Extremal generalized hexagons and triple intersection numbers

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Abstract

A triple intersection number in a finite generalized hexagon $\mathcal{S}$ of order (s, t) is a number of the form $|\Gamma_i(x) \cap \Gamma_j(y) \cap \Gamma_k(\omega)|$, where x, y are two points and ω is either a point z or a line L . The fact that $\mathcal{S}$ is extremal, i.e. satisfies $t = s^3 > 1$, implies that certain combinatorial properties regarding triple intersection numbers hold. The earliest result in this direction is due to Haemers who showed that the number $|\Gamma_i(x) \cap \Gamma_j(y) \cap \Gamma_k(L)|$ only depends on s, i, j, k and the configuration determined by (x, y, L) . In this paper, we prove a similar result for the triple intersection numbers that involve three points, with exception of the case where the points are mutually opposite. In this case, we show that the 64 triple intersection numbers only depend on s and one extra parameter. We also show how Haemers' original results can be deduced from the results obtained here, leading to a more elementary entirely combinatorial treatment where the use of any eigenvalue techniques and other non-elementary matrix theory are avoided.

Keywords: (extremal) generalized hexagon, Haemers-Roos inequality, triple intersection numbers

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1 Introduction

A point-line geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \text{I})$ with point set $\mathcal{P}$, line set $\mathcal{L}$ and incidence relation $\text{I} \subseteq \mathcal{P} \times \mathcal{L}$ is called a *generalized d -gon* with $d \geq 2$ if the incidence graph of $\mathcal{S}$ has diameter d and girth $2d$. A generalized d -gon is said to have *order* (s, t) if every line is incident with exactly $s + 1$ points and if every point is incident with precisely $t + 1$ lines. Under the assumption that $s, t \in \mathbb{N} \setminus \{0, 1\}$, we know from the Feit-Higman theorem [12] that $d \in \{2, 3, 4, 6, 8\}$. Inequalities of Higman [15, 16] imply that $t \leq s^2$ for generalized quadrangles and octagons of order (s, t) with $s \neq 1$ and an inequality of Haemers-Roos [14] states that $t \leq s^3$ for generalized hexagons of order (s, t) with $s \neq 1$. In case t attains the upper bound, the generalized d -gon with $d \in \{4, 6, 8\}$ is called *extremal*.

A *triple intersection number* in a finite generalized polygon $\mathcal{S}$ is a number of the form $|\Gamma_i(x) \cap \Gamma_j(y) \cap \Gamma_k(\omega)|$, where x, y are two points and ω is either a point z or a line

L . Triple intersection numbers have made their appearance in the study of extremal generalized polygons. Bose-Shrikhande [5] and Cameron [7] proved in the 70's that a finite generalized quadrangle of order (s, t) , $s, t > 1$, is extremal (i.e. satisfies $t = s^2$) if and only if a certain triple intersection number is constant. Triple intersection numbers also made their appearance in the study of extremal generalized hexagons. Indeed, it is known that extremal generalized hexagons satisfy certain combinatorial properties that involve triple intersection numbers. The earliest result in that respect is one of Willem Haemers [13] who showed that for an extremal generalized hexagon the triple intersection number $|\Gamma_i(x) \cap \Gamma_j(y) \cap \Gamma_k(L)|$ only depends on i, j and k and the “configuration induced by L, x and y ”, where the latter was somewhat informally defined as the configuration formed by the points and lines which are on a shortest path between L and x , L and y , or x and y . We give a more explicit definition in Section 5 where we also provide pictures (see Table 2). The reader can feel uncertain on how the informal definition of configuration must precisely be interpreted (especially when x and y are opposite and connected by many shortest paths). A full inspection of the proof in [13] however reveals more. In the case that $d(x, L) = d(y, L) = 2$, it was proved that in an extremal generalized hexagon all numbers $P_{ijk}(x, y, L) := |\Gamma_i(x) \cap \Gamma_j(y) \cap \Gamma_k(L)|$ can be expressed in terms of s and the numbers $P_{ijk}(x, y, L)$ with $\min(i, j, k + 1) \leq 1$. A configuration in the sense of [13] must therefore be interpreted as an arrangement of x, y and L that is sufficiently well described so that all the numbers $P_{ijk}(x, y, L)$ with $\min(i, j, k + 1) \leq 1$ only depend on s and t , and not on the particular choice of the points $x, y \in \Gamma_2(L)$. One can verify that this is indeed the case for the configurations of x, y and L depicted in Table 2 and described in the beginning of Section 5. The case that $d(x, L) \leq 1$ or $d(y, L) \leq 1$ is left to the reader in [13] but in this case the claim that the numbers $P_{ijk}(x, y, L)$ only depend on the configuration of x, y and L would hold in an arbitrary (not necessarily extremal) generalized hexagon (again, after a suitable interpretation of the notion of configuration).

The proof in [13] heavily relies on non-elementary matrix theory. It involves eigenvalue techniques, rank arguments, tensor products and other results from matrix theory like interlacing theorems. Two interesting questions however remained open since then.

(Q1) Does the converse of Haemers' result also hold, namely is a finite generalized hexagon of order (s, t) , $s \neq 1$, extremal if the mentioned intersection numbers only depend on s, t, i, j, k and the configuration determined by the line and the two points? If this is true, this would give rise to a combinatorial characterization of extremal generalized hexagons.

(Q2) Is it possible to give a more elementary proof, e.g. one that only involves counting?

In this paper, we will provide a positive answer to the question (Q2). As of today, the answer to the question (Q1) remains open.

The literature contains three other results about triple intersection numbers in extremal generalized hexagons. If $\mathcal{S}$ is a finite generalized hexagon of order (s, s^3) , $s \neq 1$, then F. Vanhove and one of the authors (see [10, Corollary 3.9] and [11, Corollary 3.5]) showed that the following three properties hold:

- (P1) If x and y are two points at distance 2 and z is a point opposite to x, y and at distance 2 from the unique neighbour of x and y , then $|\Gamma_2(x) \cap \Gamma_2(y) \cap \Gamma_1(z)| = s^2 + 1$.
- (P2) If x and y are two points at distance 2 and z is a point opposite to x, y and the unique neighbour of x and y , then $|\Gamma_2(x) \cap \Gamma_2(y) \cap \Gamma_1(z)| = s^2 + s + 1$.
- (P3) $|\Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_2(z)| = |\Gamma_2(x) \cap \Gamma_1(y) \cap \Gamma_2(z)|$ for any three mutually opposite points x, y and z .

In the present paper, we obtain many more results about triple intersection numbers in extremal generalized hexagons. Similarly to [13] we consider different configurations for three points a, b and c (41 in total starting from 16 basic configurations; see Section 3) and show with exception of one case that the triple intersection number $|\Gamma_i(a) \cap \Gamma_j(b) \cap \Gamma_k(c)|$ only depends on s, i, j and k for each of these configurations (Section 4). The exceptional configuration is the one where the three points a, b and c are mutually opposite. In this case, we show that the 64 possible triple intersection numbers $|\Gamma_i(a) \cap \Gamma_j(b) \cap \Gamma_k(c)|$ can be expressed in terms of s and one extra parameter. This case was more puzzling to us than the others and we were only able to deal with it completely at the end of the paper (Section 7) after obtaining some additional results in Section 6. We should also mention that one cannot get rid of the extra parameter in the exceptional case. Inspection of a particular extremal generalized hexagon (of order $(2, 8)$) reveals that it can take different values depending on the choice of the mutually opposite points a, b and c (see Section 8).

In Section 6, we show that Haemers' original result about triple intersection numbers can be deduced from the results obtained here, leading to an alternative more elementary proof of that result which avoids the use of (non-elementary) matrix theory. In fact, all the results of the present paper are derived from Properties (P1), (P2), (P3) by means of purely combinatorial arguments. We would like to note that Properties (P1), (P2), (P3) themselves can also be proved by means of purely combinatorial arguments, see e.g. [9]. The results of the present paper also provide an answer to a question of Bill Kantor (private communication to the first author) whether the combinatorial properties obtained in [10, Corollary 3.9] and [13] are independent from each other.

In the whole range of combinatorial results that are satisfied by extremal generalized hexagons, the results about triple intersection numbers that involve three points (type PPP) seem to play a more fundamental role than those results about triple intersection numbers that involve two points and a line (type PPL). We give three motivations for this claim.

- As shown in Section 6, all type PPL results can be derived from the type PPP results, but we were not at all able to show the converse.
- Although it seems currently not possible to combinatorially characterize extremal generalized hexagons by means of triple intersection numbers of type PPL (see question (Q1)), it is certainly possible to do that by means of triple intersection numbers of type PPP, see [9].

- Also for finite generalized quadrangles of order (s, t) , the triple intersection numbers of type PPP play a more fundamental role than those of type PPL when it comes to their connection with the property of being extremal. In fact, for an arbitrary generalized quadrangle of order (s, t) with $s, t \geq 2$, those of type PPL only depend on s, t and the configuration, while this is only the case for those of type PPP when the generalized quadrangle is extremal. We refer to Section 8 for more details.

In the present paper, we intended to offer a complete picture about what is currently known about triple intersection numbers in extremal generalized hexagons. We hope that the results obtained in this paper can be useful for obtaining additional classification and characterization results for extremal generalized hexagons, especially when combined with other techniques. One such possible application is the classification of near polygons (in particular, extremal generalized hexagons) containing a given near polygon as a full subgeometry. The theory of valuations has proved useful for attacking such problems, see e.g. [2, 3, 4], and has also lead to the discovery of new near polygons related to the Suzuki chain of finite simple groups [1]. In certain problems where the theory of valuations failed to obtain any progress in the past, like in the case of generalized hexagons $\mathcal{S}$ of order $(3, 3^3)$ containing a subhexagon $\mathcal{S}'$ of order $(3, t)$ with $t \in \{1, 3\}$, significant progress seems now possible when one combines this theory with the results of the present paper. Indeed, by taking at least two of the points a, b, c inside the subhexagon $\mathcal{S}'$, the results about triple intersection numbers of type PPP obtained in this paper (for the various configurations of three points) gives information about the valuations of $\mathcal{S}'$ that are induced by the points of $\mathcal{S}$, information which previously seemed impossible to obtain. This problem is currently further under investigation by A. Bishnoi and the first author.

Many results discussed in this paper were obtained during the master thesis project "Combinatorische eigenschappen van extremale veralgemeende zeshoeken" (in dutch) which the second author performed under the supervision of the first author while being a student at Ghent University in the academic year 2018-2019. This work has subsequently been thoroughly revised by the authors resulting in the present paper.

2 Preliminaries

Suppose $\mathcal{S}$ is a finite generalized hexagon of order (s, t) . If x and y are two distinct collinear points of $\mathcal{S}$, then we write $x \sim y$ and we denote by xy or $L(x, y)$ the unique line through x and y . Distances between objects of $\mathcal{S}$ will always be measured in its collinearity graph. If ω is a point or line of $\mathcal{S}$ and $i \in \mathbb{N}$, then $\Gamma_i(\omega)$ denotes the set of points at distance i from ω . The distance function in $\mathcal{S}$ will be denoted by $d(\cdot, \cdot)$. If x, y and z are points, then $d(x, y) + d(y, z) \geq d(x, z)$ (triangle inequality). We now mention several known properties of $\mathcal{S}$ which we will often implicitly use throughout this paper. We refer to [19] for proofs and more background information.

$\mathcal{S}$ is a *near hexagon*, i.e. the maximal distance between two points is equal to 3 and for every point-line pair (x, L) , there exists a unique point $\pi_L(x)$ on L nearest to x . Two points at distance 3 are called *opposite*, while a point and a line are called *opposite* when

they lie at maximal distance 2 from each other. Every two points x and y at distance 2 have a unique common neighbour, which we will denote by $x * y$, and it holds that $\Gamma_1(x) \cap \Gamma_2(y) = L(x, x * y) \setminus \{x, x * y\}$. If x, y, z are three points such that $x \sim y \sim z$ and $xy \neq yz$, then $d(x, z) = 2$. If x, y, z, v are four points such that $x \sim y \sim z \sim v$ and $xy \neq yz \neq zv$, then $d(x, v) = 3$.

The maximal distance between two lines is 2, in which case these lines are called *opposite*. If L_1 and L_2 are two distinct non-opposite lines of $\mathcal{S}$, then there exist unique points $x_1 \in L_1$ and $x_2 \in L_2$ such that $d(x_1, x_2) = d(L_1, L_2)$.

The collinearity graph of a generalized hexagon is a so-called distance-regular graph [6]. The following lemma is a special case of a more general result from the theory of distance-regular graphs.

Lemma 2.1. *There exist constants p_{ij}^k , $i, j, k \in \{0, 1, 2, 3\}$, such that $|\Gamma_i(x) \cap \Gamma_j(y)| = p_{ij}^k$ if x and y are two points at distance k from each other.*

The precise values of the p_{ij}^k 's follow from the theory of distance-regular graphs or by direct counting. Taking into account that $p_{ij}^k = p_{ji}^k$, they are:

- $p_{ij}^0 = \delta_{ij}n_i$ with $n_0 = 1$, $n_1 = s(t+1)$, $n_2 = s^2t(t+1)$ and $n_3 = s^3t^2$;
- $p_{00}^1 = p_{02}^1 = p_{03}^1 = p_{13}^1 = 0$, $p_{01}^1 = 1$, $p_{11}^1 = s-1$, $p_{12}^1 = st$, $p_{22}^1 = s(s-1)t$, $p_{23}^1 = s^2t^2$ and $p_{33}^1 = s^2(s-1)t^2$;
- $p_{00}^2 = p_{01}^2 = p_{03}^2 = 0$, $p_{02}^2 = 1$, $p_{11}^2 = 1$, $p_{12}^2 = s-1$, $p_{13}^2 = st$, $p_{22}^2 = s(t^2+t-1)$, $p_{23}^2 = s(s-1)t(t+1)$ and $p_{33}^2 = st((s^2-s+1)t-s)$;
- $p_{00}^3 = p_{01}^3 = p_{02}^3 = p_{11}^3 = 0$, $p_{03}^3 = 1$, $p_{12}^3 = t+1$, $p_{13}^3 = (s-1)(t+1)$, $p_{22}^3 = (s-1)(t+1)^2$, $p_{23}^3 = (t+1)((s^2-s+1)t-s)$ and $p_{33}^3 = (s^2+1)(s-1)t(t+1) - (s^3-1)t$.

3 The 41 configurations for three points

We describe an exhaustive list of basic configurations for three points a, b and c in an arbitrary (not necessarily extremal) generalized hexagon $\mathcal{S}$. By basic, we mean here that the condition $d(a, b) \leq d(a, c) \leq d(b, c)$ holds. Every other configuration for a, b and c can be obtained from one of the basic ones by applying a certain permutation to the points a, b and c . The exhaustive list of basic configurations is as follows:

- **Configuration 1:** $d(a, b) = d(a, c) = d(b, c) = 0$;
- **Configuration 2:** $d(a, b) = 0$ and $d(a, c) = d(b, c) = 1$;
- **Configuration 3:** $d(a, b) = 0$ and $d(a, c) = d(b, c) = 2$;
- **Configuration 4:** $d(a, b) = 0$ and $d(a, c) = d(b, c) = 3$;
- **Configuration 5:** $d(a, b) = d(a, c) = d(b, c) = 1$;
- **Configuration 6:** $d(a, b) = d(a, c) = 1$ and $d(b, c) = 2$;
- **Configuration 7:** $d(a, b) = 1$ and $d(a, c) = d(b, c) = 2$;
- **Configuration 8:** $d(a, b) = 1$, $d(a, c) = 2$ and $d(b, c) = 3$;
- **Configuration 9:** $d(a, b) = 1$, $d(a, c) = d(b, c) = 3$;
- **Configuration 10:** $d(a, b) = d(a, c) = d(b, c) = 2$ and $d(a * b, c) = 1$;
- **Configuration 11:** $d(a, b) = d(a, c) = d(b, c) = 2$ and $d(a * b, c) = 3$;

Configuration 1	Configuration 2	Configuration 3	Configuration 4	Configuration 5
Configuration 6	Configuration 7	Configuration 8	Configuration 9	Configuration 10
Configuration 11	Configuration 12	Configuration 13	Configuration 14	Configuration 15

Table 1: Fifteen configurations of three points

- **Configuration 12:** $d(a, b) = d(a, c) = 2$, $d(b, c) = 3$ and $d(a * b, a * c) = 1$;
- **Configuration 13:** $d(a, b) = d(a, c) = 2$, $d(b, c) = 3$ and $d(a * b, a * c) = 2$;
- **Configuration 14:** $d(a, b) = 2$, $d(a, c) = d(b, c) = 3$ and $d(a * b, c) = 2$;
- **Configuration 15:** $d(a, b) = 2$, $d(a, c) = d(b, c) = 3$ and $d(a * b, c) = 3$;
- **Configuration 16:** $d(a, b) = d(a, c) = d(b, c) = 3$.

Configurations 1 to 15 are also shown in Table 1. The diagrams in this table depict points and lines in the generalized hexagon rather than vertices and edges in the collinearity graph. Collinearity is indicated by straightness. E.g., each of the configurations 5, 7, 9, 12, 13 and 15 has a line containing (at least) three points.

With exception of configuration 4, the points and lines that occur in each of the diagrams of Table 1 are uniquely determined by the points a , b and c . It seems not possible to make a natural drawing for configuration 16. Indeed, as two opposite points in a generalized hexagon of order (s, t) are connected by $t + 1$ shortest paths, there are $3(t + 1)$ paths of length 3 connecting two of the three points a , b and c . It seems not possible to determine the mutual distances between the various points and lines that are defined by these $3(t + 1)$ paths.

Lemma 3.1. *If a , b and c are three points of $\mathcal{S}$ such that $d(a, b) \leq d(a, c) \leq d(b, c)$, then a , b and c are as in one of the configurations 1 to 16.*

Proof. Suppose first that $d(a, b) = 0$. Then we have configurations 1, 2, 3 or 4, depending on whether $d(a, c)$ is 0, 1, 2 or 3.

Suppose $d(a, b) = 1$. The condition that $d(a, c) \leq d(b, c)$ then implies that $\pi_{ab}(c) = a$ or $\pi_{ab}(c) \in ab \setminus \{a, b\}$. If $\pi_{ab}(c) = a$, then the fact that $1 = d(a, b) \leq d(a, c) < d(b, c) =$

$d(a, c) + 1$ implies that $d(a, c) \in \{1, 2\}$. So, either configuration 6 or 8 occurs depending on whether $d(a, c)$ is 1 or 2. If $\pi_{ab}(c) \in ab \setminus \{a, b\}$, then either configuration 5, 7 or 9 occurs depending on whether $d(c, ab)$ is 0, 1 or 2.

Suppose $d(a, b) = 2$. If $d(a, c) = 3$, then also $d(b, c) = 3$ and so we have configurations 14 or 15 depending on whether $d(c, a * b)$ is 2 or 3. Suppose therefore that $d(a, c) = 2$. Put $L_1 = L(a, a * b)$ and $L_2 = L(a, a * c)$. If $L_1 = L_2$, then either configuration 10 or 12 occurs depending on whether $a * b = a * c$ or not. If $L_1 \neq L_2$, then either configuration 11 or 13 occurs depending on whether $d(b, c)$ is 2 or 3.

Suppose $d(a, b) = 3$. Then also $d(a, c) = d(b, c) = 3$, and configuration 16 occurs. $\square$

By permuting points, we can get new configurations. There are 25 new configurations that arise in this way. These are listed in the following table, together with the basic configuration (FROM) and the permutation (PERM.) from which they arise.

CONF.	FROM	PERM.	CONF.	FROM	PERM.	CONF.	FROM	PERM.	CONF.	FROM	PERM.
17	2	(cba)	24	9	(cba)	30	3	(abc)	36	8	(ab)
18	3	(cba)	25	12	(cba)	31	4	(abc)	37	9	(abc)
19	4	(cba)	26	13	(cba)	32	6	(abc)	38	12	(abc)
20	6	(cba)	27	14	(cba)	33	7	(abc)	39	13	(abc)
21	7	(cba)	28	15	(cba)	34	8	(abc)	40	14	(abc)
22	8	(cba)	29	2	(abc)	35	8	(ac)	41	15	(abc)
23	8	(bc)	—	—	—	—	—	—	—	—	—

4 The triple intersection numbers of type PPP

If a, b and c are points of a generalized hexagon $\mathcal{S}$ of order (s, t) , then the triple intersection number $N_{ijk}(a, b, c) := |\Gamma_i(a) \cap \Gamma_j(b) \cap \Gamma_k(c)|$ will shortly be denoted by N_{ijk} . In this section, we determine the N_{ijk} 's for the 16 basic configurations. The N_{ijk} 's for the other configurations can easily be obtained from these numbers (by applying suitable permutations to the subindices as indicated by the table in Section 3). We initially make no assumption about $\mathcal{S}$ being extremal and the first results we derive here (till Corollary 4.16) are valid for general $\mathcal{S}$. From Lemma 4.17 on, we assume that $\mathcal{S}$ is extremal.

If at least one of i, j, k is 0, then N_{ijk} is easily determined for each of the configurations.

Lemma 4.1. • For all $j, k \in \{0, 1, 2, 3\}$, we have $N_{0jk} = \begin{cases} 1 & \text{if } d(a, b) = j \text{ and } d(a, c) = k, \\ 0 & \text{otherwise.} \end{cases}$

• For all $i \in \{1, 2, 3\}$ and $k \in \{0, 1, 2, 3\}$, we have $N_{i0k} = \begin{cases} 1 & \text{if } d(a, b) = i \text{ and } d(b, c) = k, \\ 0 & \text{otherwise.} \end{cases}$

• For all $i, j \in \{1, 2, 3\}$, we have $N_{ij0} = \begin{cases} 1 & \text{if } d(a, c) = i \text{ and } d(b, c) = j, \\ 0 & \text{otherwise.} \end{cases}$

We now determine the N_{ijk} 's where $i, j, k \in \{1, 2\}$. This is realized in Lemmas 4.2 to 4.14.

Lemma 4.2. *If configuration 1, 2, 3 or 4 occurs, then for all $i, j, k \in \{0, 1, 2, 3\}$, we have $N_{ijk} = 0$ if $i \neq j$ and $N_{ijk} = p_{ik}^\delta$ with $\delta = d(a, c)$ if $i = j$. In particular, we have:*

- *If configuration 1 occurs, then $N_{111} = s(t + 1)$, $N_{112} = 0$, $N_{121} = 0$, $N_{122} = 0$, $N_{211} = 0$, $N_{212} = 0$, $N_{221} = 0$, $N_{222} = s^2 t(t + 1)$.*
- *If configuration 2 occurs, then $N_{111} = s - 1$, $N_{112} = st$, $N_{121} = 0$, $N_{122} = 0$, $N_{211} = 0$, $N_{212} = 0$, $N_{221} = st$, $N_{222} = s(s - 1)t$.*
- *If configuration 3 occurs, then $N_{111} = 1$, $N_{112} = s - 1$, $N_{121} = 0$, $N_{122} = 0$, $N_{211} = 0$, $N_{212} = 0$, $N_{221} = s - 1$, $N_{222} = s(t^2 + t - 1)$.*
- *If configuration 4 occurs, then $N_{111} = 0$, $N_{112} = t + 1$, $N_{121} = 0$, $N_{122} = 0$, $N_{211} = 0$, $N_{212} = 0$, $N_{221} = t + 1$, $N_{222} = (s - 1)(t + 1)^2$.*

Proof. Note that $a = b$ in each of these four cases. So, $\Gamma_i(a) \cap \Gamma_j(b) \cap \Gamma_k(c) = \emptyset$ if $i \neq j$. If $i = j$, then $\Gamma_i(a) \cap \Gamma_j(b) \cap \Gamma_k(c) = \Gamma_i(a) \cap \Gamma_k(c)$. $\square$

Lemma 4.3. *If configuration 5 occurs, then $N_{111} = s - 2$, $N_{112} = 0$, $N_{121} = 0$, $N_{122} = st$, $N_{211} = 0$, $N_{212} = st$, $N_{221} = st$, $N_{222} = (s - 2)st$.*

Proof. If $x \in \Gamma_i(a) \cap \Gamma_j(b) \cap \Gamma_k(c)$ with $1 \leq i, j, k \leq 2$, then x has distance at most 2 from at least two points of $L := ab$ and hence distance at most 1 from L . The considerations below lead to the mentioned triple intersection numbers.

Each of the $s - 2$ points of $L \setminus \{a, b, c\}$ belongs to $\Gamma_1(a) \cap \Gamma_1(b) \cap \Gamma_1(c)$.

Let $u \in \{a, b, c\}$. If x is one of the st points of $\Gamma_1(L) \cap \Gamma_1(u)$, then $d(x, u) = 1$ and $d(x, v) = 2$ for every $v \in \{a, b, c\} \setminus \{u\}$.

If $x \in \Gamma_1(L)$ such that $\pi_L(x) \notin \{a, b, c\}$, then $x \in \Gamma_2(a) \cap \Gamma_2(b) \cap \Gamma_2(c)$. There are $s - 2$ points $w \in L \setminus \{a, b, c\}$ and for each such w , there are st points $x \in \Gamma_1(L)$ for which $\pi_L(x) = w$. $\square$

Lemma 4.4. *If configuration 6 occurs, then $N_{111} = 0$, $N_{112} = s - 1$, $N_{121} = s - 1$, $N_{122} = s(t - 1)$, $N_{211} = 0$, $N_{212} = 0$, $N_{221} = 0$, $N_{222} = 0$.*

Proof. As $b * c = a$, we have $N_{111} = N_{211} = 0$. As $\Gamma_1(b) \cap \Gamma_2(c) \subseteq L(b, b * c) = L(b, a)$ and $L(b, a) \cap \Gamma_2(a) \cap \Gamma_1(b) \cap \Gamma_2(c) = \emptyset$, we have $N_{212} = 0$. By symmetry, we then also know that $N_{221} = 0$.

As $\Gamma_1(a) \cap \Gamma_1(b) = ab \setminus \{a, b\}$, we have $\Gamma_1(a) \cap \Gamma_1(b) \cap \Gamma_2(c) = ab \setminus \{a, b\}$ and thus $N_{112} = s - 1$. By symmetry, we then also know that $N_{121} = s - 1$.

If $x \in \Gamma_1(a) \cap \Gamma_2(b) \cap \Gamma_2(c)$, then x lies on one of the $t - 1$ lines through a distinct from ab and ac . If L is one of these $t - 1$ lines through a , then $L \setminus \{a\} \subseteq \Gamma_1(a) \cap \Gamma_2(b) \cap \Gamma_2(c)$, implying that $N_{122} = s(t - 1)$.

If $x \in \Gamma_2(a) \cap \Gamma_2(b) \cap \Gamma_2(c)$, then x is collinear with a unique point $x' \in ab \setminus \{a, b\}$ and a unique point $x'' \in ac \setminus \{a, c\}$. This is impossible as x and a would then have two neighbours, namely x' and x'' . We conclude that $N_{222} = 0$. $\square$

Lemma 4.5. *If configuration 7 occurs, then $N_{111} = 1$, $N_{112} = s - 2$, $N_{121} = 0$, $N_{122} = 0$, $N_{211} = 0$, $N_{212} = 0$, $N_{221} = s - 1$, $N_{222} = s(t - 1)$.*

Proof. Each point of $\Gamma_1(a)$ on a line through a distinct from ab lies at distance 3 from c . Among the points of $\Gamma_1(a) \cap ab$, there is a unique one at distance 1 from c , namely $c * a = c * b$, and all remaining points lie at distance 2 from c . This shows that $N_{111} = 1$, $N_{112} = s - 2$ and $N_{121} = N_{122} = 0$. By symmetry, we also have $N_{211} = N_{212} = 0$.

We have $\Gamma_2(b) \cap \Gamma_1(c) \subseteq L(c, b * c)$. As $L(c, b * c) \cap \Gamma_2(a) \cap \Gamma_2(b) \cap \Gamma_1(c) = L(c, b * c) \setminus \{c, b * c\}$, we have $N_{221} = s - 1$.

If $x \in \Gamma_2(a) \cap \Gamma_2(b) \cap \Gamma_2(c)$, then x is collinear with a unique point x' of $ab \setminus \{a, b\}$. If $x' \neq c * a = c * b$, then necessarily $d(c, x) = 3$, a contradiction. So, $x' = c * a = c * b$ and x lies on one of the $t - 1$ lines through $c * a$ distinct from ab and $L(c, a * c)$. Conversely, each point of such a line distinct from $a * c$ lies at distance 2 from a , b and c , showing that $N_{222} = s(t - 1)$. $\square$

Lemma 4.6. *If configuration 8 occurs, then $N_{111} = 0$, $N_{112} = 0$, $N_{121} = 1$, $N_{122} = s - 1$, $N_{211} = 0$, $N_{212} = t$, $N_{221} = 0$, $N_{222} = (s - 1)t$.*

Proof. By the triangle inequality, we know that $N_{111} = N_{211} = 0$. As $a * c \in \Gamma_2(b)$, we have $N_{121} = 1$. If $x \in \Gamma_1(a) \cap \Gamma_1(b)$, then $x \in ab \setminus \{a, b\}$ and so $d(c, x) = 3$, implying that $N_{112} = 0$. As $\Gamma_2(a) \cap \Gamma_1(c) \subseteq L(c, a * c)$ and $L(c, a * c) \cap \Gamma_2(a) \cap \Gamma_2(b) \cap \Gamma_1(c) = \emptyset$, we have $N_{221} = 0$. As $\Gamma_1(a) \cap \Gamma_2(c) \subseteq L(a, a * c)$ and $L(a, a * c) \cap \Gamma_1(a) \cap \Gamma_2(b) \cap \Gamma_2(c) = L(a, a * c) \setminus \{a, a * c\}$, we have $N_{122} = s - 1$.

If $x \in \Gamma_2(a) \cap \Gamma_1(b) \cap \Gamma_2(c)$, then x lies on one of the t lines through b distinct from ab . Conversely, as $d(b, c) = 3$, each of the t lines through b distinct from ab contains a unique point at distance 2 from c , and this point also lies at distance 2 from a . It follows that $N_{212} = t$.

If $x \in \Gamma_2(a) \cap \Gamma_2(b) \cap \Gamma_2(c)$, then x lies at distance 1 from a unique point of $ab \setminus \{a, b\}$. Conversely, through each of the $s - 1$ points $y \in ab \setminus \{a, b\}$, there are t lines distinct from ab and as $d(c, y) = 3$ each of these t lines contains a unique point of $\Gamma_2(c)$ which moreover belongs to $\Gamma_2(a) \cap \Gamma_2(b)$. This implies that $N_{222} = (s - 1)t$. $\square$

Lemma 4.7. *If configuration 9 occurs, then $N_{111} = 0$, $N_{112} = 1$, $N_{121} = 0$, $N_{122} = t$, $N_{211} = 0$, $N_{212} = t$, $N_{221} = 1$, $N_{222} = s - 1 + (s - 2)t$.*

Proof. By the triangle inequality, we know that $N_{111} = N_{121} = N_{211} = 0$. Put $c' = \pi_{ab}(c)$. As $\Gamma_1(a) \cap \Gamma_1(b) \subseteq ab$ and $\Gamma_2(c) \cap ab = \{c'\}$, we have $\Gamma_1(a) \cap \Gamma_1(b) \cap \Gamma_2(c) = \{c'\}$ and hence $N_{112} = 1$.

The set $\Gamma_2(a) \cap \Gamma_2(b)$ contains all points $x \in \Gamma_1(ab)$ for which $\pi_{ab}(x) \notin \{a, b\}$. If $x \in \Gamma_2(a) \cap \Gamma_2(b) \cap \Gamma_1(c)$, then $\pi_{ab}(x) \sim x \sim c$ implies that $\pi_{ab}(x) = c'$ and that $x = c' * c \in \Gamma_2(a) \cap \Gamma_2(b) \cap \Gamma_1(c)$. Hence, $N_{221} = 1$.

If $x \in \Gamma_1(a) \cap \Gamma_2(b) \cap \Gamma_2(c)$, then $ax \neq ab$. As $d(a, c) = 3$, each of the t lines through a distinct from ab contains a unique point at distance 2 from c , and this point also has distance 2 from b . This shows that $N_{122} = t$. By symmetry, also $N_{212} = t$.

If $x \in \Gamma_2(a) \cap \Gamma_2(b) \cap \Gamma_2(c)$, then x lies at distance 1 from a unique point x' of $ab \setminus \{a, b\}$. If $x' = c'$, then $x \in L(c', c' * c) \setminus \{c', c' * c\}$. Conversely, each of the $s - 1$ points of $L(c', c' * c) \setminus \{c', c' * c\}$ lies at distance 2 from a , b and c . Suppose now that x' is one of the $s - 2$ points of $ab \setminus \{a, b, c'\}$. Then x lies on one of the t lines through x'

distinct from ab . As $d(x', c) = 3$, each of these t lines contains a unique point at distance 2 from c , and this point also lies at distance 2 from a and b . In conclusion, we can say that $N_{222} = s - 1 + (s - 2)t$. $\square$

Lemma 4.8. *If configuration 10 occurs, then $N_{111} = 1$, $N_{112} = 0$, $N_{121} = 0$, $N_{122} = s - 1$, $N_{211} = 0$, $N_{212} = s - 1$, $N_{221} = s - 1$, $N_{222} = N_{222}$.*

Proof. As $a * b \in \Gamma_1(c)$, $a * c \in \Gamma_1(b)$ and $b * c \in \Gamma_1(a)$, we have $N_{111} = 1$ and $N_{112} = N_{121} = N_{211} = 0$. As $\Gamma_1(a) \cap \Gamma_2(b) \subseteq L(a, a * b)$ and $L(a, a * b) \cap \Gamma_1(a) \cap \Gamma_2(b) \cap \Gamma_2(c) = L(a, a * b) \setminus \{a, a * b\}$, we have $N_{122} = s - 1$. By symmetry, also $N_{212} = N_{221} = s - 1$. $\square$

Lemma 4.9. *If configuration 11 occurs, then $N_{111} = 0$, $N_{112} = 0$, $N_{121} = 0$, $N_{122} = 0$, $N_{211} = 0$, $N_{212} = 0$, $N_{221} = 0$, $N_{222} = N_{222}$.*

Proof. As $a * b \in \Gamma_3(c)$, we have $N_{111} = N_{112} = 0$. As $\Gamma_1(a) \cap \Gamma_2(b) \subseteq L(a, a * b)$ and $L(a, a * b) \cap \Gamma_1(a) \cap \Gamma_2(b) \cap \Gamma_2(c) = \emptyset$, we have $N_{122} = 0$. Using symmetry, we then also know that $N_{121} = N_{211} = N_{212} = N_{221} = 0$. $\square$

Lemma 4.10. *If configuration 12 occurs, then $N_{111} = 0$, $N_{112} = 1$, $N_{121} = 1$, $N_{122} = s - 2$, $N_{211} = 0$, $N_{212} = 0$, $N_{221} = 0$, $N_{222} = N_{222}$.*

Proof. By the triangle inequality, we have $N_{111} = N_{211} = 0$. We also know the following:

- (i) $a * b \in \Gamma_2(c)$ and $a * c \in \Gamma_2(b)$;
- (ii) $\Gamma_1(a) \cap \Gamma_2(b) \subseteq L(a * b, a * c)$ and $L(a * b, a * c) \cap \Gamma_1(a) \cap \Gamma_2(b) \cap \Gamma_2(c) = L(a * b, a * c) \setminus \{a, a * b, a * c\}$.
- (iii) $\Gamma_2(a) \cap \Gamma_1(b) \subseteq L(b, a * b)$ and $L(b, a * b) \cap \Gamma_2(a) \cap \Gamma_1(b) \cap \Gamma_2(c) = \emptyset$.

These properties imply that $N_{112} = N_{121} = 1$, $N_{122} = s - 2$ and $N_{212} = 0$. By symmetry, we also have $N_{221} = 0$. $\square$

Lemma 4.11. *If configuration 13 occurs, then $N_{111} = 0$, $N_{112} = 0$, $N_{121} = 0$, $N_{122} = 0$, $N_{211} = 0$, $N_{212} = 1$, $N_{221} = 1$, $N_{222} = N_{222}$.*

Proof. By the triangle inequality, we have $N_{111} = N_{211} = 0$. We also have:

- (i) $a * b \in \Gamma_3(c)$ and $a * c \in \Gamma_3(b)$;
- (ii) $\Gamma_1(a) \cap \Gamma_2(b) \subseteq L(a, a * b)$ and $L(a, a * b) \cap \Gamma_1(a) \cap \Gamma_2(b) \cap \Gamma_2(c) = \emptyset$;
- (iii) $\Gamma_2(a) \cap \Gamma_1(b) \subseteq L(b, a * b)$ and $L(b, a * b) \cap \Gamma_2(a) \cap \Gamma_1(b) \cap \Gamma_2(c) = \{\pi_{L(b, a * b)}(c)\}$.

These properties imply that $N_{112} = N_{121} = N_{122} = 0$ and $N_{212} = 1$. By symmetry, we also have $N_{221} = 1$. $\square$

Lemma 4.12. *If configuration 14 occurs, then $N_{111} = 0$, $N_{112} = 1$, $N_{121} = 0$, $N_{122} = 0$, $N_{211} = 0$, $N_{212} = 0$, $N_{221} = N_{221}$, $N_{222} = N_{222}$.*

Proof. By the triangle inequality, we know that $N_{111} = N_{121} = N_{211} = 0$. We also have:

- (i) $a * b \in \Gamma_2(c)$;
- (ii) $\Gamma_1(a) \cap \Gamma_2(b) \subseteq L(a, a * b)$ and $L(a, a * b) \cap \Gamma_1(a) \cap \Gamma_2(b) \cap \Gamma_2(c) = \emptyset$.

These properties imply that $N_{112} = 1$ and $N_{122} = 0$. By symmetry, also $N_{212} = 0$. $\square$

Lemma 4.13. *If configuration 15 occurs, then $N_{111} = 0$, $N_{112} = 0$, $N_{121} = 0$, $N_{122} = 1$, $N_{211} = 0$, $N_{212} = 1$, $N_{221} = N_{221}$, $N_{222} = N_{222}$.*

Proof. By the triangle inequality, we have $N_{111} = N_{121} = N_{211} = 0$. We also have:

- (i) $a * b \in \Gamma_3(c)$;
- (ii) $\Gamma_1(a) \cap \Gamma_2(b) \subseteq L(a, a * b)$ and $L(a, a * b) \cap \Gamma_1(a) \cap \Gamma_2(b) \cap \Gamma_2(c) = \{\pi_{L(a, a * b)}(c)\}$.

These properties imply that $N_{112} = 0$ and $N_{122} = 1$. By symmetry, also $N_{212} = 1$. $\square$

Lemma 4.14. *If configuration 16 occurs, then $N_{111} = 0$, $N_{112} = 0$, $N_{121} = 0$, $N_{122} = N_{122}$, $N_{211} = 0$, $N_{212} = N_{212}$, $N_{221} = N_{221}$, $N_{222} = N_{222}$.*

Proof. This follows from the triangle inequality. $\square$

The following is obvious from the definition of the N_{ijk} 's and p_{ij}^k 's.

Lemma 4.15. *For all $i, j \in \{0, 1, 2, 3\}$, we have:*

- $N_{ij0} + N_{ij1} + N_{ij2} + N_{ij3} = p_{ij}^l$, where $l = d(a, b)$;
- $N_{i0j} + N_{i1j} + N_{i2j} + N_{i3j} = p_{ij}^m$, where $m = d(a, c)$;
- $N_{0ij} + N_{1ij} + N_{2ij} + N_{3ij} = p_{ij}^n$, where $n = d(b, c)$.

Lemma 4.15 implies that all N_{ijk} 's are known as soon as we know those with $i, j, k \leq 2$. Indeed, those with $i, j, k > 0$ and at least one of i, j, k equal to 3 can be computed by means of the formulas in the following corollary (respecting the order in which they are mentioned).

Corollary 4.16. • *If $i, j \in \{1, 2\}$, then $N_{ij3} = p_{ij}^\delta - N_{ij0} - N_{ij1} - N_{ij2}$, where $\delta = d(a, b)$.*

- *If $i, k \in \{1, 2\}$, then $N_{i3k} = p_{ik}^\delta - N_{i0k} - N_{i1k} - N_{i2k}$, where $\delta = d(a, c)$.*
- *If $j, k \in \{1, 2\}$, then $N_{3jk} = p_{jk}^\delta - N_{0jk} - N_{1jk} - N_{2jk}$, where $\delta = d(b, c)$.*
- *If $i \in \{1, 2\}$, then $N_{i33} = p_{i3}^\delta - N_{i30} - N_{i31} - N_{i32}$, where $\delta = d(a, b)$.*
- *If $j \in \{1, 2\}$, then $N_{3j3} = p_{3j}^\delta - N_{3j0} - N_{3j1} - N_{3j2}$, where $\delta = d(a, b)$.*
- *If $k \in \{1, 2\}$, then $N_{33k} = p_{3k}^\delta - N_{03k} - N_{13k} - N_{23k}$, where $\delta = d(b, c)$.*
- *$N_{333} = p_{33}^\delta - N_{330} - N_{331} - N_{332}$, where $\delta = d(a, b)$.*

The results obtained so far allow us to draw a number of conclusions for triple intersection numbers of type PPP in finite generalized hexagons of order (s, t) . These numbers only depend on s and t for configurations 1 to 9, on s, t and one extra parameter (N_{222}) for configurations 10 to 13, on s, t and two extra parameters (N_{221} and N_{222}) for configurations 14 and 15, and on s, t and four extra parameters ($N_{122}, N_{212}, N_{221}, N_{222}$) for configuration 16.

We now show that in a generalized hexagon of order (s, s^3) , $s \geq 2$, all triple intersection numbers only depend on s for configurations 1 to 15. By Properties (P1) and (P2), we know that N_{221} equals $s^2 + 1$ for configuration 14 and $s^2 + s + 1$ for configuration 15. So, it remains to prove that N_{222} only depends on s for each of the configurations 10 to 15. For configuration 16, we know that $N_{122} = N_{212} = N_{221}$ by Property (P3) and so the triple intersection numbers depend on s and two extra parameters. These two parameters are

related to each other, but we will only be able to prove this in Section 7 after we have also studied the triple intersection numbers of type PPL in Section 6.

In the remainder of this section, we suppose that $\mathcal{S}$ is extremal. The properties (P1), (P2) and the above lemmas imply that if u, v, w are three not mutually opposite points, then the numbers $N_{ij0}(u, v, w)$ and $N_{ij1}(u, v, w)$ only depend on s, i, j and the configuration number δ . We will denote these numbers by $N_{ij0}^{(\delta)}$ and $N_{ij1}^{(\delta)}$, respectively.

Lemma 4.17. *Suppose $\mathcal{S}$ is extremal, $d(a, b) = 2$ and that configuration $j \in \{10, 11, \dots, 15\}$ occurs. For every $i \in \{1, 2, \dots, 41\}$, let A_i denote the number of points $y \in \Gamma_1(c)$ for which (a, b, y) determines configuration i . Then $N_{222} = \left(\sum_{i=1}^{41} A_i \cdot N_{221}^{(i)} \right) - (s-1) \cdot N_{221}^{(j)} - s(s^3 + 1) \cdot N_{220}^{(j)}$.*

Proof. We count in two different ways the number N of pairs (y, x) where $y \in \Gamma_1(c)$ and $x \in \Gamma_1(y) \cap \Gamma_2(a) \cap \Gamma_2(b)$. Counting with respect to y , we see that $N = \sum_{i=1}^{41} A_i \cdot N_{221}^{(i)}$. On the other hand, if (y, x) is a suitable pair, then $c \sim y \sim x$ implies that $d(c, x) \leq 2$. For given x , there are $s(s^3 + 1)$, $s - 1$ or 1 possibilities for y depending on whether $d(c, x)$ is equal to 0, 1 or 2. So, we also have $N = N_{222} + (s - 1) \cdot N_{221}^{(j)} + s(s^3 + 1) \cdot N_{220}^{(j)}$. $\square$

The numbers A_i , $i \in \{1, 2, \dots, 41\}$, can be determined for each of the configurations 10 to 15. This will be realized in the following sequence of lemmas, where we only mention those A_i 's that are not always 0. The proofs rely on the properties of $\mathcal{S}$ mentioned in Section 2 (which we will no longer recall in each proof).

Lemma 4.18. *Suppose $\mathcal{S}$ is extremal and configuration 10 occurs. Then $A_{10} = s - 1$, $A_{14} = s^4$ and $A_{20} = 1$.*

Proof. Let L be a line through c , $x \in L \setminus \{c\}$, and let i denote the configuration number for (a, b, x) . If $L = L(c, a * b)$ and $x = a * b$, then $i = 20$. If $L = L(c, a * b)$ and x is one of the $s - 1$ points of $L \setminus \{c, a * b\}$, then $i = 10$. If L is one of the s^3 lines through c distinct from $L(c, a * b)$, then $i = 14$ for all s points $x \in L \setminus \{c\}$. $\square$

Lemma 4.19. *Suppose $\mathcal{S}$ is extremal and configuration 11 occurs. Then $A_{13} = s - 1$, $A_{14} = s^3 - 1$, $A_{15} = (s - 1)(s^3 - 1)$, $A_{23} = 1$, $A_{34} = 1$ and $A_{39} = s - 1$.*

Proof. Let L be a line through c , $x \in L \setminus \{c\}$, and let i denote the configuration number for (a, b, x) . If $L = L_1 := L(c, a * c)$, then $i = 23$ or $i = 13$ depending on whether x is $a * c$ or one of the $s - 1$ points of $L_1 \setminus \{c, a * c\}$. If $L = L_2 := L(c, c * b)$, then $i = 34$ or $i = 39$ depending on whether x is $c * b$ or one of the $s - 1$ points of $L \setminus \{c, c * b\}$. If L is one of the $s^3 - 1$ lines through c distinct from L_1 and L_2 , then $i = 14$ or $i = 15$ depending on whether x is $\pi_L(a * b)$ or one of the $s - 1$ points of $L \setminus \{c, \pi_L(a * b)\}$. $\square$

Lemma 4.20. *Suppose $\mathcal{S}$ is extremal and configuration 12 occurs. Then $A_{12} = s - 1$, $A_{15} = s^3(s - 1)$, $A_{21} = 1$ and $A_{39} = s^3$.*

Proof. Let L be a line through c , $x \in L \setminus \{c\}$, and let i denote the configuration number for (a, b, x) . If $L = L(c, a * c)$, then $i = 21$ or $i = 12$ depending on whether x is $a * c$ or one of the $s - 1$ points of $L \setminus \{c, a * c\}$. If L is one of the s^3 lines through c distinct from $L(c, a * c)$, then $i = 39$ or $i = 15$ depending on whether x is $\pi_L(b)$ or one of the $s - 1$ points of $L \setminus \{c, \pi_L(b)\}$. $\square$

Lemma 4.21. *Suppose $\mathcal{S}$ is extremal and configuration 13 occurs. Then $A_{11} = 1$, $A_{13} = s - 2$, $A_{14} = s^3 - 1$, $A_{15} = s^4 - 2s^3 + 1$, $A_{23} = 1$, $A_{38} = 1$ and $A_{39} = s^3 - 1$.*

Proof. Let L be a line through c , $x \in L \setminus \{c\}$, and let i denote the configuration number for (a, b, x) . Put $L_1 := L(c, a * c)$ and $L_2 = L(c, \pi_K(c) * c)$ where $K = L(b, a * b)$. If $L = L_1$, then $i = 23$, $i = 11$ or $i = 13$, depending on whether x is $a * c$, $\pi_L(b)$ or one of the $s - 2$ points of $L_1 \setminus \{c, a * c, \pi_L(b)\}$. If $L = L_2$, then $i = 38$ or $i = 15$ depending on whether x is $\pi_K(c) * c$ or one of the $s - 1$ points of $L \setminus \{c, \pi_K(c) * c\}$. If L is one of the $s^3 - 1$ lines through c distinct from L_1 and L_2 , then $i = 39$, $i = 14$ or $i = 15$, depending on whether x is $\pi_L(b)$, $\pi_L(a * b)$ or one of the $s - 2$ points of $L \setminus \{c, \pi_L(b), \pi_L(a * b)\}$. $\square$

Lemma 4.22. *Suppose $\mathcal{S}$ is extremal and configuration 14 occurs. Then $A_{10} = 1$, $A_{11} = s^2$, $A_{13} = s^3 - s^2$, $A_{14} = s - 1$, $A_{15} = s^4 - 2s^3 + s^2$ and $A_{39} = s^3 - s^2$.*

Proof. Let L be a line through c , $x \in L \setminus \{c\}$, and let i denote the configuration number for (a, b, x) . Let L' be the line through c containing the point $(a * b) * c$. If $L = L'$, then $i = 10$ or $i = 14$ depending on whether x is $(a * b) * c$ or one of the $s - 1$ points of $L \setminus \{c, (a * b) * c\}$. If L is one of the s^2 lines through c distinct from L' containing a point of $\Gamma_2(a) \cap \Gamma_2(b)$ (recall (P1)), then $i = 11$ or $i = 15$ depending on whether x is $\pi_L(a) = \pi_L(b)$ or one of the $s - 1$ points of $L \setminus \{c, \pi_L(a)\}$. If L is one of the $s^3 - s^2$ lines through c disjoint from $\Gamma_2(a) \cap \Gamma_2(b)$, then $i = 13$, $i = 39$ or $i = 15$, depending on whether x is $\pi_L(a)$, $\pi_L(b)$ or one of the $s - 2$ points of $L \setminus \{c, \pi_L(a), \pi_L(b)\}$. $\square$

Lemma 4.23. *Suppose $\mathcal{S}$ is extremal and configuration 15 occurs. Then $A_{11} = s^2 + s + 1$, $A_{12} = 1$, $A_{13} = s^3 - s^2 - s - 1$, $A_{14} = s^3 - 1$, $A_{15} = s^4 - 3s^3 + s^2 + 2s$, $A_{38} = 1$ and $A_{39} = s^3 - s^2 - s - 1$.*

Proof. Let L be a line through c , $x \in L \setminus \{c\}$, and let i denote the configuration number for (a, b, x) . Put $K_1 = L(a, a * b)$, $K_2 = L(b, a * b)$ and $L_j = L(c, c * \pi_{K_j}(c))$, $j \in \{1, 2\}$. If $L = L_1$, then $i = 12$, $i = 39$ or $i = 15$ depending on whether x is $\pi_L(a)$, $\pi_L(b)$ or one of the $s - 2$ points of $L \setminus \{c, \pi_L(a), \pi_L(b)\}$. If $L = L_2$, then $i = 38$, $i = 13$ or $i = 15$ depending on whether x is $\pi_L(b)$, $\pi_L(a)$ or one of the $s - 2$ points of $L \setminus \{c, \pi_L(a), \pi_L(b)\}$. If L is one of the $s^2 + s + 1$ lines through c meeting $\Gamma_2(a) \cap \Gamma_2(b)$ (recall (P2)), then $i = 11$, $i = 14$ or $i = 15$ depending on whether x is $\pi_L(a) = \pi_L(b)$, $\pi_L(a * b)$ or one of the $s - 2$ points of $L \setminus \{c, \pi_L(a), \pi_L(a * b)\}$. If L is one of the $s^3 - s^2 - s - 2$ lines through c distinct from L_1 , L_2 and disjoint from $\Gamma_2(a) \cap \Gamma_2(b)$, then i equals 13, 39, 14 or 15 depending on whether x is $\pi_L(a)$, $\pi_L(b)$, $\pi_L(a * b)$ or one of the $s - 3$ points of $L \setminus \{c, \pi_L(a), \pi_L(b), \pi_L(a * b)\}$. $\square$

Based on the previous lemmas, we can compute the value of N_{222} for configurations 10 to 15. For this we need to use some N_{221} 's that were already determined earlier, namely

$N_{221}^{(10)} = s - 1$, $N_{221}^{(11)} = N_{221}^{(12)} = 0$, $N_{221}^{(13)} = 1$, $N_{221}^{(14)} = s^2 + 1$, $N_{221}^{(15)} = s^2 + s + 1$,
 $N_{221}^{(20)} = N_{122}^{(6)} = s^4 - s$, $N_{221}^{(21)} = N_{122}^{(7)} = 0$, $N_{221}^{(23)} = N_{212}^{(8)} = s^3$, $N_{221}^{(34)} = N_{212}^{(8)} = s^3$,
 $N_{221}^{(38)} = N_{212}^{(12)} = 0$ and $N_{221}^{(39)} = N_{212}^{(13)} = 1$. Invoking Lemma 4.17, we find:

Corollary 4.24. *Suppose $\mathcal{S}$ is extremal.*

- *If configuration 10 occurs, then $N_{222} = s^6 + s^4 - 2s$.*
- *If configuration 11 occurs, then $N_{222} = s^6 + s^5 - s^4 + s^3 - s^2 + s - 2$.*
- *If configuration 12 occurs, then $N_{222} = s^6$.*
- *If configuration 13 occurs, then $N_{222} = s^6 - s^4 + s^3 + s - 2$.*
- *If configuration 14 occurs, then $N_{221} = s^2 + 1$ and $N_{222} = s^6 - s^5 + s^3 - s^2 + s - 1$.*
- *If configuration 15 occurs, then $N_{221} = s^2 + s + 1$ and $N_{222} = s^6 - s^5 - s^4 + 2s^3 - 2$.*

The results obtained in this section allow us to conclude the following.

Corollary 4.25. *If x , y and z are three not mutually opposite points in an extremal generalized hexagon of order (s, s^3) , $s \geq 2$, then the number $|\Gamma_i(x) \cap \Gamma_j(y) \cap \Gamma_k(z)|$ only depends on s , i , j , k and the configuration determined by x , y and z .*

5 The 39 configurations for two points and a line

We first describe a list of 28 configurations for two points x , y and one line L of a generalized hexagon $\mathcal{S}$.

- **Configuration 1:** $x = y$ and $x \in L$;
- **Configuration 2:** $x = y$ and $d(x, L) = 1$;
- **Configuration 3:** $x = y$ and $d(x, L) = 2$;
- **Configuration 4:** $d(x, y) = 1$ and $L = xy$;
- **Configuration 5:** $d(x, y) = 1$, $x \in L$ and $L \neq xy$;
- **Configuration 6:** $d(x, y) = 1$ and L meets xy in a point distinct from x and y ;
- **Configuration 7:** $d(x, y) = d(xy, L) = d(x, L) = 1$;
- **Configuration 8:** $d(x, y) = d(xy, L) = 1$ and $d(x, L) = d(y, L) = 2$;
- **Configuration 9:** $d(x, y) = 1$ and L, xy are two opposite lines;
- **Configuration 10:** $d(x, y) = 2$, $x \in L$ and $y \in \Gamma_1(L)$;
- **Configuration 11:** $d(x, y) = 2$, $x \in L$ and $y \in \Gamma_2(L)$;
- **Configuration 12:** $d(x, y) = 2$, $x, y \in \Gamma_1(L)$ and $\pi_L(x) = \pi_L(y)$;
- **Configuration 13:** $d(x, y) = d(y, L) = 2$, $d(x, L) = 1$ and $x \in L(\pi_L(y), \pi_L(y) * y)$;
- **Configuration 14:** $d(x, y) = 2$, $d(x, L) = 1$ and $L(x, \pi_L(x)) \neq L(x, x * y)$;
- **Configuration 15:** $d(x, y) = d(x, L) = d(y, L) = 2$ and $x * y \in \Gamma_1(L)$;
- **Configuration 16:** $d(x, y) = d(x, L) = d(y, L) = 2$, $\pi_L(x) = \pi_L(y)$ and the lines $L(x, x * y)$, $L(y, x * y)$ are opposite to L ;
- **Configuration 17:** $d(x, y) = d(x, L) = d(y, L) = 2$, $\pi_L(x) \neq \pi_L(y)$ and the lines $L(x, x * y)$, $L(y, x * y)$ are opposite to L ;
- **Configuration 18:** $d(x, y) = d(x, L) = d(y, L) = 2$, $x * y \in \Gamma_2(L)$ and the line $L(y, x * y)$ is not opposite to L ;

- **Configuration 19:** $d(x, y) = 3$ and $x \in L$;
- **Configuration 20:** $d(x, y) = 3$ and $d(x, L) = d(y, L) = 1$;
- **Configuration 21:** $d(x, y) = 3$, $d(x, L) = 1$, $d(y, L) = 2$ and $\pi_L(x) = \pi_L(y)$;
- **Configuration 22:** $d(x, y) = 3$, $d(x, L) = 1$, $d(y, L) = 2$ and $\pi_L(x) \neq \pi_L(y)$;
- **Configuration 23:** $d(x, y) = 3$, $d(x, L) = d(y, L) = 2$, $\pi_L(x) = \pi_L(y)$ and $L(\pi_L(x), \pi_L(x) * x) \neq L(\pi_L(y), \pi_L(y) * y)$;
- **Configuration 24:** $d(x, y) = 3$, $d(x, L) = d(y, L) = 2$, $\pi_L(x) = \pi_L(y)$ and $L(\pi_L(x), \pi_L(x) * x) = L(\pi_L(y), \pi_L(y) * y)$;
- **Configuration 25:** $d(x, y) = 3$, $d(x, L) = d(y, L) = 2$, $\pi_L(x) \neq \pi_L(y)$ and the lines $L(x, \pi_L(x) * x)$, $L(y, \pi_L(y) * y)$ lie at distance 1 from each other;
- **Configuration 26:** $d(x, y) = 3$, $d(x, L) = d(y, L) = 2$, $\pi_L(x) \neq \pi_L(y)$, the lines $L(x, \pi_L(x) * x)$, $L(y, \pi_L(y) * y)$ are opposite, and $d(x, y * \pi_L(y)) = d(y, x * \pi_L(x)) = 2$;
- **Configuration 27:** $d(x, y) = 3$, $d(x, L) = d(y, L) = 2$, $\pi_L(x) \neq \pi_L(y)$, the lines $L(x, \pi_L(x) * x)$, $L(y, \pi_L(y) * y)$ are opposite, and $d(x, y * \pi_L(y)) = 2$, $d(y, x * \pi_L(x)) = 3$;
- **Configuration 28:** $d(x, y) = 3$, $d(x, L) = d(y, L) = 2$, $\pi_L(x) \neq \pi_L(y)$, the lines $L(x, \pi_L(x) * x)$, $L(y, \pi_L(y) * y)$ are opposite, and $d(x, y * \pi_L(y)) = d(y, x * \pi_L(x)) = 3$.

Configurations 1 to 28 are also shown in Table 2. Each configuration has a dual configuration obtained by interchanging the points x and y .

Lemma 5.1. *Let x, y be two points and L a line of $\mathcal{S}$. Then x, y and L are as in one of the configurations 1 to 28 or as in one of their dual configurations.*

Proof. Put $x' := \pi_L(x)$ and $y' := \pi_L(y)$. Without loss of generality, we may suppose $d(x, L) \leq d(y, L)$.

Suppose $x = y$. Then configuration 1, 2 or 3 occurs depending on whether $d(x, L)$ is equal to 0, 1 or 2.

Suppose $d(x, y) = 1$. If $xy = L$, then configuration 4 occurs. If xy and L are opposite lines, then configuration 9 occurs. Suppose therefore that xy, L are two distinct non-opposite lines, and let z denote the unique point on xy nearest to L . The fact that $d(x, L) \leq d(y, L)$ implies that $z \neq y$. If $z = x$, then configuration 5 or 7 occurs depending on whether $d(xy, L)$ is 0 or 1. If $z \in xy \setminus \{x, y\}$, then configuration 6 or 8 occurs depending on whether $d(xy, L)$ is 0 or 1.

Suppose $d(x, y) = 2$ and $x \in L$. Then configuration 10 or 11 occurs depending on whether $d(y, L)$ is 1 or 2.

Suppose $d(x, y) = 2$ and $d(x, L) = 1$. Then $d(y, L) \geq 1$. If $d(y, L) = 1$, then the fact that $d(x, y) \neq 3$ implies that $\pi_L(x) = \pi_L(y)$, showing that configuration 12 occurs. If $d(y, L) = 2$ and $\pi_L(x) = \pi_L(y)$, then considering the sequence $x \sim \pi_L(x) \sim \pi_L(x) * y \sim y$ and taking into account that $d(x, y) = 2$, we see that $\pi_L(x) = \pi_L(y)$, x and $y * \pi_L(y)$ are on the same line, showing that configuration 13 occurs. If $d(y, L) = 2$ and $\pi_L(x) \neq \pi_L(y)$, then the six points $x, \pi_L(x), \pi_L(y), y * \pi_L(y), y, x * y$ determine a 6-gon, proving that configuration 14 occurs.

Suppose $d(x, y) = d(x, L) = 2$. Then also $d(y, L) = 2$. If $x * y \in \Gamma_1(L)$, then configuration 15 occurs. Suppose therefore that $x * y \in \Gamma_2(L)$. If the line $L(y, x * y)$ is not opposite to L , then configuration 18 occurs. If the line $L(x, x * y)$ is not opposite to

Configuration 1	Configuration 2	Configuration 3	Configuration 4
Configuration 5	Configuration 6	Configuration 7	Configuration 8
Configuration 9	Configuration 10	Configuration 11	Configuration 12
Configuration 13	Configuration 14	Configuration 15	Configuration 16
Configuration 17	Configuration 18	Configuration 19	Configuration 20
Configuration 21	Configuration 22	Configuration 23	Configuration 24
Configuration 25	Configuration 26	Configuration 27	Configuration 28

Table 2: Twenty eight configurations for two points and a line

L , then the dual of configuration 18 occurs. If $L(x, x * y)$ and $L(y, x * y)$ are opposite to L , then configuration 16 or 17 occurs depending on whether $\pi_L(x) = \pi_L(y)$ or not.

From now on, we assume that $d(x, y) = 3$. If $d(x, L) = 0$, then configuration 19 necessarily occurs. If $d(x, L) = 1$, then $d(y, L) \in \{1, 2\}$. If $d(x, L) = d(y, L) = 1$, then the fact that $d(x, y) = 3$ implies that $\pi_L(x) \neq \pi_L(y)$ and so configuration 20 occurs. If $d(x, L) = 1$ and $d(y, L) = 2$, then configuration 21 or 22 occurs depending on whether $\pi_L(x) = \pi_L(y)$ or not. So, we may suppose that $d(x, L) = 2$. Then also $d(y, L) = 2$. If $\pi_L(x) = \pi_L(y)$, then configuration 23 or 24 occurs, depending on whether the lines $L(\pi_L(x), \pi_L(x) * x)$ and $L(\pi_L(x), \pi_L(x) * y)$ are distinct or equal. We therefore suppose $\pi_L(x) \neq \pi_L(y)$. If the lines $L(x, x * \pi_L(x))$ and $L(y, y * \pi_L(y))$ have distance 1, then configuration 25 occurs. Suppose therefore that these lines are opposite. Putting $d_1 := d(x, y * \pi_L(y))$ and $d_2 := d(y, x * \pi_L(x))$, there are then four possibilities. If $d_1 = d_2 = 2$, then configuration 26 occurs. If $d_1 = 2$ and $d_2 = 3$, then configuration 27 occurs. If $d_1 = 3$ and $d_2 = 2$, then the dual of configuration 27 occurs. Finally, if $d_1 = d_2 = 3$, then configuration 28 occurs. $\square$

The dual of configuration i is denoted by i^* . The duals of certain configurations are the same as the original configurations. This is the case for configurations 1, 2, 3, 4, 6, 8, 9, 12, 15, 16, 17, 20, 23, 24, 25, 26 and 28. Besides the 28 basic configurations, we thus have 11 additional ones: $5^*, 7^*, 10^*, 11^*, 13^*, 14^*, 18^*, 19^*, 21^*, 22^*, 27^*$. Note that the configurations $1, 2, \dots, 28$ comprise all cases where $d(x, L) \leq d(y, L)$ with exception of configurations 18^* and 27^* .

6 The triple intersection numbers of type PPL

In this section, we give an alternative proof of Haemers' result on triple intersection numbers of type PPL in extremal generalized hexagons. One of the results (Lemma 6.6) we derive along the way will also be essential in Section 7 to prove a connection between the triple intersection numbers $N_{122} = N_{212} = N_{221}$ and N_{222} for PPP configuration 16. It turns out that the set of PPL configurations can be divided into two families such that a unified treatment can be given for all the configurations inside one family. The larger of the two families comprises 32 configurations, namely configurations 1 to 22 and their duals (among which 10 new configurations arise). In fact, for these configurations Haemers' result is an almost immediate consequence of our results on PPP triple intersection numbers and the observation made in the next lemma. For the remaining 7 configurations ($23, 24, \dots, 28, 27^*$), some more work is necessary. Throughout this section, x and y are two points of a finite generalized hexagon $\mathcal{S}$ of order (s, t) and L is a line of $\mathcal{S}$.

Lemma 6.1. *There exist constants $\Omega_l(m)$, $m \in \{1, 2, \dots, 28, 5^*, 7^*, \dots, 27^*\}$ and $l \in \{1, 2, \dots, 41\}$ such that if (x, y, L) determines configuration m , then there are precisely $\Omega_l(m)$ points $z \in L$ for which (x, y, z) determines configuration l . The $\Omega_l(m)$'s are as in Table 3.*

m	$\Omega_l = \Omega_l(m)$	m	$\Omega_l = \Omega_l(m)$
$1 = 1^*$	$\Omega_1 = 1, \Omega_2 = s$	$15 = 15^*$	$\Omega_{10} = 1, \Omega_{14} = s$
$2 = 2^*$	$\Omega_2 = 1, \Omega_3 = s$	$16 = 16^*$	$\Omega_{11} = \Omega_{14} = 1, \Omega_{15} = s - 1$
$3 = 3^*$	$\Omega_3 = 1, \Omega_4 = s$	$17 = 17^*$	$\Omega_{13} = \Omega_{14} = \Omega_{39} = 1, \Omega_{15} = s - 2$
$4 = 4^*$	$\Omega_{17} = \Omega_{29} = 1, \Omega_5 = s - 1$	18	$\Omega_{13} = \Omega_{38} = 1, \Omega_{15} = s - 1$
5	$\Omega_{17} = 1, \Omega_6 = s$	18^*	$\Omega_{12} = \Omega_{39} = 1, \Omega_{15} = s - 1$
5^*	$\Omega_{29} = 1, \Omega_{32} = s$	19	$\Omega_{19} = \Omega_{22} = 1, \Omega_{24} = s - 1$
$6 = 6^*$	$\Omega_5 = 1, \Omega_7 = s$	19^*	$\Omega_{31} = \Omega_{35} = 1, \Omega_{37} = s - 1$
7	$\Omega_6 = 1, \Omega_8 = s$	$20 = 20^*$	$\Omega_{22} = \Omega_{35} = 1, \Omega_{25} = s - 1$
7^*	$\Omega_{32} = 1, \Omega_{36} = s$	21	$\Omega_{22} = 1, \Omega_{27} = s$
$8 = 8^*$	$\Omega_7 = 1, \Omega_9 = s$	21^*	$\Omega_{35} = 1, \Omega_{40} = s$
$9 = 9^*$	$\Omega_8 = \Omega_{36} = 1, \Omega_9 = s - 1$	22	$\Omega_{24} = \Omega_{26} = 1, \Omega_{28} = s - 1$
10	$\Omega_{18} = \Omega_{20} = 1, \Omega_{21} = s - 1$	22^*	$\Omega_{26} = \Omega_{37} = 1, \Omega_{41} = s - 1$
10^*	$\Omega_{20} = \Omega_{30} = 1, \Omega_{33} = s - 1$	$23 = 23^*$	$\Omega_{26} = 1, \Omega_{16} = s$
11	$\Omega_{18} = 1, \Omega_{23} = s$	$24 = 24^*$	$\Omega_{25} = 1, \Omega_{16} = s$
11^*	$\Omega_{30} = 1, \Omega_{34} = s$	$25 = 25^*$	$\Omega_{28} = \Omega_{41} = 1, \Omega_{16} = s - 1$
$12 = 12^*$	$\Omega_{20} = 1, \Omega_{10} = s$	$26 = 26^*$	$\Omega_{27} = \Omega_{40} = 1, \Omega_{16} = s - 1$
13	$\Omega_{21} = 1, \Omega_{12} = s$	27	$\Omega_{28} = \Omega_{40} = 1, \Omega_{16} = s - 1$
13^*	$\Omega_{33} = 1, \Omega_{38} = s$	27^*	$\Omega_{27} = \Omega_{41} = 1, \Omega_{16} = s - 1$
14	$\Omega_{11} = \Omega_{23} = 1, \Omega_{13} = s - 1$	$28 = 28^*$	$\Omega_{28} = \Omega_{41} = 1, \Omega_{16} = s - 1$
14^*	$\Omega_{11} = \Omega_{34} = 1, \Omega_{39} = s - 1$	—	—

Table 3: The Ω -values

Proof. For every $z \in L$, let $l_z \in \{1, 2, \dots, 41\}$ such that (x, y, z) determines configuration l_z . The number l_z can easily be determined using the various properties of $\mathcal{S}$ mentioned in Section 2 (which we will not recall for each case). We then have the following.

- If $m = 1$, then $l_z = 1$ if $z = x$ and $l_z = 2$ otherwise.
- If $m = 2$, then $l_z = 2$ if $z = \pi_L(x)$ and $l_z = 3$ otherwise.
- If $m = 3$, then $l_z = 3$ if $z = \pi_L(x)$ and $l_z = 4$ otherwise.
- If $m = 4$, then $l_z = 17$ if $z = x$, $l_z = 29$ if $z = y$ and $l_z = 5$ otherwise.
- If $m = 5$, then $l_z = 17$ if $z = x$ and $l_z = 6$ otherwise.
- If $m = 6$, then $l_z = 5$ if $z = \pi_L(x)$ and $l_z = 7$ otherwise.
- If $m = 7$, then $l_z = 6$ if $z = \pi_L(x)$ and $l_z = 8$ otherwise.
- If $m = 8$, then $l_z = 7$ if $z = \pi_L(x)$ and $l_z = 9$ otherwise.
- If $m = 9$, then $l_z = 8$ if $z = \pi_L(x)$, $l_z = 36$ if $z = \pi_L(y)$ and $l_z = 9$ otherwise.
- If $m = 10$, then $l_z = 18$ if $z = x$, $l_z = 20$ if $z = \pi_L(y)$ and $l_z = 21$ otherwise.
- If $m = 11$, then $l_z = 18$ if $z = x$ and $l_z = 23$ otherwise.
- If $m = 12$, then $l_z = 20$ if $z = x * y$ and $l_z = 10$ otherwise.
- If $m = 13$, then $l_z = 21$ if $z = \pi_L(x)$ and $l_z = 12$ otherwise.
- If $m = 14$, then $l_z = 11$ if $z = \pi_L(y)$, $l_z = 23$ if $z = \pi_L(x)$ and $l_z = 13$ otherwise.
- If $m = 15$, then $l_z = 10$ if $z = \pi_L(x)$ and $l_z = 14$ otherwise.
- If $m = 16$, then $l_z = 11$ if $z = \pi_L(x)$, $l_z = 14$ if $z = \pi_L(x * y)$ and $l_z = 15$ otherwise.
- If $m = 17$, then $l_z = 13$ if $z = \pi_L(x)$, $l_z = 39$ if $z = \pi_L(y)$, $l_z = 14$ if $z = \pi_L(x * y)$ and $l_z = 15$ otherwise.
- If $m = 18$, then $l_z = 13$ if $z = \pi_L(x)$, $l_z = 38$ if $z = \pi_L(y)$ and $l_z = 15$ otherwise.
- If $m = 19$, then $l_z = 19$ if $z = x$, $l_z = 22$ if $z = \pi_L(y)$ and $l_z = 24$ otherwise.
- If $m = 20$, then $l_z = 22$ if $z = \pi_L(x)$, $l_z = 35$ if $z = \pi_L(y)$ and $l_z = 25$ otherwise.
- If $m = 21$, then $l_z = 22$ if $z = \pi_L(x)$ and $l_z = 27$ otherwise.
- If $m = 22$, then $l_z = 24$ if $z = \pi_L(x)$, $l_z = 26$ if $z = \pi_L(y)$ and $l_z = 28$ otherwise.
- If $m = 23$, then $l_z = 26$ if $z = \pi_L(x)$ and $l_z = 16$ otherwise.
- If $m = 24$, then $l_z = 25$ if $z = \pi_L(x)$ and $l_z = 16$ otherwise.
- If $m = 25$, then $l_z = 28$ if $z = \pi_L(x)$, $l_z = 41$ if $z = \pi_L(y)$ and $l_z = 16$ otherwise.
- If $m = 26$, then $l_z = 27$ if $z = \pi_L(x)$, $l_z = 40$ if $z = \pi_L(y)$ and $l_z = 16$ otherwise.
- If $m = 27$, then $l_z = 28$ if $z = \pi_L(x)$, $l_z = 40$ if $z = \pi_L(y)$ and $l_z = 16$ otherwise.
- If $m = 28$, then $l_z = 28$ if $z = \pi_L(x)$, $l_z = 41$ if $z = \pi_L(y)$ and $l_z = 16$ otherwise.

This gives rise to the constants mentioned in Table 3 (sometimes after dualizing). $\square$

For all $i, j \in \{0, 1, 2, 3\}$ and every $k \in \{0, 1, 2\}$, put $P_{ijk} := P_{ijk}(x, y, L) = |\Gamma_i(x) \cap \Gamma_j(y) \cap \Gamma_k(L)|$. We suppose that (x, y, L) determines configuration m . If $\mathcal{S}$ is extremal and $z \in L$ such that (x, y, z) determines configuration $l \neq 16$, then we know by Corollary 4.25 that the triple intersection number $N_{ijk}(l) := N_{ijk}(x, y, z) = |\Gamma_i(x) \cap \Gamma_j(y) \cap \Gamma_k(z)|$ only depends on s, l, i, j and k . Note that if x, y, L are not mutually opposite, or equivalently, $m \notin \{23, 24, 25, 26, 27, 28, 27^*\}$, then x, y, z can never be mutually opposite and so l is then always distinct from 16. The following result implies Haemers' result for all the configurations where x, y and L are not mutually opposite.

Lemma 6.2. *If $\mathcal{S}$ is extremal and x, y, L are not mutually opposite, then $P_{ij0} = \sum_{l=1}^{41} \Omega_l(m) \cdot N_{ij0}(l)$, $P_{ij1} = -s \cdot P_{ij0} + \sum_{l=1}^{41} \Omega_l(m) \cdot N_{ij1}(l)$ and $P_{ij2}(x, y, L) = p_{ij}^k - P_{ij0} - P_{ij1}$, where $k = d(x, y)$.*

Proof. Obviously, P_{ij0} is equal to $\sum_{z \in L} N_{ij0}(x, y, z) = \sum_{l=1}^{41} \Omega_l(m) \cdot N_{ij0}(l)$. In the sum $\sum_{z \in L} N_{ij1}(x, y, z)$, we count the points of $\Gamma_i(x) \cap \Gamma_j(y) \cap \Gamma_1(L)$ precisely once, and the points of $\Gamma_i(x) \cap \Gamma_i(y) \cap L$ exactly s times. Hence, $P_{ij1} = -s \cdot P_{ij0} + \sum_{z \in L} N_{ij1}(x, y, z) = -s \cdot P_{ij0} + \sum_{l=1}^{41} \Omega_l(m) \cdot N_{ij1}(l)$. The latter equality follows from the fact that $P_{ij0} + P_{ij1} + P_{ij2} = |\Gamma_i(x) \cap \Gamma_j(y)| = p_{ij}^k$. $\square$

In the sequel, we suppose that x, y and L are mutually opposite, i.e. $m \in \{23, 24, 25, 26, 27, 28, 27^*\}$. We put $\alpha := P_{121} \in \{0, 1\}$, $\beta := P_{211} \in \{0, 1\}$, $\gamma := P_{220} \in \{0, 1\}$ and $\delta := P_{221}$.

Lemma 6.3. • $P_{122} = t + 1 - \alpha$, $P_{212} = t + 1 - \beta$, $P_{222} = (s - 1)(t + 1)^2 - \gamma - \delta$;
• $P_{ijk} = 0$ if $i, j, k \in \{0, 1, 2\}$ with $(i, j, k) \notin \{(1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 0), (2, 2, 1), (2, 2, 2)\}$.

Proof. The last claim follows from the triangle inequality. The first claim follows from the facts that $P_{120} + P_{121} + P_{122} = |\Gamma_1(x) \cap \Gamma_2(y)| = p_{12}^3 = t + 1$, $P_{210} + P_{211} + P_{212} = |\Gamma_2(x) \cap \Gamma_1(y)| = p_{21}^3 = t + 1$ and $P_{220} + P_{221} + P_{222} = |\Gamma_2(x) \cap \Gamma_2(y)| = p_{22}^3 = (s - 1)(t + 1)^2$. $\square$

Lemma 6.4. *There exist constants q_{ij}^k , $i \in \{0, 1, 2, 3\}$ and $k, j \in \{0, 1, 2\}$, such that if (u, M) is a point-line pair with $d(u, M) = k$, then $|\Gamma_i(u) \cap \Gamma_j(M)| = q_{ij}^k$.*

Proof. Suppose first that $j = 0$. Then $|\Gamma_i(u) \cap \Gamma_j(M)| = |\Gamma_i(u) \cap M|$ is equal to 1 if $i = k$, equal to s if $i = k + 1$ and equal to 0 otherwise. We denote this number by q_{i0}^k .

In the sum $\sum_{v \in M} |\Gamma_i(u) \cap \Gamma_1(v)|$, the points of $\Gamma_i(u) \cap \Gamma_1(M)$ are counted once, while those of $\Gamma_i(u) \cap M$ are counted s times. So, $|\Gamma_i(u) \cap \Gamma_1(M)| = -s \cdot q_{i0}^k + \sum_{v \in M} |\Gamma_i(u) \cap \Gamma_1(v)| = -s \cdot q_{i0}^k + p_{i1}^k + s \cdot p_{i1}^{k+1} =: q_{i1}^k$ and $|\Gamma_i(u) \cap \Gamma_2(M)| = |\Gamma_i(u)| - |\Gamma_i(u) \cap M| - |\Gamma_i(u) \cap \Gamma_1(M)| = p_{ii}^0 - q_{i0}^k - q_{i1}^k =: q_{i2}^k$. $\square$

By Lemma 6.4, we know all P_{ijk} 's as soon as we know those with $i, j, k \in \{0, 1, 2\}$. Indeed:

Lemma 6.5. *We have $P_{3jk} = q_{jk}^2 - P_{0jk} - P_{1jk} - P_{2jk}$ for all $j, k \in \{0, 1, 2\}$, and $P_{i3k} = q_{ik}^2 - P_{i0k} - P_{i1k} - P_{i2k}$ for every $i \in \{0, 1, 2, 3\}$ and every $k \in \{0, 1, 2\}$.*

For the configurations 23, 24, 25, 26, 27, 27* and 28, we know the values of α, β and γ :

- $\alpha = 0, \beta = 0$ and $\gamma = 1$ for configuration 23;
- $\alpha = 1, \beta = 1$ and $\gamma = 1$ for configuration 24;
- $\alpha = 0, \beta = 0$ and $\gamma = 0$ for configuration 25;
- $\alpha = 1, \beta = 1$ and $\gamma = 0$ for configuration 26;
- $\alpha = 0, \beta = 1$ and $\gamma = 0$ for configuration 27;
- $\alpha = 1, \beta = 0$ and $\gamma = 0$ for configuration 27*;
- $\alpha = 0, \beta = 0$ and $\gamma = 0$ for configuration 28.

This in combination with Lemmas 6.3 and 6.5 imply that for each of the configurations 23, 24, 25, 26, 27, 27* and 28, all triple intersection numbers can be expressed in terms of s, t and the parameter δ . For extremal generalized hexagons, one can do better.

Lemma 6.6. *Suppose $\mathcal{S}$ is an extremal generalized hexagon. Then the following hold:*

- if $\gamma = 0$, then $\delta = (s-1)^2(s+1) + (s-1)(\alpha + \beta)$;
- if $\gamma = 1$, then $\delta = s^3 - s + (s-1)(\alpha + \beta)$.

Proof. Put $Z_1 := L \cap \Gamma_3(x) \cap \Gamma_3(y)$, $Z_2 := L \setminus Z_1$ and let (i, j, k) be one of $(2, 2, 1)$, $(1, 2, 2)$. In the sum $\sum_{z \in L} |\Gamma_i(x) \cap \Gamma_j(y) \cap \Gamma_k(z)|$, the points of $\Gamma_i(x) \cap \Gamma_j(y) \cap \Gamma_k(L)$ are counted only once, while those of $\Gamma_i(x) \cap \Gamma_j(y) \cap \Gamma_{k-1}(L)$ are counted s times. So, we have

$$\begin{aligned} P_{221} &= -s \cdot P_{220} + \sum_{z \in Z_1} |\Gamma_2(x) \cap \Gamma_2(y) \cap \Gamma_1(z)| + \sum_{z \in Z_2} |\Gamma_2(x) \cap \Gamma_2(y) \cap \Gamma_1(z)|, \\ P_{122} &= -s \cdot P_{121} + \sum_{z \in Z_1} |\Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_2(z)| + \sum_{z \in Z_2} |\Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_2(z)|. \end{aligned}$$

In the right hands sides, the second terms coincide by Property (P3). Taking into account that $P_{122} = s^3 + 1 - \alpha$, $P_{220} = \gamma$ and $P_{121} = \alpha$, we thus find:

$$\delta = P_{221} = s^3 + 1 + (s-1)\alpha - s\gamma + \Omega_1 - \Omega_2,$$

where $\Omega_1 = \sum_{z \in Z_2} \Omega_1(z)$ and $\Omega_2 = \sum_{z \in Z_2} \Omega_2(z)$, with $\Omega_1(z) = |\Gamma_2(x) \cap \Gamma_2(y) \cap \Gamma_1(z)|$ and $\Omega_2(z) = |\Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_2(z)|$ for every $z \in Z_2$.

Suppose now that $\gamma = 0$. Then $Z_2 = \{\pi_L(x), \pi_L(y)\}$.

If $z = \pi_L(x)$, then every point of $\Gamma_2(x) \cap \Gamma_1(z)$ belongs to $L(z, x * x) \setminus \{z, x * z\}$ and as $d(z, y) = 3$, we then see that $|\Gamma_2(x) \cap \Gamma_2(y) \cap \Gamma_1(z)|$ equals 0 or 1 depending on whether $x * z$ belongs to $\Gamma_2(y)$ or not. Hence, $\Omega_1(z) = |\Gamma_2(x) \cap \Gamma_2(y) \cap \Gamma_1(z)| = 1 - \alpha$. Similarly, as $\Gamma_1(x) \cap \Gamma_2(z) = L(x, x * z) \setminus \{x, x * z\}$, $|\Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_2(z)|$ equals 0 or 1 depending on whether $x * z$ belongs to $\Gamma_2(y)$ or not. Again, we have $\Omega_2(z) = 1 - \alpha$.

Suppose $z = \pi_L(y)$. The computation of $\Omega_1(z)$ is similar to the computation of $\Omega_1(z)$ above (just interchange the roles of x and y). We thus see that $\Omega_1(z) = 1 - \beta$. The triple (y, z, x) now determines a configuration of type 14 or 15 depending on whether β is 1 or 0, implying that $\Omega_2(z) = N_{221}(y, z, x) = (s^2 + s + 1) - s\beta$ by properties (P1) and (P2).

In the case $\gamma = 0$, we thus find that $\delta = (s-1)^2(s+1) + (s-1)(\alpha + \beta)$.

Suppose $\gamma = 1$. Then $Z_2 = \{\pi_L(x)\} = \{\pi_L(y)\}$ and configuration m occurs with $m \in \{23, 24\}$. Put $z := \pi_L(x) = \pi_L(y)$. We have $\Gamma_2(x) \cap \Gamma_1(z) = L(z, z * z) \setminus \{z, x * z\}$. If $m = 23$, then all these points lie at distance 3 from y , showing that $\Omega_1 = 0$. If $m = 24$, then all these points, with exception of $y * z$ lie at distance 2 from y , showing that $\Omega_1 = s - 2$. We also have $\Gamma_1(x) \cap \Gamma_2(z) = L(x, x * z) \setminus \{x, x * z\}$. If $m = 23$, then precisely one of these points lies at distance 2 from y , showing that $\Omega_2 = 1$. If $m = 24$, then none of these points lies at distance 2 from y , showing that $\Omega_2 = 0$.

If $m = 23$, then $\alpha = \beta = 0$ and one computes that $\delta = s^3 - s$. If $m = 24$, then $\alpha = \beta = 1$ and one computes that $\delta = s^3 + s - 2$. In any case, we have $\delta = s^3 - s + (s-1)(\alpha + \beta)$. $\square$

Combining all results of this section, we thus have:

Corollary 6.7. *If $\mathcal{S}$ is extremal, then the number $|\Gamma_i(x) \cap \Gamma_j(y) \cap \Gamma_k(L)|$ only depends on s, i, j, k and the configuration determined by x, y and L .*

Note that since configurations 25 and 28 have the same values for α , β and γ , they have the same intersection numbers (in the extremal case).

7 Triple intersection numbers of type PPP revisited

In this section, we prove an additional relationship between the triple intersection numbers involving three mutually opposite points. Its proof relies on Lemma 6.6.

Proposition 7.1. *Let a , b and c be three mutually opposite points in an extremal generalized hexagon $\mathcal{S}$. Then $N_{222} = s^6 - s^5 - s^4 + 2s^3 - s^2 - s + 1 + (s^3 - s^2 - 2s + 3)N$, where $N = N_{122} = N_{212} = N_{221}$.*

Proof. Let $\mathcal{L}$ denote the set of lines through c and let $\mathcal{L}_1$ denote the set of lines through c containing a point of $\Gamma_2(a) \cap \Gamma_2(b)$. Put $\mathcal{L}_2 = \mathcal{L} \setminus \mathcal{L}_1$. Recall that $|\mathcal{L}_1| = N$ and $|\mathcal{L}_2| = t + 1 - N = s^3 + 1 - N$. For every $L \in \mathcal{L}$, we put $\alpha_L := P_{121}(a, b, L)$ and $\beta_L := P_{211}(a, b, L)$. As any two points at distance 2 have a unique neighbour, we have

$$\sum_{L \in \mathcal{L}} \alpha_L = |\Gamma_1(a) \cap \Gamma_2(b) \cap \Gamma_2(c)| = N, \quad \sum_{L \in \mathcal{L}} \beta_L = |\Gamma_2(a) \cap \Gamma_1(b) \cap \Gamma_2(c)| = N.$$

In the sum $\sum_{L \in \mathcal{L}} |\Gamma_2(a) \cap \Gamma_2(b) \cap \Gamma_1(L)|$, the points of $\Gamma_2(a) \cap \Gamma_2(b) \cap \Gamma_2(c)$ are counted once (as any two points at distance 2 have a unique neighbour) and the N points of $\Gamma_2(a) \cap \Gamma_2(b) \cap \Gamma_1(c)$ are counted $t = s^3$ times. By relying on Lemma 6.6, we find

$$\begin{aligned} N_{222} &= -s^3 N + \sum_{L \in \mathcal{L}} |\Gamma_2(a) \cap \Gamma_2(b) \cap \Gamma_1(L)| \\ &= -s^3 N + \sum_{L \in \mathcal{L}_1} N_{221}(a, b, L) + \sum_{L \in \mathcal{L}_2} N_{221}(a, b, L) \\ &= -s^3 N + (s^3 - s)N + (s - 1)^2(s + 1)(s^3 + 1 - N) + (s - 1) \cdot \sum_{L \in \mathcal{L}} (\alpha_L + \beta_L) \\ &= -s^3 N + (s^3 - s)N + (s - 1)^2(s + 1)(s^3 + 1 - N) + 2N(s - 1) \\ &= s^6 - s^5 - s^4 + 2s^3 - s^2 - s + 1 - (s^3 - s^2 - 2s + 3)N. \end{aligned} \quad \square$$

8 Final discussion and alternative approach

In a generalized hexagon of order (s, t) with $s, t \geq 2$ all 41 configurations of three points can occur. If $s \geq 3$, then also all 39 configurations of two points and a line can occur. If $s = 2$, then one of them, namely configuration 28, cannot occur for the following reason. If K_x and K_y denote the two opposite lines containing respectively x and y (as in Table 2), then the fact that each point of K_x lies at distance 2 from a unique point of K_y (and conversely) would imply that $d(x, y) = 2$ (if $s = 2$), contrary to our assumption $d(x, y) = 3$. If $(s, t) = (2, 8)$, then the extremal generalized hexagon is unique [8] and computer computations show in this case that the automorphism group has 44 orbits on

the ordered triples of three points and 39 orbits on the ordered triples (x, y, L) , where x, y are two points and L is a line. The 44 orbits on the ordered triples of three points correspond to the 41 mentioned configurations where configuration 16 comprises four orbits for which the respective values for $N = N_{221}$ are equal to 1, 3, 5 and 9. With regard to the 39 orbits of ordered triples of two points and a line, we know that configuration 28 cannot occur, but there are also two orbits corresponding to configuration 24.

Above, we showed with the exception of one configuration that all triple intersection numbers in an extremal generalized hexagon of order (s, s^3) , $s \neq 1$, only depend on s for each of the considered configurations of type PPP. In the exceptional case, where three mutually opposite points are involved, we showed that the triple intersection numbers depend on s and one extra parameter. Not only did we prove that, but the various lemmas and corollaries we proved yield explicit algorithms for determining these (constant) triple intersection numbers.

The results of this paper show that an extremal generalized hexagon satisfies many combinatorial properties that need not be satisfied by an arbitrary finite generalized hexagon of order (s, t) . As already mentioned, we hope that this variety of combinatorial identities can be useful in the study and characterization of certain extremal generalized hexagons. We note that currently one infinite family of extremal generalized hexagons is known [18], and that the problem whether this family contains all extremal generalized hexagons is still open as of today.

It is possible to verify several of our results on triple intersection numbers by means of results from [17]. The Haemers-Roos inequality for generalized hexagons $\mathcal{S}$ of order (s, t) , $s \neq 1$, is equivalent to the Krein inequality $q_{33}^2 \geq 0$ for the distance-regular collinearity graph of $\mathcal{S}$. If $\mathcal{S}$ is extremal, then $q_{33}^2 = 0$. By [17] (see also [6]) we know that if a certain Krein parameter of a distance-regular graph is 0, then certain numbers $q(a, b, c)$ are 0 for all choices of the vertices a, b and c . For the Krein parameter q_{33}^2 in the collinearity graph Γ of a generalized hexagon $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ of order (s, t) , it can be verified that $q(a, b, c)$ is equal to

$$\frac{f_3^2 f_2}{v^3} \sum_{x \in \mathcal{P}} \left(-\frac{1}{s} \right)^{d(a, x) + d(b, x)} \cdot f(d(c, x)) = \frac{f_3^2 f_2}{v^3} \sum_{i, j, k=0}^3 |\Gamma_i(a) \cap \Gamma_j(b) \cap \Gamma_k(c)| \cdot \left(-\frac{1}{s} \right)^{i+j} \cdot f(k).$$

Here, $v = |\mathcal{P}|$ is the total number of points and f_2, f_3 are the multiplicities of the respective eigenvalues $\lambda_2 = s - 1 - \sqrt{st}$ and $\lambda_3 = -(t + 1)$ of Γ . The map f is defined as follows:

$$f(0) = 1, \quad f(1) = \frac{s - 1 - \sqrt{st}}{s(t + 1)}, \quad f(2) = \frac{-\sqrt{st}(s - 1) - s}{s^2 t(t + 1)}, \quad f(3) = \frac{\sqrt{st}}{s^2 t^2}.$$

Expressing that $q(a, b, c)$ equals 0 leads to certain conditions for each of the configurations. As $q(a, b, c)$ is symmetric in a and b , we can restrict our discussion to configurations 1 to 28 as all other configurations are obtained from one of these by interchanging the points a and b . Assuming that $t = s^3$, it can be shown after lengthy computations (which we performed with the aid of a Computer Algebra System) that the condition $q(a, b, c) = 0$ always holds, with exception of the configurations mentioned in the following table where for each case we also mention the condition that arises.

Configuration	Condition
10	$N_{222} = s^6 + s^4 - 2s$
11	$N_{222} = s^6 + s^5 - s^4 + s^3 - s^2 + s - 2$
12 and 25	$N_{222} = s^6$
13 and 26	$N_{222} = s^6 - s^4 + s^3 + s - 2$
14	$(s^3 - s^2 + 1)N_{221} + N_{222} = s^6 - s^4 + 2s^3 - s^2 + s$
15	$(s^3 - s^2 + 1)N_{221} + N_{222} = s^6 - s^4 + 2s^3 + s - 1$
16	$N_{222} + (s^3 - s^2 + 1)N_{221} - (s - 1)(N_{122} + N_{212}) = s^6 - s^5 - s^4 + 2s^3 - s^2 - s + 1$
27	$N_{222} - (s - 1)N_{212} = s^6 - s^5$
28	$N_{222} - (s - 1)N_{212} = s^6 - s^5 - s^4 + s^3 - 1$

By permuting points in certain configurations and using the information of the table, it is now possible to compute the value N_{222} for configurations 14, 15, 27 and 28, the value N_{221} for configurations 14 and 15, the value N_{212} for configurations 27 and 28, and to find the connection between the triple intersection numbers mentioned in Proposition 7.1. We illustrate this by means of one example, but the treatments of the other cases are similar.

As mentioned in Section 3, configuration 27 is obtained from configuration 14 by applying the permutation (cba) to its points. Condition $N_{222} - (s - 1)N_{212} = s^6 - s^5$ for configuration 27 thus implies condition $N_{222} - (s - 1)N_{221} = s^6 - s^5$ for configuration 14. Together with the condition $(s^3 - s^2 + 1)N_{221} + N_{222} = s^6 - s^4 + 2s^3 - s^2 + s$, this then implies that $N_{221} = s^2 + 1$ and $N_{222} = s^6 - s^5 + s^3 - s^2 + s - 1$ for configuration 14. Recall that we already obtained this information in Corollary 4.24.

Repeating such an argument for other configurations would then provide alternative proofs for Proposition 7.1 and the results obtained at the end of Section 4 (Lemma 4.17 to Corollary 4.24). It must however be mentioned that this method involves quite some computations (which we only performed by means of a CAS) and would not qualify as a proof based on counting. Indeed, the proof in [17] is very algebraic combinatorial in nature which among other things relies on the minimal idempotents of the Bose-Mesner algebra of the distance-regular collinearity graph.

Remark. In finite generalized quadrangles one can also consider different configurations for three points and different configurations for two points and a line. These correspond to configurations 1, 2, 3, 5, 6, 7, 10/11 in Table 1 and configurations 1, 2, 4, 5, 6, 10, 12 and 20 in Table 2, where configuration 20 splits in two subcases corresponding to $d(x, y) = 1$ or $d(x, y) = 2$. In an arbitrary finite generalized quadrangle of order (s, t) , $s, t \geq 2$, the triple intersection numbers $|\Gamma_i(x) \cap \Gamma_j(y) \cap \Gamma_k(L)|$ only depend on s, t, i, j, k and the configuration (of x, y and L). A similar remark holds for triple intersection numbers involving three points, with exception of the case where the three points a, b, c are mutually opposite. For that last configuration, we know from the results of Bose-Shrikhande [5] and Cameron [7] that the 27 triple intersection numbers $|\Gamma_i(a) \cap \Gamma_j(b) \cap \Gamma_k(c)|$ only depend on s and t if and only if the generalized quadrangle is extremal.

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