

Finite subunitals of the Hermitian unitals

Theo Grundhöfer, Markus J. Stroppel, Hendrik Van Maldeghem

Abstract

Every finite subunital of any generalized hermitian unital is itself a hermitian unital; the embedding is given by an embedding of quadratic field extensions. In particular, a generalized hermitian unital with a finite subunital is a hermitian one (i.e., it originates from a separable field extension).

A hermitian unital in a pappian projective plane consists of the absolute points of a unitary polarity of that plane, with blocks induced by secant lines (see Section 1). The finite hermitian unitals of order q are the classical examples of $2-(q^3 + 1, q + 1, 1)$ -designs, cp. [6, II.8, pp. 57–63 and p. 246], or [2, Ch. 2]. We consider generalized hermitian unitals $\mathcal{H}(C|R)$ where $C|R$ is any quadratic extension of fields; separable extensions $C|R$ yield the hermitian unitals, inseparable extensions give certain projections of quadrics.

If one has an embedding of field extensions from $E|F$ into $C|R$ (in the sense of 1.5) then clearly the unital $\mathcal{H}(E|F)$ is embedded into $\mathcal{H}(C|R)$. Our Main Theorem asserts, in the converse direction, that every embedding of a finite unital into one of those generalized hermitian unitals comes from an embedding of field extensions. This also implies that no finite subunitals exist in generalized hermitian unitals that are not hermitian unitals.

If \mathbb{U} is a unital of order q embedded in the hermitian unital $\mathcal{H}(\mathbb{F}_{r^2}|\mathbb{F}_r)$ of order r then our result says that \mathbb{U} is the hermitian unital of order q , and there is an odd integer e such that $r = q^e$.

We remark that an analogous result holds for generalized polygons (where the Moufang property singles out the classical examples), see [12, 5.2.2]: any (thick) subpolygon of a Moufang polygon is Moufang, as well.

1 Generalized hermitian unitals

Let $C|R$ be any quadratic (possibly inseparable) extension of fields; so $C = R + \varepsilon R$, with $\varepsilon \in C \setminus R$. There exist $t, d \in R$ such that $\varepsilon^2 - t\varepsilon + d = 0$, since $\varepsilon^2 \in R + \varepsilon R$. The mapping

$$\sigma: C \rightarrow C: x + \varepsilon y \mapsto (x + ty) - \varepsilon y \quad \text{for } x, y \in R$$

is a field automorphism which generates $\text{Aut}_R C$: if $C|R$ is separable, then σ has order 2; if $C|R$ is inseparable, then σ is the identity.

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Now we introduce our geometric objects. We consider the pappian projective plane $\text{PG}(2, C)$ arising from the 3-dimensional vector space C^3 over C , and we use homogeneous coordinates $[X, Y, Z] := (X, Y, Z)C$ for the points of $\text{PG}(2, C)$.

Definition 1.1. The *generalized hermitian unital* $\mathcal{H}(C|R)$ is the incidence structure (H, \mathcal{M}) with the point set $H := \{[X, Y, Z] \mid X^\sigma Y + Z^\sigma Z \in \varepsilon R\}$, and the set \mathcal{M} of *blocks* consists of the intersections of U with secant lines, i.e. lines of $\text{PG}(2, C)$ containing more than one point of H .

Remarks 1.2. The name “generalized hermitian unital” is motivated by the following observations.

If $C|R$ is separable, then $\mathcal{H}(C|R)$ is the hermitian unital arising from the skew-hermitian form $h: C^3 \times C^3 \rightarrow C$ defined by

$$h((X, Y, Z), (X', Y', Z')) = \varepsilon^\sigma X^\sigma Y' - \varepsilon Y^\sigma X' + (\varepsilon^\sigma - \varepsilon) Z^\sigma Z',$$

see [4, 2.2]. So the elements of H are the absolute points of a polarity of $\text{PG}(2, C)$, in this case.

If $C|R$ is inseparable, then $\mathcal{H}(C|R)$ is the projection of an ordinary quadric Q in some projective space of dimension at least 3 from a subspace of codimension 1 in the nucleus of Q , see [4, 2.2]. In this case, there is no polarity of $\text{PG}(2, C)$ such that the elements of H are absolute points.

In any case, we have for each point p of $\mathcal{H}(C|R)$ a unique line of $\text{PG}(2, C)$ passing through p and containing no other point of $\mathcal{H}(C|R)$, see [4, 2.3]. We will refer to this line as the tangent line to $\mathcal{H}(C|R)$ at p in $\text{PG}(2, C)$.

The unital $\mathcal{H}(C|R)$ admits all conceivable translations (i.e., automorphisms that fix a point and every block through it), and these are induced by elations of $\text{PG}(2, C)$, see [4, 2.13].

Each block of $\mathcal{H}(C|R)$ is a Baer subline in $\text{PG}(2, C)$, see [4, 2.8], cp. [5].

By an *O’Nan configuration*, we mean four blocks intersecting in six points of the unital (i.e., a $(6_2 4_3)$ configuration). Naming this configuration in honor of O’Nan [8] is customary in the context of unitals, see [2, p. 87]; the configuration is named after Veblen and Young in the axiomatics of projective spaces, or after Pasch in the context of ordered (Euclidean) geometry.

We will use the following properties of generalized hermitian unitals:

Lemma 1.3. *Let $C|R$ be a quadratic field extension.*

(NON) *There are no O’Nan configurations in $\mathcal{H}(C|R)$.*

(ALL) *For any three distinct points x, y, z on a block, there is a translation of $\mathcal{H}(C|R)$ with center z mapping x to y .*

Let p be a point of $\mathcal{H}(C|R)$, and let X, Y, Z be three distinct blocks through p .

(TRA) *If X, Y, Z intersect a block B not through p , then for each point $z \neq p$ on either of the three blocks, there exists a (unique) block B' containing z and intersecting the three blocks in three distinct points. The block B' is the image of B under a translation of $\mathcal{H}(C|R)$ with center p , and each block through p intersecting either of B, B' intersects both B and B' .*

(TAN) *If X, Y, Z intersect two disjoint blocks B and B' not containing p , then the intersection point of the two lines containing the blocks B and B' in the projective plane $\text{PG}(2, C)$ is on the tangent line at p .*

Proof. Assertions (NON) and (ALL) have been proved in [4, Proposition 2.7, Remark 2.12]. For Assertion (TRA), we apply the translation with center p that maps the intersection of the block joining p and z with B to the point z . Uniqueness of B' is a consequence of (NON), and the rest of Assertion (TRA) follows from the fact that B' is the image of B under a translation with center p .

It remains to verify Assertion (TAN). We have just seen that B' is the image of B under a translation of $\mathcal{H}(C|R)$ with center p . That translation is the restriction of an elation of $\text{PG}(2, C)$, the center is p and the axis is the tangent to $\mathcal{H}(C|R)$ at p in $\text{PG}(2, C)$. The intersection of the lines containing the blocks B and B' in the projective plane $\text{PG}(2, C)$ is a point fixed by that elation, and thus contained in the tangent line at p . \square

Definition 1.4. If (X, \mathcal{B}) and (Y, \mathcal{D}) are linear spaces then an embedding of (X, \mathcal{B}) into (Y, \mathcal{D}) is a pair of injective maps $\alpha: X \rightarrow Y$ and $\beta: \mathcal{B} \rightarrow \mathcal{D}$ such that $x \in X$ and $B \in \mathcal{B}$ are incident in (X, \mathcal{B}) exactly if x^α and B^β are incident in (Y, \mathcal{D}) . We call (X, \mathcal{B}) a *subunital* of (Y, \mathcal{D}) if (id, id) is an embedding of a unital (X, \mathcal{B}) into (Y, \mathcal{D}) .

Definition 1.5. Let $E|F$ and $C|R$ be field extensions. An *embedding* of field extensions from $E|F$ into $C|R$ is a field monomorphism $\eta: E \rightarrow C$ such that $F^\eta \subseteq R$. An embedding (α, β) of $\mathcal{H}(E|F)$ into $\mathcal{H}(C|R)$ is called *standard* if there is an embedding η of field extensions from $E|F$ into $C|R$ such that $((x, y, z)E)^\alpha = (x^\eta, y^\eta, z^\eta)C$ holds for each point $(x, y, z)E$ of $\mathcal{H}(E|F)$, up to some projective transformation of $\text{PG}(2, C)$.

2 Main Result

Lemma 2.1. *Let (P, \mathcal{L}) be a linear space containing no O'Nan configuration, let (U, \mathcal{B}) be a unital of order q , and let (α, β) be an embedding of (U, \mathcal{B}) in (P, \mathcal{L}) . If two blocks B_1, B_2 of (U, \mathcal{B}) have no point of U in common, then B_1^β, B_2^β are disjoint blocks of (P, \mathcal{L}) .*

Proof. Aiming at a contradiction, suppose that two blocks $B_1, B_2 \in \mathcal{B}$ are disjoint in U , but that B_1^β, B_2^β contain a common point x . The absence of O'Nan configurations in (P, \mathcal{L}) implies that two arbitrary blocks of (U, \mathcal{B}) both intersecting $B_1 \cup B_2$ in exactly two points have no points off $B_1 \cup B_2$ in common. Hence the number of points in U lying on a block intersecting $B_1 \cup B_2$ in exactly two points is greater than $(q+1)^2(q-1) \geq q^3 + 1$; this is a contradiction. \square

Theorem 2.2. *Let (U, \mathcal{B}) be a finite subunital of the generalized hermitian unital $\mathcal{H}(C|R)$. Then (U, \mathcal{B}) is a hermitian unital, isomorphic to $\mathcal{H}(E|F)$ for some quadratic extension $E|F$ of finite fields, and the embedding of $(U, \mathcal{B}) \cong \mathcal{H}(E|F)$ into $\mathcal{H}(C|R)$ is standard, coming from an embedding of field extensions from $E|F$ into $C|R$. Moreover, the extension $C|R$ is separable, and $\mathcal{H}(C|R)$ is a hermitian unital.*

Proof. Let q denote the order of (U, \mathcal{B}) . In this proof, we suppose $q > 2$; the case $q = 2$ is treated separately in 3.1 below.

Let $p \in U$ be arbitrary, and let $B \in \mathcal{B}$ be such that $p \notin B$. Let B_0, B_1, \dots, B_q be the blocks of (U, \mathcal{B}) containing p and intersecting B nontrivially, say in x_0, x_1, \dots, x_q , respectively. Let x be an arbitrary point on $B_0 \setminus \{p, x_0\}$. We claim that at least one block of (U, \mathcal{B}) contains x and intersects $B_1 \cup B_2 \cup \dots \cup B_q$ in at least two points (different from p). Indeed, if not, then there are q^2 blocks through x different from B_0 , which is a contradiction. So let B_x be a block of (U, \mathcal{B}) containing at least three points (including x) of $B_0 \cup B_1 \cup \dots \cup B_q$. We note that B_x and B are disjoint by (NON). For the same reason (or by 2.1) their extensions to $\mathcal{H}(C|R)$ are also disjoint. It then follows from (TRA) that B_x intersects B_i for each $i \in \{0, 1, \dots, q\}$, and 2.1 yields that these intersection points belong to U . Hence we have shown that (TRA) holds in the subunital (U, \mathcal{B}) .

Now let τ be the translation of $\mathcal{H}(C|R)$ with center p mapping x_0 to x . Let y be any point of U not on B_0 . Since B was arbitrary, we may assume that $y \in B$, so without loss of generality $y = x_1$. By the uniqueness in (TRA), τ maps x_1 to the intersection $B_x \cap B_1$. Since this intersection point belongs to U , it follows that τ preserves U . Hence (U, \mathcal{B}) admits all translations and hence is hermitian by the main result of [3].

Now consider the (standard) embedding of $\mathcal{H}(C|R)$ in the projective plane $\text{PG}(2, C)$. Then also (U, \mathcal{B}) is embedded in $\text{PG}(2, C)$ and so by the Main Theorem of [4] there is a subfield $E \leq C$ of order q^2 and a subplane $\pi \cong \text{PG}(2, E)$ containing U . (Here we use $q > 2$, for $q = 2$ the Main Theorem of [4] does not apply.) Hence there is a polarity ρ_π of π with absolute point set U . We now show that ρ_π extends to a polarity ρ of $\text{PG}(2, C)$ where the absolute points are the points of $\mathcal{H}(C|R)$.

Consider the lines extending blocks of (U, \mathcal{B}) that meet at least three blocks through p . By (TAN), any two of those lines have an intersection point on the tangent line T to U at p in π . Varying the blocks to be extended, we obtain more than one point on T . The same description applies, *mutatis mutandis*, to points on the tangent to $\mathcal{H}(C|R)$ at p in $\text{PG}(2, C)$. So that line extends the tangent to (U, \mathcal{B}) in π . This already implies that not all tangent lines to $\mathcal{H}(C|R)$ contain the same point and so $C|R$ is separable by [4, 2.3(2)]. Hence there is a polarity ρ of $\text{PG}(2, C)$ associated to $\mathcal{H}(C|R)$. Since U contains a quadrangle, and points of U are mapped to lines of π under the action of ρ , we see that ρ preserves π . Since tangent lines to (U, \mathcal{B}) and $\mathcal{H}(C|R)$ coincide in π , we see that $\rho|_\pi = \rho_\pi$. Hence the generator of the Galois group of $C|R$ preserves E and induces $x \mapsto x^q$ in E .

This proves our main result completely for $q \neq 2$. For $q = 2$, see 3.1. \square

If C is finite of order r^2 then E is unique with given order q^2 and the field automorphism $x \mapsto x^r$ is not trivial on E , which means that E is not contained in the unique subfield R of order r ; hence C is an extension of E of odd degree. Thus 2.2 gives the following.

Corollary 2.3. *If \mathbb{U} is a unital of order q embedded in the hermitian unital $\mathcal{H}(\mathbb{F}_{r^2}|\mathbb{F}_r)$ of order r , then \mathbb{U} is the hermitian unital $\mathcal{H}(\mathbb{F}_{q^2}|\mathbb{F}_q)$, there is an odd integer e such that $r = q^e$, and the embedding of the unital is standard.*

Up to projective equivalence, the embedded unital is obtained by using a hermitian equation over $\mathbb{F}_{q^2}|\mathbb{F}_q$ to define the large unital in $\text{PG}(2, \mathbb{F}_{r^2})$, and restricting coordinates to \mathbb{F}_{q^2} to define the small unital.

Remarks 2.4. Wilbrink [13] has characterized the finite hermitian unitals by three conditions. His condition (I) is our (NON) in 1.3. Under the assumption (NON), his condition (II) is equivalent to our (TRA). For unitals of even order, these two conditions alone suffice to characterize the hermitian unitals, see [7]. Wilbrink's condition (III) is too technical to state it here; compare [5, p. 299].

3 The smallest unital

Theorem 3.1. *Let $C|R$ be a quadratic field extension. The unital of order 2 is embedded in the generalized hermitian unital $\mathcal{H}(C|R)$ if, and only if, R has characteristic 2 and $C \cong R[X]/(X^2 + X + 1)$. The embedding is then a standard one.*

Proof. Each unital of order 2 is isomorphic to the hermitian unital $\mathcal{H}(\mathbb{F}_4|\mathbb{F}_2)$, and is clearly embedded in $\mathcal{H}(C|R)$ if R has characteristic 2 and $C \cong R[X]/(X^2 + X + 1)$.

The unital of order 2 is also isomorphic to the affine plane of order 3 (see [11, 10.16]), and we know the embeddings of the latter into Moufang planes ([10, 3.7], cp. [1, 5.2], [9]): Such an embedding is possible only if the coordinatizing alternative field contains an element u with $u^2 + u + 1 = 0$; the points of the embedded plane are then given as $(0, 0)$, $(1, 0)$, $(0, 1)$, $(-u, 1)$, $(1, -u^2)$, $(-u, -u^2)$, $(0, u)$, (∞) , in suitable inhomogeneous coordinates.

Consider a separable extension $C|R$ first, and assume that there is an embedding of the unital of order 2 into the hermitian unital $\mathcal{H}(C|R)$. We introduce homogeneous coordinates such that the points in question are $[1, 0, 0]$, $[1, 1, 0]$, $[1, 0, 1]$, $[1, -u, 1]$, $[1, 1, -u^2]$, $[1, -u, -u^2]$, $[0, 1, 0]$, $[0, 1, u]$, and $[0, 0, 1]$. In those homogeneous coordinates, the hermitian unital consists of all points $[x, y, z]$ satisfying the equation $(\bar{x}, \bar{y}, \bar{z})M(x, y, z)^T = 0$, where $x \mapsto \bar{x}$ is the generator of the Galois group of $C|R$, and M is a non-singular hermitian 3×3 matrix over C . (Multiplying M by a skew-symmetric field element swaps hermitian with skew-hermitian forms but preserves the unital, so we are in accordance with 1.2.) Now $[1, 0, 0]$, $[0, 1, 0]$, $[0, 0, 1] \in \mathcal{H}(C|R)$ implies that each diagonal entry of M

is zero. Thus $M = \begin{pmatrix} 0 & a & \bar{b} \\ \bar{a} & 0 & c \\ b & \bar{c} & 0 \end{pmatrix}$ and the equation becomes

$$\operatorname{tr}(\bar{x}ay) + \operatorname{tr}(\bar{z}bx) + \operatorname{tr}(\bar{y}cz) = 0;$$

where $\operatorname{tr}(w) := w + \bar{w}$ is the trace of w .

Evaluating this equation for $[1, 1, 0]$, $[1, 0, 1]$, and $[0, 1, u]$, respectively, we obtain $\operatorname{tr}(a) = \operatorname{tr}(b) = \operatorname{tr}(cu) = 0$.

If $\bar{u} = u$ then $0 = \operatorname{tr}(cu) = (\bar{c} + c)u$ yields $\operatorname{tr}(c) = 0$ and then $\det M = \operatorname{tr}(abc) = ab \operatorname{tr}(c) = 0$, a contradiction. So $\bar{u} \neq u$ is another root of $X^2 + X + 1$. This entails $\bar{u} = -u - 1 = u^2$ and $\bar{u}^2 = u$.

Now we evaluate the equation above for $[1, 1, -u^2] = [1, 1, -\bar{u}]$, $[1, -u, 1]$, and $[1, -u, -u^2] = [1, -u, -\bar{u}]$, respectively, and obtain

$$\begin{aligned} 0 &= \operatorname{tr}(a) - \operatorname{tr}(\bar{u}b) - \operatorname{tr}(\bar{u}c) &= b(\bar{u} - u) - \operatorname{tr}(\bar{u}c) \\ 0 &= -\operatorname{tr}(au) + \operatorname{tr}(b) - \operatorname{tr}(\bar{u}c) &= a(\bar{u} - u) - \operatorname{tr}(\bar{u}c) \\ 0 &= -\operatorname{tr}(au) - \operatorname{tr}(ub) + \operatorname{tr}(c\bar{u}^2) &= (a + b)(\bar{u} - u). \end{aligned}$$

As $\bar{u} - u \neq 0$, these equations give $a = b$ and then $0 = 2a$. So R has characteristic 2, and $\mathbb{F}_2(u) \cong \mathbb{F}_4$ is contained in C but not in R . We obtain an embedding of the field extension $\mathbb{F}_4|\mathbb{F}_2$ into $C|R$, and the embedding of the unital $\mathcal{H}(\mathbb{F}_4|\mathbb{F}_2)$ of order 2 in $\mathcal{H}(C|R)$ is a standard embedding; see 1.5.

It remains to show that the unital of order 2 is not embedded in $\mathcal{H}(C|R)$ if $C|R$ is inseparable. As above, we know from [10, 3.7] that any embedding of $\mathcal{H}(\mathbb{F}_4|\mathbb{F}_2)$ into $\operatorname{PG}(2, C)$ is projectively equivalent to the one where the embedded points have coordinates in $\{0, 1, u, u^2\}$, with $u \in C$ and $u^2 = u + 1$; we use that C has characteristic 2 because $C|R$ is inseparable. Now the set $\{0, 1, u, u^2\}$ is a subfield of order 4 in C , and it is contained in R because otherwise the extension $C|R$ would be separable. Thus the embedded unital of order 2 is contained in a finite subplane $\pi \cong \operatorname{PG}(2, \mathbb{F}_4)$ of $\operatorname{PG}(2, C)$.

Let B be a block of $\mathcal{H}(C|R)$ joining two points of the embedded unital of order 2. Then $B \cap \pi$ is a set of three points. We introduce coordinates for the line L of $\operatorname{PG}(2, C)$ containing B in such a way that 0, 1, and ∞ are the coordinates of the three points in $B \cap \pi$. Then B consists of the points with coordinates in $R \cup \{\infty\}$ because the blocks of $\mathcal{H}(C|R)$ are Baer sublines with respect to $C|R$, see [4, 2.8]. Thus $B \cap \pi$ contains all points of $L \cap \pi$ (i.e., those with coordinates 0, 1, u , u^2 , and ∞), contradicting the fact that $|B \cap \pi| = 3$. This contradiction yields that there is no embedding of $\mathcal{H}(\mathbb{F}_4|\mathbb{F}_2)$ in $\mathcal{H}(C|R)$ if $C|R$ is inseparable. \square

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