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# Moufang sets generated by translations in unitals

**Theo Grundhöfer**[1] | **Markus J. Stroppel**[2] | **Hendrik Van Maldeghem**[3]

[1]Institut für Mathematik, Universität Würzburg, Würzburg, Germany

[2]Fachbereich Mathematik, Fakultät für Mathematik und Physik, Universität Stuttgart, Stuttgart, Germany

[3]Department of Mathematics, Ghent University, Gent, Belgium

**Correspondence**
Markus J. Stroppel, Fachbereich Mathematik, Fakultät für Mathematik und Physik, Universität Stuttgart, Stuttgart 70550, Germany.
Email: stroppel@mathematik.uni-stuttgart.de

**Abstract**

We consider unitals of order $q$ with two points which are centers of translation groups of order $q$. The group $G$ generated by these translations induces a Moufang set on the block joining the two points. We show that $G$ is either $\mathrm{SL}(2,\mathbb{F}_q)$ (as in all classical unitals and also in some nonclassical examples), or $\mathrm{PSL}(2,\mathbb{F}_q)$, or a Suzuki, or a Ree group. Moreover, $G$ is semiregular outside the special block.

**KEYWORDS**

automorphism, design, Moufang set, Ree group, Suzuki group, translation, two-transitive group, unital, unitary group

**MATHEMATICAL SUBJECT CLASSIFICATION**

05E20, 05B30, 51A10

In [9] we considered unitals admitting all possible translations (see Section 1 for definitions) and characterized the classical (hermitian) unitals by this property. The present paper takes a more general view: We only assume translations with centers on a single block, and prove the following.

**Main Theorem.** *Let $\mathbb{U}$ be a unital of order $q$ with two points which are centers of translation groups of order $q$. Then the group $G$ generated by these two translation groups is isomorphic to one of the following.*

(a) $\mathrm{SL}(2, \mathbb{F}_q)$ *or* $\mathrm{PSL}(2, \mathbb{F}_q)$*, where* $q$ *is a prime power.*
(b) *The Suzuki group* $\mathrm{Sz}(q)$*, where* $q = 2^{2s} \geq 2^6$ *for some odd integer* $s \geq 3$.
(c) *The Ree group* $\mathrm{Ree}(q)$*, where* $q = 3^{3r} \geq 3^3$ *for some odd integer* $r \geq 1$.

*Moreover, the group* $G$ *acts semiregularly on the set of points outside the block containing the translation centers.*

In the classical (i.e., hermitian) unital of order $q$, the group $G$ as above is isomorphic to $\mathrm{SL}(2, \mathbb{F}_q)$, compare [10, 3.1, 4.1]. It seems that no unital of odd order $q$ is known where $G \cong \mathrm{PSL}(2, \mathbb{F}_q)$; for $q = 3$ there is no such unital by Proposition 2.1 (this uses [9, 2.3]). There exists a nonclassical unital of order $q = 4$ such that $G \cong \mathrm{SL}(2, \mathbb{F}_4) \cong A_5$, see [10, 4.1]. More examples (of order 8) have been found by Möhler in her Ph.D. thesis [20, Section 6], see [21].

The unitals of order $q$ with two points which are centers of translation groups of order $q$ are studied also by Rizzo in his Ph.D. thesis [25]. The last chapter of this thesis contains results about embeddings of such unitals into projective planes of order $q^2$. For generalized quadrangles, a situation analogous to the one considered in the present paper is treated in a series of papers, see [17,34–36].

Our Main Theorem resembles results for projective planes (instead of unitals) obtained by Hering [11,12], who considered groups generated by elations. The following statement is a very special case of [12, Theorem 3.1]: If a projective plane of finite order $q$ contains a triangle $p, z_1, z_2$ such that the group of all elations with center $z_j$ and axis $pz_j$ has order $q$ for $j = 1, 2$, then the plane is desarguesian and the group generated by these two elation groups is isomorphic to $\mathrm{SL}(2, \mathbb{F}_q)$.

## 1 | UNITALS, TRANSLATIONS, AND MOUFANG SETS

A unital $\mathbb{U} = (U, \mathcal{B})$ of order $q > 1$ is a 2-$(q^3 + 1, q + 1, 1)$-design. In other words, $\mathbb{U}$ is an incidence structure such that any two points in $U$ are joined by a unique block in $\mathcal{B}$, there are $|U| = q^3 + 1$ points, and every block has exactly $q + 1$ points. It follows that every point is on exactly $q^2$ blocks.

**Lemma 1.1.** *Let* $\mathbb{V}$ *be a unital of order* $q$*, and let* $\varphi \in \mathrm{Aut}(\mathbb{V})$ *be an automorphism of* $\mathbb{V}$*. If* $\varphi$ *fixes more than* $q^2 + q - 2$ *points then* $\varphi$ *is trivial. In particular, if* $\varphi$ *fixes a point* $x$ *and a block* $B$ *not through* $x$ *and all points on blocks joining* $x$ *to a point on* $B$ *then* $\varphi$ *is trivial.*

*Proof.* Let $y$ be a point that is moved by $\varphi$. Joining $y$ with each one of the fixed points yields a set of blocks through $y$. At most one of those blocks can be a fixed block of $\varphi$, and a nonfixed block contains at most one fixed point. If a fixed block through $y$ exists then that block contains at most $q - 1$ fixed points. For the number $f$ of fixed points we obtain $q^2 - 1 \geq f - (q - 1)$ and $f \leq q^2 + q - 2$. If no block through $y$ is fixed then $f \leq q^2 \leq q^2 + q - 2$.

The second assertion follows from the fact that the point set in question contains $(q + 1)q + 1 = q^2 + q + 1$ points. □

An automorphism of $\mathbb{U}$ is called a *translation* of $\mathbb{U}$ with center $z$ if it fixes each block through the point $z$. The set of all translations with center $z$ is denoted by $\Gamma_{[z]}$.

A *Moufang set* is a set $X$ together with a collection of groups $(R_x)_{x\in X}$ of permutations of $X$ such that each $R_x$ fixes $x$ and acts regularly (i.e., sharply transitively) on $X\setminus\{x\}$, and such that the collection $\{R_y|y \in X\}$ is invariant under conjugation by the *little projective group* $\langle R_x|x \in X\rangle$ of the Moufang set. The groups $R_x$ are called *root groups*.

The finite Moufang sets are known explicitly:

**Theorem 1.2.** *The little projective group of a finite Moufang set is either sharply two-transitive, or it is permutation isomorphic to one of the following two-transitive permutation groups of degree* $q+1$: $\mathrm{PSL}(2,\mathbb{F}_q)$ *with a prime power* $q>3$, $\mathrm{PSU}(3,\mathbb{F}_{f^2}|\mathbb{F}_f)$ *with a prime power* $q=f^3\geq 3^3$, *a Suzuki group* $\mathrm{Sz}(2^s)={}^2B_2(2^s)$ *with* $q=2^{2s}\geq 2^6$, *or a Ree group* $\mathrm{Ree}(3^r)={}^2G_2(3^r)$ *with* $q=3^{3r}$, *where* $r$ *and* $s$ *are positive odd integers.*

This was proved (in the context of split BN-pairs of rank one) by Suzuki [31] and Shult [27] for even $q$, and by Hering, Kantor and Seitz [13] for odd $q$; these papers rely on deep results on finite groups, but not on the classification of all finite simple groups. See also Peterfalvi [24]. Note that $\mathrm{PSL}(2,\mathbb{F}_2)\cong\mathrm{AGL}(1,\mathbb{F}_3)$, $\mathrm{PSL}(2,\mathbb{F}_3)\cong A_4\cong\mathrm{AGL}(1,\mathbb{F}_4)$, $\mathrm{PSU}(3,\mathbb{F}_4|\mathbb{F}_2)\cong\mathrm{ASL}(2,\mathbb{F}_3)$, and $\mathrm{Sz}(2)\cong\mathrm{AGL}(1,\mathbb{F}_5)$ are sharply two-transitive. The smallest Ree group $\mathrm{Ree}(3)\cong\mathrm{P\Gamma L}(2,\mathbb{F}_8)$ is almost simple, but not simple.

Let $\mathbb{U}=(U,\mathcal{B})$ be a unital of order $q$, and let $\Gamma=\mathrm{Aut}(\mathbb{U})$ be its automorphism group. Throughout this paper, we assume that $\mathbb{U}$ contains two points $\infty$ and $o$ such that for $z\in\{\infty,o\}$ the translation group $\Gamma_{[z]}$ has order $q$. Then $\Gamma_{[z]}$ acts transitively on $B\setminus\{z\}$, for any block $B$ through $z$ (see [9, 1.3]). In particular, $\Gamma_{[x]}$ has that transitivity property for each point $x$ on the block $B_\infty$ joining $\infty$ and $o$. The group $G$ generated by $\Gamma_{[\infty]}\cup\Gamma_{[o]}$ contains the translation group $\Gamma_{[x]}$ for each $x\in B_\infty$, and $(B_\infty,(\Gamma_{[x]}|_{B_\infty})_{x\in B_\infty})$ is a Moufang set, with little projective group $G^\dagger := G|_{B_\infty}\cong G/G_{[B_\infty]}$, where $G_{[B_\infty]}$ is the kernel of the action on $B_\infty$. This kernel coincides with the center $Z$ of $G$, see [9, 3.1.2] or [11, 2.11]. So $G$ is a central extension of the little projective group $G^\dagger$.

**Corollary 1.3.** *The kernel* $G_{[B_\infty]}=Z$ *acts semiregularly on* $U\setminus B_\infty$.

*Proof.* If $\varphi\in G_{[B_\infty]}=Z$ fixes $x\in U\setminus B_\infty$, then it fixes also $x^g$ for every $g\in G$, hence all points on blocks joining $x$ to a point on $B_\infty$. Thus Lemma 1.1 implies that $\varphi$ is trivial. □

The following fact was observed in the proof of [12, Theorem 2.4].

**Lemma 1.4.** *Let* $(X,\{\Delta_x|x\in X\})$ *be a finite Moufang set. If the little projective group* $\Phi=\langle\Delta_x|x\in X\rangle$ *is simple then* $\Delta_x=[\Delta_x,\Phi_x]$ *for every* $x\in X$, *where* $\Phi_x$ *denotes the stabilizer of* $x$ *in* $\Phi$.

*Proof.* By the classification of finite Moufang sets, see 1.2, the simple group $\Phi$ is isomorphic to $\mathrm{PSL}(2,\mathbb{F}_q)$, $\mathrm{PSU}(3,\mathbb{F}_{t^2}|\mathbb{F}_t)$, $\mathrm{Sz}(2^s)$, or $\mathrm{Ree}(3^r)$, where $q>3$ with $q+1=|X|$, $t>2$ with $t^3+1=|X|$, $s>1$ with $2^{2s}+1=|X|$, or $r>1$ with $3^{3r}+1=|X|$, respectively. We have $[\Delta_x,\Phi_x]\leq\Delta_x$ since $\Phi_x$ normalizes $\Delta_x$; it remains to show that $\Delta_x\leq[\Delta_x,\Phi_x]$. This inclusion is an ingredient in simplicity proofs for $\Phi$ that use the Iwasawa criterion:

For $\Phi=\mathrm{PSL}(2,\mathbb{F}_q)$ the necessary commutators are computed in the proof of [33, Theorem 4.4, p. 23]. The case where $\Phi=\mathrm{PSU}(3,\mathbb{F}_{t^2}|\mathbb{F}_t)$ is covered by [15, Proof of II.10.13, p. 244]. For $\Phi=\mathrm{Sz}(2^s)$ the assertion follows from the commutator formula in

[33, p. 205], and for $\Phi = \mathrm{Ree}(3^r)$ the three commutator formulas in [4, Section 5, p. 36/37] yield the assertion. $\square$

*Remark* 1.5. The references in the proof of 1.4 yield the following sharper conclusion: For every $y \in X\backslash\{x\}$ there exists an element $\varphi \in \Phi_{x,y}$ such that $\Delta_x$ is equal to the set $\{[\delta, \varphi] | \delta \in \Delta_x\}$ of commutators. See also the proof of [11, 2.11b].

**Proposition 1.6.** *If $G^\dagger$ is simple then $G$ is a perfect central extension of $G^\dagger$, that is, $G$ coincides with its commutator group $G'$, and $G_{[B_\infty]}$ is isomorphic to a quotient of the Schur multiplier of $G^\dagger$.*

*Proof.* If $z \in B_\infty$, $\tau \in \Gamma_{[z]}$, and $\gamma \in G_z$, then $\gamma^{-1}\tau\gamma \in \Gamma_{[z]}$ and $[\tau, \gamma] = \tau^{-1}\gamma^{-1}\tau\gamma \in \Gamma_{[z]}$. Since $\Gamma_{[z]}$ acts regularly on $B_\infty\backslash\{z\}$, every element of $\Gamma_{[z]}$ is determined by its action on $B_\infty$, that is, by its image in $G^\dagger$. By 1.4 every element of $\Gamma_{[z]}$ is a product of elements in $\Gamma_{[z]}$ that are commutators. Hence $\Gamma_{[z]} \le G'$ for every $z \in B_\infty$, and therefore $G' = G$.

The kernel $G_{[B_\infty]}$ of the action on $B_\infty$ is the center of $G$, so the perfect group $G$ is a central extension of $G^\dagger = G/G_{[B_\infty]}$. Therefore $G_{[B_\infty]}$ is isomorphic to a quotient of the Schur multiplier of $G^\dagger$; see [18, 2.1.7], [15, V.23.3, p. 629], or [2, 33.8 (4), p. 169]. $\square$

## 2 | SHARPLY TWO-TRANSITIVE GROUPS

**Proposition 2.1.** *If $q \le 3$ then $\mathbb{U}$ is the hermitian unital of order $q$.*

*Proof.* Every unital of order 2 is isomorphic to the hermitian one, see, for example, [33, 10.16]. Now let $q = 3$. Since $G^\dagger \le S_4$ is generated by elements of order 3, we have $G^\dagger = A_4 \cong \mathrm{PSL}(2, \mathbb{F}_3)$; in particular, $G^\dagger$ is sharply two-transitive. By [9, 2.3] the center $Z = G_{[B_\infty]}$ has even order, so there exists a central involution $\zeta$ in $G$.

The product $G'Z$ induces the commutator group $(G^\dagger)' \cong {C_2}^2$. Thus $G'Z$ has index 3, and $G'$ acts transitively on $B_\infty$. For $z \in B_\infty$, the translation group $\Gamma_{[z]}$ is not contained in $G'$. We obtain $G = G'\Gamma_{[z]}G' = G'\Gamma_{[z]}$, and $G'$ has index 3 in $G$. This means that $Z \le G'$, and $G$ is a covering group of $A_4$. Then $G \cong \mathrm{SL}(2, \mathbb{F}_3)$ by [18, 2.12.5]. This group acts regularly on $U\backslash B_\infty$, see [9, 3.5].

We verify that $\mathbb{U}$ is obtained by the construction described in [10, 2.1]. The central involution $\zeta$ does not fix any point apart from those on $B_\infty$. Therefore, the point set $U\backslash B_\infty$ is partitioned by fixed blocks of $\zeta$; these are obtained as the blocks joining $x \in U\backslash B_\infty$ with its image under $\zeta$. The group $G$ acts on this set of fixed blocks. There are six such blocks, and at least one of them is fixed by a subgroup $S$ of order 4 in $G$. We pick a point $a$ on that block and identify the elements of $G$ with the affine points via $\gamma \mapsto a^\gamma$. Then the block in question is $S$.

There are four blocks through $a$ that join $a$ to points on $B_\infty$, their intersections with $U\backslash B_\infty$ are identified with the Sylow 3-subgroups (viz., the translation groups) in $G$. Let $D$ be any one of the remaining four blocks through $a$. Then $D$ is not stabilized by any translation, and not stabilized by $\zeta$. As $\mathrm{SL}(2, \mathbb{F}_3)$ contains only one involution, we infer that the stabilizer of $D$ in $G$ is trivial. Therefore, the set $\mathcal{D} := \{D\delta^{-1} | \delta \in D\}$ consists of four different blocks through $a$.

It has been proved in [10, 3.3] that the subgroup $S$ and the set $\mathcal{D}$ are unique, up to conjugation. The points at infinity are the centers of translations. Therefore each such point is incident with those blocks whose points outside $B_\infty$ form an orbit under the corresponding translation group. This completes the proof that $\mathbb{U}$ is isomorphic to the hermitian unital $\mathbb{U}_{\mathcal{H}_3}$, see [10, 3.3]. □

Now we determine certain central extensions of finite sharply two-transitive permutation groups; the following result is a variation of [12, Lemma 1.1] that is suitable for our purpose.

**Theorem 2.2.** *Let $(G, X)$ be a finite sharply two-transitive permutation group with $|X| > 1$ and let $p$ be the prime dividing $|X|$. If $E$ is a central extension of $G = E/Z$ by a group $Z$ of order $p$, then $E$ splits over $Z$ (as a direct product $E = Y \times Z$ with $Y \cong G$), or we have one of the following*:

(a) $|X| = 2 = |G|$ *and $E$ is cyclic of order* 4.
(b) $|X| = 4$, $G = A_4 \cong \mathrm{PSL}(2, \mathbb{F}_3)$, *and* $E \cong \mathrm{SL}(2, \mathbb{F}_3)$.
(c) $|X| = p^2 \in \{3^2, 5^2, 7^2, 11^2\}$ *and $E = P \rtimes H$ where $P$ is the Heisenberg group of order $p^3$ and $H$ is isomorphic to* $Q_8$, $\mathrm{SL}(2, \mathbb{F}_3)$, $2^-S_4$, *or* $\mathrm{SL}(2, \mathbb{F}_5)$, *respectively.*

We describe the groups in item (c). The Heisenberg group of order $p^3$ consists of all unipotent upper triangular matrices in $\mathrm{GL}(3, \mathbb{F}_p)$. By $Q_8$ we denote the quaternion group of order 8, and $2^-S_4$ is the binary octahedral group, that is, the double cover of $S_4$ containing just one involution, see [32, 3.2.21, p. 301] or [16, XII.8.4]; this double cover is isomorphic to the normalizer of $\mathrm{SL}(2, \mathbb{F}_3)$ in $\mathrm{SL}(2, \mathbb{F}_9)$. The extension groups $E$ in item (c) do not split over $Z$ since $P$ is not abelian.

*Proof of* 2.2. It is well known that $|X| = p^n$ is a power of a prime $p$ and that the Sylow $p$-subgroup of $G$ is an elementary abelian normal subgroup of order $p^n$ in $G$; see, for example, [26, 7.3.1] or [23, 8.4] or [16, XII.9.1].

Let $P$ be a Sylow $p$-subgroup of $E$. Then $|P| = p^{n+1}$ and $Z \leq P$; moreover $P/Z$ is the regular normal subgroup of $G$, hence $P$ is normal in $E$. Each point stabilizer (or Frobenius complement) $G_x$ has order $p^n - 1$, and its preimage $E_x \leq E$ has order $(p^n - 1)p$. The group $E_x$ splits as a direct product $H \times Z$ with $H \cong G_x$ by the abelian (in fact, central) case of the Schur–Zassenhaus theorem; see [26, 9.1.2 or 11.4.12] or [23, 10.3] or [15, I.17.5, p. 122]. Then

$$E = P \rtimes H$$

and $H$ acts (by conjugation) sharply transitively on the set of nontrivial elements of $P/Z$.

If $H$ is trivial, then $|X| = 2 = |G|$, and $E$ splits or is cyclic as in item (a). From now on let $|H| > 1$. Then $C_P(H)/Z$ is a proper $H$-invariant subgroup of $P/Z$, hence trivial. This means that $C_P(H) = Z$. If $P$ is abelian, then $P = C_P(H) \times [P, H] = Z \times [P, H]$, see [6, 5.2.39] or [26, 10.1.6] or [15, III.13.4, p. 350], and then $E = P \rtimes H = Z \times ([P, H] \rtimes H)$ splits over $Z$.

Now let $P$ be nonabelian. Then $P'$ is a nontrivial subgroup of $Z$, as $P/Z$ is (elementary) abelian, hence $P' = Z$. The center of $P$ yields a proper $H$-invariant subgroup of $P/Z$; this subgroup is trivial, hence $Z$ is the center of $P$ (and the Frattini subgroup is $\Phi(P) = P^pP' = Z$). Thus $P$ is an extraspecial $p$-group.

The commutator map gives a nonzero symplectic form $f$ on $P/Z$ with values in the prime field $\mathbb{F}_p$, and $f$ is not degenerate, hence $n = 2m$ is even; see [26, p. 140] or [15, III.13.7, p. 353]. The automorphism group $\overline{H}$ induced by $H$ on $P$ has trivial intersection with the group of inner automorphisms of $P$ and acts trivially on $Z$, hence $H \cong \overline{H}$ is isomorphic to a subgroup of the symplectic group $\mathrm{Sp}(2m, \mathbb{F}_p)$ by Winter [41, Theorem 1 or (3A), p. 161].

First we assume that the permutation group $(G, X)$ is of type I (in the notation of [16, XII.9.2]), which entails that $H \leq \Gamma L(1, \mathbb{F}_{p^n}) = \mathrm{GL}(1, \mathbb{F}_{p^n}) \rtimes \mathrm{Aut}(\mathbb{F}_{p^n})$; see [16, XII.9.2]. In another terminology, this means that the corresponding nearfield (with multiplicative group $H$) is a Dickson nearfield, compare [8, p. 834]. The cyclic group $H \cap \mathrm{GL}(1, \mathbb{F}_{p^n}) \leq \mathbb{F}_{p^n}^*$ has order at least $(p^n - 1)/n$. If this cyclic group is reducible on $\mathbb{F}_p^n$, then it is contained in a proper subfield of $\mathbb{F}_{p^n}$, hence $(p^n - 1)/n \leq p^{n/2} - 1$ and therefore $p^{n/2} + 1 \leq n$; if $H \cap \mathrm{GL}(1, F_{p^n})$ is irreducible, then its order divides $p^{n/2} + 1$ by [41, Cor. 2] or [15, Satz 9.23, p. 228] as $H \leq \mathrm{Sp}(n, \mathbb{F}_p)$. In both cases we have $2^{n/2} - 1 \leq p^{n/2} - 1 \leq n$, which is false for $n \geq 6$. If $n = 4$ then $p = 2$ and $|H| = 15$, hence $H$ is cyclic and irreducible, but 15 does not divide $2^2 + 1$. As $n = 2m$ is even, there only remains the case where $n = 2$, and $p \in \{2, 3\}$ follows.

If $p = 2$ then $|X| = 4$ and $G = A_4 \cong \mathrm{PSL}(2, \mathbb{F}_3)$; moreover, $E$ is a covering group of $A_4$ since $Z = P' \leq E'$, hence $E \cong \mathrm{SL}(2, \mathbb{F}_3)$ by [18, 2.12.5], as in item (b). For $p = 3$ we have $|X| = 9$ and $|H| = 8$. Each involution $h \in H$ induces on $P/Z \cong \mathbb{F}_3^2$ a diagonalizable linear transformation $\bar{h}$ without eigenvalue 1, hence $\bar{h} = -\mathrm{id}$. Thus $H$ contains just one involution, and $H$ is cyclic or $H \cong Q_8$ (these two possibilities correspond to the two nearfields of order 9, one of them being the field $\mathbb{F}_9$). The cyclic case is ruled out because $\mathrm{Sp}(2, \mathbb{F}_3) = \mathrm{SL}(2, \mathbb{F}_3)$ contains no element of order 8. Thus $H \cong Q_8$ as in the first case of item (c).

Now we assume that $(G, X)$ is not of type I. Then $n = 2$ and there are just seven possibilities for the isomorphism type of $H$, with $p \in \{5, 7, 11, 23, 29, 59\}$: see [16, XII.9.4] or [23, 20.3] or [8, 2.4]. This rephrases a famous result of Zassenhaus, which says that there are only seven finite nearfields which are not Dickson nearfields. The condition $H \leq \mathrm{Sp}(2, \mathbb{F}_p) = \mathrm{SL}(2, \mathbb{F}_p)$ excludes four of these seven possibilities (those where $H$ contains central elements other than $\pm\mathrm{id}$), see [16, XII.9.4, XII.9.5] or [8, 2.4]. This leads to the three cases for $H$ in item (c) with $p \in \{5, 7, 11\}$.

For all our odd primes $p$, the extraspecial group $P$ of order $p^3$ has exponent $p$: Otherwise the exponent is $p^2$ and some nontrivial element of $P/Z$ is fixed by every automorphism of $P$ by [41, Cor. 1]; this is a contradiction to the action of $H$ on $P/Z$. Therefore $P$ is isomorphic to the Heisenberg group of order $p^3$, see [6, 5.5.1] or [15, p. 355]. □

**Theorem 2.3.** *If $G^\dagger$ is sharply two-transitive on $B_\infty$, then $q \leq 3$ and $\mathbb{U}$ is the hermitian unital of order $q$.*

*Proof.* By Proposition 2.1 it suffices to show that $q \leq 3$. Thus we assume that $q > 3$ and aim for a contradiction.

The degree $q + 1$ of $G^\dagger$ is a power of some prime $r$, say $q + 1 = r^n$. By [9, 3.1] the kernel $G_{[B_\infty]}$ is the center $Z$ of $G$, and $r$ divides $|Z|$ by [9, 2.3]. Thus we can choose a subgroup $U$ of index $r$ in $Z$; then $G/U$ is a central extension of $G^\dagger$ by the group $Z/U$ of order $r$. Such an extension $G/U$ does not split: If $G/U = Z/U \times Y/U$ then $Y$ contains all Sylow $s$-subgroups of $G$ with $s \neq r$, hence all translation groups $\Gamma_{[x]}$ with $x \in B_\infty$; thus $Y = G$, which is a contradiction to $|Z/U| = r$. Theorem 2.2 implies that $n = 2 \neq r$ and that the Sylow $r$-subgroup of $G/U$ is not abelian (and more, as in item (c), but we do not need more). Let $R$ be a Sylow $r$-subgroup of $G$ and let $H := \Gamma_{[o]}$. Then $RZ/Z$ is the regular normal subgroup of $G^\dagger$, and $R$ is characteristic in $RZ$, which is normal in $G$; hence $R$ is normal in $G$. The group $RHR = RH$ contains all conjugates of $H$ in $G$, hence $G = RH = R \rtimes H$. Thus $Z = G_{[B_\infty]} = G_{o,\infty} = (R_o H)_\infty = (R \cap Z)H_\infty = R \cap Z$, which gives $Z \leq R$. The group $R$ is not abelian, but $R/Z$ is abelian and has order $r^2$; thus $Z$ is the center of $R$. Now a (special case of a) result of Wiegold says that $|R'|$ divides $r$; see [39, Theorem 2.1], [32, p. 261], [18, Lemma 3.1.1, p. 113], or [15, p. 637]. We claim that $R' = Z$. Otherwise we can choose $U$ as above with $R' \leq U < Z$, and then $R/U$ is an abelian Sylow $r$-subgroup of $G/U$, contrary to Theorem 2.2.

Thus $R' = Z$ has order $r$, and $R$ is an extraspecial group of order $r^3$. Since $r \neq 2$ the group $H = \Gamma_{[o]}$ contains an involution $\alpha$ inducing inversion on $R/R' = R/Z$, hence $\alpha$ fixes each subgroup between $Z$ and $R$.

Each subgroup of order $r^2$ is normal in $R$ with abelian quotient, and thus contains $R' = Z$. As the group $H$ acts transitively on the set of nontrivial elements of $R/R'$, it also acts transitively on the set of subgroups of order $r^2$ in $R$. If $S$ is one of those subgroups then $R$ acts transitively on the set of noncentral subgroups of order $r$ in $S$. There are $r$ such subgroups, and the involution $\alpha$ (which leaves $S$ invariant) fixes at least one of them, say $T$.

The number of points not on $B_\infty$ is $q^3 + 1 - (q + 1) = r^2(r^2 - 1)(r^2 - 2)$, and not divisible by $r^3 = |R|$. Therefore, there exists some subgroup of order $r$ fixing at least one point $x$ not on $B_\infty$. That subgroup is not contained in the center because the latter acts semiregularly on $U \backslash B_\infty$, see [9, 1.7]. We have noted in the previous paragraph that the noncentral subgroups of order $r$ form a single conjugacy class in $HR$. Thus the group $T$ fixes some affine point $x$. Then $T = T^\alpha$ fixes also $x^\alpha$, and the point $o$ where $B_\infty$ meets the block joining $x$ and $x^\alpha$. This contradicts the fact that $T$ induces a subgroup of order $r$ in the regular normal subgroup on $B_\infty$. □

## 3 | UNITARY GROUPS

**Lemma 3.1.** *Let $r$ be prime power, and let $d$ be a divisor of $r + 1$. Then the following hold:*

(a) *Every element of order $d$ in $\mathrm{GL}(3, \mathbb{F}_{r^2})$ is diagonalizable over $\mathbb{F}_{r^2}$.*
(b) *If $A$ is an element of order $d$ in $\mathrm{SU}(3, \mathbb{F}_{r^2}|\mathbb{F}_r)$ then the characteristic polynomial of $A$ is $X^3 - t_A X^2 + \overline{t_A} X - 1$, where $t_A$ is the trace of $A$.*
(c) *Two elements of order $d$ in $\mathrm{SU}(3, \mathbb{F}_{r^2}|\mathbb{F}_r)$ are conjugates under $\mathrm{SU}(3, \mathbb{F}_{r^2}|\mathbb{F}_r)$ if, and only if, they have the same trace.*

*Proof.*

(a) Let $A \in \mathrm{GL}(3, \mathbb{F}_{r^2})$ be an element of order $d$. The minimal polynomial of $A$ then divides $X^{r+1} - 1$, and every characteristic root is a root of that polynomial. These roots lie in $\mathbb{F}_{r^2}$ because $r+1$ divides the order of the multiplicative group of $\mathbb{F}_{r^2}$. As the minimal polynomial has only simple roots, the matrix $A$ is diagonalizable in $\mathrm{GL}(3,\mathbb{F}_{r^2})$.

(b) Now assume $A \in \mathrm{SU}(3,\mathbb{F}_{r^2}|\mathbb{F}_r)$. Let $\lambda$ be one of the characteristic roots of $A$, then $\overline{\lambda}\,\lambda = \lambda^r\lambda = 1$. In the characteristic polynomial $\det(X\cdot\mathrm{id}-A) = X^3+c_2X^2+c_1X+c_0$, the constant $c_0$ equals $-\det A = -1$. The coefficient $c_2$ equals $-t_A$, where $t_A$ is the trace of $A$. Expanding the product of the linear factors, we obtain $t_A = -c_2$ as the sum $\lambda_0+\lambda_1+\lambda_2$ of all characteristic roots of $A$. The coefficient $c_1$ is obtained as $\lambda_0\lambda_1+\lambda_1\lambda_2+\lambda_2\lambda_0 = \lambda_2^{-1}+\lambda_0^{-1}+\lambda_1^{-1} = \overline{\lambda_2}+\overline{\lambda_0}+\overline{\lambda_1} = \overline{t_A}$.

(c) Let $A$ and $B$ be elements of order $d$ in $\mathrm{SU}(3,\mathbb{F}_{r^2}|\mathbb{F}_r)$. Clearly $t_A = t_B$ holds if $A$ and $B$ are conjugates. Conversely, assume $t_A = t_B$. We have seen above that $A$ and $B$ have the same characteristic polynomial. Therefore, they are conjugates in $\mathrm{GL}(3,\mathbb{F}_{r^2})$. According to [29, I, 3.5, III, 3.22] (or [37, Case A(ii), p. 34] or [5, Lemma 5 with remarks on p. 12]) they are also conjugates in the unitary group $U(3,\mathbb{F}_{r^2}|\mathbb{F}_r)$.

Finally, the group $U(3,\mathbb{F}_{r^2}|\mathbb{F}_r)$ contains diagonal elements of arbitrary determinant in $\{s \in \mathbb{F}_{q^2} | s\overline{s} = 1\}$. As such diagonal matrices centralize each other diagonal matrix, we can adapt the conjugating element of $U(3,\mathbb{F}_{r^2}|\mathbb{F}_r)$ in such a way that the conjugation is achieved by an element of $\mathrm{SU}(3,\mathbb{F}_{r^2}|\mathbb{F}_r)$. □

The following lemma is a consequence of results in [22, Thm. 1.6, Thm. 1.3]; we give a direct proof for the reader's convenience.

**Lemma 3.2.** *Let $r = 2^e$ and let $A \in \mathrm{SU}(3,\mathbb{F}_{r^2}|\mathbb{F}_r)$ be noncentral with $A^{r+1} = 1$. Then $A^2$ is the product of two elements of $\mathrm{SU}(3,\mathbb{F}_{r^2}|\mathbb{F}_r)$ with orders dividing 4.*

*Proof.* We use coordinates such that the hermitian form is described by $x_0\overline{y_2} + x_1\overline{y_1} + x_2\overline{y_0}$. The element $J := \begin{pmatrix}1&0&0\\0&1&0\\1&0&1\end{pmatrix} \in \mathrm{SU}(3,\mathbb{F}_{r^2}|\mathbb{F}_r)$ is an involution, and $F_{u,v} := \begin{pmatrix}1&u&v\\0&1&\overline{u}\\0&0&1\end{pmatrix}$ belongs to $\mathrm{SU}(3,\mathbb{F}_{r^2}|\mathbb{F}_r)$ if $v+\overline{v} = u\overline{u}$. Note also that $F_{u,v}^4 = \mathrm{id}$, and $F_{u,v}^2 = \mathrm{id}$ holds if $u = 0$ (then $v \in \mathbb{F}_r$). The product $JF_{u,v} = \begin{pmatrix}1&u&v\\0&1&\overline{u}\\1&u&v+1\end{pmatrix}$ has trace $v+1$, and its characteristic polynomial is $X^3+(v+1)X^2+(\overline{v}+1)X+1$.

Let $t_A$ be the trace of the given matrix $A$ and put $v := t_A+1$. The norm map $N:\mathbb{F}_{r^2}\to\mathbb{F}_r : x \mapsto x\overline{x} = x^{r+1}$ is surjective, hence we find $u \in \mathbb{F}_{r^2}$ such that $u\overline{u} = v+\overline{v}$. We abbreviate $F := F_{u,v}$ and infer from 3.1 that $JF$ and the diagonalizable matrix $A$ have the same characteristic polynomial, hence also the same set of eigenvalues. If $JF$ is diagonalizable, then $JF$ has the same order as $A$, and 3.1 implies that $A$ is conjugate to $JF$ in $\mathrm{SU}(3,\mathbb{F}_{r^2}|\mathbb{F}_r)$. Now $A^2$ is conjugate to $(JF)^2 = JFJF = (JFJ^{-1})F$.

It remains to consider the case where $JF$ is not diagonalizable. Then the characteristic polynomial has a root $\lambda$ with multiplicity 2 (not 3 since $A$ is not central), and $A$ is conjugate to the diagonal matrix $\mathrm{diag}(\lambda,\lambda,\lambda^{-2})$ where $N(\lambda) = \lambda^{r+1} = 1 \neq \lambda^3$. Thus $JF$ is similar (i.e., conjugate in $\mathrm{GL}(3,\mathbb{F}_{r^2})$) to its Jordan normal form

$$\begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{pmatrix},$$

hence $(JF)^2$ is similar to $\mathrm{diag}(\lambda^2, \lambda^2, \lambda^{-4})$ which is similar to $A^2$. The matrix $(JF)^2 = JFJF = (JFJ^{-1})F$ is conjugate to $A^2$ in $\mathrm{SU}(3, \mathbb{F}_{r^2}|\mathbb{F}_r)$ by 3.1. □

*Remark* 3.3. The assumption that $A$ is not central is needed in 3.2. Indeed, for any field $F$ of characteristic two, nontrivial central elements of $\mathrm{GL}(n, F)$ are never products of two elements in Sylow 2-subgroups. In fact, a nontrivial central element is of the form $u$ id with $u \in F$. The elements of Sylow 2-subgroups are unipotent (i.e., they have 1 as their only characteristic root). If the product of unipotent elements $S, T$ equals $u$ id then $S = uT^{-1}$ is a unipotent element with characteristic root $u$, so $u = 1$ and the product is trivial, indeed.

**Theorem 3.4.** *The little projective group $G^\dagger$ is not isomorphic to* $\mathrm{PSU}(3, \mathbb{F}_{r^2}|\mathbb{F}_r)$*, for any $r$.*

*Proof.* If $G^\dagger$ is isomorphic to $\mathrm{PSU}(3, \mathbb{F}_{r^2}|\mathbb{F}_r)$ then the translation groups are the root subgroups, that is, the (Sylow) subgroups of order $r^3$ in $\mathrm{PSU}(3, \mathbb{F}_{r^2}|\mathbb{F}_r)$. In particular, we have $q = r^3$. For $r = 2$ we have $q = 8$, and $G^\dagger$ is (isomorphic to) the sharply two-transitive group $\mathrm{PSU}(3, \mathbb{F}_4|\mathbb{F}_2) \cong Q_8 \ltimes \mathbb{F}_3^2$; this is excluded by 2.3.

From now on, let $r > 2$. The group $\mathrm{PSU}(3, \mathbb{F}_{r^2}|\mathbb{F}_r)$ is perfect, and $G$ is a perfect central extension of $\mathrm{PSU}(3, \mathbb{F}_{r^2}|\mathbb{F}_r)$, see 1.6 or [9, 3.1]. For the case at hand, we know that $\mathrm{SU}(3, \mathbb{F}_{r^2}|\mathbb{F}_r)$ is the universal cover of $\mathrm{PSU}(3, \mathbb{F}_{r^2}|\mathbb{F}_r)$, see [7, Thm. 2]. So we assume that $\mathrm{SU}(3, \mathbb{F}_{r^2}|\mathbb{F}_r)$ acts (not necessarily faithfully) on the unital $\mathbb{U}$ such that the root subgroups induce transitive groups of translations with center on $B_\infty$. Assume first that $r$ is odd, and let $2^a$ be the highest power of 2 dividing $|U \setminus B_\infty| = (r^3 + 1)r^3(r^3 - 1)$. Then $2^a$ divides $(r^3 + 1)(r - 1)$ and $2^{a+1}$ divides $(r^3 + 1)r^3(r - 1)(r + 1) = |\mathrm{SU}(3, \mathbb{F}_{r^2}|\mathbb{F}_r)|$. So some point in $U \setminus B_\infty$ is fixed by some involution $\gamma \in \mathrm{SU}(3, \mathbb{F}_{r^2}|\mathbb{F}_r)$. We use coordinates such that the hermitian form defining $\mathrm{SU}(3, \mathbb{F}_{r^2}|\mathbb{F}_{r^2})$ is given by $x_0\overline{y_2} + x_1\overline{y_1} + x_2\overline{y_0}$. Then the matrices $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1/2 & -1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & -4 & -8 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$ belong to root groups of $\mathrm{SU}(3, \mathbb{F}_{r^2}|\mathbb{F}_r)$, and their product $\begin{pmatrix} 1 & -4 & -8 \\ 1 & -3 & -4 \\ -1/2 & 1 & 1 \end{pmatrix}$ is an involution (and represents an involution in $\mathrm{PSU}(3, \mathbb{F}_{r^2}|\mathbb{F}_r)$). All involutions in $\mathrm{SU}(3, \mathbb{F}_{r^2}|\mathbb{F}_r)$ are conjugate by 3.1, hence $\gamma$ is a product of two root elements and does not fix any point outside $B_\infty$; this is a contradiction.

Therefore $r$ is even. Let $p$ be a prime dividing $r + 1$, and let $m$ be the largest integer such that $p^m$ divides $r + 1$. Then $p$ is odd (because $r$ is even), and $p^{2m}$ divides $(r^3 + 1)r^3(r^2 - 1) = |\mathrm{SU}(3, \mathbb{F}_{r^2}|\mathbb{F}_r)|$.

If $p > 3$ then $p$ does not divide $r^2 - r + 1$, and $p^{m+1}$ does not divide $|U \setminus B_\infty| = (r^3 + 1)r^3(r^2 + r + 1)(r - 1)$. So there exists at least one orbit whose length is not divisible by $p^{m+1}$, and there exists an element $\gamma$ of order $p$ in the stabilizer of some point not in $B_\infty$. If $\gamma$ is not central in $\mathrm{SU}(3, \mathbb{F}_{r^2}|\mathbb{F}_r)$ then $\gamma^2$ is a product of two root elements (see 3.2) and does not fix any point outside $B_\infty$. So $\gamma$ is a central element of order $p > 3$ in $\mathrm{SU}(3, \mathbb{F}_{r^2}|\mathbb{F}_r)$, contradicting the fact that the center of $\mathrm{SU}(3, \mathbb{F}_{r^2}|\mathbb{F}_r)$ has order 3 or is trivial.

There remains the case where $p = 3$ is the only prime divisor of $r + 1$. Then $r + 1 = 2^d + 1 = 3^m$ for positive integers $d$ and $m$. We infer that $r = 2^d \in \{2, 8\}$, see, for example, [23, Lemma 19.3]; this is an old result of Levi ben Gerson from 1343, see [3, Section 4, pp. 169 ff]. Since $r > 2$ we have $r = 8$ and $m = 2$. Then $3^3 = 3^{m+1}$ is the highest power of 3 dividing $|U\backslash B_\infty| = (r^3 + 1)r^3(r^2 + r + 1)(r - 1) = 2^9 \cdot 3^3 \cdot 7 \cdot 19 \cdot 73$ but $3^5 = 3^{2m+1}$ divides $|\mathrm{SU}(3, \mathbb{F}_{64}|\mathbb{F}_8)| = (r^3 + 1)r^3(r^2 - 1) = 2^9 \cdot 3^5 \cdot 7 \cdot 19$. We now find an element $\gamma$ of order 3 in the stabilizer of a point not in $B_\infty$. If $\gamma$ is not central in $\mathrm{SU}(3, \mathbb{F}_{64}|\mathbb{F}_8)$ then $\gamma = \gamma^{-2}$ is a product of two root elements (see 3.2) and does not fix any point outside $B_\infty$. So $\gamma$ is a central element of order 3 in $\mathrm{SU}(3, \mathbb{F}_{64}|\mathbb{F}_8)$ and fixes every point in $\mathbb{U}$, see 1.3. This means that $\mathrm{SU}(3, \mathbb{F}_{64}|\mathbb{F}_8)$ induces on $\mathbb{U}$ a group isomorphic to $\mathrm{PSU}(3, \mathbb{F}_{64}|\mathbb{F}_8)$, of order $(r^3 + 1)r^3(r^2 - 1)/3 = (8^3 + 1)8^3(8^2 - 1)/3 = 2^9 \cdot 3^4 \cdot 7 \cdot 19$. Since $3^4$ does not divide $|U\backslash B_\infty|$ we still find an element of order 3 in the stabilizer of a point not on $B_\infty$, and reach a contradiction using 3.2 again. □

## 4 | SUZUKI GROUPS AND REE GROUPS

**Theorem 4.1.** *If $G^\dagger$ is a Suzuki group then $q \geq 2^6$ and $G = G^\dagger$, and $G$ acts semiregularly on $U\backslash B_\infty$.*

*Proof.* We have $G^\dagger = \mathrm{Sz}(2^s)$ for some odd integer $s \geq 1$, and the unital has order $q = 2^{2s}$. The smallest Suzuki group $\mathrm{Sz}(2) \cong \mathrm{AGL}(1, \mathbb{F}_5)$ is sharply two-transitive, and excluded by 2.3.

The Schur multiplier of $\mathrm{Sz}(2^3)$ is elementary abelian of order 4, see [1], compare [40, 4.2.4] and [18, 7.4.2]. If $\zeta$ is a central involution in $G$ then $\zeta$ acts trivially on $B_\infty$, and joining any point $x$ with $x^\zeta$ gives a block $B$ fixed by $\zeta$. If that block does not meet $B_\infty$ then $\zeta$ fixes at least one of the $q + 1 = 65$ points on $B$. This contradicts 1.3. So $B$ contains a point $z$ of $B_\infty$. Then there exists a translation $\tau$ with center $z$ such that $x^\zeta = x^\tau$. The translations have order dividing 4, hence $\tau\zeta$ is an element of order 2 or 4 fixing $x$. If $\tau$ has order 4 then $(\tau\zeta)^2 = \tau^2$ is nontrivial translation fixing $x$. This is impossible, so $\tau$ is an involution. The automorphisms $\zeta\tau$ and $\tau$ induce the same action on $B_\infty$. In particular, the involution $\zeta\tau$ fixes no point on $B_\infty$ apart from $z$. For each point $y \in U\backslash B_\infty$, the block joining $y$ and $y^{\tau\zeta}$ is fixed by $\tau\zeta$, and meets $B_\infty$ in a fixed point of $\tau\zeta$; that point has to be $z$. This means that $\tau\zeta$ fixes every block through $z$, and is a translation with center $z$. Now $\zeta = \tau(\tau\zeta) \in \Gamma_{[z]}$ is a translation fixing every point on $B_\infty$. This contradicts the fact that a nontrivial translation fixes only one point. So $G = G^\dagger$ holds if $G^\dagger = \mathrm{Sz}(2^3)$.

If $G^\dagger = \mathrm{Sz}(2^s)$ with $s > 3$ then $G = G^\dagger$ because the Schur multiplier is trivial; see [1], compare [40, 4.2.4] and [18, 7.4.2]. Thus we have $G = G^\dagger = \mathrm{Sz}(2^s)$ for $s \geq 3$. Consequently, each element of order 2 or 4 in $G$ is a translation. Apart from the elements of order 4, every element in $\mathrm{Sz}(2^s)$ is strongly real, that is, a product of two involutions; see, for example, [19, 24.7, 24.6]. In particular, every nontrivial element is the product of two translations (viz., elements of order dividing 4), and does not fix any point in $U\backslash B_\infty$. So the action of $G$ on $U\backslash B_\infty$ is semiregular. □

The following result is contained in [12, 2.6]; we give a more detailed proof.

**Lemma 4.2.** *In the Ree group* $\mathrm{Ree}(r)$ *with* $r = 3^{2e+1} \geq 3$, *every element of prime order is the product of two elements with orders dividing* 9.

*Proof.* All involutions in $\mathrm{Ree}(r)$ are conjugate (also for $r = 3$), so each of them is contained in a subgroup isomorphic to $\mathrm{Ree}(3) \cong P\Gamma L(2, \mathbb{F}_8) \cong \mathrm{SL}(2, \mathbb{F}_8) \rtimes C_3$. The factorization $\begin{pmatrix} 1 & u+1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & u \end{pmatrix}$ in $\mathrm{SL}(2, \mathbb{F}_8)$, where $u \in \mathbb{F}_8$ satisfies $u^3 + u + 1 = 0$, shows that every involution is the product of an element of order 3 with an element of order 9 (it is also the product of two elements of order 9, see [9, Case (6), p. 429]).

The root elements of $\mathrm{Ree}(r)$ have orders dividing 9; thus it remains to consider elements with prime order $p > 3$. We have

$$|\mathrm{Ree}(r)| = (r^3 + 1)r^3(r - 1) = r^3(r^2 - 1)(r + \sqrt{3r} + 1)(r - \sqrt{3r} + 1),$$

and $\mathrm{Ree}(r)$ contains subgroups isomorphic to $\mathrm{PSL}(2, \mathbb{F}_r)$, namely, subgroups of index 2 in centralizers of involutions, see [38, p. 62]. If $p$ divides $r^2 - 1$, then $\mathrm{PSL}(2, \mathbb{F}_r)$ contains a Sylow $p$-subgroup of $\mathrm{Ree}(r)$, and every element of $\mathrm{PSL}(2, \mathbb{F}_r)$ is a product of two elements (transvections) of order 3 by [9, 3.4] or [11, 2.7].

It remains to consider primes $p > 3$ that divide $r \pm \sqrt{3r} + 1$; this includes the prime divisor 7 of $|\mathrm{Ree}(3)|$. The corresponding Sylow $p$-subgroups are cyclic, hence all subgroups of order $p$ are conjugate, and $\mathrm{Ree}(r)$ contains the Frobenius group $C_p \rtimes C_3$ of order $3p$, see [38, IV.3, p. 83]. The inclusion $C_p \rtimes C_3 \leq \mathrm{AGL}(1, \mathbb{F}_p)$ yields that every element of order $p$ in $C_p \rtimes C_3$ is a commutator, hence it is the product of two conjugate elements of order 3. □

**Theorem 4.3.** *If* $G^\dagger$ *is a Ree group then* $G = G^\dagger$, *and the action of* $G$ *on* $U \backslash B_\infty$ *is semiregular.*

*Proof.* We have $G^\dagger = \mathrm{Ree}(r)$ with $r = 3^{2e+1} \geq 3$, and the unital has order $q = r^3$.

We first prove that $G = G^\dagger$ if $r = 3$; then $G^\dagger = \mathrm{Ree}(3) \cong P\Gamma L(2, \mathbb{F}_8) \cong \mathrm{SL}(2, \mathbb{F}_8) \rtimes C_3$. As in [9, Case (6), p. 429], we note that the final term $D$ of the commutator series of $G$ is a cover of $\mathrm{SL}(2, \mathbb{F}_8)$, which has no proper cover (see [15, V.25.7] or [30]), hence $D \cong \mathrm{SL}(2, \mathbb{F}_8)$. There exists a translation $\alpha \in G \backslash D$ of order 3 such that $\langle \alpha, D \rangle = \langle \alpha \rangle \ltimes D$ induces $G^\dagger \cong G/G_{[B_\infty]}$ on $B_\infty$, as in [9, Case (6), p. 429]. Hence $G$ is the direct product of $\langle \alpha \rangle \ltimes D$ with the center $G_{[B_\infty]}$ of $G$. Each Sylow 3-subgroup of $\langle \alpha \rangle \ltimes D$ has order $3^3$ and acts faithfully on $B_\infty$, hence it is a full translation group $\Gamma_{[z]}$ for some $z \in B_\infty$. There exist at least two such Sylow 3-subgroups (as $D$ is simple), and together they generate $G$. Hence $G = \langle \alpha \rangle \ltimes D$ and $G_{[B_\infty]}$ is trivial.

For $r > 3$, the Ree group $\mathrm{Ree}(r)$ is simple and has trivial Schur multiplier; see [1]. So $G = G^\dagger$ holds for every $r \geq 3$, and the Sylow 3-subgroups of $G$ are the translation groups. By 4.2 the stabilizer $G_c$ of a point $c \in U \backslash B_\infty$ cannot contain any element of prime order, hence $G_c$ is trivial. □

## 5 | PROOF OF THE MAIN THEOREM

Let $\mathbb{U}$ be a unital of order $q$ with two points which are centers of translation groups of order $q$. Then the group $G$ generated by these two translation groups induces a Moufang set (as defined in Section 1) on the block $B_\infty$ containing the translation centers (see [9, 3.1], where our present group $G$ is called $\hat{G}$). We have listed the possibilities for the little projective group $G^\dagger$ in 1.2. The group $G^\dagger$ cannot be a unitary group, see 3.4. If $G^\dagger$ is sharply two-transitive then $q \in \{2, 3\}$ and $\mathbb{U}$ is the hermitian unital of order $q$, see 2.3; thus $G \cong \mathrm{SL}(2, \mathbb{F}_q)$.

Now assume that $G^\dagger$ is isomorphic to $\mathrm{PSL}(2, \mathbb{F}_q)$ but not sharply two-transitive. Then $q > 3$, the group $G^\dagger$ is simple, and $G$ is a perfect central extension of $G^\dagger$ by 1.6. In most cases, the Schur multiplier of $\mathrm{SL}(2, \mathbb{F}_q)$ is trivial. In those cases, we have $G \cong \mathrm{SL}(2, \mathbb{F}_q)$ or $G \cong \mathrm{PSL}(2, \mathbb{F}_q)$. The Schur multiplier of $\mathrm{SL}(2, \mathbb{F}_q)$ is not trivial only if $q \in \{4, 9\}$. In these cases, the arguments in [9, p. 428, (2)] show that $G \cong \mathrm{SL}(2, \mathbb{F}_4)$ if $q = 4$ and $G \cong \mathrm{SL}(2, \mathbb{F}_9)$ or $G \cong \mathrm{PSL}(2, \mathbb{F}_9)$ if $q = 9$. By [9, 3.5], the action of $G$ on $U \backslash B_\infty$ is semiregular (this also applies if $q \leq 3$).

If $G^\dagger$ is a Suzuki or Ree group then $G = G^\dagger$, and the action on $U \backslash B_\infty$ is semiregular, see 4.1 and 4.3. The smallest Suzuki group $\mathrm{Sz}(2) \cong \mathrm{AGL}(1, \mathbb{F}_5)$ is sharply two-transitive, and excluded by 2.3.

In each one of the cases discussed above, the order $q$ of the unital turns out to be a prime power (thanks to the restriction $q \in \{2, 3\}$ in the sharply two-transitive case).

## 6 | SIMPLIFICATIONS OF A PREVIOUS PAPER

The present paper yields some simplifications of the classification of the unitals admitting all translations in [9], as we explain now. The elimination of the sharply two-transitive groups in 2.3 leaves only Moufang sets which are determined uniquely by the isomorphism type of their root groups, see [9, 3.3]. Thus the mapping $g : \mathcal{L}_c \to \mathbb{N}$ considered in [9, p. 430] is constant, and Proposition 4.2 in [9] is not needed anymore; the proof of that proposition depends on the classification of the finite simple groups. By 3.4 one can omit the consideration of unitary groups.

The classification of the finite simple groups is still involved at the very end of the proof in [9, p. 430], when we quote a result by Kantor which uses the classification of finite doubly transitive groups. If the order $q$ of the unital is a power of 2, then the classification of finite simple groups can be avoided, because the doubly transitive groups of degree $q^3 + 1$ are classified in [14, Theorem 2]; see also [28].

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### ORCID

*Markus J. Stroppel* https://orcid.org/0000-0002-3163-903X

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