ELLIPTICITY AND FREDHOLMNESS OF PSEUDO-DIFFERENTIAL OPERATORS ON $\ell^2(\mathbb{Z}^n)$

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ABSTRACT. The minimal operator and the maximal operator of an elliptic pseudo-differential operator with symbols on $\mathbb{Z}^n \times \mathbb{T}^n$ are proved to coincide and the domain is given in terms of a Sobolev space. Ellipticity and Fredholmness are proved to be equivalent for pseudo-differential operators on $\mathbb{Z}^n$. The index of an elliptic pseudo-differential operator on $\mathbb{Z}^n$ is also computed.

1. INTRODUCTION

The aim of this paper is to investigate the ellipticity and Fredholmness of pseudo-differential operators on $\ell^2(\mathbb{Z}^n)$ in the context of minimal and maximal operators, the domains of elliptic pseudo-differential operators and Fredholm operators. The pseudo-differential operators on the lattice $\mathbb{Z}^n$, as in this paper, are suitable for solving difference equations on $\mathbb{Z}^n$. Such equations naturally appear in various problems of modelling and in the discretisation of continuous problems. Several attempts of developing a suitable theory of pseudo-differential operators on the lattice $\mathbb{Z}^n$ have been done in the literature, see e.g. [9, 10, 14, 15, 16] but with no symbolic calculus. Recently, a global symbolic calculus has been developed in [2]. The symbol classes therein exhibit improvements when differences are taken with respect to the space (lattice) variable, thus resembling the behaviour of the so-called SG pseudo-differential operators on $\mathbb{R}^n$, first developed by Cordes [6], but with a twist in variables. Ellipticity is a condition on the operators at infinity where the lower-order terms cannot be neglected. On the other hand, Fredholmness measures the almost invertibility of the operators and depends very much on the space on which the operators are defined. For a good theory of pseudo-differential operators on a given function space, a relation between of ellipticity and Fredholmness property is crucial. In [7], the ellipticity of the Fredholm pseudo-differential operator with arbitrary SG-symbols has been proved. For some recent developments on the theory of pseudo-differential operators on $\mathbb{Z}^n$ and its applications, we refer to [4, 5, 3]. In [8, 31], the equivalence of ellipticity and Fredholmness and the applications of SG pseudo-differential operators with positive order on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, has been investigated. The maximal and minimal extension for another class of elliptic pseudo-differential operators on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, can be found in [30]. We would like to emphasize here that Rabinowich and Roch [17] have also studied Fredholmness of discrete pseudo-differential operator associated to symbols which are bounded on $\mathbb{Z}^n \times \mathbb{T}^n$ together with their

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derivative with respect to second variable. They used limit operator method [18] together with pseudo-differential calculus for aforementioned symbol classes to establish their results. In this paper, to study the Fredholmness of the discrete pseudo-differential operators, we consider the operators associated global Hörmander symbol classes and make use of full global symbolic calculus recently developed in [2] along with the maximal and minimal operators of an elliptic pseudo-differential operators which we will develop here in our setting.

In Section 2, we recall symbol classes and the corresponding calculus of the pseudo-difference operators on $\mathbb{Z}^n$. An interesting difference with the usual theory of pseudo-differential operators is that since the space $\mathbb{Z}^n$ is discrete, the Schwartz kernels of the corresponding pseudo-differential operators do not have singularities at the diagonal. Also the phase space is $\mathbb{Z}^n \times \mathbb{T}^n$ with the frequencies being elements of the compact space $\mathbb{T}^n$ (the torus $\mathbb{T}^n := \mathbb{R}^n / \mathbb{Z}^n$). The usual theory of pseudo-differential operators works with symbol classes with increasing decay of symbols after taking their derivatives in the frequency variable. The discrete calculus is essentially different since one can not construct it using standard methods relying on the decay properties in the frequency component of the phase space since the frequency space, $\mathbb{T}^n$ is compact, so no improvement with respect to the decay of the frequency variable is possible. Also it is not suitable to work with derivatives with respect to the space variable $k \in \mathbb{Z}^n$. Therefore, it is replaced by working with appropriate difference operators on the lattice. The global theory of discrete symbolic calculus has been developed in [2] similar to the theory developed in [21, 23, 22, 25, 20, 24, 26, 27] but with a twist, swapping the order of the space and frequency variables. In Section 2, we developed the family of discrete Sobolev spaces, which are an essential tools to study the maximal and minimal extensions of the discrete operators. Furthermore, in Section 3, we investigate the minimal and maximal operators of pseudo-differential operators on $\mathbb{Z}^n$ with symbols in S^m , $m > 0$ and show that they are equal for elliptic operators. The main ingredient for this is an analogue of the Agmon–Douglis–Nirenberg inequalities for elliptic discrete pseudo-differential operators, which we also establish in Section 3. In Section 4, we first recall the Fredholm pseudo-differential operators and study Fredholmness property of discrete pseudo-differential operators and calculated the index of a Fredholm pseudo-differential operator on $\ell^2(\mathbb{Z}^n)$.

2. PRELIMINARIES

In this section, we recapitulate basic facts, notation and definitions of discrete Fourier analysis and pseudo-differential operator on $\mathbb{Z}^n$. We begin with recalling the definition of Fourier transform on lattice $\mathbb{Z}^n$.

The discrete Fourier transform $\hat{f}$ of a function f in $\ell^1(\mathbb{Z}^n)$ is defined by

$$\hat{f}(x) = \sum_{k \in \mathbb{Z}^n} e^{-2\pi i k \cdot x} f(k)$$

for all $x \in \mathbb{T}^n$, where $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$, $k \cdot x = \sum_{j=1}^n k_j x_j$ for $k = (k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$ and $x = (x_1, x_2, \dots, x_n) \in \mathbb{T}^n$. The discrete Fourier transform can be extended to $\ell^2(\mathbb{Z}^n)$ using the usual density arguments. We normalize the Haar measures on $\mathbb{Z}^n$ and $\mathbb{T}^n$ in such a way that the Plancherel formula to the effect that

$$\sum_{k \in \mathbb{Z}^n} |f(k)|^2 = \int_{\mathbb{T}^n} |\widehat{f}(x)|^2 dx.$$

is valid. Next, the inverse discrete Fourier transform is given by

$$f(k) = \int_{\mathbb{T}^n} e^{2\pi i k \cdot x} \widehat{f}(x) dx, \quad k \in \mathbb{Z}^n.$$

We recall the discrete calculus developed in [21]. Let f be a function on $\mathbb{Z}^n$ and $e_j \in \mathbb{N}^n$ be such that e_j has 1 in the j^{th} entry and zeros elsewhere. The 1 difference operator Δ_{k_j} is defined by $\Delta_{k_j} f(k) = f(k + e_j) - f(k)$ and set $\Delta_k^\alpha = \Delta_{k_1}^{\alpha_1} \Delta_{k_2}^{\alpha_2} \dots \Delta_{k_n}^{\alpha_n}$ for all $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}_0^n = \mathbb{N}^n \cup \{0\}$ and all $k = (k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$. We have the following formulae from [2], that are useful in the sequel.

(i) Let $f : \mathbb{Z}^n \rightarrow \mathbb{C}$. Then

$$(\Delta_k^\alpha \sigma)(k) = \sum_{\beta \leq \alpha} (-1)^{|\alpha - \beta|} \binom{\alpha}{\beta} f(k + \beta), \quad k \in \mathbb{Z}^n.$$

(ii) Let $f, g : \mathbb{Z}^n \rightarrow \mathbb{C}$. Then

$$(\Delta_k^\alpha (fg))(k) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (\Delta_k^\beta f)(k) (\bar{\Delta}_k^{\alpha - \beta} g)(k + \beta), \quad k \in \mathbb{Z}^n,$$

for all multi-indices α . This is a special case of the Leibniz formula.

(iii) Let $f, g : \mathbb{Z}^n \rightarrow \mathbb{C}$. Then

$$\sum_{k \in \mathbb{Z}^n} f(k) (\Delta_k^\alpha g)(k) = (-1)^{|\alpha|} \sum_{k \in \mathbb{Z}^n} (\bar{\Delta}_k^\alpha f)(k) g(k), \quad k \in \mathbb{Z}^n,$$

where $(\bar{\Delta}_{k_j} f)(k) = f(k) - f(k - e_j)$, with higher order differences iteratively defined.

We will be using the usual notations, $D_x^\alpha = D_{x_1}^{\alpha_1} \dots D_{x_n}^{\alpha_n}$, $D_{x_j} = \frac{1}{2\pi i} \frac{\partial}{\partial x_j}$ and $D_x^{(\alpha)} = D_{x_1}^{(\alpha_1)} \dots D_{x_n}^{(\alpha_n)}$, $D_{x_j^l} = \prod_{m=0}^l \left(\frac{1}{2\pi i} \frac{\partial}{\partial x_j} - m \right)$, $l \in \mathbb{N}$. The operators $D_x^{(\alpha)}$ are useful in the analysis in torus and details can be found in [21]. The symbol classes are then defined as follows (see [2] for more detail):

Definition 2.1. [2, Definition 2.2] For $-\infty < m < \infty$, we say that a function $\sigma : \mathbb{Z}^n \times \mathbb{T}^n \rightarrow \mathbb{C}$ belongs to $S^m(\mathbb{Z}^n \times \mathbb{T}^n)$ if $\sigma(k, \cdot) \in C^\infty(\mathbb{T}^n)$ for all $k \in \mathbb{Z}^n$ and for all multi-indices α, β there exists a positive constant $C_{\alpha, \beta}$ such that

$$|(D_x^{(\beta)} \Delta_k^\alpha \sigma)(k, x)| \leq C_{\alpha, \beta} (1 + |k|)^{m - |\alpha|}, \quad (k, x) \in \mathbb{Z}^n \times \mathbb{T}^n.$$

The corresponding discrete pseudo-differential operator with symbol σ is given by

$$(T_\sigma f)(k) = \int_{\mathbb{T}^n} e^{2\pi i k \cdot x} \sigma(k, x) \widehat{f}(x) dx, \quad k \in \mathbb{Z}^n.$$

The following theorem gives the product of two discrete pseudo-differential operators.

Theorem 2.2. [2, Theorem 3.1] *Let $\sigma \in S^{m_1}(\mathbb{Z}^n \times \mathbb{T}^n)$ and $\tau \in S^{m_2}(\mathbb{Z}^n \times \mathbb{T}^n)$. Then the product $T_\sigma T_\tau$ of the pseudo-differential operators T_σ and T_τ is a pseudo-differential operator with symbol in $S^{m_1+m_2}(\mathbb{Z}^n \times \mathbb{T}^n)$.*

Definition 2.3. A symbol $\sigma \in S^m(\mathbb{Z}^n \times \mathbb{T}^n)$ is called *elliptic* (of order m) if there exist positive constants C and $M > 0$ such that

$$|\sigma(k, x)| \geq C(1 + |k|)^m,$$

for all $k \in \mathbb{Z}^n$ and all $x \in \mathbb{T}^n$ with $|k| > M$.

The corresponding pseudo-differential operator T_σ is called *elliptic*.

The following theorem gives the parametrix for an elliptic pseudo-differential operators.

Theorem 2.4. [2, Theorem 3.6] *Let $\sigma \in S^m(\mathbb{Z}^n \times \mathbb{T}^n)$, $-\infty < m < \infty$, be elliptic. Then there exists a symbol $\tau \in S^m(\mathbb{Z}^n \times \mathbb{T}^n)$ such that $T_\sigma T_\tau = I + R$ and $T_\tau T_\sigma = I + S$, where R and S are infinitely smoothing in the sense that they are pseudo-differential operators with symbol in $\bigcap_{\nu \in \mathbb{R}} S^\nu(\mathbb{Z}^n \times \mathbb{T}^n)$.*

Next we recall the *Schwartz space* $\mathcal{S}(\mathbb{Z}^n)$, on the lattice $\mathbb{Z}^n$, the space of all functions $\varphi : \mathbb{Z}^n \rightarrow \mathbb{C}$ such that for all multi-indices α and β ,

$$\sup_{k \in \mathbb{Z}^n} |k^\alpha (\Delta_k^\beta \varphi)(k)| < \infty.$$

A sequence $\{\varphi_j\}$ of functions in $\mathcal{S}(\mathbb{Z}^n)$ is said to converge to 0 in $\mathcal{S}(\mathbb{Z}^n)$ if for all multi-indices α, β , we have $\sup_{k \in \mathbb{Z}^n} |k^\alpha (\Delta_k^\beta \varphi_j)(k)| \rightarrow 0$ as $j \rightarrow \infty$.

A linear functional T on $\mathcal{S}(\mathbb{Z}^n)$ is called a tempered distribution if for any sequence $\{\varphi_j\}$ of function in $\mathcal{S}(\mathbb{Z}^n)$ converging to 0, we have $T(\varphi_j) \rightarrow 0$ as $j \rightarrow \infty$.

For $-\infty < s < \infty$, we denote by J_s the pseudo-differential operator of which the symbol σ_s is given by $\sigma_s(k) = (1 + |k|^2)^{s/2}$, $k \in \mathbb{Z}^n$. Note that the symbol of J_s is in $S^s(\mathbb{Z}^n \times \mathbb{T}^n)$. The pseudo-differential operator J_s is often called the discrete Bessel potential of order s .

Now for $-\infty < s < \infty$, we define l^2 -Sobolev space, $H^{s,2}(\mathbb{Z}^n)$, to be the set of all tempered distributions u for which $J_{-s}u$ is in $\ell^2(\mathbb{Z}^n)$. Then $H^{s,2}(\mathbb{Z}^n)$ is a Banach space with respect to the norm $\|\cdot\|_{s,2}$ given by

$$\|u\|_{s,2} = \|J_{-s}u\|_2, \quad u \in H^{s,2}(\mathbb{Z}^n).$$

It is easy to see that for $-\infty < s, t < \infty$, J_t is an isometry from $H^{s,2}(\mathbb{Z}^n)$ onto $H^{s+t,2}(\mathbb{Z}^n)$.

3. MINIMAL AND MAXIMAL PSEUDO-DIFFERENTIAL OPERATORS ON $\mathbb{Z}^n$

This section is devoted to investigation of the minimal and maximal operators of pseudo-differential operators on $\mathbb{Z}^n$ with symbols in S^m , $m > 0$. and show that they are equal for elliptic operators. For this purpose, we prove for an analogue of the Agmon–Douglis–Nirenberg inequalities for elliptic discrete pseudo-differential operators.

The following theorem gives the relation between lattice and toroidal quantizations proved in [2, Theorem 4.1].

Theorem 3.1. *Let $\sigma : \mathbb{Z}^n \times \mathbb{T}^n \rightarrow \mathbb{C}$ be a measurable function such that the pseudo-differential operator $T_\sigma : \ell^2(\mathbb{Z}^n) \rightarrow \ell^2(\mathbb{Z}^n)$ is a bounded linear operator. If we define $\tau : \mathbb{T}^n \times \mathbb{Z}^n \rightarrow \mathbb{C}$ by*

$$\tau(x, k) = \overline{\sigma(-k, x)}, \quad k \in \mathbb{Z}^n, x \in \mathbb{T}^n,$$

then $T_\sigma = \mathcal{F}_{\mathbb{Z}^n} T_\tau^ \mathcal{F}_{\mathbb{Z}^n}^{-1}$, where T_τ^* is the adjoint of T_τ .*

Remark 1. Let us observe that if $\sigma \in S^0(\mathbb{Z}^n \times \mathbb{T}^n)$ then it follows from [2, Corollary 5.6] that the operator $T_\sigma : \ell^2(\mathbb{Z}^n) \rightarrow \ell^2(\mathbb{Z}^n)$ is a bounded linear operator.

Corollary 3.2. *Let $\sigma \in S^m(\mathbb{Z}^n \times \mathbb{T}^n)$, $-\infty < m < \infty$. Then $T_\sigma : H^{s,2}(\mathbb{Z}^n) \rightarrow H^{s-m,2}(\mathbb{Z}^n)$ is a bounded linear operator.*

Proof. Let $u \in H^{s,2}$. Then $J_{-s}u \in \ell^2(\mathbb{Z}^n)$ and

$$\|T_\sigma u\|_{s-m,2} = \|J_{-s+m}(T_\sigma u)\|_{\ell^2(\mathbb{Z}^n)} = \|(J_{-s+m}T_\sigma J_s)(J_{-s}u)\|_{\ell^2(\mathbb{Z}^n)}.$$

Since $J_{-s+m}T_\sigma J_s$ is a pseudo-differential operator with symbol in S^0 by Theorem 2.2 and therefore using Remark 1, there exists a positive constant C such that

$$\|T_\sigma u\|_{s-m,2} \leq C \|J_{-s}u\|_{\ell^2(\mathbb{Z}^n)} = C \|u\|_{s,2}.$$

Hence $T_\sigma : H^{s,2}(\mathbb{Z}^n) \rightarrow H^{s-m,2}(\mathbb{Z}^n)$ is a bounded linear operator. $\square$

Proposition 3.3. *Let $s \leq t$. Then $H^{t,2}(\mathbb{Z}^n) \subseteq H^{s,2}(\mathbb{Z}^n)$. In fact, there exists a positive constant C such that*

$$\|u\|_{s,2} \leq C \|u\|_{t,2}, \quad u \in H^{t,2}(\mathbb{Z}^n).$$

Proof. Let $u \in H^{t,2}(\mathbb{Z}^n)$. Then we have $\|u\|_{s,2} = \|J_{-s}u\|_{\ell^2(\mathbb{Z}^n)} = \|J_{t-s}J_{-t}u\|_{\ell^2(\mathbb{Z}^n)}$. Since $s \leq t$, it follows that J_{t-s} is a pseudo-differential operator with symbol in S^0 . So, by Remark 1, there exists a positive constant C such that

$$\|J_{-s}u\|_{\ell^2(\mathbb{Z}^n)} = \|J_{t-s}J_{-t}u\|_{\ell^2(\mathbb{Z}^n)} \leq C \|J_{-t}u\|_{\ell^2(\mathbb{Z}^n)} = C \|u\|_{t,2}, \quad u \in H^{t,2}(\mathbb{Z}^n).$$

Thus, $\|u\|_{s,2} \leq C \|u\|_{t,2}$, $u \in H^{t,2}(\mathbb{Z}^n)$. $\square$

Theorem 3.4. *Let $s, t \in (-\infty, \infty)$ be such that $s < t$. Then the inclusion $i : H^{t,2}(\mathbb{Z}^n) \rightarrow H^{s,2}(\mathbb{Z}^n)$ is a compact operator.*

To prove the above theorem, we first observe that the discrete counterpart of Grushin operator, [11], on $\mathbb{R}^n$ of order zero is equivalent to $S^0(\mathbb{Z}^n \times \mathbb{T}^n)$ in this setting. So the following discrete ℓ^2 -version of [29, Theorem 3.2] for discrete pseudo-differential operators can be proved using similar techniques as in [29].

Theorem 3.5. *Let $\sigma \in S^{-m}(\mathbb{Z}^n \times \mathbb{T}^n)$, $0 \leq m < \infty$. Then for every positive number ε , $T_\sigma : H^{s-m,2}(\mathbb{Z}^n) \rightarrow H^{s-\varepsilon,2}(\mathbb{Z}^n)$ is a compact operator for $-\infty < s < \infty$.*

As a simple consequence of Theorem 3.5, we have the following corollary.

Corollary 3.6. *For every positive number ϵ , $J_\epsilon : \ell^2(\mathbb{Z}^n) \rightarrow \ell^2(\mathbb{Z}^n)$ is a compact operator.*

The proof of Theorem 3.4 again follows from the above Corollary 3.6 using the same techniques as in Theorem 2.5 of [8].

Proof. of Theorem 3.4 Let $\epsilon > 0$, be such that $t-s-\epsilon > 0$. Since $J_\epsilon^{-1}J_{-s}$ is a discrete pseudo-differential operator of order $s + \epsilon$, it follows that the composition $J_\epsilon i J_\epsilon^{-1} J_{-s}$ of the mappings

$$J_\epsilon^{-1} J_{-s} : H^{t,2} \rightarrow H^{t-s-\epsilon,2}$$

$$i : H^{t-s-\epsilon,2} \hookrightarrow \ell^2(\mathbb{Z}^n),$$

and

$$J_\epsilon : \ell^2(\mathbb{Z}^n) \rightarrow \ell^2(\mathbb{Z}^n)$$

is compact since $J_\epsilon^{-1} J_{-s} : H^{t,2} \rightarrow H^{t-s-\epsilon,2}$ is a bounded linear operator by Theorem 3.5 and $J_\epsilon : \ell^2(\mathbb{Z}^n) \rightarrow \ell^2(\mathbb{Z}^n)$ is a compact operator by Corollary 3.6. Thus the linear operator

$$H^{t,2} \ni u \rightarrow J_\epsilon i J_\epsilon^{-1} J_{-s} u = J_{-s} \in \ell^2(\mathbb{Z}^n)$$

is compact and this completes the proof. $\square$

Proposition 3.7. *Let $\sigma \in S^m(\mathbb{Z}^n \times \mathbb{T}^n)$. Then the pseudo-differential operator $T_\sigma : \mathcal{S}(\mathbb{Z}^n) \rightarrow \ell^2(\mathbb{Z}^n)$ is closable.*

Proof. Let $\{\varphi_j\}$ be a sequence of functions in $\mathcal{S}(\mathbb{Z}^n)$ such that $\varphi_j \rightarrow 0$ and $T_\sigma \varphi_j \rightarrow f$ in $\ell^2(\mathbb{Z}^n)$ as $j \rightarrow \infty$. Then for any function $\psi \in \mathcal{S}(\mathbb{Z}^n)$, we have

$$(T_\sigma \varphi_j, \psi)_{\ell^2(\mathbb{Z}^n)} = (\varphi_j, T_\sigma^* \psi)_{\ell^2(\mathbb{Z}^n)}, \quad j = 1, 2, \dots,$$

where T_σ^* is the formal adjoint of T_σ . Let $j \rightarrow \infty$. Then $(f, \psi)_{\ell^2(\mathbb{Z}^n)} = 0$ for all functions $\psi \in \mathcal{S}(\mathbb{Z}^n)$. Since $\mathcal{S}(\mathbb{Z}^n)$ is dense in $\ell^2(\mathbb{Z}^n)$, it follows that $f = 0$. Hence $T_\sigma : \mathcal{S}(\mathbb{Z}^n) \rightarrow \ell^2(\mathbb{Z}^n)$ is closable. $\square$

As a consequence of Proposition 3.7, the minimal operator $T_{\sigma,0}$ of T_σ exists. Let us recall that the domain $\mathcal{D}(T_{\sigma,0})$ of $T_{\sigma,0}$ consist of all functions u in $\ell^2(\mathbb{Z}^n)$ for which a sequence $\{\varphi_j\}$ in $\mathcal{S}(\mathbb{Z}^n)$ can be found such that $\varphi_j \rightarrow u$ in $\ell^2(\mathbb{Z}^n)$ and $T_\sigma \varphi_j \rightarrow f$ in $\ell^2(\mathbb{Z}^n)$ for some $f \in \ell^2(\mathbb{Z}^n)$ as $j \rightarrow \infty$. Also, we have $T_{\sigma,0} u = f$.

Definition 3.8. Let u and f be two functions in $\ell^2(\mathbb{Z}^n)$. We say that u lies in $\mathcal{D}(T_{\sigma,1})$ and $T_{\sigma,1} u = f$ if and only if

$$(3.1) \quad (u, T_\sigma^* \varphi)_{\ell^2(\mathbb{Z}^n)} = (f, \varphi)_{\ell^2(\mathbb{Z}^n)}, \quad \varphi \in \mathcal{S}(\mathbb{Z}^n),$$

where T_σ^* is the formal adjoint of T_σ .

Proposition 3.9. *Let $u \in \mathcal{D}(T_{\sigma,1})$. Then $T_{\sigma,1} u = T_\sigma u$ in the distribution sense.*

Proof. By Definition 3.8, we have

$$(u, T_\sigma^* \varphi)_{\ell^2(\mathbb{Z}^n)} = (T_{\sigma,1} u, \varphi)_{\ell^2(\mathbb{Z}^n)}, \quad \varphi \in \mathcal{S}(\mathbb{Z}^n).$$

So, by considering u and $T_{\sigma,1} u$ as tempered distributions, we get

$$(3.2) \quad (T_{\sigma,1} u)(\bar{\varphi}) = u(\overline{T_\sigma^* \varphi}), \quad \varphi \in \mathcal{S}(\mathbb{Z}^n).$$

On the other hand, we have

$$(3.3) \quad (T_\sigma u)(\bar{\varphi}) = u(\overline{T_\sigma^* \varphi}), \quad \varphi \in \mathcal{S}(\mathbb{Z}^n).$$

Hence, by (3.2) and (3.3), $T_{\sigma,1} u = T_\sigma u$ in the distribution sense. $\square$

Theorem 3.10. $T_{\sigma,1}$ is a closed linear operator from $\ell^2(\mathbb{Z}^n)$ into $\ell^2(\mathbb{Z}^n)$ with domain $\mathcal{D}(T_{\sigma,1})$ containing $\mathcal{S}(\mathbb{Z}^n)$.

Proof. It is clear from the definition of the formal adjoint T_σ^* and (3.1) that $\mathcal{S}(\mathbb{Z}^n) \subset \mathcal{D}(T_{\sigma,1})$. It is easy to prove the linearity. Now we prove that $T_{\sigma,1}$ is closed. Let $\{u_j\}$ be a sequence of functions in $\mathcal{D}(T_{\sigma,1})$ such that $u_j \rightarrow u$ in $\ell^2(\mathbb{Z}^n)$ and $T_{\sigma,1} u_j \rightarrow f$ in $\ell^2(\mathbb{Z}^n)$ for some u and f in $\ell^2(\mathbb{Z}^n)$ as $j \rightarrow \infty$. Then by (3.1)

$$(3.4) \quad (u_j, T_\sigma^* \varphi)_{\ell^2(\mathbb{Z}^n)} = (T_{\sigma,1} u_j, \varphi)_{\ell^2(\mathbb{Z}^n)}$$

for all $\varphi \in \mathcal{S}(\mathbb{Z}^n)$ and $j = 1, 2, \dots$. By letting $j \rightarrow \infty$ in (3.4), we have

$$(u, T_\sigma^* \varphi)_{\ell^2(\mathbb{Z}^n)} = (f, \varphi)_{\ell^2(\mathbb{Z}^n)}, \quad \varphi \in \mathcal{S}(\mathbb{Z}^n).$$

Hence by Definition 3.8, $u \in \mathcal{D}(T_{\sigma,1})$ and $T_{\sigma,1} u = f$. This proves that $T_{\sigma,1}$ is closed. $\square$

Proposition 3.11. $T_{\sigma,1}$ is an extension of $T_{\sigma,0}$.

Proof. Let $u \in \mathcal{D}(T_{\sigma,0})$ and $T_{\sigma,0} u = f$. Then there exists a sequence $\{\varphi_j\}$ of functions in $\mathcal{S}$ for which $\varphi_j \rightarrow u$ and $T_\sigma \varphi_j \rightarrow f$ in $\ell^2(\mathbb{Z}^n)$ as $j \rightarrow \infty$. By the definition of T_σ^* , we have

$$(\varphi_j, T_\sigma^* \psi)_{\ell^2(\mathbb{Z}^n)} = (T_\sigma \varphi_j, \psi)_{\ell^2(\mathbb{Z}^n)}$$

for all $\psi \in \mathcal{S}(\mathbb{Z}^n)$ and $j = 1, 2, \dots$. By letting $j \rightarrow \infty$, we get

$$(u, T_\sigma^* \psi)_{\ell^2(\mathbb{Z}^n)} = (f, \psi)_{\ell^2(\mathbb{Z}^n)}, \quad \psi \in \mathcal{S}(\mathbb{Z}^n).$$

So, by Definition 3.8, $u \in \mathcal{D}(T_{\sigma,1})$ and $T_{\sigma,1} u = f$. $\square$

Lemma 3.12. $T_\sigma^t \varphi = T_\sigma^* \varphi$ for all $\varphi \in \mathcal{S}(\mathbb{Z}^n)$, where T_σ^t is the true adjoint of T_σ . In other words, the true adjoint and the formal adjoint coincide on the space $\mathcal{S}(\mathbb{Z}^n)$.

Proof. Let $\varphi \in \mathcal{S}(\mathbb{Z}^n)$. Then by the definition of T_σ^* , we have

$$(T_\sigma^* \varphi, \psi)_{\ell^2(\mathbb{Z}^n)} = (\varphi, T_\sigma \psi)_{\ell^2(\mathbb{Z}^n)}, \quad \psi \in \mathcal{S}(\mathbb{Z}^n).$$

So, by the definition of T_σ^t and the duality of $\ell^2(\mathbb{Z}^n)$, $\varphi \in \mathcal{D}(T_\sigma^t)$ and $T_\sigma^t \varphi = T_\sigma^* \varphi$. $\square$

Proposition 3.13. *$T_{\sigma,1}$ is the largest closed extension of T_σ in the sense that if B is any extension of T_σ such that $\mathcal{S}(\mathbb{Z}^n) \subseteq \mathcal{D}(B^t)$, then $T_{\sigma,1}$ is an extension of B .*

Proof. Let $u \in \mathcal{D}(B)$. Then for all $\psi \in \mathcal{S}(\mathbb{Z}^n)$, we have $\psi \in \mathcal{D}(B^t)$. Hence by the definition of B^t ,

$$(3.5) \quad (\psi, Bu)_{\ell^2(\mathbb{Z}^n)} = (B^t \psi, \varphi)_{\ell^2(\mathbb{Z}^n)}.$$

Since B is an extension of T_σ , it follows from [31, Proposition 13.9] that T_σ^t is an extension of B^t . Hence by (3.5),

$$(3.6) \quad (\psi, Bu)_{\ell^2(\mathbb{Z}^n)} = (T_\sigma^t \psi, u)_{\ell^2(\mathbb{Z}^n)}.$$

By Lemma 3.12, $T_\sigma^t = T_\sigma^*$ on $\mathcal{S}(\mathbb{Z}^n)$. Hence by (3.6), we have

$$(\psi, Bu)_{\ell^2(\mathbb{Z}^n)} = (T_\sigma^* \psi, u)_{\ell^2(\mathbb{Z}^n)}, \quad \psi \in \mathcal{S}(\mathbb{Z}^n).$$

Therefore by Definition 3.8, we have $u \in \mathcal{D}(T_{\sigma,1})$ and $T_{\sigma,1}u = Bu$. $\square$

In view of Proposition 3.13, we call $T_{\sigma,1}$ the *maximal operator* of T_σ .

The proof of following proposition is standard and can be found in [2].

Proposition 3.14. *Let $\sigma \in S^m(\mathbb{Z}^n \times \mathbb{T}^n)$. Then the pseudo-differential operator T_σ maps $\mathcal{S}(\mathbb{Z}^n)$ into $\mathcal{S}(\mathbb{Z}^n)$.*

Lemma 3.15. *The space $\mathcal{S}(\mathbb{Z}^n)$ is dense in $H^{s,2}(\mathbb{Z}^n)$.*

Proof. Let $u \in H^{s,2}(\mathbb{Z}^n)$. By the definition of $H^{s,2}(\mathbb{Z}^n)$, $J_{-s}u \in \ell^2(\mathbb{Z}^n)$. Since $\mathcal{S}(\mathbb{Z}^n)$ is dense in $\ell^2(\mathbb{Z}^n)$, it follows that there exists a sequence $\{\varphi_k\}$ in $\mathcal{S}(\mathbb{Z}^n)$ such that $\varphi_k \rightarrow J_{-s}u$ in $\ell^2(\mathbb{Z}^n)$ as $k \rightarrow \infty$. Let $\psi_k = J_s \varphi_k$, $k = 1, 2, \dots$. By Proposition 3.14, $\psi_k \in \mathcal{S}(\mathbb{Z}^n)$ because J_s is a pseudo-differential operator. By the definition of $H^{s,2}(\mathbb{Z}^n)$, we have

$$\|\psi_k - u\|_{s,2} = \|J_{-s}\psi_k - J_{-s}u\|_2 = \|\varphi_k - J_{-s}u\|_2 \rightarrow 0$$

as $k \rightarrow \infty$. This shows that $\psi_k \rightarrow u$ in $H^{s,2}(\mathbb{Z}^n)$ as $k \rightarrow \infty$, and hence $\mathcal{S}(\mathbb{Z}^n)$ is dense in $H^{s,2}(\mathbb{Z}^n)$. $\square$

The following theorem is a discrete version of Agmon–Douglis–Nirenberg estimate for pseudo-differential operators on $\mathbb{Z}^n$.

Proposition 3.16. *Let $m > 0$ and let σ be an elliptic symbol in $S^m(\mathbb{Z}^n \times \mathbb{T}^n)$. Then there exist positive constants C_1 and C_2 such that*

$$C_1 \|u\|_{m,2} \leq (\|T_\sigma u\|_2 + \|u\|_2) \leq C_2 \|u\|_{m,2}, \quad u \in H^{m,2}(\mathbb{Z}^n).$$

Proof. By Corollary 3.2 and Proposition 3.3, there exists a positive constant C_2 such that

$$(3.7) \quad (\|T_\sigma u\|_2 + \|u\|_2) \leq C_2 \|u\|_{m,2}, \quad u \in H^{m,2}(\mathbb{Z}^n).$$

Now, by Theorem 2.4 we have

$$(3.8) \quad u = T_\tau T_\sigma u - Ru, \quad u \in H^{m,2}(\mathbb{Z}^n),$$

where $\tau \in S^{-m}(\mathbb{Z}^n \times \mathbb{T}^n)$ and R is a pseudo-differential operator with symbol in $\cap_{\nu \in \mathbb{R}} S^\nu(\mathbb{Z}^n \times \mathbb{T}^n)$. Hence it follows from Corollary 3.2 that there exists a positive constant C_1 such that

$$C_1 \|u\|_{m,2} \leq (\|T_\sigma u\|_2 + \|u\|_2).$$

This complete the proof. $\square$

Proposition 3.17. *Let $m > 0$ and let σ be an elliptic symbol in $S^m(\mathbb{Z}^n \times \mathbb{T}^n)$. Then $\mathcal{D}(T_{\sigma,0}) = H^{m,2}(\mathbb{Z}^n)$.*

Proof. Let $u \in H^{m,2}(\mathbb{Z}^n)$. Then by Lemma 3.15, we can find a sequence $\{\varphi_k\}$ in $\mathcal{S}(\mathbb{Z}^n)$ such that $\varphi_k \rightarrow u$ in $H^{m,2}(\mathbb{Z}^n)$ as $k \rightarrow \infty$. By Lemma 3.15 and Proposition 3.16, $\{T_\sigma \varphi_k\}$ and $\{\varphi_k\}$ are Cauchy sequences in $\ell^2(\mathbb{Z}^n)$ and hence $\varphi_k \rightarrow u$ and $T_\sigma \varphi_k \rightarrow f$ in $\ell^2(\mathbb{Z}^n)$ for some u and f in $\ell^2(\mathbb{Z}^n)$ as $k \rightarrow \infty$. Hence by the definition of $T_{\sigma,0}$, $u \in \mathcal{D}(T_{\sigma,0})$ and $T_{\sigma,0}u = f$. On the other hand, let $u \in \mathcal{D}(T_{\sigma,0})$. Then by the definition of $T_{\sigma,0}$ again, we can find a sequence $\{\varphi_k\}$ in $\mathcal{S}(\mathbb{Z}^n)$ such that $\varphi_k \rightarrow u$ and $T_\sigma \varphi_k \rightarrow f$ in $\ell^2(\mathbb{Z}^n)$ for some $f \in \ell^2(\mathbb{Z}^n)$ as $k \rightarrow \infty$. Therefore $\{\varphi_k\}$ and $\{T_\sigma \varphi_k\}$ are Cauchy sequences in $L^2(\mathbb{Z}^2)$. So, by Proposition 3.16 and Lemma 3.15, $\{\varphi_k\}$ is a Cauchy sequence in $H^{m,2}(\mathbb{Z}^n)$. Since $H^{m,2}(\mathbb{Z}^n)$ is complete, it follows that $\varphi_k \rightarrow v$ in $H^{m,2}(\mathbb{Z}^n)$ for some $v \in H^{m,2}(\mathbb{Z}^n)$ as $k \rightarrow \infty$. Then by Proposition 3.3, $\phi_k \rightarrow v$ in $\ell^2(\mathbb{Z}^n)$ as $k \rightarrow \infty$. Hence $u = v$ and consequently, $u \in H^{m,2}(\mathbb{Z}^n)$. $\square$

The following theorem is the main theorem of this section.

Theorem 3.18. *Let $m > 0$ and let σ be an elliptic symbol in $S^m(\mathbb{Z}^n \times \mathbb{T}^n)$. Then $T_{\sigma,0} = T_{\sigma,1}$.*

Proof. Since $T_{\sigma,0}$ is the smallest closed extension of T_σ , it follows from Proposition 3.17 that it is sufficient to prove that $\mathcal{D}(T_{\sigma,0}) \subseteq H^{m,2}(\mathbb{Z}^n)$. Let $u \in \mathcal{D}(T_{\sigma,1})$. Then Theorem 2.4 gives

$$(3.9) \quad u = T_\sigma T_\tau u - Ru,$$

where $\tau \in S^{-m}(\mathbb{Z}^n \times \mathbb{T}^n)$ and R is a pseudo-differential operator with symbol in $\cap_{k \in \mathbb{R}} S^k(\mathbb{Z}^n \times \mathbb{T}^n)$. By Proposition 3.9, $T_{\sigma,1}u = T_\sigma u$ in the distribution sense. Thus, by the definition of $T_{\sigma,1}$, $u \in \ell^2(\mathbb{Z}^n)$. Since $\tau \in S^{-m}(\mathbb{Z}^n \times \mathbb{T}^n)$, it follows from Theorem 2.2 and Corollary 3.2 that $T_\sigma T_\tau u \in H^{m,2}(\mathbb{Z}^n)$. Since $u \in \ell^2(\mathbb{Z}^n)$ and R is a pseudo-differential operator with symbol in $S^{-m}(\mathbb{Z}^n \times \mathbb{T}^n)$, it follows from Corollary 3.2 again that $Ru \in H^{m,2}(\mathbb{Z}^n)$. Hence $u \in H^{m,2}(\mathbb{Z}^n)$. $\square$

4. FREDHOLMNESS AND ELLIPTICITY

In this section we study the Fredholmness and ellipticity of a pseudo-differential operator. We prove that a pseudo-differential operator of order 0 is elliptic if and only if Fredholm. We also calculate the index of such a pseudo-differential operator.

Let us first recall that a closed linear operator A from a complex Banach space X into a complex Banach space Y with dense domain $\mathcal{D}(A)$ is said to be Fredholm if the range $R(A)$ of A is a closed subspace of Y , the null space $N(A)$ of A and the null space $N(A^t)$ of the true adjoint A^t of A are finite dimensional. For a Fredholm operator A , the index $i(A)$ of A is defined by

$$i(A) = \dim N(A) - \dim N(A^t).$$

The following criterion for a closed linear operator to be Fredholm is usually attributed to Atkinson [1].

Theorem 4.1. *Let A be a closed linear operator from a complex Banach space X into a complex Banach space Y with dense domain $\mathcal{D}(A)$. Then A is Fredholm if and only if we can find a bounded linear operator $B : Y \rightarrow X$, a compact operator $K_1 : X \rightarrow X$ and a compact operator $K_2 : Y \rightarrow Y$ such that $BA = I + K_1$ on $\mathcal{D}(A)$ and $AB = I + K_2$ on Y .*

The main result in this section is the following theorem.

Theorem 4.2. *Let $\sigma \in S^0(\mathbb{Z}^n \times \mathbb{T}^n)$. Then $T_\sigma : \ell^2(\mathbb{Z}^n) \rightarrow \ell^2(\mathbb{Z}^n)$ is Fredholm if and only if $T_\sigma : \ell^2(\mathbb{Z}^n) \rightarrow \ell^2(\mathbb{Z}^n)$ is elliptic.*

Proof. Suppose that $T_\sigma : \ell^2(\mathbb{Z}^n) \rightarrow \ell^2(\mathbb{Z}^n)$ is a Fredholm operator. Then $T_\tau = \mathcal{F}_{\mathbb{Z}^n} T_\sigma^t \mathcal{F}_{\mathbb{Z}^n}^{-1}$ is also a Fredholm operator from $L^2(\mathbb{T}^n)$ into $L^2(\mathbb{T}^n)$, where

$$\tau(x, k) = \overline{\sigma(-k, x)}, \quad x \in \mathbb{T}^n, k \in \mathbb{Z}^n.$$

It is easy to see that $\sigma \in S^0(\mathbb{Z}^n \times \mathbb{T}^n)$ implies that $\tau \in S^0(\mathbb{T}^n \times \mathbb{Z}^n)$. Therefore by [13, Theorem 3.9], τ is elliptic which is equivalent to say that σ is elliptic. Conversely, suppose that σ is elliptic. Then there exists a symbol $\tau \in S^0(\mathbb{Z}^n \times \mathbb{T}^n)$ such that

$$(4.10) \quad T_\sigma T_\tau = I + R$$

and

$$(4.11) \quad T_\tau T_\sigma = I + S,$$

where R and S are pseudo-differential operators with symbol in $\cap_{\nu \in \mathbb{R}} S^\nu(\mathbb{Z}^n \times \mathbb{T}^n)$. So, for any positive number k , the pseudo-differential operator $R : \ell^2(\mathbb{Z}^n) \rightarrow \ell^2(\mathbb{Z}^n)$ is the same as the composition of the pseudo-differential operator $R : \ell^2(\mathbb{Z}^n) \rightarrow H^{k,2}(\mathbb{Z}^n)$ and the inclusion $i : H^{k,2}(\mathbb{Z}^n) \rightarrow \ell^2(\mathbb{Z}^n)$. Since $R : \ell^2(\mathbb{Z}^n) \rightarrow H^{k,2}(\mathbb{Z}^n)$ is a bounded linear operator and from Theorem 3.4 $i : H^{k,2}(\mathbb{Z}^n) \rightarrow \ell^2(\mathbb{Z}^n)$ is

compact, it follows that $R : \ell^2(\mathbb{Z}^n) \rightarrow \ell^2(\mathbb{Z}^n)$ is compact. Similarly $S : \ell^2(\mathbb{Z}^n) \rightarrow \ell^2(\mathbb{Z}^n)$ is compact. Hence by Atkinson's theorem, T_σ is Fredholm. $\square$

Remark 2. We would like mention here that Hörmander [12, Page 227] showed that for pseudo-differential operators of order m on compact manifolds, ellipticity is equivalent to Fredholmness in L^2 -Sobolev spaces. Molahajloo and Wong [13] presented a new proof of the fact that, for pseudo-differential operators on $\mathbb{T}$ of order zero, ellipticity is equivalent to Fredholmness on $L^2(\mathbb{T})$, which is a special case of the known Hörmander's result in [12]. In view of Hörmander's result and using the analysis done in [2], one can also directly deduce a proof of Theorem 4.2.

We end this section with an index formula for Fredholm pseudo-differential operators on $\mathbb{Z}^n$. First, we prove the following lemma.

Lemma 4.1. Let $\mathcal{S}(\mathbb{Z}^n \times \mathbb{T}^n)$ be the Schwartz space on $\mathbb{Z}^n \times \mathbb{T}^n$. Then

$$\bigcap_{\nu \in \mathbb{R}} S^\nu(\mathbb{Z}^n \times \mathbb{T}^n) = \mathcal{S}(\mathbb{Z}^n \times \mathbb{T}^n).$$

Proof. It is easy to check that $\mathcal{S}(\mathbb{Z}^n \times \mathbb{T}^n) \subseteq \bigcap_{\nu \in \mathbb{R}} S^\nu(\mathbb{Z}^n \times \mathbb{T}^n)$. Now, let $\sigma \in \bigcap_{\nu \in \mathbb{R}} S^\nu(\mathbb{Z}^n \times \mathbb{T}^n)$. Let α, β, γ and δ be multi-indices. Then there exists a real number ν_1 such that

$$|\alpha| + \nu_1 - |\gamma| \leq 0.$$

Since $\sigma \in S^{\nu_1}$, we can find a positive constant $C_{\nu_1, \gamma, \delta}$ such that

$$\begin{aligned} \sup_{(k, x) \in \mathbb{Z}^n \times \mathbb{T}^n} |k^\alpha x^\beta (D_x^\delta \Delta_k^\gamma \sigma)(k, x)| &\leq \sup_{(k, x) \in \mathbb{Z}^n \times \mathbb{T}^n} |k|^{|\alpha|} |(D_x^\delta \Delta_k^\gamma \sigma)(k, x)| \\ (4.12) \qquad \qquad \qquad &\leq C_{\nu_1, \gamma, \delta} \sup_{k \in \mathbb{Z}^n, x \in \mathbb{T}^n} (1 + |k|)^{\nu_1 + |\alpha| - |\gamma|} < \infty \end{aligned}$$

Therefor $\sigma \in \mathcal{S}(\mathbb{Z}^n \times \mathbb{T}^n)$. $\square$

Let $\sigma \in S^0(\mathbb{Z}^n \times \mathbb{T}^n)$ be elliptic. Let $\tau \in S^0(\mathbb{Z}^n \times \mathbb{T}^n)$ be such that $T_\tau T_\sigma = I - T_1$ and $T_\sigma T_\tau = I - T_2$, where T_1 and T_2 are pseudo-differential operators with symbols τ_1 and τ_2 respectively in $\bigcap_{\nu \in \mathbb{R}} S^\nu(\mathbb{Z}^n \times \mathbb{T}^n)$. Let $f \in \mathcal{S}$. Then for $j = 1, 2$,

$$(T_j f)(k) = \int_{\mathbb{T}^n} e^{2\pi i k \cdot x} \tau_j(k, x) \widehat{f}(x) dx$$

for all $k \in \mathbb{Z}^n$. By Lemma 4.1, for $j = 1, 2$, $\tau_j \in \mathcal{S}(\mathbb{Z}^n \times \mathbb{T}^n)$. Then by Theorem 5.3 in [2], T_1 and T_2 are trace class operators and

$$\text{tr}(T_j) = \sum_{k \in \mathbb{Z}^n} \int_{\mathbb{T}^n} \tau_j(k, x) dx.$$

Then by Theorem 20.13 in [31], we get

$$i(T_\sigma) = \text{tr}(T_1) - \text{tr}(T_2) = \sum_{k \in \mathbb{Z}^n} \int_{\mathbb{T}^n} (\tau_1 - \tau_2)(k, x) dx.$$

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