while the Grassmann-embedding has projective dimension 13. By Corollary 2 of Ronan [24], the hyperplanes of
$DW(5,2)$ are precisely the hyperplanes of $DW(5,2)$ which arise from its universal embedding. Pralle [23] has classified all hyperplane classes of $DW(5,2)$ with the aid of a computer. He found 12 isomorphism classes, including the 6 above-mentioned classes. In the final section, we will give a computer free proof for this classification.

**Remark.** In De Bruyn and Pralle [13] and [14], all hyperplanes of the dual polar space $DH(5,q^2)$ were determined and it was shown that they all arise from embedding.

## 2 Some properties of the automorphism group of $DW(5,q)$

**Lemma 2.1** For every quad $Q$ of $\Delta = DW(5,q)$, there exists a group $A$ of automorphisms of $\Delta$ satisfying the following properties:

(a) every element of $A$ fixes $Q$ pointwise and every line meeting $Q$ setwise;

(b) if $L$ is a line meeting $Q$ in a unique point $x$, then $A$ acts regularly on $L \setminus \{x\}$.

**Proof.** Let $(\cdot, \cdot)$ denote a symplectic form of a 6-dimensional vector space defining the polar space $W(5,q)$, dual of $\Delta$. Let $\langle x_Q \rangle$ denote the point of $W(5,q)$ corresponding with the quad $Q$. For every $k \in F_q$, the map $\bar{y} \mapsto \bar{y} - k \cdot (x_Q, \bar{y}) \cdot x_Q$ defines an automorphism of $W(5,q)$. The corresponding automorphism $\theta_k$ of $\Delta$ fixes $Q$ pointwise and every line meeting $Q$ setwise. It is straightforward to verify that $A := \{\theta_k | k \in F_q\}$ is a group acting regularly on each set $L \setminus \{x\}$, where $L$ is a line of $\Delta$ meeting $Q$ in a unique point $x$. ■

**Definition.** If $q = 2$, then the unique nontrivial element of the group $A$ defined in Lemma 2.1 is called the **reflection about $Q$**.

**Lemma 2.2** The automorphism group of $\Delta = DW(5,q)$ acts transitively on the pairs $(G,x)$, where $G$ is a $(q+1) \times (q+1)$-subgrid and $x$ is a point of $\Delta_2(G)$.

**Proof.** The automorphism group of $\Delta$ acts transitively on the set of $(q+1) \times (q+1)$-grids. Now, fix a certain grid $G$ and let $Q$ denote the unique quad containing $G$. Since $G \cup \Delta_1(G)$ is a hyperplane, $\Delta_2(G)$ is connected.
by Lemma 6.1 of Shult [26]. So, we need to show that for any two collinear points \( x_1, x_2 \in \Delta_2(G) \), there exists an automorphism \( \theta \) of \( \Delta \) fixing \( G \) setwise and mapping \( x_1 \) to \( x_2 \). There are two possibilities:

- the line \( x_1x_2 \) meets \( Q \) in a unique point. Then the claim readily follows from Lemma 2.1.

- the line \( x_1x_2 \) meets \( \Delta_1(G) \) in a point \( x \notin Q \). Let \( Q' \) denote one of the two quads through \( x \) meeting \( Q \) in a line of \( G \). By Lemma 2.1, there exists an automorphism \( \theta \) fixing \( Q' \) pointwise, every line meeting \( Q' \) setwise and mapping \( x_1 \) to \( x_2 \). Obviously, \( \theta \) also fixes the grid \( G \) setwise.

**Lemma 2.3** Let \( x_1 \) and \( x_2 \) be two points of \( \Delta = DW(5,q) \), \( q = 2^h \), at distance 3 from each other. Then there exists a line \( L \) in \( \Delta \) satisfying the following: (i) \( d(x_1, L) = d(x_2, L) = 2 \); (ii) \( \pi_L(x_1) \neq \pi_L(x_2) \); (iii) for any pair of points \( y_1, y_2 \in L \setminus \{\pi_L(x_1), \pi_L(x_2)\} \), there exists an automorphism \( \theta \) of \( \Delta \) fixing \( x_1 \) and \( x_2 \), stabilizing \( L \) and mapping \( y_1 \) to \( y_2 \).

**Proof.** Suppose the polar space \( W(5,q) \) associated with \( \Delta \) is described by the following symplectic form:

\[
(X_0Y_3 - X_3Y_0) + (X_1Y_4 - X_4Y_1) + (X_2Y_5 - X_5Y_2).
\]

Without loss of generality, we may suppose that \( x_1 \leftrightarrow X_3 = X_4 = X_5 = 0 \) and \( x_2 \leftrightarrow X_0 = X_1 = X_2 = 0 \). Let \( L \) be the following line of \( \Delta \): \( L \leftrightarrow X_0 - X_3 = X_1 - X_4 = X_2 = X_5 = 0 \). The points \( p_1 \leftrightarrow X_0 - X_3 = X_1 - X_4 = X_5 = 0 \) and \( p_2 \leftrightarrow X_0 - X_3 = X_1 - X_4 = X_2 = 0 \) belong to \( L \). Moreover, \( d(x_1, p_1) = d(x_2, p_2) = 2 \) and \( d(x_1, p_2) = d(x_2, p_1) = 3 \). The other \( q-1 \) points of \( L \) are given by the equations \( X_0 - X_3 = X_1 - X_4 = X_2 - \mu X_5 = 0, \mu \in \mathbb{F}_q^* \), and lie at distance 3 from \( x_1 \) and \( x_2 \). Now, choose two arbitrary points \( y_1 \) and \( y_2 \) on the line \( L \setminus \{p_1, p_2\} \). So, there exist \( \mu_1, \mu_2 \in \mathbb{F}_q^* \) such that \( y_i \leftrightarrow X_0 - X_3 = X_1 - X_4 = X_2 - \mu_i X_5 = 0, i \in \{1, 2\} \). Now, choose a \( k \in \mathbb{F}_q^* \) such that \( k^2 = \frac{\mu_2}{\mu_1} \). The map \( (X_0, X_1, X_2, X_3, X_4, X_5) \mapsto (X_0, X_1, kX_2, X_3, X_4, \frac{X_5}{k}) \) induces an automorphism \( \theta \) of \( \Delta \) fixing \( x_1 \) and \( x_2 \), stabilizing \( L \) and mapping \( y_1 \) to \( y_2 \). This proves the lemma.

**Lemma 2.4** Let \( G \) be a \((q + 1) \times (q + 1)\)-subgrid of \( \Delta = DW(5,q) \), \( q = 2^h \), let \( Q \) denote the unique quad containing \( G \) and let \( p_1 \) be a point outside \( Q \) such that \( \pi_Q(p_1) \notin G \). Then there exists a line \( L \) satisfying the following properties:
(i) $L$ intersects $Q$ in a point $p_2$ of $\Delta_3(p_1) \setminus G$;

(ii) for every two points $y_1, y_2 \in L \setminus \{p_2, \pi_L(p_1)\}$, there exists an automorphism $\theta$ of $\Delta$ fixing $p_1$, stabilizing $G$ and $L$, and mapping $y_1$ to $y_2$.

Proof. Suppose first that $q = 2$. Let $p_2$ be a point of $Q \setminus (G \cup \pi_Q(p_1))$ and let $L$ denote an arbitrary line through $p_2$ not contained in $Q$. Then $|L \setminus \{p_2, \pi_L(p_1)\}| = 1$ and so condition (ii) holds: since $y_1 = y_2$, we can take for $\theta$ the trivial automorphism.

Suppose $q \geq 4$. The point $p_1$ corresponds with a totally isotropic plane $\alpha_1$ of $\text{PG}(5, q)$ (with respect to the symplectic polarity defining $\Delta$). The grid $G$ corresponds with a nonisotropic plane $\alpha_2$, see De Bruyn [10]. The quad $Q$ corresponds with the singular point $x_{\alpha_2}$ of $\alpha_2$ and the $(q+1)^2$ points of $G$ correspond with the $q+1$ totally isotropic planes intersecting $\alpha_2$ in a line through $x_{\alpha_2}$. Since $\pi_Q(p_1) \not\in G$ and $p_1 \not\in Q$, $\alpha_1$ and $\alpha_2$ are disjoint.

Now, suppose that the polar space $W(5, q)$ associated with $\Delta$ is described by the following symplectic form:

$$(X_0Y_3 - X_3Y_0) + (X_1Y_4 - X_4Y_1) + (X_2Y_5 - X_5Y_2).$$

By Lemma 2.2, we may suppose that $\alpha_1 \leftrightarrow X_0 = X_1 = X_2 = 0$ and $\alpha_2 \leftrightarrow X_3 = X_4 = X_0 - X_5 = 0$. One easily checks that $\alpha_2$ is a nonisotropic plane and that the point $(0, 1, 0, 0, 0, 0)$ is its singular point. Now, choose a $\delta \in \mathbb{F}_q \setminus \{0, 1\}$ and let $L$ be the following line of $\Delta$: $X_0 - \delta X_5 = X_2 - \delta X_3 = X_1 = X_4 = 0$. Put $L \cap Q = \{p_2\}$. Then $p_2$ is the following point of $Q$:

$X_0 - \delta X_5 = \delta X_3 - X_2 = X_4 = 0$. Obviously, $d(p_1, p_2) = 3$. Since the system $X_3 = X_4 = X_0 - X_5 = 0$, $X_0 - \delta X_5 = \delta X_3 - X_2 = X_4 = 0$ has only the point $(0, 1, 0, 0, 0, 0)$ as solution, $p_2 \not\in G$. The point $\pi_L(p_1)$ has the following equation:

$X_0 - \delta X_5 = X_2 - \delta X_3 = X_1 = 0$. A point $y$ of $L \setminus \{p_2, \pi_L(p_1)\}$ has the following equation for a certain $\mu \in \mathbb{F}_q^*$:

$X_0 - \delta X_5 = X_2 - \delta X_3 = X_1 - \mu X_4 = 0$. Now, let $y_1, y_2$ be arbitrary points of $L \setminus \{p_2, \pi_L(p_1)\}$ and let $\mu_1, \mu_2 \in \mathbb{F}_q^*$ such that $y_i \leftrightarrow X_0 - \delta X_5 = X_2 - \delta X_3 = X_1 - \mu_i X_4 = 0$ for every $i \in \{1, 2\}$. Let $k \in \mathbb{F}_q^*$ such that $k^2 = \frac{\mu_2}{\mu_1}$, then the map $(X_0, X_1, X_2, X_3, X_4, X_5) \mapsto (X_0, kX_1, X_2, X_3, \frac{X_4}{k}, X_5)$ induces an automorphism of $\Delta$ satisfying all required properties. 

\vspace{1cm}

9
A proposition regarding hyperplanes of general dual polar spaces

Definition. A set $W$ of hyperplanes of a dual polar space $\Delta$ with point-set $P$ is called a pencil of hyperplanes if the following properties hold:

(i) $P = \bigcup_{H \in W} H$;

(ii) $H_1 \cap H_2 = H_1 \cap H_3 = H_2 \cap H_3$ for any three distinct hyperplanes $H_1$, $H_2$ and $H_3$ of $W$.

Proposition 3.1 Let $\Delta$ be a thick dual polar space and let $M$ denote a max of $\Delta$. Let $G_1$ and $G_2$ be two distinct hyperplanes of $M$ and let $H_i$, $i \in \{1, 2\}$, be the hyperplane of $\Delta$ which arises by extending $G_i$. Then the following holds: if $W$ is a pencil of hyperplanes of $\Delta$ containing $H_1$ and $H_2$, then there exists a pencil $W'$ of hyperplanes of $M$ containing $G_1$ and $G_2$ such that the elements of $W$ are the extensions of the elements of $W'$.

Proof. For every point $x$ of $M$, let $L_x$ denote the set of lines through $x$ not contained in $M$. Let $H$ denote an arbitrary hyperplane of $W \setminus \{H_1, H_2\}$. If $x$ is a point of $M$, then one of the following holds:

(i) $x \in G_1 \cap G_2$. Then every line of $L_x$ is contained in $H$.

(ii) $x \in (G_1 \setminus G_2) \cup (G_2 \setminus G_1)$. Then no line of $L_x$ is contained in $H$.

(iii) $x \in M \setminus (G_1 \cup G_2)$. Suppose that there exists a line $L \in L_x$ contained in $H$. Let $L'$ denote an arbitrary line of $L_x$ different from $L$ and let $Q$ be the quad $\langle L, L' \rangle$. The quad $Q$ intersects $M$ in a line $L''$ which contains a unique point $y$ of $G_1$. If $y \in G_1 \cap G_2$, then $y^\perp \cap Q \subseteq H$ and $L \subseteq H$. It follows that $Q$ is deep and that $L' \subseteq H$. If $y \in G_1 \setminus G_2$, then $(y^\perp \cap H) \cap Q = L''$ and $L \subseteq H$. It follows that $Q$ is singular with deep point $x$ and that $L' \subseteq H$.

By (i), (ii) and (iii), it follows that either all lines of $L_x$ are contained in $H$ or no line of $L_x$ is contained in $H$. Let $G_H$ denote the set of points $x$ of $M$ with the property that every line of $L_x$ is contained in $H$. Then

$$H = M \cup \bigcup_{x \in G_H} x^\perp.$$
Let $M'$ denote a max disjoint from $M$ not contained in $H$. Then $M' \cap H$ is a hyperplane of $M'$. Since $G_H = \pi_M(M' \cap H)$, $G_H$ is a hyperplane of $M$ and hence $H = \bigcup_{x \in G_H} x^\perp$. So, we have shown that every hyperplane $H$ of $W$ is the extension of a (necessarily unique) hyperplane $G_H$ of $M$. Put $W' := \{G_H \mid H \in W\}$. Again, let $M'$ denote a max which is disjoint from $M$. Then $W'' := \{M' \cap H \mid H \in W\}$ is a pencil of hyperplanes of $M'$. Since $W'' = \{\pi_M(M' \cap H) \mid H \in W\}$, $W''$ is a pencil of hyperplanes of $M$. ■

4 Extensions of grids and classical ovoids

Lemma 4.1 Let $Q$ be the generalized quadrangle $Q(4, q)$ and let $e$ be the absolutely universal embedding of $Q$ in $PG(4, q)$. Let $\mathcal{H}$ denote the set of all hyperplanes of $Q$ which arise from the embedding $e$. Then the following holds: if $H_1$ and $H_2$ are two distinct hyperplanes of $\mathcal{H}$, then there exists a unique pencil $W$ of hyperplanes of $\mathcal{H}$ containing $H_1$ and $H_2$.

Proof. Let $\Pi_i, i \in \{1, 2\}$, denote the hyperplane of $PG(4, q)$ for which $H_i := e^{-1}(e(Q) \cap \Pi_i)$. Then $W' := \{e^{-1}(\Pi \cap e(Q)) \mid \dim(\Pi) = 3 \text{ and } \Pi_1 \cap \Pi_2 \subseteq \Pi\}$ is a pencil of hyperplanes satisfying all required properties. We will show that this is the unique pencil. There are four possibilities for $\Pi_1 \cap \Pi_2$:

(i) $\Pi_1 \cap \Pi_2 \cap e(Q)$ is a nondegenerate conic,

(ii) $\Pi_1 \cap \Pi_2 \cap e(Q)$ is the union of two distinct lines,

(iii) $\Pi_1 \cap \Pi_2 \cap e(Q)$ is a line;

(iv) $\Pi_1 \cap \Pi_2 \cap e(Q)$ is a point.

In cases (i) and (ii), $(\Pi_1 \cap \Pi_2 \cap e(Q)) = \Pi_1 \cap \Pi_2$ and it readily follows that $W'$ is the unique pencil of hyperplanes satisfying the required properties.

Suppose case (iii) occurs. For every $i \in \{1, 2\}$, $H_i$ is either a singular hyperplane, a subgrid or a classical ovoid. Since $H_1 \cap H_2$ is a line $L$, $H_1$ and $H_2$ must be singular hyperplanes whose deepest points lie on $L$. If $H$ is a hyperplane of $\mathcal{H}$ such that $H \cap H_1 = H_1 \cap H_2 = H_2 \cap H = L$, then $H$ must also be a singular hyperplane with deepest point on $L$. Hence, there exists a unique pencil of hyperplanes of $\mathcal{H}$ containing $H_1$ and $H_2$. This pencil of hyperplanes contains all the singular hyperplanes whose deepest points lie on $L$. 

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Suppose case (iv) occurs. Let $x$ denote the unique point of $H_1 \cap H_2$. Then $H_i$, $i \in \{1, 2\}$, is either the singular hyperplane with deepest point $x$ or is an ovoid through $x$. Without loss of generality, we may suppose that $H_1$ is an ovoid through $x$. The plane $\Pi_1 \cap \Pi_2$ is the unique plane of $\Pi_1$ through $e(x)$ intersecting $e(H_1)$ in only the point $e(x)$. Now, let $H$ be a hyperplane of $\mathcal{H}$ such that $H \cap H_1 = H_1 \cap H_2 = H_2 \cap H = \{x\}$ and let $\Pi$ be the hyperplane of $\text{PG}(4, q)$ giving rise to $H$. Since $H \cap H_1 = \{x\}$, $\Pi \cap \Pi_1$ is a plane of $\Pi_1$ intersecting $e(H_1)$ in $e(x)$. It follows that $\Pi \cap \Pi_1 = \Pi_1 \cap \Pi_2$, i.e., $\Pi$ contains $\Pi_1 \cap \Pi_2$. Again, it follows that $\mathcal{W}'$ is the unique pencil of hyperplanes satisfying the required properties.

\begin{lemma}
Let $O$ be a classical ovoid of $Q(4, q)$ and let $G$ denote a $(q+1) \times (q+1)$-grid of $Q(4, q)$. Let $\mathcal{W}$ denote the unique pencil of $q+1$ hyperplanes containing $O$ and $G$ such that every element of $\mathcal{W}$ arises from the absolutely universal embedding of $Q(4, q)$. Then there exists an element $H \in \mathcal{W} \setminus \{G\}$ which is not an ovoid.
\end{lemma}

\begin{proof}
If this were not the case, then since $|G \cap O| = q+1$, $|Q(4, q)| = (q+1) + [(q^2+1)-(q+1)] \cdot q + [(q+1)^2-(q+1)] = q^3 + 2q + 1 \neq (q+1)(q^2+1)$, a contradiction.
\end{proof}

\begin{proposition}
(i) Every singular hyperplane of $DW(5, q)$ arises from the Grassmann-embedding of $DW(5, q)$.

(ii) Let $Q$ be a $Q(4, q)$-quad of the dual polar space $DW(5, q)$ and let $G$ be a $(q+1) \times (q+1)$-subgrid of $Q$. Then the extension of $G$ arises from the Grassmann-embedding of $DW(5, q)$.
\end{proposition}

\begin{proof}
Claim (i) says that the Grassmann-embedding is polarized. Claim (ii) is a consequence of Propositions 8 and 10 of De Bruyn [10].
\end{proof}

\begin{proposition}
Let $Q$ be a $Q(4, q)$-quad of the dual polar space $DW(5, q)$ and let $O$ be a classical ovoid of $Q$. Then the extension of $O$ arises from the Grassmann-embedding of $DW(5, q)$.
\end{proposition}

\begin{proof}
By Lemma 4.2, there exists a pencil $\mathcal{W}'$ of hyperplanes of $Q$ such that the following holds:

(i) every element of $\mathcal{W}'$ arises from the absolutely universal embedding of $Q$;

(ii) $O \in \mathcal{W}'$;

\end{proof}
(iii) there exist two hyperplanes $G_1$ and $G_2$ in $\mathcal{W}'$ which are not ovoids.

Let $\mathcal{W}$ denote the set of hyperplanes of $\Delta$ which arise as extensions of elements of $\mathcal{W}'$. By Proposition 4.3, the extension $\mathcal{G}_i$, $i \in \{1, 2\}$, of $G_i$ arises from the Grassmann-embedding. Let $\Pi_i$, $i \in \{1, 2\}$, denote the unique hyperplane of $\Sigma_{gr}$ giving rise to $\mathcal{G}_i$ and let $\mathcal{W}$ denote the pencil of hyperplanes of $\Delta$ which arise from the hyperplanes of $\Sigma_{gr}$ through $\Pi_1 \cap \Pi_2$. By Proposition 3.1, there exists a pencil $\mathcal{W}'$ of hyperplanes of $Q$ such that the hyperplanes of $\mathcal{W}$ arise as extensions of elements of $\mathcal{W}'$. Obviously, $G_1, G_2 \in \mathcal{W}'$. Now, since the extensions of elements of $\mathcal{W}'$ arise from the Grassmann-embedding, every element of $\mathcal{W}$ is either a singular hyperplane, a $(q + 1) \times (q + 1)$-grid or a classical ovoid. By Lemma 4.1, $\mathcal{W}$ coincides with $\mathcal{W}'$. Hence, $\mathcal{W} = \mathcal{W}'$. The proposition now follows from the fact that the extension of $O$ belongs to $\mathcal{W}$ and every hyperplane of $\mathcal{W}$ arises from the Grassmann-embedding. ■

5 General properties of hyperplanes arising from the Grassmann-embedding

**Definition.** Let $H$ be a hyperplane of $DW(5, q)$ and let $x$ be a point of $H$. Then $\Lambda_H(x)$ denotes the set of lines through $x$ contained in $H$. If no confusion is possible, we will write $\Lambda(x)$ instead of $\Lambda_H(x)$. We can regard $\Lambda(x)$ as a set of points of the projective plane $Res_\Delta(x)$. If $\Lambda(x)$ consists of all lines through $x$, then $x$ is called deep with respect to $H$.

**Lemma 5.1** Let $H$ be a hyperplane of $DW(5, q)$. Then the set of deep points with respect to $H$ is a subspace of $DW(5, q)$.

**Proof.** If $x_1$ and $x_2$ are two distinct collinear deep points, then any quad $Q$ through the line $x_1x_2$ must be deep since $x_1^\perp \cap Q \subseteq H$ and $x_2^\perp \cap Q \subseteq H$. It is then easily seen that also every point of $x_1x_2 \setminus \{x_1, x_2\}$ must be deep. ■

Take now a point $x$ of $DW(5, q)$ and consider the quotient space $\langle e_{gr}(x^+) / e_{gr}(x) \rangle$, where $e_{gr}$ is the Grassmann-embedding of $DW(5, q)$. If $L$ is a point of $Res_\Delta(x)$, i.e. a line through $x$, then $e_{gr}(L) / e_{gr}(x)$ is a point of $\langle e_{gr}(x^+) / e_{gr}(x) \rangle$. Pasini [19] and Cardinali, De Bruyn [5] showed that this embedding is isomorphic to the veronese embedding ([16, chapter 25]) of $PG(2, q)$ into $PG(5, q)$. Hence, the following holds (cf. Theorem 25.1.3 of [16]):
Lemma 5.2 Let $H$ be a hyperplane of $DW(5, q)$ arising from the Grassmann-embedding and let $x$ be a point of $H$. Then $\Lambda_H(x)$ is one of the following sets of points of $\text{Res}_\Delta(x)$: (1) a point; (2) a line; (3) the union of two distinct lines; (4) a nondegenerate conic, (5) the whole set of points of $\text{Res}_\Delta(x)$.

Lemma 5.3 Let $H_1$ and $H_2$ be two distinct hyperplanes of $DW(5, q)$ arising from the universal embedding, and let $Q$ be a quad of $DW(5, q)$. Then the following are equivalent:

- there exists a hyperplane $H$ in $[H_1, H_2] \setminus \{H_1, H_2\}$ containing $Q$;
- $H_1 \cap Q = H_2 \cap Q$.

Proof. Suppose $H_1 \cap Q = H_2 \cap Q = Q$. Then $Q$ is contained in any hyperplane of $[H_1, H_2] \setminus \{H_1, H_2\}$.

Suppose $H_1 \cap Q = H_2 \cap Q = Q$. Let $x$ denote an arbitrary point of $Q \setminus (H_1 \cap Q)$. Then there exists a unique hyperplane $H$ in $[H_1, H_2] \setminus \{H_1, H_2\}$ containing $H_1 \cap Q$ and $x$. Since $H_1 \cap Q$ is a maximal subspace of $Q$, $H$ contains $Q$.

Suppose $H_1 \cap Q \neq H_2 \cap Q$ and that there exists an $i \in \{1, 2\}$ such that $H_i \cap Q = Q$. Then $H \cap Q = H_{3-i} \cap Q \neq Q$ for every hyperplane $H$ of $[H_1, H_2] \setminus \{H_1, H_2\}$.

Suppose $Q \neq H_1 \cap Q \neq H_2 \cap Q \neq Q$. Then $\{H \cap Q \mid H \in [H_1, H_2]\}$ is a pencil of hyperplanes of $Q$ containing $H_1 \cap Q$ and $H_2 \cap Q$. It follows that $H \cap Q \neq Q$ for every hyperplane $H$ of $[H_1, H_2] \setminus \{H_1, H_2\}$.

Lemma 5.4 Let $x_1$ and $x_2$ be two points of $\Delta = DW(5, q)$ at distance 3 from each other. Let $H_i, i \in \{1, 2\}$, denote the singular hyperplane of $\Delta$ with deepest point $x_i$. Then no hyperplane of $[H_1, H_2] \setminus \{H_1, H_2\}$ has a deep quad.

Proof. Let $Q$ be a quad of $\Delta$. If $Q$ contains the point $x_i, i \in \{1, 2\}$, then $Q \cap H_i = Q$ and $Q \cap H_{3-i}$ is a singular hyperplane of $Q$. By Lemma 5.3, no hyperplane of $[H_1, H_2] \setminus \{H_1, H_2\}$ contains $Q$.

Suppose therefore that $\{x_1, x_2\} \cap Q = \emptyset$. Let $y_i, i \in \{1, 2\}$, denote the point of $Q$ collinear with $x_i$. Then $H_i \cap Q$ is a singular hyperplane with deepest point $y_i$. Since $d(x_1, x_2) = 3$, $y_1 \neq y_2$ and hence $Q \cap H_1 \neq Q \cap H_2$. By Lemma 5.3, no hyperplane of $[H_1, H_2] \setminus \{H_1, H_2\}$ contains $Q$. ■
Lemma 5.5 Let \( H_1 \) be a hyperplane of \( \Delta = DW(5, q) \) which is the extension of a \((q + 1) \times (q + 1)\)-grid \( G \) in a quad \( R \). Let \( x \) denote a point of \( \Delta \) not contained in \( H_1 \) and let \( H_2 \) be the singular hyperplane with deepest point \( x \). Then no hyperplane of \( \{H_1, H_2\} \setminus \{H_1, H_2\} \) has a deep quad.

Proof. Let \( Q \) denote an arbitrary quad of \( \Delta \). If \( H_1 \cap Q \neq H_2 \cap Q \), then no \( H \in \{H_1, H_2\} \setminus \{H_1, H_2\} \) contains \( Q \) by Lemma 5.3. So, suppose \( H_1 \cap Q = H_2 \cap Q \). Then \( x \not\in Q \) and \( Q \cap H_2 \) is singular with deepest point \( y \) collinear with \( x \). As \( H_1 \cap Q = H_2 \cap Q \), \( Q \) is also singular for \( H_1 \) with deepest point \( y \). This forces \( y \in G \). However, if so, then \( x \in H_1 \), a contradiction. \( \blacksquare \)

6 Hyperplanes of \( DW(5, q) \) with a deep point

Definition. Let \( H \) be a hyperplane of \( DW(5, q) \) and let \( x \) be a point of \( H \). Then \( N_H(x) \) denotes the set of quads through \( x \) which are contained in \( H \). If no confusion is possible, then we will also write \( N(x) \) instead of \( N_H(x) \).

Lemma 6.1 Let \( H \) be a hyperplane of \( DW(5, q) \), let \( x \in H \) be a deep point with respect to \( H \) and let \( y \) be a point of \( H \cap \Delta_3(x) \). Then there exists an isomorphism from \( Res_\Delta(y) \) to the dual of \( Res_\Delta(x) \) mapping \( \Lambda(y) \) to \( N(x) \).

Proof. There exists a natural isomorphism \( \theta \) between \( Res_\Delta(y) \) and the dual of \( Res_\Delta(x) \): for a line \( L \) (respectively quad \( Q \)) through \( y \), let \( L^\theta \) (respectively \( Q^\theta \)) denote the unique quad (respectively line) through \( x \) meeting \( L \) (respectively \( Q \)). Now, a line through \( y \) is contained in \( H \) if and only if \( L^\theta \) is a deep quad through \( x \). Hence, \( \Lambda(y)^\theta = N(x) \). \( \blacksquare \)

Corollary 6.2 Let \( H \) be a hyperplane of \( \Delta = DW(5, q) \), let \( x \) be a deep point and let \( y_1, y_2 \) be points of \( H \cap \Delta_3(x) \). Then there exists an isomorphism from \( Res_\Delta(y_1) \) to \( Res_\Delta(y_2) \) mapping \( \Lambda(y_1) \) to \( \Lambda(y_2) \).

If \( H \) is a hyperplane of \( DW(5, q) \) arising from the Grassmann-embedding with a deep point \( x \), then by Lemma 5.2 and Corollary 6.2, there are five possibilities:

(i) \( \Delta_3(x) \cap H = \emptyset \);

(ii) \( \Delta_3(x) \cap H \neq \emptyset \) and \( \Lambda(y) \) is a point of \( Res_\Delta(y) \) for every point \( y \in H \cap \Delta_3(x) \);
(iii) $\Delta_3(x) \cap H \neq \emptyset$ and $\Lambda(y)$ is a line of $\text{Res}_\Delta(y)$ for every point $y \in H \cap \Delta_3(x)$;

(iv) $\Delta_3(x) \cap H \neq \emptyset$ and $\Lambda(y)$ is the union of two distinct lines of $\text{Res}_\Delta(y)$ for every point $y \in H \cap \Delta_3(x)$;

(v) $\Delta_3(x) \cap H \neq \emptyset$ and $\Lambda(y)$ is a conic of $\text{Res}_\Delta(y)$ for every point $y \in H \cap \Delta_3(x)$.

We will treat each of these possibilities in the following propositions.

**Proposition 6.3** Let $H$ be a hyperplane of $\Delta = \text{DW}(5, q)$ arising from the Grassmann-embedding and suppose $x$ is a deep point of $H$. If $\Delta_3(x) \cap H = \emptyset$, then $H$ is the singular hyperplane with deepest point $x$.

**Proof.** If $H_x$ denotes the singular hyperplane with deepest point $x$, then $H \subseteq H_x$. Since $H$ is a maximal subspace, we necessarily have $H = H_x$. ■

**Proposition 6.4** Let $H$ be a hyperplane of $\Delta = \text{DW}(5, q)$ arising from the Grassmann-embedding and suppose $x$ is a deep point of $H$. If $\Delta_3(x) \cap H \neq \emptyset$ and $\Lambda(y)$ is a line of $\text{Res}_\Delta(y)$ for every point $y$ of $H$ at distance 3 from $x$, then $H$ is a singular hyperplane whose deepest point lies at distance 1 from $x$.

**Proof.** By Lemma 6.1, there exists a line $L$ through $x$ such that $\mathcal{N}(x)$ consists of the $q + 1$ quads through $L$. For every point $y \in H \cap \Delta_3(x)$, let $Q_y$ denote the unique quad through $y$ intersecting $L$ in a point $f(y)$. Since $f(y)^\perp \cap Q_y \subseteq H$ and $y \in H$, the quad $Q(y)$ is deep and the lines of $H$ through $y$ are precisely the $q + 1$ lines through $y$ contained in $Q_y$. If $f(y_1) \neq f(y_2)$ for two points $y_1$ and $y_2$ of $H \cap \Delta_3(x)$, then the unique line through $y_1$ meeting $Q_{y_2}$ is contained in $H$ but not in $Q_{y_1}$, a contradiction. So, there exists a point $y^* \in L$ such that $f(y) = y^*$ for all $y \in H \cap \Delta_3(x)$. So, $H \subseteq H_{y^*}$ and hence $H = H_{y^*}$ since $H$ is a maximal subspace. ■

**Proposition 6.5** Let $H$ be a hyperplane of $\Delta = \text{DW}(5, q)$ arising from the Grassmann-embedding and suppose $x$ is a deep point of $H$. If $\Delta_3(x) \cap H \neq \emptyset$ and $\Lambda(y)$ is a point of $\text{Res}_\Delta(y)$ for every point $y$ of $H$ at distance 3 from $x$, then $H$ is the extension of a classical ovoid in a quad.

**Proof.** By Lemma 6.1, there exists a unique deep quad $Q$ through $x$. For every point $y$ of $Q$, let $X_y$ denote the set of lines through $y$ contained in
$H$, but not in $Q$. Let $O$ denote the set of points $y$ of $Q$ for which $X_y \neq \emptyset$. Obviously, $x \in O$.

Suppose $y$ is a point of $O$ collinear with $x$ and let $L_y$ denote a line of $X_y$. Let $R$ denote the quad $\langle x, L_y \rangle$. Then $x^+ \cap R \subseteq H$ and $L_y \subseteq H$. It follows that $R$ is deep, contradicting the fact that $Q$ is the only deep quad through $x$. So, no point of $O \setminus \{x\}$ is collinear with $x$.

Suppose $y$ is a point of $O$ at distance 2 from $x$, let $L_y$ denote a line of $X_y$ and let $z \in L_y \setminus \{y\}$. If $R$ is a quad through $L_y$, then $R$ is singular with deep point $y$ since $z^+ \cap R \cap H = L_y$ and $R \cap Q \subseteq H$. It follows that $y^+ \subseteq H$. So, $H = Q \cup \left( \bigcup_{y \in O} y^+ \right)$.

Now, let $Q'$ denote a quad disjoint from $Q$. Then $Q' \cap H$ is a hyperplane of $Q'$. Since $\pi_Q(Q' \cap H) = O$, $O$ is a hyperplane of $Q$. Since no line through $x \in O$ is contained in $O$, $O$ is an ovoid. Since $H$ arises from the Grassmann-embedding, the ovoid $O$ is classical. This proves the proposition.

\textbf{Proposition 6.6} Let $H$ be a hyperplane of $\Delta = DW(5, q)$ arising from the Grassmann-embedding and suppose $x$ is a deep point of $H$. If $\Delta_3(x) \cap H \neq \emptyset$ and $\Lambda(y)$ is the union of two distinct lines of $Res_\Delta(y)$ for every point $y$ of $H$ at distance 3 from $x$, then $H$ is the extension of a $(q + 1) \times (q + 1)$-grid in a quad of $\Delta$.

\textbf{Proof.} By Lemma 6.1, there exist two lines $L_1, L_2$ through $x$ contained in a quad $Q$ such that the deep quads through $x$ are the quads containing at least one of the lines $L_1, L_2$. Obviously, every point of $L_1 \cup L_2$ is deep and $Q \subseteq H$. Now, let $y$ denote an arbitrary point of $\Delta_3(x) \cap H$. The two quads through the line $y\pi_Q(y)$ meeting $L_1 \cup L_2$ are deep and also the quad $Q$ is deep. By Lemma 5.2, $\pi_Q(y)$ must be a deep point. Now, let $G$ denote the $(q + 1) \times (q + 1)$-grid of $Q$ containing $L_1$, $L_2$ and $\pi_Q(y)$. Since the set of deep points of $\Delta$ is a subspace, see Lemma 5.1, every point of $G$ is deep. Now, let $H^*$ denote the hyperplane of $\Delta$ obtained by extending $G$. Then $H^* \subseteq H$ and hence $H^* = H$, since $H^*$ is a maximal subspace.

We divide the hyperplanes of $DW(5, q)$ which arise from the Grassmann-embedding and which contain a deep point $x$ into the following classes:

- **TYPE I:** the singular hyperplanes;
• **Type II**: the extensions of the $(q + 1) \times (q + 1)$-grids in quads;

• **Type III**: the extensions of the classical ovoids in quads;

• **Type IV**: the non-singular hyperplanes with a (necessarily unique) deep point $x$ such that $\Lambda(y)$ is a conic of $\text{Res}_\Delta(y)$ for every point $y$ of the hyperplane at distance 3 from $x$.

Obviously, there are up to isomorphism unique hyperplanes of type I, II and III. We will now show that up to isomorphism there also exists a unique hyperplane of type IV.

In the following proposition, we will show the existence of hyperplanes of type IV.

**Proposition 6.7** Let $H_1$ be the hyperplane of $\Delta = DW(5, q)$ arising as extension of a $(q + 1) \times (q + 1)$-grid $G$ in a quad $Q$, let $x \in H_1 \setminus Q$ and let $H_2$ denote the singular hyperplane with deepest point $x$. Then every hyperplane of $[H_1, H_2] \setminus \{H_1, H_2\}$ is a hyperplane of type IV.

**Proof.** Let $L_1$ and $L_2$ denote the two lines through $\pi_Q(x)$ contained in $G$. The following holds for every hyperplane $H \in [H_1, H_2] \setminus \{H_1, H_2\}$:

1. the point $\pi_Q(x)$ is deep with respect to $H$;
2. the quads $\langle x\pi_Q(x), L_1 \rangle$ and $\langle x\pi_Q(x), L_2 \rangle$ are contained in $H$ and none of the remaining quads through $x\pi_Q(x)$ are contained in $H$.

It follows that every hyperplane of $[H_1, H_2] \setminus \{H_1, H_2\}$ is a hyperplane of type II or IV. Suppose $H$ is a hyperplane of $[H_1, H_2] \setminus \{H_1, H_2\}$ of type II. Then by (i), there exists a line $L$ through $\pi_Q(x)$ with the property that every quad through $L$ is contained in $H$. By (ii), $L \neq x\pi_Q(x)$.

If $L$ is contained in $Q$, then $Q$ is deep with respect to $H_1$ and $H$ and hence also with respect to $H_2 \in [H_1, H]$, a contradiction.

Hence, $L \neq x\pi_Q(x)$ and $L$ is not contained in $Q$. The quad $\langle L, L_i \rangle$, $i \in \{1, 2\}$, is deep with respect to $H$ and $H_1$ and hence also with respect to $H_2$. It follows that $x\pi_Q(x) \subseteq \langle L, L_i \rangle$, i.e. $L_i \subseteq \langle L, x\pi_Q(x) \rangle$. Hence, $L_1 = \langle L, x\pi_Q(x) \rangle \cap Q = L_2$, a contradiction.

So, no hyperplane of $[H_1, H_2] \setminus \{H_1, H_2\}$ is of type II. Hence, every hyperplane of $[H_1, H_2] \setminus \{H_1, H_2\}$ is of type IV. 

\[\blacksquare\]
Lemma 6.8  Let $x$ be a point of $\Delta = DW(5, q)$, let $V$ denote a set of quads through $x$ which form a nondegenerate conic in the dual of $Res_\Delta(x)$, and let $y$ be a point of $\Delta_3(x)$. Then there exists a unique hyperplane of type IV containing $\{y\} \cup \bigcup_{Q \in V} Q$.

Proof.  Let $V'$ denote the set of lines through $y$ meeting the quads of $V$. Then the hyperplanes through $\{y\} \cup \bigcup_{Q \in V} Q$ are precisely the hyperplanes through $x^\perp \cup \bigcup_{L \in V'} L$.

Now, put $\alpha_1 = \langle e_{gr}(x^\perp) \rangle$ and $\alpha_2 = \langle e_{gr}(y^\perp) \rangle$, then $\Sigma_{gr} = \langle \alpha_1, \alpha_2 \rangle$ and $\dim(\alpha_1) = \dim(\alpha_2) = 6$ by Cardinali and De Bruyn [5]. The set $V'$ is a conic of $Res_{\Delta}(y)$ and also by [5], it follows that $\langle e(\bigcup_{L \in V'} L) \rangle$ is a hyperplane $\alpha'_2$ of $\alpha_2$ (see also the discussion before Lemma 5.2). The lemma now readily follows. The unique hyperplane of type IV containing $\{y\} \cup \bigcup_{Q \in V} Q$ arises from the hyperplane $\langle \alpha_1, \alpha'_2 \rangle$ of $\Sigma_{gr}$. ■

Corollary 6.9  Let $x$ be a point of $\Delta = DW(5, q)$ and let $V$ denote a set of quads through $x$ which form a nondegenerate conic in the dual of $Res_\Delta(x)$. Then there are precisely $q$ hyperplanes of type IV containing $\bigcup_{Q \in V} Q$.

Proof.  Let $L$ denote a line at distance 2 from $x$ disjoint from $\bigcup_{Q \in V} Q$. Then by Lemma 6.8, for each of the $q$ points $y$ of $L \setminus \{\pi_L(x)\}$, there exists a unique hyperplane of type IV containing $\{y\} \cup \bigcup_{Q \in V} Q$. Each hyperplane of type IV containing $\bigcup_{Q \in V} Q$ can be obtained in this way. The corollary follows. ■

Proposition 6.10  Up to isomorphism, there exists a unique hyperplane of type IV in $DW(5, q)$.

Proof.  This follows from the following facts:

(i) The automorphism group of $DW(5, q)$ acts transitively on the pairs of points at distance 3 from each other.

(ii) The stabilizer of two points $x$ and $y$ at distance 3 induces in $Res_\Delta(x)$ its full automorphism group.

(iii) Lemma 6.8.

(iv) If $H$ is a hyperplane of type IV and if $\theta$ is an automorphism of $DW(5, q)$, then from Lemma 1.1, it readily follows that $H^\theta$ is also a hyperplane of type IV. ■
The locally singular hyperplanes of $DW(5, 2^h)$

By Shult [25] (or Pralle [22]), there are two types of locally singular hyperplanes in $DW(5,q)$, $q = 2^h$:

- the singular hyperplanes,
- the hexagonal hyperplanes.

The points and lines contained in a hexagonal hyperplane define a split-Cayley hexagon $H(q)$. The singular hyperplanes are precisely the locally singular hyperplanes with a deep quad.

Lemma 7.1 (Shult and Thas [27]; De Bruyn [11]) The locally singular hyperplanes of the dual polar space $DW(5, 2^h)$ are precisely those hyperplanes of $DW(5, 2^h)$ which arise from the spin-embedding.

Lemma 7.2 Let $H_1$ and $H_2$ be two distinct singular hyperplanes of $DW(5, 2^h)$, and let $x_i$, $i \in \{1,2\}$, denote the deepest point of $H_i$. If $d(x_1,x_2) \in \{1,2\}$, then every hyperplane of $[H_1,H_2]$ is singular. If $d(x_1,x_2) = 3$, then every hyperplane of $[H_1,H_2] \setminus \{H_1,H_2\}$ is a hexagonal hyperplane.

Proof. Since $H_1$ and $H_2$ arise from the spin-embedding, also every hyperplane of $[H_1,H_2]$ arises from the spin-embedding. Hence, every hyperplane of $[H_1,H_2]$ is singular or hexagonal.

If $d(x_1,x_2) \leq 2$, then every quad through $x_1$ and $x_2$ is contained in $H_1 \cap H_2$ and hence also in every hyperplane of $[H_1,H_2]$. It follows that every hyperplane of $[H_1,H_2]$ is singular.

If $d(x_1,x_2) = 3$, then by Lemma 5.4, no hyperplane of $[H_1,H_2] \setminus \{H_1,H_2\}$ contains a deep quad. This proves that every hyperplane of $[H_1,H_2] \setminus \{H_1,H_2\}$ is hexagonal.

Lemma 7.3 Let $H_1$ be a hexagonal hyperplane of $DW(5, 2^h)$ and let $H_2$ be a singular hyperplane of $DW(5, 2^h)$ whose deepest point $x$ does not belong to $H_1$. Then $[H_1,H_2] \setminus \{H_1,H_2\}$ contains $q-2$ hexagonal hyperplanes and 1 singular hyperplane. The deepest point of the unique singular hyperplane of $[H_1,H_2] \setminus \{H_1,H_2\}$ lies at distance 3 from $x$.

Proof. As in Lemma 7.2, one can conclude that every hyperplane of $[H_1,H_2]$ is singular or hexagonal. Suppose that $[H_1,H_2]$ contains $\alpha$ singular hyperplanes and $q+1-\alpha$ hexagonal hyperplanes. There are $(q+1)(q^2+1)(q^3+1)$
points in $DW(5, q)$. Since $|H_2| = 1 + (q^2 + q + 1)q + (q^2 + q + 1)q^3 = q^5 + q^4 + 2q^3 + q^2 + q + 1$, $|H_1| = (q + 1)(q^3 + q^4 + 1) = q^5 + q^4 + q^3 + q^2 + q + 1$, and $|H_1 \cap H_2| = (q^3 + q + 1) + (q^2 + q + 1)q^2 = q^4 + q^3 + 2q^2 + q + 1$, 

$$(q + 1)(q^3 + 1) = (q^2 + q + 1)(q^2 + q + 1) + \alpha(q^5 + q^3 - q^2) + (q + 1 - \alpha)(q^5 - q^2).$$

Hence, $\alpha = 2$. So, there exists a unique singular hyperplane $H_3$ in $[H_1, H_2] \setminus \{H_1, H_2\}$ and the remaining hyperplanes are of hexagonal type. If the deepest point of $H_3$ lies at distance at most 2 from $x$, then every hyperplane of $[H_1, H_2] = [H_1, H_3]$ would be singular by Lemma 7.2, a contradiction.

**Remark.** In a similar way one can show that if $H_1$ is a hexagonal hyperplane and if $H_2$ is a singular hyperplane with deepest point belonging to $H_1$, then every hyperplane of $[H_1, H_2] \setminus \{H_1, H_2\}$ is hexagonal.

**Proposition 7.4** Up to isomorphism, there exists a unique hexagonal hyperplane in $DW(5, q)$, $q = 2^h$.

**Proof.** This follows from the following facts:

(i) If $H$ is a hexagonal hyperplane of $DW(5, q)$, then by Lemma 7.3, there exist singular hyperplanes $H_1$ and $H_2$ for which the following holds: (a) the deepest points of $H_1$ and $H_2$ lie at distance 3 from each other; (b) $H \in [H_1, H_2]$.

(ii) The automorphism group of $DW(5, q)$ acts transitively on the pairs of points at distance 3 from each other.

(iii) If $H_1$ and $H_2$ are two distinct hyperplanes whose deepest points lie at distance 3 from each other, then by Lemma 1.1, $[H_1, H_2]^{\theta} = [H_1^\theta, H_2^\theta]$ for every automorphism $\theta$ of $DW(5, q)$.

(iv) Let $H_1$ and $H_2$ be two singular hyperplanes whose deepest points lie at distance 3 from each other. Then by (iii) and Lemma 2.3, the automorphism group of $DW(5, q)$ acts transitively on the hyperplanes of the set $[H_1, H_2] \setminus \{H_1, H_2\}$.

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**8 Proof of the main theorem**

We consider the following two extra classes of hyperplanes in the dual polar space $\Delta = DW(5, q)$, $q = 2^h$:

- **Type V**: the hexagonal hyperplanes;
• TYPE VI: The hyperplanes of $\mathcal{H}_{gr}$ without deep points which are not hexagonal.

As in Section 1.3, let $\mathcal{H}_{gr}$, respectively $\mathcal{H}_{sp}$, denote the set of hyperplanes of $\Delta$ arising from the Grassmann-embedding, respectively the spin-embedding. By Proposition 7.4, there is up to isomorphism a unique hyperplane of type V. We will now show that there exists up to isomorphism a unique hyperplane of type VI. We start with counting the number of hyperplanes of each type.

**Lemma 8.1** The number of hyperplanes of type I, II, III, IV, V and VI of $DW(5,q)$, $q = 2^h$, is as given in the following table:

<table>
<thead>
<tr>
<th>Type</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$(q + 1)(q^2 + 1)(q^3 + 1)$</td>
</tr>
<tr>
<td>II</td>
<td>$\frac{1}{2}q^2(q^2 + 1)(q^2 + q + 1)(q^3 + 1)$</td>
</tr>
<tr>
<td>III</td>
<td>$\frac{1}{2}q^2(q + 1)(q^2 - 1)$</td>
</tr>
<tr>
<td>IV</td>
<td>$q^2(q + 1)(q^2 + 1)(q^h - 1)$</td>
</tr>
<tr>
<td>V</td>
<td>$q^3(q^4 - 1)$</td>
</tr>
<tr>
<td>VI</td>
<td>$q^4(q^5 - 1)(q^h - 1)$</td>
</tr>
</tbody>
</table>

**Proof.** The number of singular hyperplanes is equal to the number of points of $\Delta$. The number of hexagonal hyperplanes is equal to $(\# \text{ hyperplanes of } \mathcal{H}_{sp}) - (\# \text{ singular hyperplanes})$. The number of hyperplanes of type II is equal to $(\# \text{ quads}) \times (\# (q + 1) \times (q + 1)\text{-subgrids in a given quad})$. The number of hyperplanes of type III is equal to $(\# \text{ quads}) \times (\# \text{ classical ovoids in a quad})$. The number of hyperplanes of type IV is equal to $(\# \text{ points of } \Delta) \times (\# \text{ nondegenerate conics in a plane}) \times q$ (cf. Corollary 6.9). The number of hyperplanes of type VI is equal $(\# \text{ number of hyperplanes of } \mathcal{H}_{gr}) - (\# \text{ hyperplanes of type I, II, III, IV or V})$. Recall also that $|\mathcal{H}_{sp}| = \frac{q^8 - 1}{q - 1}$ and $|\mathcal{H}_{gr}| = \frac{q^{14} - 1}{q - 1}$.

We will now count the number of hyperplanes of type IV in another way. This will allow us to prove our main theorem.

Recall that the set $\mathcal{H}_{gr}$ has the structure of a 13-dimensional projective space, with the subsets $[H_1, H_2]$, $H_1, H_2 \in \mathcal{H}_{gr}$ and $H_1 \neq H_2$, as lines. $\mathcal{H}_{sp}$ is a 7-dimensional subspace of $\mathcal{H}_{gr}$. For every hyperplane $H$ of $\mathcal{H}_{gr} \setminus \mathcal{H}_{sp}$, $\mathcal{H}_H := \langle \mathcal{H}_{sp}, H \rangle$ is an 8-dimensional subspace of $\mathcal{H}_{gr}$.
Lemma 8.2 Let $H$ be a hyperplane of type II. Let $G$ denote the subgrid of $\Delta$ consisting of all deep points with respect to $H$ and let $Q$ denote the unique quad of $DW(5, q)$ containing $G$. Let $x$ be a point of $\Delta$ and let $H'$ be the singular hyperplane with deepest point $x$.

(i) If $x \in H \setminus Q$, then every hyperplane of $[H, H'] \setminus \{H, H'\}$ is a hyperplane of type IV.

(ii) If $x \notin H$, then every hyperplane of $[H, H'] \setminus \{H, H'\}$ is a hyperplane of type VI.

(iii) If $x \in G$, then every hyperplane of $[H, H'] \setminus \{H, H'\}$ is a hyperplane of type II.

(iv) If $x \in Q \setminus G$, then $[H, H'] \setminus \{H, H'\}$ contains $\frac{q^2}{2} - 1$ hyperplanes of type II and $\frac{q^2}{2}$ hyperplanes of type III.

Proof.

(i) This is precisely Proposition 6.7.

(ii) A hyperplane of $[H, H'] \setminus \{H, H'\}$ cannot contain hyperplanes of type I, II, III or IV by Lemma 5.5. Such a hyperplane cannot be hexagonal either since $H \notin H_{sp}$ and $H' \in H_{sp}$.

(iii) Let $G'$ denote the singular hyperplane of $Q$ with deep point $x$. By Lemma 4.1, there exists a unique pencil $W$ of hyperplanes of $Q$ satisfying the following properties: (i) $G, G' \in W$, (ii) every hyperplane of $W$ arises from the absolutely universal embedding of $Q$. One easily sees that every hyperplane of $W \setminus \{G'\}$ is a subgrid of $Q$. Now, by Proposition 3.1, the hyperplanes of $[H, H']$ are the extensions of the hyperplanes of $W$. This proves claim (iii).

(iv) Let $G'$ denote the singular hyperplane of $Q$ with deep point $x$ and let $W$ denote the unique pencil of hyperplanes of $Q$ satisfying the following property: (i) $G, G' \in W$; (ii) every hyperplane of $W$ arises from the absolutely universal embedding of $Q$. Since $G'$ is singular and $G$ is not singular, no hyperplane of $W \setminus \{G, G'\}$ is singular. Suppose $W \setminus \{G, G'\}$ contains $\alpha$ grids and $q - 1 - \alpha$ ovoids. From $|G| = (q + 1)^2$, $|G'| = q^2 + q + 1$ and $|G \cap G'| = q + 1$,

$$q^3 + q^2 + q + 1 = |Q| = (q+1)+q^2+(q^2+q)+\alpha(q^2+q)+(q-1-\alpha)(q^2-q).$$
It follows that $\alpha = \frac{q}{2} - 1$. By Proposition 3.1, the hyperplanes of $[H, H']$ are extensions of elements of $W$. This proves claim (iv).

**Lemma 8.3** Let $H$ be a hyperplane of type II and let $H'$ be a hexagonal hyperplane. Then every hyperplane of $[H, H'] \setminus \{H, H'\}$ is either of type IV or of type VI.

**Proof.** Let $G$ be the subgrid of $\Delta$ consisting of all deep points with respect to $H$ and let $Q$ denote the unique quad of $\Delta$ containing $G$. The grid $G$ is not contained in $H'$. Let $x_1$ be a point of $G \setminus H'$. Then by Lemma 7.3, there exists a unique point $x_2$ at distance 3 from $x_1$ such that $H' \in [H_1, H_2]$, where $H_i$, $i \in \{1, 2\}$, denotes the singular hyperplane with deepest point $x_i$. Let $H_3$ denote a hyperplane of $[H, H'] \setminus \{H, H'\}$. Then $H_3$ belongs to the plane of $H_{gr}$ spanned by $H$, $H_1$ and $H_2$. Hence, $H_3 \in [H_2, H_4]$, where $H_4$ is some hyperplane of the line of $H_{gr}$ spanned by $H$ and $H_1$. By Lemma 8.2 (iii), $H_4$ is the extension of a subquadrangle of $Q$. By Lemma 8.2 (i) and (ii), $H_3$ is either of type IV or VI.

**Lemma 8.4** If $H'$ is an 8-dimensional subspace of $H_{gr}$ through $H_{sp}$ containing a hyperplane of type II, then $H'$ contains precisely $\frac{1}{2}q^2(q^2 + 1)$ hyperplanes of type II.

**Proof.** By Lemma 8.2, $H'$ contains $1 + (q + 1)^2(q - 1) + [(q + 1)(q^2 + 1) - (q + 1)^2](\frac{q}{2} - 1) = \frac{1}{2}q^2(q^2 + 1)$ hyperplanes of type II.

**Lemma 8.5** If $H'$ is an 8-dimensional subspace of $H_{gr}$ through $H_{sp}$, then $H'$ contains precisely $\frac{1}{2}q^2(q^2 + 1)$ hyperplanes of type II.

**Proof.** There are $\frac{1}{2}q^2(q^2 + 1)(q^2 + q + 1)(q^3 + 1)$ hyperplanes of type II. Hence, by Lemma 8.4, there are $(q^2 + q + 1)(q^3 + 1)$ 8-dimensional subspaces through $H_{sp}$ containing a hyperplane of type II. The latter number is equal to the total number of 8-dimensional subspaces of $H_{gr}$ through $H_{sp}$. Hence, every 8-dimensional subspace of $H_{gr}$ through $H_{sp}$ contains precisely $\frac{1}{2}q^2(q^2 + 1)$ hyperplanes of type II.

**Proposition 8.6** If $H$ is a hyperplane of type VI of $DW(5, 2^h)$, then there exists a hyperplane $H_1$ of type II and a singular hyperplane $H_2$ such that the following holds:

(i) $H \in [H_1, H_2]$;

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(ii) the deepest point of $H_2$ does not belong to $H_1$.

**Proof.** Let $H$ be a hyperplane of type VI. By Lemma 8.5, the 8-dimensional subspace $H_H = \langle H_{sp}, H \rangle$ of $H_{gr}$ contains a hyperplane $H'_1$ of type II. The line $[H, H'_1]$ of $H_H$ intersects $H_{sp}$ in a point $H'_2$. Obviously, $H \in [H'_1, H'_2]$. If $H'_2$ is a singular hyperplane, then its deepest point cannot be contained in $H'_1$ by Lemma 8.2. In this case, the claim follows with $H_1 = H'_1$ and $H_2 = H'_2$.

If $H'_2$ is a hexagonal hyperplane, then by the proof of Lemma 8.3, we know that $H \in [H_1, H_2]$, where $H_1$ is the extension of a certain $(q+1) \times (q+1)$-grid $G$ and $H_2$ is a singular hyperplane whose deepest point does not belong to the quad carrying $G$. Since $H \in [H_1, H_2]$ is of type VI, the deepest point of $H_2$ does not belong to $H_1$. Hence, also the claim holds in this case. ■

**Proposition 8.7** Up to isomorphism, there exist a unique hyperplane of type VI in $DW(5, 2^h)$.

**Proof.** This follows from the following facts:

(i) If $H$ is a hyperplane of type VI, then by Proposition 8.6, there exists a hyperplane $H_1$ of type II and a singular hyperplane $H_2$ such that (a) $H \in [H_1, H_2]$ and (b) the deepest point of $H_2$ does not belong to $H_1$.

(ii) By Lemma 2.2, the automorphism group of $DW(5, 2^h)$ acts transitively on the pairs $(G, x)$, where $G$ is a $(q+1) \times (q+1)$-subgrid of $DW(5, 2^h)$ and $x$ is a point of $\Delta_2(G)$.

(iii) If $H_1$ and $H_2$ are two distinct hyperplanes of $H_{gr}$, then by Lemma 1.1, $[H_1, H_2]^\theta = [H_1^\theta, H_2^\theta]$ for every automorphism $\theta$ of $DW(5, 2^h)$.

(iv) If $H_1$ is a hyperplane of type II and if $H_1$ is a singular hyperplane whose deepest point $x$ does not belong to $H_1$, then by (iii) and Lemma 2.4, the automorphism group of $DW(5, 2^h)$ acts transitively on the set of hyperplanes of $[H_1, H_2] \setminus \{H_1, H_2\}$. ■

**Remark.** We have shown that there are 6 isomorphism classes of hyperplanes in $DW(5, 2^h)$ which arise from its Grassmann-embedding. The number of points in a hyperplane of type I, II, III or V follows from a straightforward calculation (the latter number is equal to the number of points of the split-Cayley hexagon $H(q)$). The number of points in a hyperplane of type IV or VI follows from Lemma 8.2 (i) + (ii). In either case, the number is equal to $\frac{1}{q-1}(|\Delta| + q \cdot |H \cap H'| - |H| - |H'|)$, where $H$ and $H'$ are hyperplanes as in Lemma 8.2 (i) or (ii). We have listed the obtained numbers in Table 1 of Section 1.4.
9 The hyperplanes of $DW(5, 2)$

The hyperplanes of the dual polar space $\Delta = DW(5, 2)$ were classified by Pralle [23] with the aid of a computer. In this section, we will present a computer free proof based on counting the sizes of the various classes of hyperplanes which we will describe. Recall that by Ronan [24], every hyperplane of $\Delta$ arises from its universal embedding. Since this universal embedding has projective dimension 14, there are $|\mathcal{H}_{un}| = 2^{15} - 1 = 32767$ distinct hyperplanes of $\Delta$. If $H_1$ and $H_2$ are two distinct hyperplanes of $\Delta$, then we denote by $H_1 \ast H_2$ the unique hyperplane in $[H_1, H_2] \setminus \{H_1, H_2\}$. Obviously, $H_1 \ast H_2$ is the complement of the symmetric difference of $H_1$ and $H_2$. By the main theorem, there are 6 isomorphism classes of hyperplanes in $\mathcal{H}_{gr}$. The number of hyperplanes of each type and the number of points in each hyperplane of $\mathcal{H}_{gr}$ were calculated before. We will list these numbers again in Table 2 at the end of this section. We will now classify the hyperplanes of $\mathcal{H}_{un} \setminus \mathcal{H}_{gr}$.

We start with describing the locally subquadrangular hyperplanes of $\Delta$. We will work with the dual polar space $DQ(6, 2)$ which is isomorphic to $\Delta$. Let $Q(6, 2)$ be a nonsingular parabolic quadric in $\text{PG}(6, 2)$ and let $\pi$ be a hyperplane of $\text{PG}(6, 2)$ intersecting $Q(6, 2)$ in a nonsingular hyperbolic quadric $Q^+(5, 2)$. The 105 generators of $Q(6, 2)$ not contained in $Q^+(5, 2)$ define a hyperplane $H_1$ of $DQ(6, 2)$. By Pasini and Shpectorov [20], every locally subquadrangular hyperplane of $DQ(6, 2)$ arises in this way. Now, there are 36 hyperplanes of $\text{PG}(6, 2)$ intersecting $Q(6, 2)$ in a hyperbolic quadric of type $Q^+(5, 2)$ giving rise to 36 locally subquadrangular hyperplanes of $DQ(6, 2)$. Through every point of a locally subquadrangular hyperplane $H$ there are 6 lines which are contained in $H$; hence, $H \not\in \mathcal{H}_{gr}$ by Lemma 5.2. One readily verifies that if $H$ is a locally subquadrangular hyperplane, then the complement of $H$ (which is a set of generators of a $Q^+(5, 2)$) is a convex set of points of $\Delta$ (but it is not a subspace).

Define now the following four classes of hyperplanes of $\Delta$:

- **TYPE VII**: the locally subquadrangular hyperplanes of $\Delta$;
- **TYPE VIII**: the hyperplanes of the form $H_1 \ast H_2$, where $H_1$ is a locally subquadrangular hyperplane and where $H_2$ is a singular hyperplane whose deepest point belongs to $H_1$;
• **Type IX**: the hyperplanes of the form $H_1 \ast H_2$, where $H_1$ is a locally subquadrangular hyperplane and where $H_2$ is a singular hyperplane whose deepest point does not belong to $H_1$;

• **Type X**: the hyperplanes of the form $H_1 \ast H_2$, where $H_1$ is a locally subquadrangular hyperplane and where $H_2$ is a hexagonal hyperplane.

**Lemma 9.1** A hyperplane of type VII contains 105 points. A hyperplane of type VIII contains 73 points. A hyperplane of type IX contains 57 points and a hyperplane of type X contains 65 points.

**Proof.** (1) We already remarked above that a hyperplane of type VII has 105 points.

(2) Let $H_1$ be a locally subquadrangular hyperplane of $\Delta$, let $x$ be a point of $H_1$ and let $H_2$ be the singular hyperplane with deepest point $x$. As $x$ is contained in 3 subquadrangular quads and 4 deep quads (with respect to $H_1$), $|H_1 \cap H_2| = 1 + 6 \cdot 2 + 3 \cdot 4 + 4 \cdot 8 = 57$. Hence, $|H_1 \ast H_2| = |\Delta| - |H_1| - |H_2| + 2 \cdot |H_1 \cap H_2| = 135 - 105 - 71 + 2 \cdot 57 = 73$.

(3) Let $H_1$ be a locally subquadrangular hyperplane of $\Delta$, let $x$ be a point of $\Delta$ not contained in $H_1$ and let $H_2$ be the singular hyperplane with deepest point $x$. As $x$ is contained in 7 subquadrangular quads (with respect to $H_1$), $|H_1 \cap H_2| = 7 + 7 \cdot 6 = 49$. Hence, $|H_1 \ast H_2| = |\Delta| - |H_1| - |H_2| + 2 \cdot |H_1 \cap H_2| = 135 - 105 - 71 + 2 \cdot 49 = 57$.

(4) Let $H_1$ be a locally subquadrangular hyperplane of $\Delta$ and let $H_2$ be a hexagonal hyperplane. We will calculate $|H_1 \cap H_2|$. There are 28 quads contained in $H_1$. Each such quad contains 7 points of $H_1 \cap H_2$. Conversely, through every point of $H_1 \cap H_2$, there are 4 quads which are contained in $H_1$. Hence, $|H_1 \cap H_2| = \frac{28 \cdot 7}{4} = 49$ and $|H_1 \ast H_2| = |\Delta| - |H_1| - |H_2| + 2 \cdot |H_1 \cap H_2| = 135 - 105 - 63 + 2 \cdot 49 = 65$. ■

**Corollary 9.2** (1) An 8-dimensional subspace through $\mathcal{H}_{sp}$ contains at most one locally subquadrangular hyperplane.

(2) There are 3780 hyperplanes of type VIII, 1080 hyperplanes of type IX and 4320 hyperplanes of type X.

**Proof.** By Lemma 9.1, the hyperplanes of type VII, VIII, IX and X define four mutually disjoint classes of hyperplanes. Claim (1) readily follows. If $H$ is a locally subquadrangular hyperplane, then $\langle \mathcal{H}_{sp}, H \rangle$ contains 105 hyperplanes of type VIII, 30 hyperplanes of type IX and 120 hyperplanes of type X.
X. If we multiply these numbers with the total number of locally subquadrangular hyperplanes (i.e. 36), we get the numbers mentioned in Claim (2).

The automorphism group of $\Delta$ fixing a subquadrangular hyperplane $H$ acts transitively on the set of points of $H$ and the set of points not contained in $H$. Hence, there are up to isomorphism unique hyperplanes of type VIII and IX. We now show that there is also a unique hyperplane of type X.

**Definition.** Let $H_1$ be a locally subquadrangular hyperplane of $\Delta$. A hexagonal hyperplane $H_2$ of $\Delta$ is called special with respect to $H_1$ if there are two points $x_1, x_2 \in \Delta \setminus H_1$ at distance 3 from each other such that $H_2 = H_{x_1} \ast H_{x_2}$. Here, $H_{x_i}, i \in \{1, 2\}$, denotes the singular hyperplane with deepest point $x_i$.

**Lemma 9.3** Let $H_1$ be a locally subquadrangular hyperplane of $\Delta$. Let $H_2$ and $H_3$ be two hexagonal hyperplanes which are special with respect to $H_1$. Then $H_2 \ast H_1$ and $H_3 \ast H_1$ are isomorphic.

**Proof.** Let $Q^+(5, 2)$ denote the hyperbolic quadric on $Q(6, 2)$ defining $H_1$. Let $x_2, x_3, y_2, y_3 \in \Delta \setminus H_1$ such that $d(x_2, y_2) = d(x_3, y_3) = 3$, $H_{x_2} \ast H_{y_2} = H_2$ and $H_{x_3} \ast H_{y_3} = H_3$. Then $x_2$ and $y_2$ (respectively $x_3$ and $y_3$) correspond with two disjoint generators $\alpha_2$ and $\beta_2$ (respectively $\alpha_3$ and $\beta_3$) of $Q^+(5, 2)$. Now, the automorphism group of the polar space $Q(6, 2)$ acts transitively on the ordered pairs of disjoint generators of $Q(6, 2)$. Hence, there exists an automorphism $\theta$ of $Q(6, 2)$ such that $\alpha_2^\theta = \alpha_3$ and $\beta_2^\theta = \beta_3$. Such an automorphism necessarily fixes $Q^+(5, 2)$ since the ambient space of $Q^+(5, 2)$ is $\langle \alpha_2, \beta_2 \rangle = \langle \alpha_3, \beta_3 \rangle$. So, $\theta$ defines an automorphism of $DQ(6, 2)$ fixing $H_1$ and mapping $H_2$ to $H_3$. This proves the lemma.

**Lemma 9.4** Let $H_1$ be a locally subquadrangular hyperplane of $\Delta$. Then every hexagonal hyperplane is special with respect to $H_1$.

**Proof.** The number of unordered pairs of disjoint generators of $Q^+(5, 2)$ is equal to 120. This is also the number of hexagonal hyperplanes of $\Delta$. So, it suffices to show the following: if $H_2$ is a hexagonal hyperplane of $\Delta$, then there is at most one pair $\{x_1, x_2\}$ of points of $\Delta \setminus H_1$ at distance 3 from each other such that $H_2 = H_{x_1} \ast H_{x_2}$. Suppose $\{x_1, x_2\}$ is such a set of points. In the proof of Lemma 9.1, we showed that $|H_1 \cap H_2| = 49$. Hence, $H_2 \setminus H_1 = 14$. Now, let $X_i, i \in \{1, 2\}$, denote the set of points collinear with
Let \( y \in X_1 \). Then \( y \in H_{x_1} \). Since \( \Delta \setminus H_1 \) is convex, also \( y \in H_{x_2} \). So, \( y \in H_2 = H_{x_1} \ast H_{x_2} \). This proves that \( X_1 \subseteq H_2 \setminus H_1 \). By symmetry, also \( X_2 \subseteq H_2 \setminus H_1 \).

We will now show how \( \{x_1, x_2\} \) can be retrieved in a unique way from the set \( H_2 \setminus H_1 \). The graph induced by \( \Delta \) on the set \( \Delta \setminus H_1 \) is isomorphic to the point graph of \( DQ^+(5, 2) \) and hence is a bipartite graph. By the previous discussion, we know that the set \( H_2 \setminus H_1 \) can be partitioned into two subsets \( Y_1 \) and \( Y_2 \) of size 7 belonging to different parts of the bipartite graph. Obviously, \( \{Y_1, Y_2\} = \{X_1, X_2\} \) and \( \{Y_1^\perp, Y_2^\perp\} = \{\{x_1\}, \{x_2\}\} \). The lemma now readily follows.

The following is an immediate corollary of Lemmas 9.3 and 9.4.

**Corollary 9.5** Up to isomorphism there exists a unique hyperplane of type \( X \).

**Definition.** A hyperplane \( H \) of \( \Delta \) is called of type \( XI \) if there exists a locally subquadrangular hyperplane \( H_1 \) and a hyperplane \( H_2 \) of type \( II \) such that:

- the quad of \( \Delta \) containing the deep points of \( H_2 \) is contained in \( H_1 \);
- \( H = H_1 \ast H_2 \).

Now,

(i) the automorphism group of \( \Delta \) fixing \( H_1 \) acts transitively on the set of quads of \( \Delta \) contained in \( H_1 \);

(ii) the automorphism group of \( \Delta \) fixing \( H_1 \) and a quad \( Q \) of \( \Delta \) contained in \( H_1 \) acts transitively on the set of \((3 \times 3)\)-subgrids of \( Q \).

[The necessary automorphisms can easily be realized by composing reflections about quads.] Hence, up to isomorphism there is a unique hyperplane of type \( XI \).

**Lemma 9.6** A hyperplane \( H \) of type \( XI \) contains 81 points.
Proof. Let $H_1$ and $H_2$ be as above. Then $|H_1| = 105$, $|H_2| = 87$, $|H_1 \cap H_2| = 15 + 9 \cdot 3 \cdot 2 = 69$ and hence $|H_1 \ast H_2| = |\Delta| - |H_1| - |H_2| + 2 \cdot |H_1 \cap H_2| = 135 - 105 - 87 + 2 \cdot 69 = 81$. ■

We will now give an alternative description of the hyperplanes of type XI, which will allow us to count them. Let $H$ be a hyperplane of type XI and let $H_1$ and $H_2$ be as above. Let $G$ denote the grid of $\Delta$ consisting of all deep points with respect to $H_2$ and let $Q_1$ denote the unique quad containing $G$. Let $L_i$, $i \in \{4, \ldots, 9\}$, denote the six lines of $G$. Without loss of generality, we may suppose that $L_4 \cup L_5 \cup L_6 = G = L_7 \cup L_8 \cup L_9$. If $Q_i$, $i \in \{4, \ldots, 9\}$, denotes the unique quad through $L_i$ different from $Q_1$ contained in $H_1$, then $Q_1 \cup Q_4 \cup Q_5 \cdots \cup Q_9 \subseteq H_1 \ast H_2$. Moreover, $Q_6$ (respectively $Q_9$) is the reflection of $Q_5$ about $Q_4$ (respectively $Q_8$ about $Q_7$). If $u \in G$, then every line of $H$ through $u$ is contained in $Q_1 \cup Q_4 \cup Q_5 \cup \cdots \cup Q_9$. If $u \in Q_1 \setminus G$, then precisely one line through $u$ is contained in $H$, but not in $Q_1$. Now, $|Q_1 \cup Q_4 \cup Q_5 \cup \cdots \cup Q_9| = 69$, while $|H_1 \ast H_2| = 81$. So, there exists a point $x$ in $H_1 \ast H_2$ not contained in $Q_1 \cup \cdots \cup Q_9$. If the points $\pi_{Q_4}(x)$ and $\pi_{Q_5}(x)$ were collinear, then $x$, $\pi_{Q_4}(x)$ and $\pi_{Q_5}(x)$ form a line and $x \in Q_6$, a contradiction. Hence, $Q_2 := \langle \pi_{Q_4}(x), \pi_{Q_5}(x) \rangle$ is the unique quad through $x$ intersecting $Q_4$ and $Q_5$ in lines. Now, $Q_2 \cap H$ contains the grid $Q_2 \cap (Q_4 \cup Q_5 \cup Q_6) \subseteq H$ and the point $x \in H$. It follows that $Q_2 \subseteq H$. Also, $Q_2$ is disjoint with $Q_1$. If $Q_3$ denotes the reflection of $Q_2$ about $Q_1$, then $Q_3 \subseteq H$. Now, $Q_1 \cup Q_2 \cup \cdots \cup Q_9 \subseteq H_1 \ast H_2$ and hence $Q_1 \cup Q_2 \cup \cdots \cup Q_9 = H_1 \ast H_2$, since both sets contain precisely 81 points.

Lemma 9.7 Let $Q_1$, $Q_2$ and $Q_3$ denote three mutually disjoint quads of $\Delta$ such that $Q_3$ is the reflection of $Q_2$ about $Q_1$. Let $G$ be a $(3 \times 3)$-subgrid of $Q_1$ and let $Q_4, Q_5, \ldots, Q_9$ denote the quads meeting $Q_1$, $Q_2$, $Q_3$ and intersecting $Q_1$ in a line of $G$. Then $Q_1 \cup Q_2 \cup \cdots \cup Q_9$ is a hyperplane of type XI.

Proof. This follows from the discussion preceding this lemma and the fact that the automorphism group of $\Delta$ acts transitively on the triples $(Q_1, Q_2, G)$, where $Q_1$ and $Q_2$ are two disjoint quads of $\Delta$ and $G$ is a $(3 \times 3)$-grid of $Q_1$. Again, the necessary automorphisms can easily be realized as composition of reflections about quads. ■

Remark. The description of the hyperplanes of type XI as given in Lemma 9.7 can also be found in [12].
Lemma 9.8 There are precisely 1120 hyperplanes of type XI.

Proof. The number of hyperplanes of type XI is equal to $\frac{1}{18} \times (\# \text{ ordered pairs of disjoint quads}) \times (\# \text{ grids in a quad}) = \frac{1}{18} \cdot 63 \cdot 32 \cdot 10 = 1120$. ■

Definition. A hyperplane $H$ of $\Delta$ is called of type XII if there exists a locally subquadrangular hyperplane $H_1$ and a hyperplane $H_2$ of type III such that

- the quad of $\Delta$ containing the deep points of $H_2$ is contained in $H_1$;
- $H = H_1 \ast H_2$.

Now,

(i) the automorphism group of $\Delta$ fixing $H_1$ acts transitively on the sets of quads of $\Delta$ contained in $H_1$;
(ii) the automorphism group of $\Delta$ fixing $H_1$ and a quad $Q$ of $\Delta$ contained in $H_1$ acts transitively on the ovoids of $Q$.

Again, the necessary automorphisms can easily be realized by composing reflections about quads. It follows that there exists up to isomorphism a unique hyperplane of type XII.

Lemma 9.9 A hyperplane $H$ of type XII contains 65 points.

Proof. Let $H_1$ and $H_2$ be as above, then $|H_1| = 105$, $|H_2| = 55$ and $|H_1 \cap H_2| = 15 + 5 \cdot 3 \cdot 2 = 45$. Hence, $|H_1 \ast H_2| = |\Delta| - |H_1| - |H_2| + 2 \cdot |H_1 \cap H_2| = 65$. ■

Lemma 9.10 If $H$ is a hyperplane of type XII, then there exists a unique pair $(H_1, H_2)$ satisfying: (i) $H_1$ is a locally subquadrangular hyperplane; (ii) $H_2$ is the extension of an ovoid $O$ in a quad $Q$; (iii) $Q \subseteq H_1$; (iv) $H = H_1 \ast H_2$.

Proof. By Lemma 5.3, every hyperplane of type XII contains a unique deep quad. So, the quad $Q$ in property (iii) has to be the unique deep quad of $H$. Also the ovoid $O$ is easy to determine: $O$ consists of the points of $Q$ which are contained in 6 lines of $H$ (the points of $Q \setminus O$ are contained in 4 lines of $H$). Knowing $Q$ and $O$, we also know $H_2$ and hence also $H_1 = H \ast H_2$. This proves the lemma. ■
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Table 2: The hyperplanes of $DW(5,2)$

**Lemma 9.11** The number of hyperplanes of type XII is equal to 6048.

**Proof.** The number of hyperplanes of type XII is equal to ($\#$ locally subquadrangular hyperplanes) $\times$ ($\#$ deep quads in a given locally subquadrangular hyperplane) $\times$ ($\#$ ovoids in a quad), i.e., equal to $36 \cdot 28 \cdot 6 = 6048$. ■

In Table 2, we give the number of hyperplanes in each of the 12 considered classes. We also mention how many points each hyperplane contains.

**Theorem 9.12** Every hyperplane of $DW(5,2)$ is a hyperplane of type I, II, ..., XI or XII.

**Proof.** By Table 2, the number of hyperplanes of type I, II, ..., XI or XII is equal to $|\mathcal{H}_{un}| = 32767$, i.e., the total number of hyperplanes of $DW(5,2)$. This proves the theorem. ■

**References**


