A full discretization for the saddle-point approach of a degenerate parabolic problem involving a moving body

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Abstract

This paper aims to study a degenerate parabolic problem for a solenoidal vector field in which the time derivative acts on a moving body. We propose a fully-discrete finite element scheme combined with backward Euler's method for the saddle-point variational formulation. The convergence of this numerical scheme is proved and error estimates for some stable finite element pairs are also established.

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1. Introduction

Let Ω be an open bounded connected polyhedral and Lipschitz domain in $\mathbb{R}^d$, with $d = 2$ or 3. Inside the domain Ω, a body $\Sigma_0 \in \mathrm{C}^{2,1}$ is considered, which occupies different regions during the time interval $[0, T]$. We refer to Σ_0 as the reference configuration and describe the motion of Σ_0 by a C^3-function

$$\mathbf{\Phi} : \Sigma_0 \times [0, T] \to \mathbb{R}^d,$$

where $\mathbf{\Phi}_t := \mathbf{\Phi}(\cdot, t)$ for each $t \in [0, T]$ is a deformation of Σ_0 to $\Sigma(t) := \mathbf{\Phi}(\Sigma_0, t)$, which is the space occupied by Σ_0 at time t, cf. [1]. We make the following assumptions throughout the article

$$\widetilde{\Sigma} := \bigcup_{t\in[0,T]} \overline{\Sigma(t)} \subset \Omega; \qquad \det \nabla\mathbf{\Phi}(\boldsymbol{x}, t) > 0, \quad \forall(\boldsymbol{x}, t) \in \Sigma_0 \times [0, T]. \tag{1}$$

The trajectory of the motion, which is a subset of the space-time domain $Q := \Omega \times (0, T)$, is specified by

$$\mathbb{T} := \{(\boldsymbol{x}, t) : \boldsymbol{x} \in \Sigma(t),\ t \in [0, T]\}.$$

Since $\mathbf{\Phi}_t$ is a bijective mapping for each t, the velocity vector of the moving body is defined by $\mathbf{v}(\boldsymbol{x}, t) := \dot{\mathbf{\Phi}}(\mathbf{\Phi}_t^{-1}(\boldsymbol{x}), t)$. From now on, we assume that there exists an extension of $\mathbf{v}$ from $\mathbb{T}$ to Q such that $\mathbf{v} \in \mathbf{C}^1(\overline{Q})$. We denote further by $\mathbf{n}$ the unit outward normal vector associated to the boundary of Ω and $\Sigma(t)$.

In this paper, we aim to investigate the following initial-boundary value problem for the solenoidal vector field $\boldsymbol{u}$

$$\begin{cases} \alpha\partial_t\boldsymbol{u} - \beta\Delta\boldsymbol{u} + \chi_\Sigma \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f} & \text{in} \quad \Omega \times (0, T), \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in} \quad \Omega \times (0, T), \\ \boldsymbol{u} = \mathbf{0} & \text{on} \quad \partial\Omega \times (0, T), \\ [\![(\nabla\boldsymbol{u})\mathbf{n}]\!] = \mathbf{0} & \text{on} \quad \partial\Sigma \times (0, T), \\ \boldsymbol{u}(\cdot, 0) = \boldsymbol{u}_0 & \text{in} \quad \Sigma_0, \end{cases} \tag{2}$$

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where $\boldsymbol{u}(\boldsymbol{x},t) = \left[u^i(\boldsymbol{x},t)\right]_{i=1}^d$,$\beta > 0$ is a constant and $\boldsymbol{f} \in \mathrm{Lip}([0,T], \mathbf{L}^2(\Omega))$. Moreover, α and $\boldsymbol{A}$ are defined by

$$\alpha(t) = \alpha_\Sigma \chi_{\Sigma(t)} = \begin{cases} \alpha_\Sigma > 0 & \text{in } \Sigma(t) \\ 0 & \text{in } \Omega \setminus \overline{\Sigma(t)} \end{cases}, \qquad \boldsymbol{A}(\boldsymbol{x},t)\boldsymbol{u}(\boldsymbol{x},t) = \left[\sum_{j,k=1}^d a^{ijk}(\boldsymbol{x},t)\frac{\partial u^j(\boldsymbol{x},t)}{\partial x^k} + \sum_{j=1}^d b^{ij}(\boldsymbol{x},t)u^j(\boldsymbol{x},t)\right]_{i=1}^d,$$

where $a^{ijk}, b^{ij} \in \mathrm{Lip}([0,T], \mathrm{L}^\infty(\Omega))$ with $i,j,k = 1,\ldots,d$. For example, the operator $\boldsymbol{A}$ can play the role of the convection, i.e. $\boldsymbol{A}\boldsymbol{u} = (\mathbf{v}\cdot\nabla)\boldsymbol{u}$, which is a part of the material derivative $\frac{D\boldsymbol{u}}{Dt} = \partial_t\boldsymbol{u} + (\mathbf{v}\cdot\nabla)\boldsymbol{u}$.

The system (2) is a degenerate parabolic problem. Moreover, it can be achieved from two different problems separately, namely a parabolic one on the trajectory $\mathbb{T}$ where $\alpha > 0$ and an elliptic one otherwise, which are combined via the interface condition. Therefore, (2) is also called a parabolic-elliptic problem. The given initial guest $\boldsymbol{u}_0$ is supposed to satisfy that $\boldsymbol{u}_0 \in \mathbf{H}^2(\Sigma_0)$ and $\nabla\cdot\boldsymbol{u}_0 = 0$. Since Σ_0 is of the class $\mathrm{C}^{2,1}$, there exists an extension of $\boldsymbol{u}_0$ such that $\boldsymbol{u}_0 \in \mathbf{H}_0^1(\Omega)\cap\mathbf{H}^2(\Omega)$ and $\nabla\cdot\boldsymbol{u}_0 = 0$ in Ω, see [2, Proposition 4.1].

The motivation of studying the system (2) comes from the eddy current model describing an electromagnetic problem with a moving non-magnetic conductor considered in [3, 4, 5]. More specifically, the paper [4] introduces a time discretization for the saddle-point formulation of a degenerate parabolic problem for the divergence-free vector potential. In this setting, α and β stand for the electrical conductivity and the magnetic permeability constant of vacuum, respectively. The operator $\boldsymbol{A}$ is defined as $\boldsymbol{A}\boldsymbol{u} = \sigma\mathbf{v}\times(\nabla\times\boldsymbol{u})$. A full space-time discretization of this problem has not been discussed yet. In [5], we propose a fully-discrete finite element scheme by incorporating a penalty term (Coulomb gauge) into the governing partial differential equation (PDE). However, the purpose of this paper is to introduce a fully-discrete finite element scheme for the saddle-point formulation of (2).

Let us mention also some other relevant recent results to the governing problem (2). In the paper [6], the author studied regularity of the solution to a parabolic-elliptic problem with moving parabolic subdomain, which was also motivated by an eddy current model with moving conductors. Nevertheless, the divergence-free condition and the convection-type term arising from the movement of conductors were not taken into account. The goal of [7] is to present an abstract framework for analyzing a family of linear degenerate parabolic mixed equations, then the paper [8] aims at introducing a fully-discrete approximation for this kind of problems. As stated in [8], the discrete inf-sup condition plays an important role for finite element analysis of the mixed problems, which allowed the authors to get quasi-optimal error estimate $O(\sqrt{\tau} + h/\sqrt{\tau})$. However, in these articles, all concerned domain and subdomains were fixed during the time process.

In the present paper, we propose a full discretization based on the finite element scheme and the backward Euler method for the variational formulation, see Section 3. The discrete inf-sup condition required for the existence of a discrete solution to the saddle-point approach together with handling terms acting on the moving body makes it challenging to establish an error estimate (with independent h and τ) for this numerical scheme, *which are the highlights of this contribution*. In the future, we aim to study the stability and to establish error estimates for the full discretization of the problem (2) with a jumping (non-Lipschitz) coefficient β, which still remains as a challenge at the moment. In the next section, we derive the mixed variational formulation for the degenerate parabolic problem (2).

2. Variational formulation

By means of the saddle-point approach, the variational formulation of the system (2) reads as follows:

Find $\boldsymbol{u}(t) \in \mathbf{H}_0^1(\Omega)$ with $\partial_t\boldsymbol{u}(t) \in \mathbf{L}^2(\Sigma(t))$ and $p(t) \in \mathrm{L}_0^2(\Omega)$ such that for a.a. $t \in (0,T)$, it holds that

$$\alpha_\Sigma\left(\partial_t\boldsymbol{u}(t),\boldsymbol{\varphi}\right)_{\Sigma(t)} + \beta\left(\nabla\boldsymbol{u}(t),\nabla\boldsymbol{\varphi}\right)_\Omega + \left(\boldsymbol{A}(t)\boldsymbol{u}(t),\boldsymbol{\varphi}\right)_{\Sigma(t)} + \left(p(t),\nabla\cdot\boldsymbol{\varphi}\right)_\Omega = \left(\boldsymbol{f}(t),\boldsymbol{\varphi}\right)_\Omega \qquad \forall\boldsymbol{\varphi}\in\mathbf{H}_0^1(\Omega), \tag{3}$$

$$\left(\nabla\cdot\boldsymbol{u}(t),q\right)_\Omega = 0 \qquad \forall q\in\mathrm{L}_0^2(\Omega). \tag{4}$$

Please note that the additional unknown p plays the role of the divergence of $\boldsymbol{u}$ (see [4] for more details on the interpretation of p). Since $p(t) \in \mathrm{L}_0^2(\Omega)$, the inf-sup condition is satisfied following from [9, Theorem 5.1 on p. 80]. Throughout this paper, we consider the following subspace of $\mathbf{H}_0^1(\Omega)$:

$$\mathbf{H}_0^1(\mathrm{div}) = \left\{\boldsymbol{\varphi}\in\mathbf{H}_0^1(\Omega) : \nabla\cdot\boldsymbol{\varphi} = 0\right\}.$$

The well-posedness of the mixed variational problem (3-4) can be proved by performing similar arguments as presented in [10, Theorem 5.1]. Thus, we can obtain the following result without the proof.

Theorem 2.1 (Well-posedness)**.** *Let* $\boldsymbol{u}_0 \in \mathbf{H}_0^1(\mathrm{div}) \cap \mathbf{H}^2(\Omega)$ *satisfying* $\Delta \boldsymbol{u}_0 = \mathbf{0}$ *on* $\Omega \setminus \overline{\Sigma_0}$, $\mathbf{v} \in \mathbf{C}^1(\overline{Q})$ *and* $\boldsymbol{f} \in \mathrm{Lip}([0,T], \mathbf{L}^2(\Omega))$*. Moreover, we assume that* $a^{ijk}, b^{ij} \in \mathrm{Lip}([0,T], \mathrm{L}^\infty(\Omega))$ *with* $i, j, k = 1, \ldots, d$*. Then the system (3-4) admits exactly one solution* $(\boldsymbol{u}, p)$ *satisfying* $p \in \mathrm{L}^2((0,T), \mathrm{L}_0^2(\Omega))$ *and* $\boldsymbol{u} \in \mathrm{C}([0,T], \mathbf{H}_0^1(\mathrm{div}))$ *with* $\partial_t \boldsymbol{u} \in \mathrm{L}^2((0,T), \mathbf{H}_0^1(\mathrm{div}))$.

The following local regularity result provided by [11, Theorem 8.8] will play a crucial role for the error estimate of the full discretization scheme in the next section.

Corollary 2.1. *Let the assumptions of Theorem 2.1 be fulfilled. Then for any subdomain* $\Sigma' \subset\subset \Omega$ *(i.e.* $\overline{\Sigma'} \subset \Omega$*), we have* $\boldsymbol{u} \in \mathrm{L}^2((0,T), \mathbf{H}^2(\Sigma'))$.

We mention here the well-known Reynolds transport theorem, which will be helpful for further analysis of PDEs with time-dependent domains. Let $\omega(t)$ be a Lipschitz moving body whose velocity vector $\mathbf{v}$ is of class $\mathbf{C}^1$ and f an abstract function satisfying $f(t) \in \mathrm{W}^{1,1}(\omega(t))$ and $\partial_t f(t) \in \mathrm{L}^1(\omega(t))$ for all $t \in (0,T)$. Then it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\omega(t)} f \,\mathrm{d}\boldsymbol{x} = \int_{\omega(t)} \partial_t f \,\mathrm{d}\boldsymbol{x} + \int_{\partial\omega(t)} f \mathbf{v} \cdot \mathbf{n} \,\mathrm{d}s. \tag{5}$$

3. Full discretization

Let $\mathbf{V}_0^h$ and V^h be two finite-dimensional subspaces of $\mathbf{H}_0^1(\Omega)$ and $\mathrm{L}_0^2(\Omega)$, respectively. These spaces are equipped with two orthogonal projection operators $\mathbf{P}^h \in \mathcal{L}(\mathbf{H}_0^1(\Omega), \mathbf{V}_0^h)$ and $\mathrm{P}^h \in \mathcal{L}(\mathrm{L}_0^2(\Omega), \mathrm{V}^h)$. The time interval $[0,T]$ is divided into n equidistant subintervals with length $\tau = \frac{T}{n}$. The fully-discrete approximations of $\boldsymbol{u}$ and p at time $t_i = i\tau$ $(0 \le i \le n)$ are denoted by $\boldsymbol{u}_i^h$ and p_i^h, respectively. We also introduce the following notations

$$\delta \boldsymbol{u}_i^h = \frac{\boldsymbol{u}_i^h - \boldsymbol{u}_{i-1}^h}{\tau}, \qquad \boldsymbol{u}_0^h = \mathbf{P}^h \boldsymbol{u}_0, \qquad \boldsymbol{A}_i = \boldsymbol{A}(t_i), \qquad \Sigma_i = \Sigma(t_i).$$

The full discretization of the mixed variational formulation (3-4) is defined as:

Find $\boldsymbol{u}_i^h \in \mathbf{V}_0^h$ and $p_i^h \in \mathrm{V}^h$ such that for any $i = 1, 2, \ldots, n$, it holds that

$$\alpha_\Sigma \left(\delta \boldsymbol{u}_i^h, \boldsymbol{\varphi}^h\right)_{\Sigma_i} + \beta \left(\nabla \boldsymbol{u}_i^h, \nabla \boldsymbol{\varphi}^h\right)_\Omega + \left(\boldsymbol{A}_i \boldsymbol{u}_i^h, \boldsymbol{\varphi}^h\right)_{\Sigma_i} + \left(p_i^h, \nabla \cdot \boldsymbol{\varphi}^h\right)_\Omega = \left(\boldsymbol{f}_i, \boldsymbol{\varphi}^h\right)_\Omega \qquad \forall \boldsymbol{\varphi}^h \in \mathbf{V}_0^h, \tag{6}$$

$$\left(\nabla \cdot \boldsymbol{u}_i^h, q^h\right)_\Omega = 0 \qquad \forall q^h \in \mathrm{V}^h. \tag{7}$$

The solvability of the system (6-7) on every time step follows from the Brezzi theorem, cf. [12, Corollary 1.1].

Lemma 3.1 (Solvability)**.** *Let the assumptions of Theorem 2.1 be fulfilled. Moreover, we assume that the discrete inf-sup condition is satisfied, i.e. there exists a constant* $C > 0$ *such that*

$$\sup_{\boldsymbol{\varphi}^h \in \mathbf{V}_0^h,\ \boldsymbol{\varphi}^h \neq \mathbf{0}} \frac{\left(\nabla \cdot \boldsymbol{\varphi}^h, q^h\right)_\Omega}{\left\|\boldsymbol{\varphi}^h\right\|_{\mathbf{H}_0^1(\Omega)}} \ge C \left\|q^h\right\|_{\mathrm{L}^2(\Omega)} \qquad \forall q^h \in \mathrm{V}^h. \tag{8}$$

Then, for any $i = 1, 2, \ldots, n$ *and any* $\tau < \tau_0$*, there exists a unique couple* $(\boldsymbol{u}_i^h, p_i^h) \in \mathbf{V}_0^h \times \mathrm{V}^h$ *solving (6-7).*

The following basic a priori estimate for iterates can be obtained in the same way as in [4, Lemma 4.3]. This estimate is crucial in obtaining the main result of this paper.

Lemma 3.2 (A priori estimate)**.** *Let the assumptions of Lemma 3.1 be fulfilled. In addition, we assume that* $\boldsymbol{u}_0^h$ *solves the equation (7). Then there exists a constant* $C > 0$ *such that the following relation holds true for any* $\tau < \tau_0$

$$\max_{1 \le l \le n} \left\|\delta \boldsymbol{u}_l^h\right\|_{\mathbf{L}^2(\Sigma_l)}^2 + \sum_{i=1}^n \left\|\nabla \delta \boldsymbol{u}_i^h\right\|_{\mathbf{L}^2(\Omega)}^2 \tau + \sum_{i=1}^n \left\|\delta \boldsymbol{u}_i^h - \delta \boldsymbol{u}_{i-1}^h\right\|_{\mathbf{L}^2(\Sigma_{i-1})}^2 + \max_{1 \le l \le n} \left\|p_l^h\right\|_{\mathrm{L}^2(\Omega)}^2 \le C. \tag{9}$$

We define the following piecewise-constant and piecewise-affine in time functions, operator and domain

$$\overline{\boldsymbol{u}}_n^h(t) = \boldsymbol{u}_i^h, \qquad \boldsymbol{u}_n^h(t) = \boldsymbol{u}_{i-1}^h + (t - t_{i-1})\delta \boldsymbol{u}_i^h, \qquad \overline{p}_n^h(t) = p_i^h, \qquad \overline{\boldsymbol{f}}_n(t) = \boldsymbol{f}_i, \qquad \overline{\boldsymbol{A}}_n(t) = \boldsymbol{A}_i, \qquad \overline{\Sigma}_n(t) = \Sigma_i,$$

for every $t \in (t_{i-1}, t_i]$, $1 \le i \le n$, with the initial data

$$\overline{\boldsymbol{u}}_n^h(0) = \boldsymbol{u}_n^h(0) = \boldsymbol{u}_0^h, \qquad \overline{p}_n^h(0) = 0, \qquad \overline{\boldsymbol{f}}_n(0) = \boldsymbol{f}(0), \qquad \overline{\boldsymbol{A}}_n(0) = \boldsymbol{A}(0), \qquad \overline{\Sigma}_n(0) = \Sigma_0.$$

The following relation between Rothe's functions $\overline{\boldsymbol{u}}_n^h$ and $\boldsymbol{u}_n^h$ comes from the a priori estimate (9):

$$\int_0^T \left\|\nabla \overline{\boldsymbol{u}}_n^h(t) - \nabla \boldsymbol{u}_n^h(t)\right\|_{\mathbf{L}^2(\Omega)}^2 \, \mathrm{d}t = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (t_i - t)^2 \left\|\nabla \delta \boldsymbol{u}_i^h\right\|_{\mathbf{L}^2(\Omega)}^2 \, \mathrm{d}t \le \sum_{i=1}^n \left\|\nabla \delta \boldsymbol{u}_i^h\right\|_{\mathbf{L}^2(\Omega)}^2 \tau^3 \overset{(9)}{\lesssim} \tau^2. \tag{10}$$

Hence, the equations (6) and (7) can be rewritten in the following form

$$\alpha_\Sigma \left(\partial_t \boldsymbol{u}_n^h(t), \boldsymbol{\varphi}^h\right)_{\overline{\Sigma}_n(t)} + \beta \left(\nabla \overline{\boldsymbol{u}}_n^h(t), \nabla \boldsymbol{\varphi}^h\right)_\Omega + \left(\overline{\boldsymbol{A}}_n(t) \overline{\boldsymbol{u}}_n^h(t), \boldsymbol{\varphi}^h\right)_{\overline{\Sigma}_n(t)} + \left(\overline{p}_n^h(t), \nabla \cdot \boldsymbol{\varphi}^h\right)_\Omega = \left(\overline{\boldsymbol{f}}_n(t), \boldsymbol{\varphi}^h\right)_\Omega \qquad \forall \boldsymbol{\varphi}^h \in \mathbf{V}_0^h, \tag{11}$$

$$\left(\nabla \cdot \overline{\boldsymbol{u}}_n^h(t), q^h\right)_\Omega = 0 \qquad \forall q^h \in \mathrm{V}^h. \tag{12}$$

Now, we are in the position to investigate the convergence of the full discretization scheme.

Theorem 3.1. *Let the assumptions of Lemma 3.2 be fulfilled. Then there exists a constant $C > 0$ such that the following relation holds true for every $\xi \in [0, T]$*

$$\int_0^\xi \left\|\partial_t \boldsymbol{u}_n^h(t) - \partial_t \boldsymbol{u}(t)\right\|_{\mathbf{L}^2(\Sigma(t))}^2 \mathrm{d}t + \left\|\nabla \boldsymbol{u}_n^h(\xi) - \nabla \boldsymbol{u}(\xi)\right\|_{\mathbf{L}^2(\Omega)}^2 + \int_0^\xi \left\|\overline{p}_n^h(t) - p(t)\right\|_{\mathrm{L}^2(\Omega)}^2 \mathrm{d}t$$
$$\le C \left(\tau + \left\|\nabla \boldsymbol{u}_0 - \nabla \mathbf{P}^h \boldsymbol{u}_0\right\|_{\mathbf{L}^2(\Omega)}^2 + \sqrt{\int_0^\xi \left\|p(t) - \mathrm{P}^h p(t)\right\|_{\mathrm{L}^2(\Omega)}^2 \mathrm{d}t} + \int_0^\xi \left\|\nabla \partial_t \boldsymbol{u}(t) - \nabla \mathbf{P}^h \partial_t \boldsymbol{u}(t)\right\|_{\mathbf{L}^2(\Omega)}^2 \mathrm{d}t \right). \tag{13}$$

Proof. Subtracting (3) for $\boldsymbol{\varphi} = \boldsymbol{\varphi}^h$ from (11), then rewriting the result by the Reynolds transport theorem (5), we get for a.a. $t \in (0, T)$ that

$$\alpha_\Sigma \left(\partial_t \boldsymbol{u}_n^h(t) - \partial_t \boldsymbol{u}(t), \boldsymbol{\varphi}^h\right)_{\Sigma(t)} + \beta \left(\nabla \overline{\boldsymbol{u}}_n^h(t) - \nabla \boldsymbol{u}(t), \nabla \boldsymbol{\varphi}^h\right)_\Omega + \left(\overline{p}_n^h(t) - p(t), \nabla \cdot \boldsymbol{\varphi}^h\right)_\Omega + \left((\overline{\boldsymbol{A}}_n(t) - \boldsymbol{A}(t)) \overline{\boldsymbol{u}}_n^h(t), \boldsymbol{\varphi}^h\right)_{\overline{\Sigma}_n(t)}$$
$$+ \left(\boldsymbol{A}(t)(\overline{\boldsymbol{u}}_n^h(t) - \boldsymbol{u}(t)), \boldsymbol{\varphi}^h\right)_{\overline{\Sigma}_n(t)} + \int_t^{\bar{t}_n} \int_{\partial\Sigma(\eta)} \left[\left(\alpha_\Sigma \partial_t \boldsymbol{u}_n^h(t) + \boldsymbol{A}(t)\boldsymbol{u}(t)\right) \cdot \boldsymbol{\varphi}^h\right] (\mathbf{v} \cdot \mathbf{n})(\eta) \, \mathrm{d}s \, \mathrm{d}\eta = \left(\overline{\boldsymbol{f}}_n(t) - \boldsymbol{f}(t), \boldsymbol{\varphi}^h\right)_\Omega, \tag{14}$$

where $\bar{t}_n = \left\lceil \frac{t}{\tau} \right\rceil \tau$. Setting $\boldsymbol{\varphi}^h = \partial_t \boldsymbol{u}_n^h(t) - \mathbf{P}^h \partial_t \boldsymbol{u}(t)$ in (14), then integrating in time over $(0, \xi) \subset (0, T)$ and rearranging the result give us that

$$\alpha_\Sigma \int_0^\xi \left\|\partial_t \boldsymbol{u}_n^h(t) - \partial_t \boldsymbol{u}(t)\right\|_{\mathbf{L}^2(\Sigma(t))}^2 \mathrm{d}t + \frac{\beta}{2} \left\|\nabla \boldsymbol{u}_n^h(\xi) - \nabla \boldsymbol{u}(\xi)\right\|_{\mathbf{L}^2(\Omega)}^2 - \frac{\beta}{2} \left\|\nabla \boldsymbol{u}_0^h - \nabla \boldsymbol{u}_0\right\|_{\mathbf{L}^2(\Omega)}^2$$
$$= -\alpha_\Sigma \int_0^\xi \left(\partial_t \boldsymbol{u}_n^h(t) - \partial_t \boldsymbol{u}(t), \partial_t \boldsymbol{u}(t) - \mathbf{P}^h \partial_t \boldsymbol{u}(t)\right)_{\Sigma(t)} \mathrm{d}t + \int_0^\xi \left(\overline{\boldsymbol{f}}_n(t) - \boldsymbol{f}(t), \partial_t \boldsymbol{u}_n^h(t) - \mathbf{P}^h \partial_t \boldsymbol{u}(t)\right)_\Omega \mathrm{d}t$$
$$- \beta \int_0^\xi \left(\nabla \boldsymbol{u}_n^h(t) - \nabla \boldsymbol{u}(t), \nabla \partial_t \boldsymbol{u}(t) - \nabla \mathbf{P}^h \partial_t \boldsymbol{u}(t)\right)_\Omega \mathrm{d}t - \beta \int_0^\xi \left(\nabla \overline{\boldsymbol{u}}_n^h(t) - \nabla \boldsymbol{u}_n^h(t), \nabla \partial_t \boldsymbol{u}_n^h(t) - \nabla \mathbf{P}^h \partial_t \boldsymbol{u}(t)\right)_\Omega \mathrm{d}t$$

$$- \int_0^{\xi} \left(\overline{p}_n^h(t) - p(t), \nabla \cdot \partial_t \boldsymbol{u}_n^h(t) - \nabla \cdot \partial_t \boldsymbol{u}(t)\right)_\Omega \mathrm{d}t - \int_0^{\xi} \left(\overline{p}_n^h(t) - p(t), \nabla \cdot \partial_t \boldsymbol{u}(t) - \nabla \cdot \mathbf{P}^h \partial_t \boldsymbol{u}(t)\right)_\Omega \mathrm{d}t$$

$$- \int_0^{\xi} \left((\overline{\boldsymbol{A}}_n(t) - \boldsymbol{A}(t))\overline{\boldsymbol{u}}_n^h(t), \partial_t \boldsymbol{u}_n^h(t) - \mathbf{P}^h \partial_t \boldsymbol{u}(t)\right)_{\overline{\Sigma}_n(t)} \mathrm{d}t - \int_0^{\xi} \left(\boldsymbol{A}(t)(\overline{\boldsymbol{u}}_n^h(t) - \boldsymbol{u}(t)), \partial_t \boldsymbol{u}_n^h(t) - \mathbf{P}^h \partial_t \boldsymbol{u}(t)\right)_{\overline{\Sigma}_n(t)} \mathrm{d}t$$

$$- \int_0^{\xi} \int_t^{\bar{t}_n} \int_{\partial\Sigma(\eta)} \left[\left(\alpha_\Sigma \partial_t \boldsymbol{u}_n^h(t) + \boldsymbol{A}(t)\boldsymbol{u}(t)\right) \cdot \left(\partial_t \boldsymbol{u}_n^h(t) - \mathbf{P}^h \partial_t \boldsymbol{u}(t)\right)\right] (\mathbf{v} \cdot \mathbf{n})(\eta) \, \mathrm{d}s \, \mathrm{d}\eta \, \mathrm{d}t =: \sum_{i=1}^{9} S_i.$$

The Cauchy-Schwarz and ε-Young inequalities are used to estimate S_1 and S_3 as follows

$$|S_1| \leq \varepsilon \int_0^{\xi} \left\|\partial_t \boldsymbol{u}_n^h(t) - \partial_t \boldsymbol{u}(t)\right\|^2_{\mathbf{L}^2(\Sigma(t))} \mathrm{d}t + C_\varepsilon \int_0^{\xi} \left\|\partial_t \boldsymbol{u}(t) - \mathbf{P}^h \partial_t \boldsymbol{u}(t)\right\|^2_{\mathbf{L}^2(\Omega)} \mathrm{d}t,$$

$$|S_3| \lesssim \int_0^{\xi} \left\|\nabla \boldsymbol{u}_n^h(t) - \nabla \boldsymbol{u}(t)\right\|^2_{\mathbf{L}^2(\Omega)} \mathrm{d}t + \int_0^{\xi} \left\|\nabla \partial_t \boldsymbol{u}(t) - \nabla \mathbf{P}^h \partial_t \boldsymbol{u}(t)\right\|^2_{\mathbf{L}^2(\Omega)} \mathrm{d}t.$$

We invoke the properties of $\boldsymbol{f}$ and a^{ijk}, b^{ij} $(i, j, k = 1, \ldots, d)$ together with Friedrichs's inequality and (10) to obtain that

$$|S_2| + |S_4| + |S_7| \overset{(10)}{\lesssim} \tau \sqrt{\int_0^{\xi} \left\|\nabla \partial_t \boldsymbol{u}_n^h(t) - \nabla \mathbf{P}^h \partial_t \boldsymbol{u}(t)\right\|^2_{\mathbf{L}^2(\Omega)} \mathrm{d}t} \overset{(9)}{\lesssim} \tau.$$

To estimate S_8, we need the following auxiliary estimate

$$\left| \int_0^{\xi} \left\|\partial_t \boldsymbol{u}_n^h(t) - \partial_t \boldsymbol{u}(t)\right\|^2_{\mathbf{L}^2(\overline{\Sigma}_n(t))} \mathrm{d}t - \int_0^{\xi} \left\|\partial_t \boldsymbol{u}_n^h(t) - \partial_t \boldsymbol{u}(t)\right\|^2_{\mathbf{L}^2(\Sigma(t))} \mathrm{d}t \right|$$

$$\overset{(5)}{=} \left| \int_0^{\xi} \int_t^{\bar{t}_n} \int_{\partial\Sigma(\eta)} \left|\partial_t \boldsymbol{u}_n^h(t) - \partial_t \boldsymbol{u}(t)\right|^2 (\mathbf{v} \cdot \mathbf{n})(\eta) \, \mathrm{d}s \, \mathrm{d}\eta \, \mathrm{d}t \right| \lesssim \tau \int_0^{\xi} \left\|\nabla \partial_t \boldsymbol{u}_n^h(t) - \nabla \partial_t \boldsymbol{u}(t)\right\|^2_{\mathbf{L}^2(\Omega)} \mathrm{d}t \overset{(9)}{\lesssim} \tau.$$

Therefore, we arrive at

$$|S_8| \overset{(10)}{\leq} C_\varepsilon \tau + C_\varepsilon \int_0^{\xi} \left\|\nabla \boldsymbol{u}_n^h(t) - \nabla \boldsymbol{u}(t)\right\|^2_{\mathbf{L}^2(\Omega)} \mathrm{d}t + \varepsilon \int_0^{\xi} \left\|\partial_t \boldsymbol{u}_n^h(t) - \partial_t \boldsymbol{u}(t)\right\|^2_{\mathbf{L}^2(\Sigma(t))} \mathrm{d}t + \varepsilon \int_0^{\xi} \left\|\partial_t \boldsymbol{u}(t) - \mathbf{P}^h \partial_t \boldsymbol{u}(t)\right\|^2_{\mathbf{L}^2(\Omega)} \mathrm{d}t.$$

The most challenges lie on handling the terms S_5, S_6 and S_9, which arise from the saddle-point approach and the movement of the body Σ_0. Using the equations (4) and (12), we have that

$$|S_5| \overset{(4)}{=} \left| \int_0^{\xi} \left(\overline{p}_n^h(t) - p(t), \nabla \cdot \partial_t \boldsymbol{u}_n^h(t)\right)_\Omega \mathrm{d}t \right| \overset{(12)}{=} \left| \int_0^{\xi} \left(p(t) - \mathrm{P}^h p(t), \nabla \cdot \partial_t \boldsymbol{u}_n^h(t)\right)_\Omega \mathrm{d}t \right| \overset{(9)}{\lesssim} \sqrt{\int_0^{\xi} \left\|p(t) - \mathrm{P}^h p(t)\right\|^2_{\mathrm{L}^2(\Omega)} \mathrm{d}t}.$$

For the term S_6, we first deduce the following estimate

$$\left\|\overline{p}_n^h(t) - \mathrm{P}^h p(t)\right\|_{\mathrm{L}^2(\Omega)} \overset{(8)}{\lesssim} \sup_{\boldsymbol{\varphi}^h \in \mathbf{V}_0^h, \, \boldsymbol{\varphi}^h \neq \mathbf{0}} \frac{\left(\nabla \cdot \boldsymbol{\varphi}^h, \overline{p}_n^h(t) - \mathrm{P}^h p(t)\right)_\Omega}{\left\|\boldsymbol{\varphi}^h\right\|_{\mathbf{H}_0^1(\Omega)}}$$

$$\overset{(14)}{\lesssim} \tau + \left\| p(t) - \mathrm{P}^h p(t) \right\|_{\mathrm{L}^2(\Omega)} + \left\| \partial_t \boldsymbol{u}_n^h(t) - \partial_t \boldsymbol{u}(t) \right\|_{\mathbf{L}^2(\Sigma(t))}$$
$$+ \left\| \nabla \overline{\boldsymbol{u}}_n^h(t) - \nabla \boldsymbol{u}(t) \right\|_{\mathbf{L}^2(\Omega)} + \left\| \nabla \partial_t \boldsymbol{u}_n^h(t) \right\|_{\mathbf{L}^2(\Omega)} \tau + \|\boldsymbol{u}(t)\|_{\mathbf{H}^2(\widetilde{\Sigma})} \tau, \tag{15}$$

which together with Corollary 2.1 and Lemma 3.2 allow us to conclude that

$$|S_6| \le \varepsilon \int_0^\xi \left\| \overline{p}_n^h(t) - p(t) \right\|^2_{\mathrm{L}^2(\Omega)} \,\mathrm{d}t + C_\varepsilon \int_0^\xi \left\| \nabla \cdot \partial_t \boldsymbol{u}(t) - \nabla \cdot \mathbf{P}^h \partial_t \boldsymbol{u}(t) \right\|^2_{\mathrm{L}^2(\Omega)} \,\mathrm{d}t$$
$$\overset{(9)}{\lesssim} \varepsilon \tau^2 + \varepsilon \int_0^\xi \left\| p(t) - \mathrm{P}^h p(t) \right\|^2_{\mathrm{L}^2(\Omega)} \,\mathrm{d}t + \varepsilon \int_0^\xi \left\| \partial_t \boldsymbol{u}_n^h(t) - \partial_t \boldsymbol{u}(t) \right\|^2_{\mathbf{L}^2(\Sigma(t))} \,\mathrm{d}t$$
$$+ \varepsilon \int_0^\xi \left\| \nabla \boldsymbol{u}_n^h(t) - \nabla \boldsymbol{u}(t) \right\|^2_{\mathbf{L}^2(\Omega)} \,\mathrm{d}t + C_\varepsilon \int_0^\xi \left\| \nabla \cdot \partial_t \boldsymbol{u}(t) - \nabla \cdot \mathbf{P}^h \partial_t \boldsymbol{u}(t) \right\|^2_{\mathrm{L}^2(\Omega)} \,\mathrm{d}t.$$

The term S_9 can be handled by applying the local regularity on the weak solution $\boldsymbol{u}$ presented in Corollary 2.1, i.e.

$$|S_9| \lesssim \tau \sqrt{\int_0^\xi \left(\left\| \nabla \partial_t \boldsymbol{u}_n^h(t) \right\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{u}(t)\|^2_{\mathbf{H}^2(\widetilde{\Sigma})} \right) \mathrm{d}t} \sqrt{\int_0^\xi \left\| \nabla \partial_t \boldsymbol{u}_n^h(t) - \nabla \mathbf{P}^h \partial_t \boldsymbol{u}(t) \right\|^2_{\mathbf{L}^2(\Omega)} \,\mathrm{d}t} \overset{(9)}{\lesssim} \tau.$$

Finally, taking all considerations into account, then fixing a sufficiently small $\varepsilon > 0$ and using a Grönwall argument, we can achieve the desired estimate for $\boldsymbol{u}_n^h$. The error estimate for $\overline{p}_n^h$ can be acquired following the relation (15). □

The convergence of the proposed full discretization scheme is a consequence of Theorem 3.1 and Céa's lemma.

Theorem 3.2. *Let the assumptions of Lemma 3.2 be fulfilled. Then the following convergences hold true:* $\partial_t \boldsymbol{u}_n^h \to \partial_t \boldsymbol{u}$ *in* $\mathbf{L}^2(\mathbb{T})$, $\boldsymbol{u}_n^h \to \boldsymbol{u}$ *in* $\mathrm{C}([0,T], \mathbf{H}_0^1(\Omega))$ *and* $\overline{p}_n^h \to p$ *in* $\mathrm{L}^2((0,T), \mathrm{L}^2(\Omega))$.

3.1. Error estimates for some finite element spaces

In this section, we examine the error estimate (13) for some finite element space pairs $(\mathbf{V}_0^h, \mathrm{V}^h)$, which satisfy the discrete inf-sup condition (8). Let $\{\mathcal{T}^h\}_{h>0}$ be a regular family of triangulations of $\overline{\Omega}$. For each integer $k \ge 0$, we denote by P_k the space of all polynomials of degree at most k. We introduce the following spaces of piecewise polynomials

$$\mathcal{P}_0 = \left\{ \varphi \in \mathrm{L}^2(\Omega) : \varphi|_K \in \mathrm{P}_0 \quad \forall K \in \mathcal{T} \right\}, \qquad \mathcal{P}_k = \left\{ \varphi \in \mathrm{C}(\overline{\Omega}) : \varphi|_K \in \mathrm{P}_k(\, k \ge 1) \quad \forall K \in \mathcal{T} \right\}.$$

There are some stable finite element pairs satisfying the discrete inf-sup condition (8), see [13, Section VI.3]. Now, we present two simple examples, namely $(\mathcal{P}_2 - \mathcal{P}_0)$ for discontinuous element space of p and the Taylor-Hood pair $(\mathcal{P}_2 - \mathcal{P}_1)$ for continuous elements. The following error estimates are the results of the relation (13) combined with Céa's lemma and the standard interpolation error of the finite element spaces.

Corollary 3.1. *Let the assumptions of Lemma 3.2 be fulfilled. Moreover, we assume that* $\mathbf{V}_0^h = (\mathcal{P}_2)^d \cap \mathbf{H}_0^1(\Omega)$ *and* $\boldsymbol{u} \in \mathrm{C}([0,T], \mathbf{H}_0^1(\mathrm{div})) \cap \mathrm{L}^2((0,T), \mathbf{H}^2(\Omega))$.

(i) If $\mathrm{V}^h = \mathcal{P}_0 \cap \mathrm{L}_0^2(\Omega)$ *and* $p \in \mathrm{L}^2((0,T), \mathrm{H}^1(\Omega) \cap \mathrm{L}_0^2(\Omega))$*, then there exists a constant* $C > 0$ *such that*

$$\int_0^\xi \left\| \partial_t \boldsymbol{u}_n^h(t) - \partial_t \boldsymbol{u}(t) \right\|^2_{\mathbf{L}^2(\Sigma(t))} \,\mathrm{d}t + \left\| \nabla \boldsymbol{u}_n^h(\xi) - \nabla \boldsymbol{u}(\xi) \right\|^2_{\mathbf{L}^2(\Omega)} + \int_0^\xi \left\| \overline{p}_n^h(t) - p(t) \right\|^2_{\mathrm{L}^2(\Omega)} \,\mathrm{d}t \le C(\tau + h).$$

(ii) If $\mathrm{V}^h = \mathcal{P}_1 \cap \mathrm{L}_0^2(\Omega)$ *and* $p \in \mathrm{L}^2((0,T), \mathrm{H}^2(\Omega) \cap \mathrm{L}_0^2(\Omega))$*, then there exists a constant* $C > 0$ *such that*

$$\int_0^\xi \left\| \partial_t \boldsymbol{u}_n^h(t) - \partial_t \boldsymbol{u}(t) \right\|^2_{\mathbf{L}^2(\Sigma(t))} \,\mathrm{d}t + \left\| \nabla \boldsymbol{u}_n^h(\xi) - \nabla \boldsymbol{u}(\xi) \right\|^2_{\mathbf{L}^2(\Omega)} + \int_0^\xi \left\| \overline{p}_n^h(t) - p(t) \right\|^2_{\mathrm{L}^2(\Omega)} \,\mathrm{d}t \le C(\tau + h^2).$$

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