A consecutive-interpolation polyhedral finite element method for solid structures

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Abstract
In this paper, we investigate the use of a consecutive-interpolation for polyhedral finite element method (CIPFEM) in the analysis of three-dimensional solid mechanics problems. A displacement-based Galerkin weak form is used, in which the nodal degrees of freedom (DOF) and their derivatives are both considered for the approximation scheme. Based on arbitrary star-convex polyhedral elements using piecewise linear shape function, the present method can have the advantage of being applicable to complicated structures. Nevertheless, the proposed interpolation technique gives higher-order continuity, greater accuracy with the same number of DOFs. The reliability and efficiency of the CIPFEM are proved by comparing the present results with those obtained by the consecutive-interpolation for tetrahedral element (CT4), conventional linear FEM using polyhedral elements (PFEM), and tetrahedral elements (T4) through numerical examples. Cantilever beam, concrete corbel, and complex hollow concrete revetment block are considered to show the excellent performance of the present approach.

KEYWORDS
averaged derivatives, CIPFEM, consecutive-interpolation, polyhedral finite element method

1 INTRODUCTION

The finite element methods (FEMs) have been well-known as the most popular and effective tool to solve a wide range of engineering problems. Due to their advantages such as more flexible in meshing complicated structures, minimizing the number of elements for a given mesh resolution and reducing mesh-distortion sensitivity compared to the standard FEM, the arbitrary polyhedral finite element (pFEM) has attracted huge attention as well as applications in different
fields of engineering, e.g. topology optimization, plate structures, solid mechanics problems, fluid flow problems, microstructures, and so forth. The development of pFEM in recent years was clearly reviewed in Reference 13. Recently, a new interpolation scheme, the so-called piecewise linear shape function, has been developed for 2D polygonal and 3D polyhedral elements (PFEM), which showed many advantages compared to conventional FEM. However, the obtained strain fields are constant over elements’ sub-domains due to the nature of the shape functions. Moreover, the gradients between integration elements are discontinuous, which require post processing to smooth the obtained results such as strain, stress over the whole domain. Therefore, the ambition of increasing the accuracy of numerical solutions have encouraged researchers to developed new formulae, technique or analysis procedure rapidly. Natarajan et al. presented an integration scheme using Schwarz–Christoffel mapping and midpoint quadrature rule over arbitrary polyhedral elements, which can reduce the mapping process by sub-dividing the element. Bishop proposed the harmonic shape functions that were used for 3D large deformation applications. The integration of bivariate polynomials using new quadrature rules over arbitrary polygons was developed. Dai et al. extended the cell-based smoothed FEM to more general cases of meshes using arbitrary n-sided polygonal elements, which showed great accuracy. The gradient correction was proposed by Talischi et al. to increase the consistency of weak form integration. Chi et al. proposed an approach based on Mean Value coordinate in the problems of nonlinear elastic material, which can overcome the drawback of large deformation and nearly incompressible materials. Recently, an effectively numerical technique that was not required the interpolation functions over the discretized elements, was named as Virtual Element Method (VEM). Gain et al. implemented the VEM to first-order polyhedral elements for elastostatic solid problems. Whereas, the obtained displacements in the linear patch test were very close to the machine precision. Beirão da Veiga et al. showed the h-convergence rate and the accuracy of the results obtained by using VEM in the analysis of diffusion–reaction problem using various orders of polynomials. Park et al. showed the applicability of VEM in elastodynamic problems with very distorted polyhedral elements. Other studies using VEM on 3D polyhedral elements can also be seen in References. Those are some examples of innovative methods, which can improve the results of solid mechanics problems using polygonal/polyhedral elements.

In this study, we improve the PFEM by using the consecutive interpolation (CI) scheme, which shows great advantages compared to the conventional linear FEM such as higher accuracy, continuous stress and strain fields without any requirement of smoothing operators, higher convergence rate with the same degree of freedoms (DOFs), high-order polynomial shape functions processing Kronecker-delta property and mesh distortion insensitivity. This method was firstly proposed for 2D triangular elements with linear shape functions, in which the unknowns of Galerkin weak form are nodal displacements analyzed via two steps. In the first step, the interpolation process is totally similar to conventional FEM. Then, in the second step, the average nodal derivatives of displacements are used to re-formulate the shape functions with high-order polynomials. CI scheme not only interpolates the unknowns of current elements but also includes those of adjacent nodes (see Figure 1), which increase the smoothness of strain field. This method was further developed to adapt various element types and problems. For instance, Bui et al. successfully applied the above consecutive interpolation scheme over 2D quadrilateral elements for problems of elasticity. Nguyen et al. developed similar interpolation technique over 3D tetrahedral elements for heat transfer in solid structures problems, and 3D hexahedral meshes for solid and composite problems.

The main target of this article is to develop a general analysis process based on consecutive interpolation scheme for three-dimensional solid structures using arbitrary star-convex polyhedral elements. The considered piecewise linear shape functions are taken from the study of Nguyen-Ngoc et al., in which the polyhedral mesh is sub-divided

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**Figure 1** Support nodes of current elements, which are the nodes connecting all adjacent elements including current nodes and adjacent nodes. Whereas, current element is the element containing point of interest x.
into sub-tetrahedra. Then, the shape functions are formulated based on these sub-cells. It should be noted that the nodal derivatives of piecewise linear shape functions are not consistent over the whole polyhedron, but only over the sub-cells. This is a major difference compared to conventional linear tetrahedral and hexahedral mesh. The efficiency of the present consecutive-interpolation for polyhedral finite element method (CIPFEM) scheme is benchmarked by comparing the obtained results with analytical results of a cantilever beam reported in Reference 31. The results of a concrete corbel and complex revetment block computed by linear T4, PFEM, 14 CT4, and ANSYS32 are also used for comparison.

This paper is constructed as follows. Section 2 explains the basic of consecutive interpolation scheme in FEM. The present CIPFEM is developed for three-dimensional arbitrary star-convex polyhedral element in Section 3, which is followed by Section 4, where the proposed method is applied to some representative solid structures. Finally, the conclusion of this article is summarized in Section 5.

2 BASIC OF CONSECUTIVE INTERPOLATION SCHEME FOR 3D FEM

Let us consider a 3D elastic domain $\Omega$ bounded by a boundary $\Gamma = \Gamma_t \cup \Gamma_u$ and $\Gamma_t \cap \Gamma_u = \emptyset$, the equilibrium equations of domain $\Omega$ can be described as follows:

$$\nabla^T \sigma + b = 0 \text{ in } \Omega$$  

(1)

where $\nabla$ is the differential operator with respect to three axes; $\sigma$ is the stress tensor: $\sigma^T = \{\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}, \sigma_{zx}\}$; and the body force $b^T = \{b_x, b_y, b_z\}$.

In the equilibrium state, Equation (1) must satisfy the natural and essential boundary conditions as given by:

$$\sigma \cdot n = t_0 \text{ on } \Gamma_t \text{ and } u = u_0 \text{ on } \Gamma_u$$  

(2)

where $n$ is a unit outward vector normal to the boundary surface; $u = \{u_x, u_y, u_z\}$ is the displacement vector; $t_0$ and $u_0$ are pre-defined traction vector and displacement vector on the boundary, respectively. The general Galerkin weak form of the static equilibrium is given as:

$$\int_{\Omega} \delta e^T D e d\Omega - \int_{\Gamma_t} \delta u^T b d\Gamma - \int_{\Gamma_t} \delta u^T t_0 d\Gamma = 0$$  

(3)

in which $D$ is the $6 \times 6$ matrix of material and $e^T = \{\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \varepsilon_{xy}, \varepsilon_{yz}, \varepsilon_{zx}\}$ is the tensor of strain.

By using the PFEM approximation method, the bounded domain $\Omega$ is divided into a set of $n$ finite non-overlapping arbitrary polyhedral elements, $\Omega \approx \Omega^h = \bigcup_{e} \Omega_e$. The displacement interpolation $u^h$ over each element $\Omega_e$ based on piecewise linear shape functions is explicitly formulated as:

$$u^h(x) = \sum_{i=1}^{ne} N_i(x) d_i$$  

(4)

where $ne$ is the number of element’s nodes; $N_i(x)$ is the shape function at point $x$; $d_i$ is the approximated displacement vector at corresponding node $i$.

As it is known that the derivatives of piecewise linear shape functions are discontinuous between the elements, which share the same node $i$. This property of shape function does not cover the physical smoothness of mechanics problems. Therefore, the CI scheme is constructed based on the fact that the nodal derivatives should be smooth at every finite node. The average nodal derivatives are used and formulated as follows:

$$\overline{\mathbf{u}}_i = \overline{\mathbf{N}}_i^T \mathbf{d}$$  

(5)

where the superscript $i$ stands for node $i$, and $\mathbf{u} = u(x_i)$, $\mathbf{N}' = \mathbf{N}(x_i)$ is the matrix of shape functions at $i$; $\overline{\mathbf{u}}_i$ is the average displacement derivative at node $i$ with respect to axis $l$ ($l = x, y, \text{ or } z$) and $\overline{\mathbf{N}}_i^T$ are the average derivatives of shape functions.
\[ \tilde{\mathbf{N}}_j = \sum_{e \in S_i} w_e \mathbf{N}^{[e]}_j \]  

where \( S_i \) is a collection of all the polyhedral elements containing node \( i \); \( w_e \) is the weight functions of each element \( e \), which is formulated in conjunction with polyhedral element in the subsequent section and \( \mathbf{N}^{[e]}_j \) are the nodal derivatives of \( \mathbf{N}_j \) in element \( e \).

To overcome the discontinuous of strain and stress over the finite elements due to the nature of piecewise linear shape functions, CI method considers the contributions of both the nodal displacements \( \tilde{\mathbf{u}} \) and the averaged displacement derivatives \( \overline{\mathbf{u}} \). The approximation \( \mathbf{u}^h \) over each element \( \Omega_e \) in Equation (4) analyzed by means of CI approach is given as follows:

\[ \tilde{\mathbf{u}}(\mathbf{x}) = \sum_{i=1}^{n_e} (\phi_i \mathbf{u}^i + \phi_{iz} \overline{u}_z^i + \phi_{iy} \overline{u}_y^i + \phi_{ix} \overline{u}_x^i) = \sum_{i=1}^{n_e} (\phi_i \mathbf{N}_i^j + \phi_{iz} \overline{N}_z^i + \phi_{iy} \overline{N}_y^i + \phi_{ix} \overline{N}_x^i) \mathbf{d} \]  

Equation (7) can be re-written as:

\[ \tilde{\mathbf{u}}(\mathbf{x}) = \tilde{\mathbf{N}}(\mathbf{x}) \mathbf{d} \]  

where \( \tilde{\mathbf{N}}(\mathbf{x}) \) is the CI shape function given by:

\[ \tilde{\mathbf{N}}_j(\mathbf{x}) = \sum_{i=1}^{n_e} (\phi_i \mathbf{N}_i^j + \phi_{iz} \overline{N}_z^i + \phi_{iy} \overline{N}_y^i + \phi_{ix} \overline{N}_x^i) \]  

and \( \phi_i, \phi_{ix}, \phi_{iy}, \) and \( \phi_{iz} \) are the functions defined based on the type of element, which will be clearly described in the following section for arbitrary polyhedral element with piecewise linear shape functions.

It should be noted that the CI scheme only modifies the shape functions and maintains the displacements as nodal unknowns compared to the conventional FEM. Therefore, the total numbers of DOFs are unchanged during the modification. However, due to the formulated process including support nodes as well as averaged nodal derivatives, the assembling time of stiffness matrix is longer than the standard PFEM.

### 3 FORMULATION BASED ON CIPFEM

#### 3.1 Piecewise linear shape functions for polyhedral element

The PFEM was formulated for 2D plate elements\textsuperscript{3} and then improved for general 3D models of solid mechanics problems.\textsuperscript{5,14} The polyhedral elements are sub-divided into sub-tetrahedra. Consequently, the piecewise linear shape functions over the arbitrary elements with \( n_{ne} \) vertices are formulated based on the standard linear shape functions of 4-node tetrahedral elements\textsuperscript{14}:

\[ N_{I}^{PL}(\mathbf{x}) = \sum_{j=1}^{4} \phi_{Ij}^{T4}(\mathbf{x}) N_{Ij}^{PL}(\mathbf{x}_j) \quad \text{for } \mathbf{x} \in \Omega^{T4} \]  

\[ \nabla N_{I}^{PL}(\mathbf{x}) = \sum_{j=1}^{4} \nabla \phi_{Ij}^{T4}(\mathbf{x}) N_{Ij}^{PL}(\mathbf{x}_j) \quad \text{for } \mathbf{x} \in \Omega^{T4} \]  

where \( N_{I}^{PL}(\mathbf{x}) \) and \( \nabla N_{I}^{PL}(\mathbf{x}) \) are the shape functions and their derivatives of node \( I \) (\( I = 1:n_{ne} \)) at the interpolation point \( \mathbf{x} \) in sub-tetrahedron \( \Omega^{T4} \); \( \phi_{Ij}^{T4}(\mathbf{x}) \) and \( \nabla \phi_{Ij}^{T4}(\mathbf{x}) \) are the standard shape functions and their derivatives of sub-tetrahedron; \( N_{I}^{PL}(\mathbf{x}_j) \) is the shape function on polyhedron at node \( \mathbf{x}_j \), which satisfies the Kronecker-delta property if \( \mathbf{x}_j \) is the vertex of the element, equals to average face’s nodes at face’s center \( \overline{\mathbf{x}} \) (\( N_{I}^{PL}(\overline{\mathbf{x}}) = 1/n_j \)) and equals to average polyhedral vertices at element center \( \mathbf{x}_{pc} \) (\( N_{I}^{PL}(\mathbf{x}_{pc}) = 1/n_{ne} \)) (Figure 2(A)). The same formulation process can be simplified for 2D cases of polygonal element by using sub-triangles (Figure 2(B)).
It should be noted that the arbitrary star-convex polyhedral element is a polyhedral element with arbitrary number of facets, edges, vertices and the surrounded facets can be non-planar. Moreover, the element can be convex or concave (star-convex). This definition is used throughout this study, if not otherwise indicated.

### 3.2 CIPFEM formulation

In each sub-domain, the PFEM uses linear shape functions and one quadrature point to integrate the weak form. Consequently, the nodal derivatives are constant within each sub-tetrahedron, which cause the discontinuously results (stress and strain) within an arbitrary polyhedral element. The CIPFEM scheme possesses higher-order shape functions, which increases not only the smooth across elements, but also the smooth within the element itself. The CIPFEM interpolation at an arbitrary point \( \mathbf{x} \) includes the effects of adjacent nodes of the current element containing \( \mathbf{x} \) as illustrated in Figure 3.

![Diagram](image)

(A) 3D arbitrary polyhedral element is sub-divided at vertex \( v_i \).

![Diagram](image)

(B) 2D polygonal element

**Figure 2** The studied element is sub-divided into sub-tetrahedral elements for 3D cases and sub-triangles for 2D polygon. (A) The representative sub-dividing at vertex \( v_i \) for 3D polyhedral element and (B) sub-division the whole polygonal element. Over each sub-domain, the nodal derivatives are discontinuous due to the usage of linear shape functions.
Unlike linear tetrahedral or hexahedral elements that contain the constant nodal derivatives despite the number of quadrature point(s), the nodal derivatives of PFEM are various due to the position of interest point \( x_{GP1}, x_{GP2}, \) and so forth, as shown in Figure 2. Therefore, the weighted functions for each vertex of the element are firstly defined within each element as follows:

\[
w_s = \frac{V_s}{\sum_{s \in S_i} V_s}, \text{ with } s \in S_i
\]  

(12)

where \( S_i \) is the collection of all sub-tetrahedral domain connecting to vertex \( v_i \) of the polyhedron, \( V_s \) is the volume of each sub-domain in \( S_i \) and \( V_\bar{s} \) is the sum of all sub-domains’ volume in \( S_i \).

The weight functions of elements connecting to node \( v_i \) are given as:

\[
w_e = \frac{V_e}{\sum_{e \in S_i} V_\bar{e}}, \text{ with } e \in R_i
\]  

(13)

**Figure 3** A support domain \( \Omega_{sup} \) of an element of interest, which is the red hexahedron with eight nodes. In general, it is a polyhedron with arbitrary number of edges and \( n_{np} \) vertices. The support regions \( R_i \) of each element’s vertex \( i \) are also presented. \( \Omega_{sup} = \bigcup_{i=1}^{n_{np}} R_i \)
Similarly, \( R_i \) is the collection of elements connecting to vertex \( \mathbf{v}_i \), \( V_e \) is the volume of element in \( R_i \), and \( V_\Omega \) is the sum of all elements’ volume in \( R_i \) (see Figure 3).

By defining \( n_s \) and \( \mathbf{d}_s \) as the number of nodes and the displacement vector at vertex \( s (s \in \{1, \ldots, n_s\}) \) in the support domain \( \Omega_{\text{sup}} \), respectively, the interpolation is analyzed using conventional FEM as follows:\(^{27}\)

\[
\mathbf{u}(\mathbf{x}) = \sum_{s=1}^{n_s} N_s(\mathbf{x}) \mathbf{d}_s 
\]

(14)

Because the unknown derivatives at point \( \mathbf{x} \) vary if \( \mathbf{x} \) belongs to different sub-tetrahedron in the polyhedral element, the average nodal derivative at vertex \( i \) is calculated through two steps. The first step is to calculate the average derivatives based on the support region \( S_i \) (see Equation (12)) within a current element as follows:

\[
\bar{\mathbf{u}}_x = \sum_{s \in S_i} \sum_{j=1}^{n_e} w_{js} N_s^{[js]} \mathbf{d}_j
\]

(15a)

or

\[
\bar{\mathbf{u}}_y = \sum_{s \in S_i} \sum_{j=1}^{n_e} w_{js} N_s^{[js]} \mathbf{d}_j
\]

(15b)

\[
\bar{\mathbf{u}}_z = \sum_{s \in S_i} \sum_{j=1}^{n_e} w_{js} N_s^{[js]} \mathbf{d}_j
\]

(15c)

where the average derivatives of shape functions are given by:

\[
\bar{N}_s^{[js]} = \sum_{s \in S_i} w_{js} N_s^{[js]} 
\]

(17)

or we can re-write in the matrix form as:

\[
\nabla \bar{\mathbf{N}}_j^{[js]} = \sum_{s \in S_i} w_{js} \nabla N_s^{[js]} 
\]

(18)

with

\[
\nabla \bar{\mathbf{N}}_j^{[js]} = (\bar{N}_s^{[js]} \cdot \bar{N}_s^{[js]} \cdot \bar{N}_s^{[js]})^T 
\]

and

\[
\nabla \mathbf{N}_j^{[js]} = (N_s^{[js]} \cdot N_s^{[js]} \cdot N_s^{[js]})^T 
\]

(19)

The second step is to calculate the average derivatives at each discretized node \( i \) based on the support region \( R_i \) as follows:

\[
\nabla \bar{\mathbf{u}}_i = \sum_{e \in R_i} w_e \nabla \bar{\mathbf{u}}^{[e]} = \sum_{e \in R_i} \left[ \sum_{l=1}^{n_s} w_e \nabla \bar{\mathbf{N}}_l^{[e]} \right] \mathbf{d}_l 
\]

(20)

where \( \nabla \bar{\mathbf{N}}_l^{[e]} \) is the derivative analyzed by Equation (18) of element \( e \in R_i \).

Similar to the previous step, Equation (20) can be re-written as:

\[
\nabla \bar{\mathbf{u}}_i = \sum_{l=1}^{n_s} \nabla \bar{\mathbf{N}}_l \mathbf{d}_l
\]

(21)
with

$$\nabla \mathbf{N}_l = \sum_{e \in E_l} w_e \nabla \mathbf{N}_l^{[e]}$$  \hspace{2cm} (22)$$

The CIPFEM formulation for arbitrary \( ne \)-vertex polyhedral elements, which uses both the nodal parameters and their average nodal derivatives into the interpolation procedure, is expressed as follows:

$$\mathbf{\tilde{u}}(x) = \sum_{i=1}^{ne} (\phi_i \mathbf{u}^i + \phi_{ix} \mathbf{\tilde{u}}^i_x + \phi_{iy} \mathbf{\tilde{u}}^i_y + \phi_{iz} \mathbf{\tilde{u}}^i_z)$$  \hspace{2cm} (23)$$

where \( \phi_i, \phi_{ix}, \phi_{iy}, \phi_{iz} \) are the functions that satisfy the following conditions\(^{27} \):

$$\begin{align*}
\phi_i(x_l) &= \delta_{il}, \quad \phi_{ix}(x_l) = 0, \quad \phi_{iy}(x_l) = 0, \quad \phi_{iz}(x_l) = 0 \\
\phi_{ix}(x_l) &= 0, \quad \phi_{ixx}(x_l) = \delta_{il}, \quad \phi_{ixy}(x_l) = 0, \quad \phi_{ixz}(x_l) = 0, \\
\phi_{iy}(x_l) &= 0, \quad \phi_{iyx}(x_l) = 0, \quad \phi_{iyy}(x_l) = \delta_{il}, \quad \phi_{iyz}(x_l) = 0, \\
\phi_{iz}(x_l) &= 0, \quad \phi_{izx}(x_l) = 0, \quad \phi_{izy}(x_l) = 0, \quad \phi_{izz}(x_l) = \delta_{il},
\end{align*}$$  \hspace{2cm} (24)$$

with \( \delta_{il} \) is the Kronecker-delta value and \( j \) is any vertex of the element \( (i, j \in \{1, \ldots, ne\}) \).

The equations of \( \phi_i, \phi_{ix}, \phi_{iy}, \phi_{iz} \) and their derivatives for PFEM elements satisfying conditions in Equation (24) are formulated as follows:

$$\begin{align*}
\phi_i &= L_i + L_i^2(\Sigma_1 - L_i) - L_i(\Sigma_2 - L_i^2); \quad i \in \{1, \ldots, ne\} \\
\nabla \phi_i &= \nabla L_i + 2(\nabla L_i)L_i(\Sigma_1 - L_i) + L_i^2(\nabla \Sigma_1 - \nabla L_i) - \nabla L_i(\Sigma_2 - L_i^2) - 2L_i(\Sigma_3 - L_i(\nabla L_i)); \quad i \in \{1, \ldots, ne\}
\end{align*}$$  \hspace{2cm} (25-26)$$

where

$$\begin{align*}
L_i &= N_i^{PL}, \quad \Sigma_1 = \sum_{j=1}^{ne} L_j; \quad \Sigma_2 = \sum_{j=1}^{ne} L_j^2; \quad \Sigma_3 = \sum_{j=1}^{ne} L_j(\nabla L_j) \\
\phi_{ix} &= \sum_{j=1, j \neq i}^{ne} (x_j - x_i)[(L_j L_i^2 + 0.5L_j L_i(\Sigma_1 - L_i - L_j))]; \quad i \in \{1, \ldots, ne\} \\
\nabla \phi_{ix} &= \sum_{j=1, j \neq i}^{ne} (x_j - x_i)[(\nabla L_j) L_i^2 + 2L_i L_j + 0.5(\nabla L_i L_j + L_i L_j \nabla L_i)(\Sigma_1 - L_i - L_j) + L_i L_j(\nabla \Sigma_1 - \nabla L_i - \nabla L_j)]; \quad i \in \{1, \ldots, ne\}
\end{align*}$$  \hspace{2cm} (27-29)$$

where \( x_i \) and \( x_j \) are the \( x \)-coordinate of nodes \( i \) and \( j \), respectively. The formulations of \( \phi_{iy} \) and \( \phi_{iz} \) are similar to the Equation (28), in which the \( x \)-coordinates are replaced by \( y \)- and \( z \)-coordinates instead.

Substituting all the corresponding variables into Equation (23), the CIPFEM is derived as:

$$\mathbf{\tilde{u}}(x) = \sum_{l=1}^{ns} \mathbf{\tilde{N}}_l(x) \mathbf{d}_l$$  \hspace{2cm} (30)$$

where \( \mathbf{\tilde{N}}_l(x) \) is the CI shape functions of node \( l \in \{1, \ldots, ns\} \) at the point of interest \( x \), which is formulated as:

$$\begin{align*}
\mathbf{\tilde{N}}_l(x) &= \sum_{i=1}^{ne} (\phi_i(x) \mathbf{N}_l^i + \phi_{ix}(x) \mathbf{\tilde{N}}_l^i_x + \phi_{iy}(x) \mathbf{\tilde{N}}_l^i_y + \phi_{iz}(x) \mathbf{\tilde{N}}_l^i_z) \\
\nabla \mathbf{\tilde{N}}_l(x) &= \sum_{i=1}^{ne} (\nabla \phi_i(x) \mathbf{N}_l^i + \nabla \phi_{ix}(x) \mathbf{\tilde{N}}_l^i_x + \nabla \phi_{iy}(x) \mathbf{\tilde{N}}_l^i_y + \nabla \phi_{iz}(x) \mathbf{\tilde{N}}_l^i_z)
\end{align*}$$  \hspace{2cm} (31-32)$$
The CI shape functions are high-order polynomials that produce continuous nodal derivatives. However, in some cases, the C0-continuity is required such as the discontinuity between material layers or on essential boundaries.\textsuperscript{27–29} The modification at those C0-nodes should be done by simply replace the average nodal derivatives by regular PFEM nodal derivatives.

\begin{equation}
\overline{u}_x = u_x^{[s]} \tag{33}
\end{equation}

The CI shape functions and their modifications on the essential boundary are illustrated in the Figures 4 and 5. For convenient of observation, the 2D quadrilateral elements are used. Figure 4 shows the CI shape functions with no modification on the essential boundary, while Figure 5 uses the correction in Equation (33).

### 3.3 Stiffness matrix using CIPFEM

It should be noted that the procedure of solving Galerkin weak form using CIPFEM is similar to conventional FEM. The discretized equivalent equation of solid structures is given by\textsuperscript{14,31}:

\begin{equation}
\tilde{K}u = F \tag{34}
\end{equation}

where \( u \) and \( F \) are the nodal displacement vector of all nodes in the analyzed domain and the matrix of external force, respectively. \( \tilde{K} \) is the CIPFEM global stiffness matrix such as:

\begin{equation}
\tilde{K} = \bigwedge_{e=1}^{ne}\tilde{K}_e \tag{35}
\end{equation}

with \( \tilde{K}_e \) is the stiffness matrix of a polyhedral element containing point of interest \( x \). The numerical integration is performed over each sub-tetrahedron of polyhedral element (Figure 6). By using Gaussian quadrature rule on each

![Figure 4](image-url)

**Figure 4** Illustration of CIPFEM shape functions for various node positions without correction on the essential boundary: (A) node at corner of essential boundary with only one connected element, (B) node on the essential boundary with two connected elements, (C) node close to the boundary, and (D) node in the middle
FIGURE 5  The corresponding CIPFEM shape functions after reducing the shape function to C0 continuity on the essential boundary

FIGURE 6  A sub-domain used to calculate stiffness matrix is sub-divided from a polyhedral element in its: (A) global coordinate system and (B) natural coordinate system

sub-tetrahedron, the approximation on quadrature point(s) is applied in assembling the stiffness matrix as follows:

$$\tilde{K}_e = \int_{\Omega_e} B^T_e(x_j)DB_e(x_j)d\Omega = \sum_{i=1}^{n_e} \sum_{j=1}^{n_{GP}} B^T_s(x_j)DB_s(x_j)||J||w_j$$  \hspace{1cm} (36)$$

where \(n_e\) is the number of sub-tetrahedra in element \(e\); \(n_{GP}\) is the number of Gauss points in each sub-domain; \(\Omega_e\) is the volume of element \(e\); \(w_j\) is the weight of the \(j\)th Gauss point \(x_{GP}\) in sub-domain \(i\); \(D\) is the matrix of material and \(J_j\) is
the Jacobian matrix of 4-node tetrahedron calculating as following\(^{33}\):

\[
\mathbf{J}_i^j = \begin{bmatrix}
\frac{\partial \phi_1}{\partial \xi} & \frac{\partial \phi_2}{\partial \xi} & \frac{\partial \phi_3}{\partial \xi} & \frac{\partial \phi_4}{\partial \xi} \\
\frac{\partial \phi_1}{\partial \eta} & \frac{\partial \phi_2}{\partial \eta} & \frac{\partial \phi_3}{\partial \eta} & \frac{\partial \phi_4}{\partial \eta} \\
\frac{\partial \phi_1}{\partial \zeta} & \frac{\partial \phi_2}{\partial \zeta} & \frac{\partial \phi_3}{\partial \zeta} & \frac{\partial \phi_4}{\partial \zeta}
\end{bmatrix}
\begin{bmatrix}
x_1 & y_1 & z_1 \\
x_2 & y_2 & z_2 \\
x_3 & y_3 & z_3 \\
x_4 & y_4 & z_4
\end{bmatrix}
\]

\(B_j\) is the displacement transformation matrix at \(x_j\) in sub-domain \(\Omega_s\). Unlike the matrix of PFEM that includes \(ne\) nodes of the element \(e\), the CIPFEM's matrix contains all the nodes (\(ns\)) in the support domain as follows:

\[
[B_{s}]^T = \begin{bmatrix}
\frac{\partial N_l}{\partial x} & 0 & 0 & \frac{\partial N_l}{\partial y} & 0 & \frac{\partial N_l}{\partial z} \\
0 & \frac{\partial N_l}{\partial y} & 0 & \frac{\partial N_l}{\partial x} & \frac{\partial N_l}{\partial z} & 0 \\
0 & 0 & \frac{\partial N_l}{\partial z} & 0 & \frac{\partial N_l}{\partial y} & \frac{\partial N_l}{\partial x}
\end{bmatrix}, \quad l = 1 \div ns
\]

As it can be seen in Equation (38), the size of the CIPFEM's transformation matrix is \((6 \times 3 \, ns)\), which is greater than that of PFEM \((6 \times 3 \, ne)\) because the present interpolation method uses all the discretized nodes (\(ns\)) within the support domain.

As described in Reference 14, because of using piecewise linear shape function, the accuracy of the PFEM solutions does not depend on the number of Gauss points in each sub-tetrahedron \((\mathbf{B} \text{ is a constant matrix with all point of interest } x \in \Omega_s)\). Nevertheless, CI scheme use high-order shape functions so that the results can be improved significantly when using higher numbers of quadrature points.\(^{28}\) In this study, four Gauss points are used for all numerical examples.

4 | NUMERICAL RESULTS

In this section, the accuracy and convergence of present CIPFEM solutions are benchmarked through linear patch test, 3D numerical examples of a cantilever beam, a bracket and a complex hollow concrete block. The results obtained from present scheme will be compared to other popular methods. The following abbreviations are used:

- T4: Conventional FEM using tetrahedral element with well-known linear shape functions.
- CT4: Conventional FEM using consecutive interpolation scheme over linear tetrahedral elements.
- PFEM: The standard polyhedral finite element method with linear piecewise shape functions.
- CIPFEM: Polyhedral finite elements with consecutive shape functions.
- Reference: The results obtained from ANSYS software.\(^{32}\)

4.1 | Linear patch test

The effectiveness of the proposed CIPFEM is evaluated through the linear patch test of a unit cube, which is meshed into nine arbitrary star-convex polyhedral elements (Figure 7). The isotropic material is applied with Young’s modulus and Poisson’s ratio respectively given as \(E = 1 \, \text{N/m}^2\) and \(\nu = 0.25\). The test is to verify that the CIPFEM can return exact solutions of linear displacements and constant strains. The displacements of vertices on the boundary are prescribed as follows\(^{37}\):

\[
\begin{pmatrix}
u_1 \\
u_2 \\
u_3
\end{pmatrix} = \begin{pmatrix}
0.1 + 0.1x + 0.2y + 0.2z \\
0.05 + 0.15x + 0.1y + 0.2z \\
0.05 + 0.1x + 0.2y + 0.2z
\end{pmatrix}
\]

where \(u_1\), \(u_2\), and \(u_3\) are the prescribed displacements in \(x\), \(y\), and \(z\) directions, respectively.
The exact solutions are analyzed by substituting coordinates of discretized nodes into the Equation (39). To evaluate the accuracy of the obtained results, the displacement error norm ($L^2$-norm) and energy error semi-norm ($H^1$-semi-norm) are investigated based on following equations:

$$e_{L^2} = \sqrt{\int_{\Omega} (\|u^h - u^{exact}\|)^2 d\Omega}$$  \hspace{1cm} (40)

and

$$e_{H^1} = \sqrt{\int_{\Omega} (\sigma^h - \sigma^{exact}) : (\varepsilon^h - \varepsilon^{exact}) d\Omega}$$  \hspace{1cm} (41)

where $u^h$ and $u^{exact}$ are the approximated displacements and analytical results, respectively; $\sigma^h$ and $\varepsilon^h$ are the approximated stress and strain; $\sigma^{exact}$ and $\varepsilon^{exact}$ are the analytical results of stress and strain, respectively.

The displacement norm error and energy norm error of the linear patch test are respectively $1.8814 \times 10^{-4}$ and $1.9418 \times 10^{-3}$. Additionally, the error of each node in the middle of the mesh is given in the Table 1.

### 4.2 Cantilever beam

A cantilever beam with rectangular cross section subjected to uniform shear load, $P$, at one end ($z = 0$) and pre-defined analytical displacements at the other end (weakly fixed) is investigated ($z = L$). The beam is assigned with the dimensions of $L = 5$ mm in length, $2a = 1$ mm in width and $2b = 1$ mm in height (Figure 8). Its homogeneous material has the Young’s modulus, $E = 2 \times 10^5$ MPa and Poisson’s ratio, $\nu = 0.3$. The displacements and stress analytical solutions of the beam are presented in References 31, 38 as follows:

$$u_x = -\frac{P\nu}{EI} xy z$$

$$u_y = \frac{P}{EI} \left[ \frac{\nu}{2} (x^2 - y^2) z - \frac{1}{6} z^3 \right]$$

$$u_z = \frac{P}{EI} \left[ \frac{1}{2} y (x^2 + z^2) + \frac{1}{6} vy^3 + (1 + \nu) \left( b^2 y - \frac{1}{3} y^3 \right) - \frac{1}{3} a^2 vy - \frac{4a^3 v}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \cos \left( \frac{n\pi x}{a} \right) \sinh \left( \frac{n\pi y}{a} \right) \cosh \left( \frac{n\pi b}{a} \right) \right]$$  \hspace{1cm} (42)
Table 1: Displacement errors of nodes inside the cube domain

<table>
<thead>
<tr>
<th>Node</th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$u_3$</th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$u_3$</th>
<th>Displacement norm error</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>0.4122</td>
<td>0.3058</td>
<td>0.3622</td>
<td>0.4120</td>
<td>0.3057</td>
<td>0.3622</td>
<td>0.00022</td>
</tr>
<tr>
<td>10</td>
<td>0.4372</td>
<td>0.3683</td>
<td>0.3872</td>
<td>0.4369</td>
<td>0.3679</td>
<td>0.3866</td>
<td>0.00076</td>
</tr>
<tr>
<td>11</td>
<td>0.3372</td>
<td>0.2933</td>
<td>0.2872</td>
<td>0.3375</td>
<td>0.2937</td>
<td>0.2877</td>
<td>0.00072</td>
</tr>
<tr>
<td>12</td>
<td>0.3122</td>
<td>0.2308</td>
<td>0.2622</td>
<td>0.3121</td>
<td>0.2307</td>
<td>0.2622</td>
<td>0.00013</td>
</tr>
<tr>
<td>13</td>
<td>0.4122</td>
<td>0.3558</td>
<td>0.3622</td>
<td>0.4120</td>
<td>0.3557</td>
<td>0.3620</td>
<td>0.00024</td>
</tr>
<tr>
<td>14</td>
<td>0.3122</td>
<td>0.2808</td>
<td>0.2622</td>
<td>0.3121</td>
<td>0.2807</td>
<td>0.2621</td>
<td>0.00015</td>
</tr>
<tr>
<td>15</td>
<td>0.4372</td>
<td>0.3183</td>
<td>0.3872</td>
<td>0.4371</td>
<td>0.3181</td>
<td>0.3869</td>
<td>0.00030</td>
</tr>
<tr>
<td>16</td>
<td>0.2372</td>
<td>0.1683</td>
<td>0.1872</td>
<td>0.2375</td>
<td>0.1687</td>
<td>0.1876</td>
<td>0.00066</td>
</tr>
<tr>
<td>17</td>
<td>0.3122</td>
<td>0.2058</td>
<td>0.2622</td>
<td>0.3122</td>
<td>0.2058</td>
<td>0.2624</td>
<td>0.00023</td>
</tr>
<tr>
<td>18</td>
<td>0.3122</td>
<td>0.1933</td>
<td>0.1872</td>
<td>0.3126</td>
<td>0.1936</td>
<td>0.1876</td>
<td>0.00067</td>
</tr>
<tr>
<td>19</td>
<td>0.3122</td>
<td>0.2308</td>
<td>0.2622</td>
<td>0.3122</td>
<td>0.2309</td>
<td>0.2621</td>
<td>0.00018</td>
</tr>
<tr>
<td>20</td>
<td>0.4372</td>
<td>0.3433</td>
<td>0.3872</td>
<td>0.4371</td>
<td>0.3431</td>
<td>0.3870</td>
<td>0.00030</td>
</tr>
</tbody>
</table>

Figure 8: Dimensions of the cantilever beam with analytical results

\[
\sigma_{xx} = \sigma_{xy} = \sigma_{yy} = 0
\]

\[
\sigma_{yz} = \frac{P}{I \pi^2} \frac{b^2 - y^2}{2} + \frac{P}{I \pi^2} \frac{\nu}{1 + \nu} \left[ \frac{3x^2 - a^2}{6} - \frac{2a^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \left( \frac{n\pi x}{a} \right) \cosh \left( \frac{n\pi y}{a} \right) \frac{\cos \left( \frac{n\pi y}{b} \right)}{1 + \nu} \right]
\] (43)

where \( I = 4ab^3/3 \).

The accuracy and convergence of the present method are shown by investigating the relative displacement norm of all the discretized domain given by\(^6\):

\[
E_d = \sqrt{\frac{\int_{\Omega} (\| u^h - u^{\text{exact}} \|)^2 d\Omega}{\int_{\Omega} (u^{\text{exact}})^2 d\Omega}}
\] (44)

where \( u^h \) and \( u^{\text{exact}} \) are the approximated and analytical displacements, respectively.

And the relative energy norm is:

\[
E_e = \sqrt{\frac{\int_{\Omega} (\sigma^h - \sigma^{\text{exact}}) : (\varepsilon^h - \varepsilon^{\text{exact}}) d\Omega}{\int_{\Omega} (\sigma^{\text{exact}}) : (\varepsilon^{\text{exact}}) d\Omega}}
\] (45)

where \( \sigma^h \) and \( \varepsilon^h \) are the approximated stress and strain; \( \sigma^{\text{exact}} \) and \( \varepsilon^{\text{exact}} \) are the analytical results of stress and strain, respectively.
The feasibility, generality, and efficiency of the CIPFEM are investigated with various element types and mesh resolutions of the cantilever beam. The pentahedral, structured polyhedral, and unstructured polyhedral elements are used. Figure 9 shows representative meshes of each element type. The relative errors of the obtained results are compared to those of CI scheme for conventional FEM using tetrahedral elements (CT4). For all studied cases, four integration points (Gauss’s points) in each sub-domain are considered.

Table 2 shows the results of relative displacement norm error in Equation (44) and relative energy norm error in Equation (45) with respect to various mesh resolutions. For each element type, the mesh size $h$ determined from the average of maximum elements’ diameter has the range of approximately from 0.55 to 0.25 mm. Figures 10–12 are the comparison graphs of those data given in the table. As it can be seen from these figures, the total errors of the displacement and energy obtained from present CIPFEM are much smaller than those of CT4 with respect to the same mesh resolutions. Furthermore, the accuracy of CIPFEM solutions is similar for various element types. Nevertheless, the unstructured polyhedral elements give the most accurate results and the highest computer cost in return. As it can be seen that the computational time of the present method are higher than CT4. Nevertheless, the CIPFEM’s solutions are far more accurate than those of CT4.

Figure 13 shows the outstanding convergence rates in terms of displacement norm error obtained by CIPFEM compared to those of CT4, while the comparisons for convergence rates of energy norm error are similar for the two methods. Figure 14 illustrates the obtained von-Mises stress distribution of the studied beam compared to analytical (exact) solutions.

### 4.3 3D concrete corbel

Compared to cantilever beam in Section 4.2, a more complicated concrete structure as showed in Figure 15 is investigated. The structure is fixed in $x$- and $y$-directions at one end and subjected to uniform shear load at the other end.

![Figure 9](image_url)

**Figure 9** Representative meshes of the cantilever beam for numerical investigation using CIPFEM and CT4. Sub-figures (A), (B), and (C) are for CIPFEM analysis, while figure (D) is for CT4 analysis.
TABLE 2  Comparison between the solutions obtained by CIPFEM and CT4 with various mesh resolutions and element types

<table>
<thead>
<tr>
<th>Tetrahedral Mesh</th>
<th>CT4</th>
<th>Pentahedral mesh</th>
<th>CIPFEM-Pen</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>No. of Ele.</td>
<td>h (mm)</td>
<td>Disp. ($E_d$)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>568</td>
<td>0.554</td>
<td>0.0901</td>
</tr>
<tr>
<td></td>
<td>1068</td>
<td>0.444</td>
<td>0.0574</td>
</tr>
<tr>
<td></td>
<td>2154</td>
<td>0.349</td>
<td>0.0266</td>
</tr>
<tr>
<td></td>
<td>3777</td>
<td>0.285</td>
<td>0.0128</td>
</tr>
<tr>
<td></td>
<td>6666</td>
<td>0.236</td>
<td>0.0074</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Structured polyhedral mesh</th>
<th>CIPFEM-Spoly</th>
<th>Unstructured polyhedral mesh</th>
<th>CIPFEM-Upoly</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of Ele.</td>
<td>h (mm)</td>
<td>Disp. ($L^2$)</td>
<td>Ener. ($H^1$)</td>
</tr>
<tr>
<td>------------------</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
</tr>
<tr>
<td></td>
<td>216</td>
<td>0.532</td>
<td>0.0192</td>
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<td>420</td>
<td>0.420</td>
<td>0.0079</td>
</tr>
<tr>
<td></td>
<td>780</td>
<td>0.340</td>
<td>0.0035</td>
</tr>
<tr>
<td></td>
<td>1344</td>
<td>0.284</td>
<td>0.0017</td>
</tr>
<tr>
<td></td>
<td>2072</td>
<td>0.245</td>
<td>0.0008</td>
</tr>
</tbody>
</table>

FIGURE 10  Convergence trend of relative displacement norm error of the cantilever beam

FIGURE 11  Convergence trend of relative energy norm error of the cantilever beam
**FIGURE 12**  Computational time of stiffness matrix assembling by CIPFEM and CT4 methods

**FIGURE 13**  Convergence rates of the displacement norm error and energy norm error obtained by the present CIPFEM and CT4
Figure 14  von-Mises stress distributions of the cantilever beams analyzed by using (A) CIPFEM and (B) analytical formulation.

Figure 15  Dimensions of a concrete structure subjected to shear force $F$ at one end and fixed in $x$- and $y$-directions at the other end. Edges A-B and C-D are targeted for extracting analyzed compressive and tensile results, respectively. Unit is in mm.

$$F = q \times a = 16 \text{kN},$$
while the other faces are traction free. The edges A-B and C-D are used for extracting the displacement distribution and comparing the results of von-Mises stress, respectively. The concrete material has Young’s modulus, $E = 30 \times 10^3 \text{MPa}$, and Poisson’s ratio, $\nu = 0.25$.

The concrete structure with four different meshes is studied (see Table 3). For PFEM and CIPFEM analysis, the mesh size of polyhedral elements is in the range from 164 to 2862 elements. Whereas, for T4 and CT4 analyses, the number of tetrahedral elements is between 308 and 3806. As it can be seen from the table, the maximum vertical ($y$) displacement of present CIPFEM reaches the reference value with only 2.2% of difference at the coarsest polyhedral mesh, while the difference gaps of that displacements obtained by T4, PFEM, and CT4 compared to reference are 17.9%, 7.5%, and 5.4%, respectively. When the structure reaches the finest meshes, the error of present CIPFEM reduces to 0.5%, while the other errors are consequently 4.6%, 1.6%, and 1.5%. The maximum von-Mises stress of CIPFEM can be considered to be converged with maximum error of 0.2 for all studied cases. Figure 16 shows the outstanding convergence of displacement and total strain energy determined by present method compared to the others.
TABLE 3 Various mesh resolutions using for investigation the convergence and accuracy of studied methods and their maximum vertical displacement errors in percentage

<table>
<thead>
<tr>
<th>Meshes</th>
<th>Polyhedra</th>
<th>Tetrahedra</th>
<th>Percentage of error (%)</th>
<th>Max-Mises stress of C-D</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>No. of Ele.</td>
<td>h (mm)</td>
<td>No. of Ele.</td>
<td>h (mm)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>164</td>
<td>150</td>
<td>150</td>
<td>17.9</td>
<td>5.4</td>
</tr>
<tr>
<td>319</td>
<td>121</td>
<td>122</td>
<td>14.3</td>
<td>4.3</td>
</tr>
<tr>
<td>1073</td>
<td>78</td>
<td>79</td>
<td>6.9</td>
<td>2.1</td>
</tr>
<tr>
<td>2862</td>
<td>64</td>
<td>64</td>
<td>4.6</td>
<td>1.5</td>
</tr>
</tbody>
</table>

FIGURE 16 Convergence of maximum vertical displacement and total strain energy analyzed by CIPFEM compared to T4, CT4, PFEM, and ANSYS’s results
The distribution of y- and x-displacements along edge A-B is orderly illustrated in Figures 17 and 18 with respect to the second mesh in Table 3. For both cases, the displacement distributions calculated by using the present CIPFEM agree very well with reference values and give the most accurate solutions compared with the other studied methods.

Similarly, Figures 19 and 20 show the distribution of von-Mises stress along edges A-B and C-D, respectively. It can be observed that, T4 and PFEM produce a little fluctuated equivalent stress even though they were averaged during extrapolation from the Gauss's points. On the contrary, the CT4 gives better solutions while the present CIPFEM can produce the most accurate and smooth von-Mises stresses. Figure 21 illustrates the von-Mises stress distributions of the whole structure with the finest studied meshes.

Shortly, from the analysis of total strain energy, displacements and equivalent stress of the concrete corbel, the present CIPFEM shows that the accuracy and smoothness of the solutions of elastostatic solid problems are outstanding compared with those calculated by T4, CT4, and PFEM.

4.4 | 3D industrial complex hollow concrete block

The hollow concrete block is designed as an element of interlocking revetment protecting a dike's slope along an industry zone in Vietnam. It has the shape as a hollow box with thin walls and made of micro-synthetic fiber concrete, which has Young's modulus of $E = 30,500$ MPa and Poisson's ratio of $\nu = 0.2$. The geometry of the block is complicated as
shown in Figure 22, which can be used to prove the efficiency of the present method in analyzing complex industrial structures. The wave load, which acts on the front surface, is considered as a distribution load varying from 26 to 28 kPa. The $y$-displacements of the block’s sides are set to 0 due to the constraint of adjacent blocks and the bottom surface of the structure is fixed. To present a clear comparison between the solutions of CIPFEM and those of CT4, the results of edge #1–2 are considered.

The structure is modeled and discretized into arbitrary star-convex polyhedral, hybrid, and tetrahedral elements (see Figure 23). In this section, the term “hybrid mesh” is used for the mesh using various element types with low number of facets including tetrahedral, pyramid, prism, and hexahedral elements.

Figures 24 and 25 show the convergence of maximum displacement of studied edge along $x$- and $z$- axes, respectively. The figures compare the solutions obtained by CIPFEM using unstructured polyhedral and hybrid elements and the solutions obtained by CT4. As it can be seen in both figures, the displacements of the present CIPFEM reached the reference results sooner compared to CT4 with respect to the mesh size $h$. The outstanding accuracy of the CIPFEM’s solutions are illustrated in Figures 26 and 27 for displacements and Figure 28 for normal stress along the investigated edge. Finally, Figure 29 shows the general distribution of von-Mises stress of the whole block analyzed by present CIPFEM with the mesh size $h \approx 93$ mm compared to solutions of CT4 and reference values.
**Figure 21** Equivalent stress of the concrete structure with the finest mesh. Unit is in MPa.

(A) CT4  
(B) PFEM  
(C) CIPFEM  
(D) Reference

**Figure 22** The analyzed thin-wall hollow concrete block forming an interlocking structure protecting a dike slope. The block has a complicated shape so that the results of edge #1–2 are used to prove the efficiency and rationality of the present method compared to CT4. Unit is in cm.

(A) The interlocking sea wall  
(B) An individual concrete block
Representative meshes of the concrete block: (A) meshing into unstructured polyhedral elements, (B) meshing into hybrid mesh including hexahedral, prism, pyramid and tetrahedral elements, and (C) meshing into tetrahedral elements for CT4 analysis.

**Figure 23**

Convergence history of maximum $x$-displacement of edge #1–2 with respect to the mesh size.

**Figure 24**

Convergence history of maximum $z$-displacement of edge #1–2 with respect to the mesh size.

**Figure 25**
FIGURE 26  Distribution of displacement in $x$-direction along edge #1–2

FIGURE 27  Distribution of vertical displacement along edge #1–2

FIGURE 28  Distribution of normal stress $\sigma_{zz}$ along edge #1–2
5 | CONCLUSION

In this study, the consecutive interpolation scheme was presented for solid structures discretized into arbitrary star-convex polyhedral elements. The feasibility of the present method was validated by comparing the CIPFEM’s solutions with those calculated by analytical formulations, CT4, PFEM, T4, and the FE commercial software ANSYS. A cantilever beam with analytical solutions, a concrete bracket and a complex hollow concrete block were used to benchmark the present CIPFEM. Some conclusions are drawn as follows:

- The present CIPFEM is a displacement-based method, which is constructed based on the linear shape functions of arbitrary star-convex polyhedral element, but it can produce remarkably high accurate solutions.
- The convergence of solutions obtained by CIPFEM is much higher than conventional FEM and CT4.
- The stress and strain obtained by CIPFEM are smooth without the need of post processing thanks to the averaged gradient terms in the CI shape functions.
- The present CIPFEM is flexible in using various element types in the analysis of complex geometrical structures.

It should be noted that, the CIPFEM scheme uses support domain to construct the interpolants, which lead to a larger bandwidth in the global stiffness matrix compared to conventional PFEM. As a result, it will impact the efficiency of the solver especially for large-scale problems.
In this study, the analysis was carried out for the problems of static elasticity. However, it can be straightforwardly applied to nonlinear and dynamic analyses. This will be the topic of our future research.

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DATA AVAILABILITY STATEMENT
Research data are not shared.

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REFERENCES