Blocking sets of external, tangent and secant lines to a quadratic cone in $\text{PG}(3, q)$

Bart De Bruyn Puspendu Pradhan Bikramaditya Sahu

October 5, 2020

Abstract

Consider a quadratic cone $\mathcal{K}$ in the 3-dimensional projective space $\text{PG}(3, q)$ over a finite field of order q , where q is a prime power. Let $\mathcal{E}$ (respectively, $\mathcal{T}$, $\mathcal{S}$) denote the set of all lines of $\text{PG}(3, q)$ that are external (respectively, tangent, secant) with respect to $\mathcal{K}$. We characterize the minimum size blocking sets in $\text{PG}(3, q)$ with respect to the line set $\mathcal{A}$, where $\mathcal{A}$ is one of $\mathcal{E}$, $\mathcal{T}$, $\mathcal{S}$, $\mathcal{E} \cup \mathcal{T}$, $\mathcal{E} \cup \mathcal{S}$ and $\mathcal{T} \cup \mathcal{S}$.

Keywords: Projective space, Blocking set, Irreducible conic, Quadratic cone

AMS 2020 subject classification: 51E21, 05B25

1 Introduction

1.1 Blocking sets

Let $\text{PG}(n, q)$ be the n -dimensional projective space over a finite field of order q , where q is a prime power. For a given nonempty set $\mathcal{A}$ of lines of $\text{PG}(n, q)$, a set X of points of $\text{PG}(n, q)$ is called an $\mathcal{A}$ -*blocking set* if each line in $\mathcal{A}$ contains at least one point of X . Blocking sets in $\text{PG}(n, q)$ with respect to various line sets have been studied by several authors. The first step in this regard has been to determine the minimum size of a blocking set with respect to a given line set and if possible, to give geometric descriptions of all blocking sets of that minimum cardinality.

When $\mathcal{A}$ is the set of all lines of $\text{PG}(n, q)$, the following fundamental result was proved by Bose and Burton in [8, Theorem 1].

Proposition 1.1 ([8]). *Let $\mathcal{A}$ be the line set of $\text{PG}(n, q)$ and let X be an $\mathcal{A}$ -blocking set in $\text{PG}(n, q)$. Then $|X| \geq (q^n - 1)/(q - 1)$, and equality holds if and only if X is a hyperplane of $\text{PG}(n, q)$.*

1.2 Quadratic cones in $\text{PG}(3, q)$

Let $\mathcal{K}$ be a quadratic cone in $\text{PG}(3, q)$ with kernel the point p^* whose base is an irreducible conic $\mathcal{C}^*$ in a plane π^* not containing p^* . We refer to [15] for the basic properties of points,

lines and planes of $\text{PG}(3, q)$ with respect to $\mathcal{K}$. The cone $\mathcal{K}$ consists of the $q^2 + q + 1$ points of $\text{PG}(3, q)$ that are contained in the $q + 1$ lines through p^* meeting $\mathcal{C}^*$. Every line of $\text{PG}(3, q)$ intersects $\mathcal{K}$ in either 0, 1, 2 or $q + 1$ points. A line of $\text{PG}(3, q)$ is called a *tangent line* if it intersects $\mathcal{K}$ in either 1 or $q + 1$ points. A line is called an *external* or *secant line* depending on whether it intersects $\mathcal{K}$ in 0 or 2 points. The set of tangent, external and secant lines will respectively be denoted by $\mathcal{T}$, $\mathcal{E}$ and $\mathcal{S}$. We have $|\mathcal{S}| = q^3(q + 1)/2$, $|\mathcal{T}| = q^3 + 2q^2 + q + 1$ and $|\mathcal{E}| = q^3(q - 1)/2$.

A plane π of $\text{PG}(3, q)$ intersects $\mathcal{K}$ either in the point p^* , in a line (through p^*), in two lines (intersecting at p^*), or in an irreducible conic. If $\pi \cap \mathcal{K}$ is an irreducible conic, then we call π a *secant plane*. The planes of $\text{PG}(3, q)$ not passing through p^* are precisely the secant planes, giving in total q^3 secant planes of $\text{PG}(3, q)$. There are q^2 secant planes through a point of $\text{PG}(3, q) \setminus \{p^*\}$, and q secant planes through a line not containing p^* .

1.3 Main result

Let $\mathcal{A}$ be one of the following sets of lines of $\text{PG}(3, q)$: $\mathcal{E}$, $\mathcal{T}$, $\mathcal{S}$, $\mathcal{E} \cup \mathcal{T}$, $\mathcal{E} \cup \mathcal{S}$, $\mathcal{T} \cup \mathcal{S}$. With $\mathcal{A}$, we associate two parameters $N_{\mathcal{A}}$ and $\epsilon_{\mathcal{A}}$ as given in the following table.

$\mathcal{A}$	$N_{\mathcal{A}}$	$\epsilon_{\mathcal{A}}$
$\mathcal{E}$	$q - 1$	0
$\mathcal{T}$	1 if q is even, and $\frac{q+1}{2}$ if q is odd	1
$\mathcal{S}$	q	0
$\mathcal{E} \cup \mathcal{T}$	q	1
$\mathcal{E} \cup \mathcal{S}$	q if q is even, 3 if $q = 3$ and $q + 1$ if $q > 3$ is odd	0
$\mathcal{T} \cup \mathcal{S}$	$q + 1$	1

Note that $N_{\mathcal{A}} \leq q + 1$ and $\epsilon_{\mathcal{A}} = 1$ if and only if $\mathcal{T} \subseteq \mathcal{A}$. Denote by $\mathcal{A}'$ the set of lines of $\mathcal{A}$ that are contained in π^* . As we will see in Lemma 2.1, the number $N_{\mathcal{A}}$ equals the smallest size of an $\mathcal{A}'$ -blocking set in π^* . We prove the following result which gives a uniform characterization of the minimum size $\mathcal{A}$ -blocking sets in $\text{PG}(3, q)$.

Theorem 1.2. *Let B be a minimum size $\mathcal{A}$ -blocking set in $\text{PG}(3, q)$, where $\mathcal{A}$ is one of the line sets $\mathcal{E}$, $\mathcal{T}$, $\mathcal{S}$, $\mathcal{E} \cup \mathcal{T}$, $\mathcal{E} \cup \mathcal{S}$ and $\mathcal{T} \cup \mathcal{S}$. Then $|B| = qN_{\mathcal{A}} + \epsilon_{\mathcal{A}}$ and the following hold:*

- (i) *If $\mathcal{A} \in \{\mathcal{E}, \mathcal{S}, \mathcal{E} \cup \mathcal{S}\}$, then $B = \mathcal{K}' \setminus \{p^*\}$, where $\mathcal{K}'$ is a cone with kernel p^* and base an $\mathcal{A}'$ -blocking set of size $N_{\mathcal{A}}$ in π^* .*
- (ii) *If $\mathcal{A} \in \{\mathcal{T}, \mathcal{E} \cup \mathcal{T}\}$, then $B = \mathcal{K}'$, where $\mathcal{K}'$ is a cone with kernel p^* and base an $\mathcal{A}'$ -blocking set of size $N_{\mathcal{A}}$ in π^* .*
- (iii) *Suppose that $\mathcal{A} = \mathcal{T} \cup \mathcal{S}$.*
 - (a) *If $p^* \notin B$, then B is a secant plane.*

- (b) If $p^* \in B$, then $B = \mathcal{K}'$, where $\mathcal{K}'$ is a cone with kernel p^* and base an $\mathcal{A}'$ -blocking set of size $N_{\mathcal{A}}$ in π^* .

Theorem 1.2 fills a gap in the existing literature on blocking sets of $\text{PG}(3, q)$ with respect to the various line sets associated with a given quadric. Indeed, the minimum size blocking sets in $\text{PG}(3, q)$ with respect to such line sets have already been characterized in the papers [5, 6, 12, 13, 18, 19, 20] for a hyperbolic quadric, and in [6, 10, 11, 20] for an elliptic quadric.

2 Preliminaries

Let $\mathcal{C}$ be an irreducible conic in $\text{PG}(2, q)$. We refer to [16] for the basic properties of points and lines of $\text{PG}(2, q)$ with respect to $\mathcal{C}$. The conic $\mathcal{C}$ consists of $q + 1$ points meeting every line of $\text{PG}(2, q)$ in at most two points. A line of $\text{PG}(2, q)$ is called *external* (respectively, *tangent*, *secant*) with respect to $\mathcal{C}$ if it meets $\mathcal{C}$ in 0 (respectively, 1, 2) points. Every point of $\mathcal{C}$ is contained in a unique tangent line, this gives exactly $q + 1$ tangent lines to $\mathcal{C}$. There are $q(q + 1)/2$ secant lines and so $q(q - 1)/2$ external lines to $\mathcal{C}$.

We note that, for a secant plane π of $\text{PG}(3, q)$ with respect to the cone $\mathcal{K}$, the set of external (respectively, tangent, secant) lines with respect to $\mathcal{K}$ in $\text{PG}(3, q)$ contained in π are precisely the set of external (respectively, tangent, secant) lines with respect to the irreducible conic $\pi \cap \mathcal{K}$ in π .

Let $\mathcal{A}$ be one of the line sets $\mathcal{E}$, $\mathcal{T}$, $\mathcal{S}$, $\mathcal{E} \cup \mathcal{T}$, $\mathcal{E} \cup \mathcal{S}$, $\mathcal{T} \cup \mathcal{S}$ in $\text{PG}(3, q)$. Recall that $\mathcal{A}'$ denotes the set of lines of $\mathcal{A}$ that are contained in π^* . Note that $\mathcal{A}'$ consists of all the lines of π^* of the same type (as that of $\mathcal{A}$) with respect to the irreducible conic $\mathcal{C}^*$.

Lemma 2.1. *Every $\mathcal{A}'$ -blocking set in π^* has size at least $N_{\mathcal{A}}$, and there exists an $\mathcal{A}'$ -blocking set in π^* of size $N_{\mathcal{A}}$.*

Proof. In the projective plane $\text{PG}(2, q)$, blocking sets of minimum size with respect to various line sets (consisting of external/tangent/secant lines) determined by a given irreducible conic have been studied in the papers [1, 2, 3, 4, 7, 9, 12, 14, 17, 21]. The present lemma is a consequence of the results obtained in these papers. In particular, for $\mathcal{A} = \mathcal{E}$, the lemma is implied by [14, Theorems 1.1 and 1.2] for q even, by [3, Theorem 1.1] for $q > 3$ odd and by [12, Theorem 2.1(iii)] for $q = 3$. For $\mathcal{A} = \mathcal{T}$, the lemma is implied by Section 3.1 of [17]. For $\mathcal{A} = \mathcal{S}$, the lemma is implied by [4, Theorem 1.1] for q even and by [1, Theorem 1.1] for q odd. For $\mathcal{A} = \mathcal{E} \cup \mathcal{T}$, the lemma is implied by [1, Theorem 1.2] for q even and by [2, Theorem 1.1] for q odd. For $\mathcal{A} = \mathcal{E} \cup \mathcal{S}$, the lemma is implied by [17, Theorem 3.2] for q even and by [17, Theorem 3.3] for q odd. Finally, for $\mathcal{A} = \mathcal{T} \cup \mathcal{S}$, the lemma is implied by Section 2.3 of [17] (and more information about $(\mathcal{T} \cup \mathcal{S})'$ -blocking sets of minimum size can be found in [7, 9, 21]). $\square$

Lemma 2.2. *Let X be an $\mathcal{A}'$ -blocking set of size $N_{\mathcal{A}}$ in π^* . Let $\mathcal{K}'$ be the cone in $\text{PG}(3, q)$ with kernel p^* and base $X \subseteq \pi^*$. We put $B' = \mathcal{K}' \setminus \{p^*\}$ if $\mathcal{A} \in \{\mathcal{E}, \mathcal{S}, \mathcal{E} \cup \mathcal{S}\}$ and $B' = \mathcal{K}'$ if $\mathcal{A} \in \{\mathcal{T}, \mathcal{E} \cup \mathcal{T}, \mathcal{T} \cup \mathcal{S}\}$. Then B' is an $\mathcal{A}$ -blocking set in $\text{PG}(3, q)$ of size $qN_{\mathcal{A}} + \epsilon_{\mathcal{A}}$.*

Proof. By construction, B' has size $qN_{\mathcal{A}} + \epsilon_{\mathcal{A}}$. We need to prove that B' is an $\mathcal{A}$ -blocking set in $\text{PG}(3, q)$.

Let L be a line of $\mathcal{A}$. If $p^* \in L$, then we necessarily have $\mathcal{A} \in \{\mathcal{T}, \mathcal{E} \cup \mathcal{T}, \mathcal{T} \cup \mathcal{S}\}$, and in this case L contains the point p^* of B' . So, we may suppose that L does not contain p^* . The plane $\sigma := \langle p^*, L \rangle$ can have three possibilities depending upon L . If L is a tangent line meeting $\mathcal{K}$ in one point, then σ contains a line of $\mathcal{K}$. If L is a secant line, then σ contains two lines of $\mathcal{K}$ through p^* . If L is an external line, then σ contains only the point p^* of $\mathcal{K}$. In each of these three cases, it follows that σ intersects π^* in a line L' that belongs to $\mathcal{A}'$. Since X is an $\mathcal{A}'$ -blocking set in π^* , L' contains a point x belonging to X . Then the unique point of L on the line p^*x belongs to B' . We conclude that B' is an $\mathcal{A}$ -blocking set in $\text{PG}(3, q)$. $\square$

3 Proof of Theorem 1.2

In the rest of the paper, we follow the notation used in the previous sections. Suppose B is an $\mathcal{A}$ -blocking set of minimum possible size. By Lemmas 2.1 and 2.2, we then know that

$$|B| \leq qN_{\mathcal{A}} + \epsilon_{\mathcal{A}}. \quad (1)$$

Lemma 3.1. *The following hold:*

- (a) *If $\mathcal{A} \in \{\mathcal{E}, \mathcal{S}, \mathcal{E} \cup \mathcal{S}\}$, then $p^* \notin B$.*
- (b) *If $\mathcal{A} \in \{\mathcal{T}, \mathcal{E} \cup \mathcal{T}, \mathcal{T} \cup \mathcal{S}\}$ and $p^* \notin B$, then $N_{\mathcal{A}} = q + 1$, $\epsilon_{\mathcal{A}} = 1$ and $|B| = q^2 + q + 1$.*

Proof. (a) If $\mathcal{A} \in \{\mathcal{E}, \mathcal{S}, \mathcal{E} \cup \mathcal{S}\}$, then $B \setminus \{p^*\}$ is also an $\mathcal{A}$ -blocking set in $\text{PG}(3, q)$. By the minimality of $|B|$, we then have $B \setminus \{p^*\} = B$, i.e. $p^* \notin B$.

(b) If $\mathcal{A} \in \{\mathcal{T}, \mathcal{E} \cup \mathcal{T}, \mathcal{T} \cup \mathcal{S}\}$ and $p^* \notin B$, then each of the $q^2 + q + 1$ (tangent) lines through p^* contains a point of B implying that $|B| \geq q^2 + q + 1$. On the other hand, (1) implies that $|B| \leq qN_{\mathcal{A}} + \epsilon_{\mathcal{A}} \leq q(q + 1) + 1 = q^2 + q + 1$. So, $|B| = q^2 + q + 1$, $N_{\mathcal{A}} = q + 1$ and $\epsilon_{\mathcal{A}} = 1$. $\square$

Lemma 3.2. *Suppose $p^* \in B$ if $\mathcal{A} \in \{\mathcal{T}, \mathcal{E} \cup \mathcal{T}, \mathcal{T} \cup \mathcal{S}\}$. Then $|B| = qN_{\mathcal{A}} + \epsilon_{\mathcal{A}}$ and every secant plane meets B in exactly $N_{\mathcal{A}}$ points.*

Proof. By Lemma 3.1(a), we have $p^* \notin B$ if $\mathcal{A} \in \{\mathcal{E}, \mathcal{S}, \mathcal{E} \cup \mathcal{S}\}$. So,

$$|B \setminus \{p^*\}| = |B| - \epsilon_{\mathcal{A}} \leq qN_{\mathcal{A}} \quad (2)$$

by (1) and the definition of $\epsilon_{\mathcal{A}}$. We count the number N of pairs (x, π) , where x is a point of $B \setminus \{p^*\}$ and π is a secant plane through x . As each point of $B \setminus \{p^*\}$ is contained in precisely q^2 secant planes, we have

$$N = |B \setminus \{p^*\}| \cdot q^2 \leq q^3 N_{\mathcal{A}} \quad (3)$$

using (2). On the other hand, if π is one of the q^3 secant planes, then π contains at least $N_{\mathcal{A}}$ points of $B \setminus \{p^*\}$ by Lemma 2.1, implying that

$$q^3 N_{\mathcal{A}} \leq N. \quad (4)$$

By (2), (3) and (4), we know that $N = q^3 N_{\mathcal{A}}$ and $|B| = qN_{\mathcal{A}} + \epsilon_{\mathcal{A}}$. Moreover, the reasoning leading to (4) then reveals that every secant plane contains exactly $N_{\mathcal{A}}$ points of B . $\square$

Lemma 3.3. *In any case, we have $|B| = qN_{\mathcal{A}} + \epsilon_{\mathcal{A}}$.*

Proof. By Lemma 3.2, we can assume that $\mathcal{A} \in \{\mathcal{T}, \mathcal{E} \cup \mathcal{T}, \mathcal{T} \cup \mathcal{S}\}$ and $p^* \notin B$. By Lemma 3.1(b), we then know that $N_{\mathcal{A}} = q+1$, $\epsilon_{\mathcal{A}} = 1$ and $|B| = q^2 + q + 1$. Hence, $|B| = qN_{\mathcal{A}} + \epsilon_{\mathcal{A}}$ also holds in this case. $\square$

Lemma 3.3 proves the first part of Theorem 1.2 regarding the minimum size of an $\mathcal{A}$ -blocking set in $\text{PG}(3, q)$. We next characterize the $\mathcal{A}$ -blocking sets of minimum size $qN_{\mathcal{A}} + \epsilon_{\mathcal{A}}$ for which following result is needed.

Lemma 3.4. *Suppose $p^* \in B$ if $\mathcal{A} \in \{\mathcal{T}, \mathcal{E} \cup \mathcal{T}, \mathcal{T} \cup \mathcal{S}\}$. If π is a plane through p^* and L is a line of π not containing p^* , then $|L \cap B| = \frac{qN_{\mathcal{A}} + |\pi \cap B| - |B|}{q} = \frac{|\pi \cap B| - \epsilon_{\mathcal{A}}}{q}$.*

Proof. By Lemma 3.2, the q secant planes through L contain in total $|L \cap B| + q(N_{\mathcal{A}} - |L \cap B|) = qN_{\mathcal{A}} - (q-1) \cdot |L \cap B|$ points of B . It follows that

$$qN_{\mathcal{A}} - (q-1) \cdot |L \cap B| = |B| - |\pi \cap B| + |L \cap B|.$$

Hence, $|L \cap B| = \frac{qN_{\mathcal{A}} + |\pi \cap B| - |B|}{q}$. By Lemma 3.2, we also know that this number is equal to $\frac{|\pi \cap B| - \epsilon_{\mathcal{A}}}{q}$. $\square$

Lemma 3.5. *Suppose $p^* \in B$ if $\mathcal{A} \in \{\mathcal{T}, \mathcal{E} \cup \mathcal{T}, \mathcal{T} \cup \mathcal{S}\}$. Then B can be obtained as in Lemma 2.2.*

Proof. By Lemma 3.1(a), it suffices to prove that if x is a point not belonging to $B \cup \{p^*\}$, then $p^*x \setminus \{p^*\}$ has no points in common with B . Consider a secant plane π through x . As $|\pi \cap B| = N_{\mathcal{A}} \leq q+1$ by Lemma 3.2, there exists a line L in π through x meeting B in $\eta \in \{0, 1\}$ points. Let π' be the plane $\langle p^*, L \rangle$. By Lemma 3.4, $B \cap \pi'$ is a set of $\eta q + \epsilon_{\mathcal{A}}$ points intersecting each line of π' not containing p^* in exactly η points. If $\eta = 0$, then necessarily $B \cap \pi' \subseteq \{p^*\}$ and the claim is valid. If $\eta = 1$, then $(B \cap \pi') \cup \{p^*\}$ is a set of $q+1$ points meeting each line and so $(B \cap \pi') \cup \{p^*\}$ is a line through p^* by Proposition 1.1. Also the claim is valid in this case. $\square$

Lemma 3.6. *Suppose $\mathcal{A} \in \{\mathcal{T}, \mathcal{E} \cup \mathcal{T}, \mathcal{T} \cup \mathcal{S}\}$ and $p^* \notin B$. Then $\mathcal{A} = \mathcal{T} \cup \mathcal{S}$ and B is a plane not containing p^* .*

Proof. By Lemma 3.1(b), we know that $N_{\mathcal{A}} = q + 1$ and $|B| = q^2 + q + 1$. As $N_{\mathcal{A}} = q + 1$, we necessarily have $\mathcal{A} = \mathcal{T} \cup \mathcal{S}$. If each external line meets B , we necessarily have that B is a plane by Proposition 1.1. So, we may assume that there exists an external line L that is disjoint from B . Each of the q secant planes through L contains at least $N_{\mathcal{A}} = q + 1$ points of B . Also, the plane $\sigma := \langle p^*, L \rangle$ (where $\sigma \cap \mathcal{K} = \{p^*\}$) through L must contain at least $q + 1$ points of B , namely at least one on each of the $q + 1$ tangent lines of σ through p^* . So, $|B| \geq (q + 1)^2 = q^2 + 2q + 1$, a contradiction. $\square$

Remark. A classification of all $\mathcal{E}$ -blocking sets of minimum size was already obtained in Theorem 4.3 of [6] in the case that $q \geq 9$. However, that theorem has counter examples in the case that q is even. Indeed, by [14, Theorem 1.2], we know that there are three types of minimum size blocking sets with respect to the external lines of an irreducible conic in $\text{PG}(2, q)$, q even. Via the construction mentioned in Lemma 2.2, we then obtain three types of $\mathcal{E}$ -blocking sets of minimum size $q^2 - q$. Theorem 4.3 of [6] only mentions one of these possibilities. The problem seems to originate from errors in the proof of [6, Proposition 4.2].

Acknowledgement

The second author is supported by a research fellowship (File No.: 09/1002(0040)/2017-EMR-I) of the Council of Scientific and Industrial Research (CSIR), Ministry of Human Resource Development, Government of India.

References

- [1] A. Aguglia and M. Giulietti. Blocking sets of certain line sets related to a conic. *Des. Codes Cryptogr.* 39 (2006), 397–405.
- [2] A. Aguglia and G. Korchmáros. Blocking sets of nonsecant lines to a conic in $\text{PG}(2, q)$, q odd. *J. Combin. Des.* 13 (2005), 292–301.
- [3] A. Aguglia and G. Korchmáros. Blocking sets of external lines to a conic in $\text{PG}(2, q)$, q odd. *Combinatorica* 26 (2006), 379–394.
- [4] A. Aguglia, G. Korchmáros and A. Siciliano. Minimal covering of all chords of a conic in $\text{PG}(2, q)$, q even. *Bull. Belg. Math. Soc. Simon Stevin* 12 (2005), 651–655.
- [5] P. Biondi and P. M. Lo Re. On blocking sets of external lines to a hyperbolic quadric in $\text{PG}(3, q)$, q even. *J. Geom.* 92 (2009), 23–27.
- [6] P. Biondi, P. M. Lo Re and L. Storme. On minimum size blocking sets of external lines to a quadric in $\text{PG}(3, q)$. *Beitr. Algebra Geom.* 48 (2007), 209–215.

- [7] E. Boros, Z. Füredi and J. Kahn. Maximal intersecting families and affine regular polygons in $\text{PG}(2, q)$. *J. Combin. Theory Ser. A* 52 (1989), 1–9.
- [8] R. C. Bose and R. C. Burton. A characterization of flat spaces in a finite geometry and the uniqueness of the Hamming and the MacDonald codes. *J. Combinatorial Theory* 1 (1966), 96–104.
- [9] A. A. Bruen and J. A. Thas. Flocks, chains and configurations in finite geometries. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat.* (8) 59 (1975) (1976), 744–748.
- [10] B. De Bruyn, P. Pradhan and B. K. Sahoo. Minimum size blocking sets of certain line sets with respect to an elliptic quadric in $\text{PG}(3, q)$. *Contrib. Discrete Math.* 15 (2020), no. 2, 132–147.
- [11] B. De Bruyn, P. Pradhan and B. K. Sahoo. Blocking sets of tangent and external lines to an elliptic quadric in $\text{PG}(3, q)$, communicated.
- [12] B. De Bruyn, B. K. Sahoo and B. Sahu. Blocking sets of tangent and external lines to a hyperbolic quadric in $\text{PG}(3, q)$. *Discrete Math.* 341 (2018), 2820–2826.
- [13] B. De Bruyn, B. K. Sahoo and B. Sahu. Blocking sets of tangent lines to a hyperbolic quadric in $\text{PG}(3, 3)$. *Discrete Appl. Math.* 266 (2019), 121–129.
- [14] M. Giulietti. Blocking sets of external lines to a conic in $\text{PG}(2, q)$, q even. *European J. Combin.* 28 (2007), 36–42.
- [15] J. W. P. Hirschfeld. *Finite projective spaces of three dimensions*. Oxford Mathematical Monographs, Oxford University Press, 1985.
- [16] G. E. Moorhouse, Incidence Geometry, 2017, available online at http://ericmoorhouse.org/handouts/Incidence_Geometry.pdf
- [17] K. L. Patra, B. K. Sahoo and B. Sahu. Minimum size blocking sets of certain line sets related to a conic in $\text{PG}(2, q)$. *Discrete Math.* 339 (2016), 1716–1721.
- [18] B. K. Sahoo and B. Sahu. Blocking sets of tangent and external lines to a hyperbolic quadric in $\text{PG}(3, q)$, q even. *Proc. Indian Acad. Sci. Math. Sci.* 129 (2019), Paper No. 4, 14 pp.
- [19] B. K. Sahoo and B. Sahu. Blocking sets of certain line sets related to a hyperbolic quadric in $\text{PG}(3, q)$. *Adv. Geom.* 19 (2019), 477–486.
- [20] B. K. Sahoo and N. S. N. Sastry. Binary codes of the symplectic generalized quadrangle of even order. *Des. Codes Cryptogr.* 79 (2016), 163–170.
- [21] B. Segre and G. Korchmáros. Una proprietà degli insiemi di punti di un piano di Galois caratterizzante quelli formati dai punti delle singole rette esterne ad una conica. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat.* (8) 62 (1977), 613–619.

Addresses:

Bart De Bruyn (E-mail: Bart.DeBruyn@Ugent.be)

Department of Mathematics: Algebra and Geometry, Ghent University, Krijgslaan
281 (S25), B-9000 Gent, Belgium

Puspendu Pradhan (Email: puspendu.pradhan@niser.ac.in)

- 1) School of Mathematical Sciences, National Institute of Science Education and Research (NISER) Bhubaneswar, P.O.- Jatni, District- Khurda, Odisha - 752050, India.
- 2) Homi Bhabha National Institute (HBNI), Training School Complex, Anushakti Nagar, Mumbai - 400094, India.

Bikramaditya Sahu (Email: sahuba@nitrkl.ac.in)

Department of Mathematics, National Institute of Technology Rourkela, Odisha -
769008, India.