ON (λ, μ)-CLASSES ON THE ENGEL GROUP

MARIANNA CHATZAKOU

Abstract. The purpose of this note is to compare the properties of the symbolic pseudo-differential calculus on the Heisenberg and on the Engel groups; nilpotent Lie groups of 2-step and 3-step, respectively. Here we provide a preliminary analysis of the structure and of the symbolic calculus with symbols parametrized by (λ, μ) on the Engel group, while for the case of the Heisenberg group we recall the analogous results on the λ-classes of symbols.

1. Introduction

In [FR16] the authors developed a global pseudo-differential calculus in the setting of a graded nilpotent Lie groups. Here we present the analogous preliminary results in the particular case of the Engel group $\mathcal{B}_4$.

We prove that the representation of $\mathcal{B}_4$ is associated with the Kohn-Nirenberg quantization on $\mathbb{R}^4$. This, together with the analogue of the Kohn-Nirenberg quantization on Lie groups (c.f [Tayl84],[RT10],[FR16]) gives rise to the development of the pseudo-differential calculus on $\mathcal{B}_4$ with scalar-valued symbols depending on the parameters (λ, μ)-the co-adjoint orbits.

In [Tayl84], M. Taylor describes a way one can develop a symbolic non-invariant calculus by defining a general quantization and the general symbols on any type-I Lie group, and explained his ideas in the setting of the Heisenberg group $\mathbb{H}_n$, with symbols defined by some asymptotic expansions. Particularising in the setting of a large class of nilpotent Lie groups; namely on the class of graded Lie groups, to the best of our knowledge, the development of a non-invariant calculus with scalar-valued symbols has been restricted to the case of the Heisenberg group (graded group of 2-step), see [BFKG12], or [FR14].

Besides the amount of work devoted to the case of the Heisenberg group, the same motivating aspects appear as well on any graded Lie group. In our consideration of $\mathcal{B}_4$ (graded group of 3-step) our approach differs from the one in [Tayl84] or in [BFKG12] in the sense that the symbols are operator valued. However, using the link between the Kohn-Nirenberg quantization and the representations on $\mathcal{B}_4$ they can be expressed on the euclidean level. Concrete formulas for the difference operators in the setting of $\mathcal{B}_4$ are provided, laying down the necessary foundation for the characterisation of the symbol classes in our setting.

2. Prelimaries on the Engel group $\mathcal{B}_4$ and its Lie algebra

We start by fixing the notation required for presenting our results. The map

$$\begin{aligned}&(x_1, x_2, x_3, x_4) \circ (y_1, y_2, y_3, y_4)\\ &:= \quad (x_1 + y_1, x_2 + y_2, x_3 + y_3 - x_1 y_2, x_4 + y_4 + \frac{1}{2} x_1^2 y_2 - x_1 y_3)\end{aligned}$$

endows $\mathbb{R}^4$ with a structure of a Lie group, and we shall refer to $\mathcal{B}_4 = (\mathbb{R}^4, \circ)$ as the *Engel group*. The Lie algebra of $\mathcal{B}_4$, say $\mathfrak{l}_4$, is, by the general theory, the vector space of (smooth) left invariant vector fields X on $\mathbb{R}^4$ characterised by the property

$$(XI)(x) = \mathcal{J}_{\tau_x}(0) \cdot (XI)(0)\,, \forall x \in \mathcal{B}_4\,,$$

where I is the identity map on $\mathbb{R}^4$ and $\mathcal{J}_{\tau_x}(0)$ denotes the Jacobian matrix at the origin of the map τ_x for $x \in \mathcal{B}_4$ where $\tau_x(y) := x \circ y$ is the left translation by x on $\mathcal{B}_4$. In particular we have

$$\mathcal{J}_{\tau_x}(0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -x_1 & 1 & 0 \\ 0 & \frac{x_1^2}{2} & -x_1 & 1 \end{pmatrix},$$

so that for example for $X_1 = \partial_{x_1}$, we can recognise that for every $x \in \mathcal{B}_4$,

$$(X_1 I)(x) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -x_1 & 1 & 0 \\ 0 & \frac{x_1^2}{2} & -x_1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \mathcal{J}_{\tau_x}(0) \cdot (X_1 I)(0)\,,$$

while similarly for the vector fields $X_2 = \partial_{x_2} - x_1\partial_{x_3} + \frac{x_1^2}{2}\partial_{x_4}$, $X_3 = \partial_{x_3} - x_1\partial_{x_4}$ and $X_4 = \partial_{x_4}$. Simple calculations show that $[X_1, X_2] = X_3$, and $[X_1, X_3] = X_4$ [1], are the only non-zero relations, so that

$$\mathfrak{l}_4 = span\{X_1, X_2, [X_1, X_2], [X_1, X_3]\}\,,$$

and X_1, X_2 satisfy the so-called *Hörmander condition*. The lower center series of $\mathfrak{l}_4$ defined inductively by

$$\mathfrak{l}_{4(1)} := \mathfrak{l}_4\,, \quad \mathfrak{l}_{4(j)} = [\mathfrak{l}_4, \mathfrak{l}_{4(j-1)}]^{2}\,,$$

terminates at 0 after 3 steps, that is $\mathcal{B}_4$ is of 3-step. In addition, $\mathcal{B}_4$ is a homogeneous Lie group on $\mathbb{R}^4$ since the mapping

$$\delta_\lambda : \mathbb{R}^4 \to \mathbb{R}^4\,, \quad \delta_\lambda(x_1, x_2, x_3, x_4) = (\lambda x_1, \lambda x_2, \lambda^2 x_3, \lambda^3 x_4)\,,$$

is an automorphism of $\mathcal{B}_4$ for every $\lambda > 0$, and so the natural gradation of its Lie algebra $\mathfrak{l}_4$ appears as

$$\mathfrak{l}_4 = V_1 \oplus V_2 \oplus V_3\,,$$

where $V_1 = span\{X_1, X_2\}\,, V_2 = span\{X_3\}$ and $V_3 = span\{X_4\}$, and is such that $[V_i, V_j] \subset V_{i+j}$, $i \neq j$.

Finally, let us note that, from the general theory of homogeneous Lie groups, the Lebesgue measure on $\mathbb{R}^4$ is invariant with respect to the left and right invariant translation on $\mathcal{B}_4$, that is the Lebesgue measure on $\mathbb{R}^4$ is the Haar measure for $\mathcal{B}_4$ and we can formulate as

$$\int_{\mathcal{B}_4} \cdots dx_1 dx_2 dx_3 dx_4 = \int_{\mathbb{R}^4} \cdots dx_1 dx_2 dx_3 dx_4\,.$$

[1] For smooth vectors X, Y in $\mathbb{R}^n$, we define the Lie-bracket $[X, Y] := YX - XY$.

[2] For V, W spaces of vector fields, we denote by $[V, W]$ the set $\{[v, w] : v \in V, w \in W\}$.

3. Group representation and quantization of the Fourier transform

The representations of the Engel group $\mathcal{B}_4$ are the infinite dimensional unitary (equivalence classes of) representations of $\mathcal{B}_4$. Parametrised by $\lambda \neq 0$ and $\mu \in \mathbb{R}$, following [Dix57, p.333], they act on $L^2(\mathbb{R}^n)$. We denote them by $\pi_{\lambda,\mu}$, and realise them as

$$\pi_{\lambda,\mu}(x_1,x_2,x_3,x_4)h(u) = exp\left(i\left(-\frac{\mu}{2\lambda}x_2 + \lambda x_4 - \lambda x_3 u + \frac{\lambda}{2}x_2 u^2\right)\right) h(u+x_1)\,,$$

for $h \in L^2(\mathbb{R}), u \in \mathbb{R}$. The group Fourier transform of a function $f \in L^1(\mathcal{B}_4)$ is by definition the linear endomorphism on $L^2(\mathbb{R})$

$$\mathcal{F}_{\mathcal{B}_4}(f)(\pi_{\lambda,\mu}) \equiv \hat{f}(\pi_{\lambda,\mu}) \equiv \pi_{\lambda,\mu}(f) := \int_{\mathcal{B}_4} f(x)\pi_{\lambda,\mu}(x)^*\,dx\,.$$

Rigorous computations show that $\hat{f}(\pi_{\lambda,\mu})h(u)$ can be written as

$$\begin{aligned}
&\int_{\mathbb{R}^4}\Bigg[f(x_1,x_2,x_3,x_4)\\
&\cdot \exp\left(i\left(\frac{\mu}{2\lambda}x_2 - \lambda x_4 + \lambda x_3(u-x_1) - \frac{\lambda}{2}x_2(u-x_1)^2\right)\right) h(u-x_1)\Bigg] dx_1\,dx_2\,dx_3\,dx_4 \qquad (1)\\
&= (2\pi)^{-2}\int_{\mathbb{R}^4}\int_{\mathbb{R}^4}\Bigg[\mathcal{F}_{\mathbb{R}^4}(f)(\xi,\eta,\tau,\omega)\cdot e^{ix_1\xi}\cdot e^{ix_2\eta}\cdot e^{ix_3\tau}\cdot e^{ix_4\omega}\\
&\cdot \exp\left(i\left(\frac{\mu}{2\lambda}x_2 - \lambda x_4 + \lambda x_3(u-x_1) - \frac{\lambda}{2}x_2(u-x_1)^2\right)\right)\\
&\cdot h(u-x_1)\Bigg] dx_1\,dx_2\,dx_3\,dx_4 d\xi\,d\eta\,d\tau\,d\omega\\
&= -(2\pi)\int_{\mathbb{R}}\int_{\mathbb{R}}\left[e^{ix_1\xi}\mathcal{F}(f)_{\mathbb{R}^4}(\xi,\frac{\lambda}{2}(u-x_1)^2 - \frac{\mu}{2\lambda},\lambda(x_1-u),\lambda)h(u-x_1)\right]dx_1\,d\xi\\
&= (2\pi)\int_{\mathbb{R}}\int_{\mathbb{R}}\left[e^{i(u-v)\xi}\mathcal{F}(f)_{\mathbb{R}^4}(\xi,\frac{\lambda}{2}v^2 - \frac{\mu}{2\lambda},-\lambda v,\lambda)h(v)\right]dv\,d\xi\,,
\end{aligned}$$

for $h \in L^2(\mathbb{R})$ and $u \in \mathbb{R}$, that is

$$\mathcal{F}_{\mathcal{B}_4}(f)(\pi_{\lambda,\mu}) = Op[a_{f,\lambda,\mu}(\cdot,\cdot)]\,, \qquad (2)$$

where

$$a_{f,\lambda,\mu}(v,\xi) = (2\pi)^2\mathcal{F}_{\mathbb{R}^4}(f)(\xi,\frac{\lambda}{2}v^2 - \frac{\mu}{2\lambda},-\lambda v,\lambda)\,.$$

Here the Fourier transform $\mathcal{F}_{\mathbb{R}^4}$ is defined via:

$$\mathcal{F}_{\mathbb{R}^4}f(\xi) = (2\pi)^{-2}\int_{\mathbb{R}^4} f(x)e^{-ix\xi}\,dx \quad (\xi \in \mathbb{R}^4, f \in L^1(\mathbb{R}^4))\,, \qquad (3.1)$$

and Op denotes the Kohn-Nirenberg quantization, that is for a smooth symbol a on $\mathbb{R}\times\mathbb{R}$ the operator

$$Op(a)f(u) = (2\pi)^{-1}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(u-v)\xi}a(v,\xi)f(v)\,dv\,d\xi\,,$$

for $f \in \mathcal{S}(\mathbb{R})$ and $u \in \mathbb{R}$.

We note that for the case of the Heisenberg group $\mathbb{H}_n$ the group Fourier transform has been computed in [FR14] as being the operator

$$\mathcal{F}_{\mathbb{H}_n}(f)(\pi_\lambda) = (2\pi)^{\frac{n}{2}} Op^W[\mathcal{F}_{\mathbb{R}^{2n+1}}(f)(\sqrt{|\lambda|}\cdot, \sqrt{\lambda}\cdot, \lambda)]\,, \tag{3}$$

where Op^W denotes the Weyl-quantization, i.e.

$$Op^W(a)f(u) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(u-v)\xi} a\left(\xi, \frac{u+v}{2}\right) f(v)\, dv\, d\xi\,,$$

for $f \in \mathcal{S}(\mathbb{R}^n)$ and $u \in \mathbb{R}^n$, where π_λ denotes the Schrödinger representations of $\mathbb{H}_n$,

Going back to our case, of one keeps the same notation $\pi_{\lambda,\mu}$ for the infinitesimal representation, we compute that:

$$\begin{aligned}
&\pi_{\lambda,\mu}(X_1) = \partial_u = Op(i\xi)\,,\\
&\pi_{\lambda,\mu}(X_2) = \tfrac{i}{2}\left(\lambda u^2 - \tfrac{\mu}{\lambda}\right) = Op\left(\tfrac{i\lambda u^2}{2} - \tfrac{i\mu}{2\lambda}\right)\,,\\
&\pi_{\lambda,\mu}(X_3) = -i\lambda u = Op(-i\lambda u)\,,\\
&\pi_{\lambda,\mu}(X_4) = i\lambda = Op(i\lambda)\,,
\end{aligned}$$

thus

$$\pi_{\lambda,\mu}(\mathcal{L}) = \pi_{\lambda,\mu}(X_1)^2 + \pi_{\lambda,\mu}(X_2)^2 = \frac{d^2}{du^2} - \frac{1}{4}\left(\lambda u^2 - \frac{\mu}{\lambda}\right)^2 = -Op\left(\xi^2 + \frac{1}{4}\left(\lambda u^2 - \frac{\mu}{\lambda}\right)^2\right)\,.$$

With our choice of notation, the Plancherel measure of the Engel group $\mathcal{B}_4$ is $(2^{-3}\pi^{-4})d\lambda\, d\mu$, in the sense that following expression for the Plancherel formula

$$\int_{\mathcal{B}_4} |f(x_1, x_2, x_3, x_4)|^2\, dx_1\, dx_2\, dx_3\, dx_4 = 2^{-3}\pi^{-4} \int_{\lambda\neq 0} \int_{\mu\in\mathbb{R}} \|\pi_{\lambda,\mu}(f)\|^2_{\text{HS}}\, d\mu\, d\lambda\,, \tag{4}$$

holds for any $f \in \mathcal{S}(\mathbb{R})$, where $\|\cdot\|_{\text{HS}}$ denotes the Hilbert-Schmidt norm of an operator on L^2, that is $\|A\|_{\text{HS}} := Tr(A^*A)$. The last allows for an extension of the group Fourier transform to $L^2(\mathcal{B}_4)$, and in particular formula (4) holds true for any $f \in L^2(\mathcal{B}_4)$.

Indeed, by using (2) the operator $\pi_{\lambda,\mu}(f)$ has integral kernel

$$\mathcal{K}_{f,\lambda,\mu}(u,v) = 2\pi \int_{\mathbb{R}} e^{i(u-v)\xi} \mathcal{F}_{\mathbb{R}^4}(f)(\xi, \frac{\lambda}{2}v^2 - \frac{\mu}{2\lambda}, -\lambda v, \lambda)\, d\xi\,,$$

or equivalently

$$\mathcal{K}_{f,\lambda,\mu}(u,v) = (2\pi)^{\frac{3}{2}} \mathcal{F}_{\mathbb{R}^3}(f)(v-u, \frac{\lambda}{2}v^2 - \frac{\mu}{2\lambda}, -\lambda v, \lambda)\,,$$

where the Fourier transform is taken with respect to the second, the third and the fourth variable of f. Integrating the $L^2(\mathbb{R}\times\mathbb{R})$-norm of $\mathcal{K}_{f,\lambda,\mu}$ (or the Hilbert-Schmidt norm of $\pi_{\lambda,\mu}(f)$) against $d\lambda, d\mu$ we obtain

$$\begin{aligned}
&\int_{\mathbb{R}\setminus\{0\}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} |\mathcal{K}_{f,\lambda,\mu}(u,v)|^2\, du\, dv\, d\mu\, d\lambda \\
&= (2\pi)^3 \int_{\mathbb{R}\setminus\{0\}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} |\mathcal{F}_{\mathbb{R}^3}(f)(u-v, \frac{\lambda}{2}v^2 - \frac{\mu}{2\lambda}, -2\lambda, \lambda)|^2\, du\, dv\, d\lambda\, d\mu \\
&= (2\pi)^3 \int_{\mathbb{R}\setminus\{0\}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} |\mathcal{F}_{\mathbb{R}^3}(f)(x_1, w_2, w_3, w_4)|^2\, \frac{1}{2}\, dw_2\, dw_3\, dw_4\, dx_1\,,
\end{aligned}$$

where the constant $\frac{1}{2}$ comes from the calculation of the determinant of the Jacobian matrix of the linear transformation $F(u,v,\lambda,\mu) = (w_1 = u - v, w_2 = \frac{\lambda}{2}v^2 - \frac{\mu}{2\lambda}, w_3 = -\lambda v, w_4 = \lambda)$. Finally, the Plancherel formula on $\mathbb{R}^3$ in the variable (w_2, w_3, w_4) with dual variable (x_2, x_3, x_4) gives

$$\int_{\mathbb{R}\setminus\{0\}}\int_{\mathbb{R}^3} |\mathcal{K}_{f,\lambda,\mu}(u,v)|^2\, dv\, du\, d\mu\, d\lambda = 2^2\pi^3 \int_{\mathbb{R}^4} |f(x_1,x_2,x_3,x_4)|^2\, dx_1\, dx_2\, dx_3\, dx_4\,,$$

and the last implies (4).

4. Difference operators

Difference operators on the setting of a compact Lie group introduced in [RT10] as acting on Fourier coefficients, while on graded Lie groups in [FR16]. In the setting of the Engel group $\mathcal{B}_4$ this yields the definition of the difference operators Δ_{x_i} as:

$$\Delta_{x_i}\hat{\kappa}(\pi_{\lambda,\mu}) := \pi_{\lambda,\mu}(x_i\kappa)\,, \quad i = 1, \cdots, 4\,,$$

for suitable distributions κ on $\mathcal{B}_4$.

To find the explicit expressions of the difference operators Δ_{x_i} we make use of the following property: For X and $\tilde{X}$ being a left and a right invariant vector field, respectively, in the Lie algebra $\mathfrak{l}_4$, and for a distribution κ on $\mathcal{B}_4$ we have

$$\pi_{\lambda,\mu}(X\kappa) = \pi_{\lambda,\mu}(X)\pi_{\lambda,\mu}(\kappa)\,, \quad \pi_{\lambda,\mu}(\tilde{X}\kappa) = \pi_{\lambda,\mu}(\kappa)\pi_{\lambda,\mu}(X)\,.$$

Notice that the right invariant vector fields that generate $\mathfrak{l}_4$ can be calculated as:

$$\tilde{X}_1 = \partial_{x_1} - x_2\partial_{x_3} - x_3\partial_{x_4}\,, \tilde{X}_2 = \partial_{x_2}\,, \tilde{X}_3 = \partial_{x_3}\,, \tilde{X}_4 = \partial_{x_4}\,.$$

Proposition 4.1. *For suitable distribution κ on $\mathcal{B}_4$ we have:*

$$\Delta_{x_1}\hat{\kappa}(\pi_{\lambda,\mu}) = \frac{i}{\lambda}(\pi_{\lambda,\mu}(X_3)\pi_{\lambda,\mu}(\kappa) - \pi_{\lambda,\mu}(\kappa)\pi_{\lambda,\mu}(X_3))\,,$$

where $\pi_{\lambda,\mu}(X_3) = -i\lambda u$, and

$$\Delta_{x_2}\hat{\kappa}(\pi_{\lambda,\mu}) = \frac{2\lambda}{i}\partial_\mu \pi_{\lambda,\mu}(\kappa)\,.$$

Proof. Since $\pi_{\lambda,\mu}(X_4) = i\lambda$, and $\tilde{X}_3 - X_3 = X_4 x_1$, we have

$$\begin{aligned}
\pi_{\lambda,\mu}(x_1\kappa) &= \frac{1}{i\lambda}\pi_{\lambda,\mu}(X_4 x_1\kappa) = \frac{1}{i\lambda}((\tilde{X}_3 - X_3)\kappa) \\
&= \frac{i}{\lambda}(\pi_{\lambda,\mu}(X_3)\pi_{\lambda,\mu}(\kappa) - \pi_{\lambda,\mu}(\kappa)\pi_{\lambda,\mu}(X_3))\,.
\end{aligned}$$

Now, for the difference operator corresponding to x_2, we differentiate the group Fourier transform of κ as in (1) at h with respect to μ and get

$$\begin{aligned}\partial_\mu\{\pi_{\lambda,\mu}(\kappa)h(u)\} = \partial_\mu\Big\{ &\int_{\mathbb{R}^4} \kappa(x)\exp\left(i\left(\frac{\mu}{2\lambda}x_2-\lambda x_4\right)\right) \\ &\cdot \exp\left(i\left(\lambda x_3(u-x_1)-\frac{\lambda}{2}x_2(u-x_1)^2\right)\right)h(u-x_1)dx\Big\} \\ &= \int_{\mathbb{R}^4} \kappa(x)\exp\left(i\left(\frac{\mu}{2\lambda}x_2-\lambda x_4\right)\right) \\ &\cdot \exp\left(i\left(\lambda x_3(u-x_1)-\frac{\lambda}{2}x_2(u-x_1)^2\right)\right)h(u-x_1)\left(\frac{i}{2\lambda}x_2\right)dx\,,\end{aligned}$$

or in terms of difference operators,

$$\partial_\mu \pi_{\lambda,\mu}(\kappa) = \pi_{\lambda,\mu}\left(\frac{i}{2\lambda}x_2\kappa\right) = \frac{i}{2\lambda}\Delta_{x_2}\pi_{\lambda,\mu}(\kappa)\,.$$

□

Proposition 4.2. *For a suitable distribution κ we have:*

$$\Delta_{x_3}\hat{\kappa}(\pi_{\lambda,\mu}) = \frac{i}{\lambda}(\Delta_{x_2}\pi_{\lambda,\mu}(\kappa)\pi_{\lambda,\mu}(X_3)+\pi_{\lambda,\mu}(\kappa)\pi_{\lambda,\mu}(X_1)-\pi_{\lambda,\mu}(X_1)\pi_{\lambda,\mu}(\kappa))\,,$$

where $\Delta_{x_2|\pi_{\lambda,\mu}}$ is given in Proposition 4.1 and $\pi_{\lambda,\mu}(X_1)=\partial_u$, $\pi_{\lambda,\mu}(X_3)=-i\lambda u$.

Proof. Since $X_1-\tilde{X}_1-x_2X_3=\partial_{x_4}x_3$ we have

$$\begin{aligned}\pi_{\lambda,\mu}(x_3\kappa) &= \frac{1}{i\lambda}(X_4x_3\kappa) = \frac{1}{i\lambda}((X_1-\tilde{X}_1-x_2\tilde{X}_3)\kappa) \\ &= \frac{1}{i\lambda}(\pi_{\lambda,\mu}(X_1)\pi_{\lambda,\mu}(\kappa)-\pi_{\lambda,\mu}(\kappa)\pi_{\lambda,\mu}(X_1)-\Delta_{x_2}\pi_{\lambda,\mu}(\kappa)\pi_{\lambda,\mu}(X_3) \\ &= \frac{i}{\lambda}(\Delta_{x_2}\pi_{\lambda,\mu}(\kappa)\pi_{\lambda,\mu}(X_3)+\pi_{\lambda,\mu}(\kappa)\pi_{\lambda,\mu}(X_1)-\pi_{\lambda,\mu}(X_1)\pi_{\lambda,\mu}(\kappa))\,,\end{aligned}$$

completing the proof. □

Proposition 4.3. *For a suitable distribution κ on $\mathcal{B}_4$ we have:*

$$\begin{aligned}(\Delta_{x_4}\pi_{\lambda,\mu}(\kappa))h(u) &= i\partial_\lambda\{\pi_{\lambda,\mu}(\kappa)h(u)\} - \left(\frac{\mu}{2\lambda^2}+\frac{u^2}{2}\right)\Big\{\Delta_{x_2}\pi_{\lambda,\mu}(\kappa)h(u)\Big\} \\ &+ u\Big\{\Delta_{x_3}\pi_{\lambda,\mu}(\kappa)h(u)\Big\} - \Big\{\Delta_{x_3}\Delta_{x_1}\pi_{\lambda,\mu}(\kappa)h(u)\Big\} \\ &+ u\Big\{\Delta_{x_2}\Delta_{x_1}\pi_{\lambda,\mu}(\kappa)h(u)\Big\} - \frac{1}{2}\Big\{\Delta_{x_2}\Delta_{x_1}^2\pi_{\lambda,\mu}(\kappa)h(u)\Big\}\,,\end{aligned}$$

where the difference operators $\Delta_{x_i|\pi_{\lambda,\mu}}$, $i=1,2,3$, are given in Propositions 4.1 and 4.2, respectively.

Proof. Differentiating the group Fourier transform of κ as in (1) at h with respect to λ yields

$$\begin{aligned}\partial_\lambda\{\pi_{\lambda,\mu}(\kappa)h(u)\} = \partial_\lambda\Big\{ &\int_{\mathbb{R}^4} \kappa(x)\exp\left(i\left(\frac{\mu}{2\lambda}x_2-\lambda x_4\right)\right)\\ &\cdot \exp\left(i\left(\lambda x_3(u-x_1)-\frac{\lambda}{2}x_2(u-x_1)^2\right)\right)h(u-x_1)\,dx\Big\}\\ &= \int_{\mathbb{R}^4}\kappa(x)\exp\left(i\left(\frac{\mu}{2\lambda}x_2-\lambda x_4+\lambda x_3(u-x_1)-\frac{\lambda}{2}x_2(u-x_1)^2\right)\right)\\ &h(u-x_1)\Big\{i\left(-\frac{\mu}{2\lambda^2}x_2-x_4+x_3(u-x_1)-\frac{x_2}{2}(u-x_1)^2\right)\Big\}dx\,.\end{aligned}$$

Rewriting the above formula in terms of difference operators we obtain

$$\begin{aligned}\partial_\lambda\{\pi_{\lambda,\mu}(\kappa)h(u)\} = i\Big[&-\left(\frac{\mu}{2\lambda^2}+\frac{u^2}{2}\right)\Big\{\Delta_{x_2}\pi_{\lambda,\mu}(\kappa)h(u)\Big\}-\Big\{\Delta_{x_4}\pi_{\lambda,\mu}(\kappa)h(u)\Big\}\\ &+u\Big\{\Delta_{x_3}\pi_{\lambda,\mu}(\kappa)h(u)\Big\}-\Big\{\Delta_{x_3}\Delta_{x_1}\pi_{\lambda,\mu}(\kappa)h(u)\Big\}\\ &+u\Big\{\Delta_{x_2}\Delta_{x_1}\pi_{\lambda,\mu}(\kappa)h(u)\Big\}-\frac{1}{2}\Big\{\Delta_{x_2}\Delta_{x_1}^2\pi_{\lambda,\mu}(\kappa)h(u)\Big\}\Big]\,,\end{aligned}$$

completing the proof. □

For example we have:

$\Delta_{x_1}\pi_{\lambda,\mu}(X_1) = -I\,, \Delta_{x_1}\pi_{\lambda,\mu}(X_2) = \Delta_{x_1}\pi_{\lambda,\mu}(X_3) = \Delta_{x_1}\pi_{\lambda,\mu}(X_4) = 0$
$\Delta_{x_2}\pi_{\lambda,\mu}(X_1) = \Delta_{x_2}\pi_{\lambda,\mu}(X_3) = \Delta_{x_2}\pi_{\lambda,\mu}(X_4) = 0\,, \Delta_{x_2}\pi_{\lambda,\mu}(X_2) = -\lambda I$
$\Delta_{x_3}\pi_{\lambda,\mu}(X_1) = \Delta_{x_3}\pi_{\lambda,\mu}(X_4) = 0\,, \Delta_{x_3}\pi_{\lambda,\mu}(X_2) = -\lambda u + u\,, \Delta_{x_3}\pi_{\lambda,\mu}(X_3) = -I$
$\Delta_{x_4}\pi_{\lambda,\mu}(X_1) = \Delta_{x_4}\pi_{\lambda,\mu}(X_4) = 0\,, \Delta_{x_4}\pi_{\lambda,\mu}(X_2) = \frac{u^2}{2}(1-\lambda)+\frac{\mu}{2\lambda}\,, \Delta_{x_4}\pi_{\lambda,\mu}(X_4) = -I,$

where the difference operators $\Delta_{x_i}\pi_{\lambda,\mu}(X_j)$ can be understood as the group Fourier transform of the distribution $x_iX_j\delta_0$.

5. Quantization and symbol classes

In this note, we may slightly change the notation of the symbol introduced in [FR16]. We keep the notation

$$x = (x_1,x_2,x_3,x_4)\in\mathcal{B}_4\,,$$

to denote the coordinates of an element in the Engel group $\mathcal{B}_4$, and we may denote by

$$\sigma(x,\lambda,\mu) := \sigma(x,\pi_{\lambda,\mu})\,,\quad (x,\lambda,\mu)\in\mathcal{B}_4\times\mathbb{R}\setminus\{0\}\times\mathbb{R}\,,$$

the symbol σ parametrised by (x,λ,μ). In addition, if the multi-index $\alpha\in\mathbb{N}_0^4$ is written as

$$\alpha = (\alpha_1,\alpha_2,\alpha_3,\alpha_4)\,,\quad \alpha_i\in\mathbb{N}_0\,,$$

then the homogeneous degree of α is given by:

$$[\alpha] = \alpha_1+\alpha_2+2\alpha_3+3\alpha_4\,.$$

For each α we may write:

$$x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}x_4^{\alpha_4}\,,$$

so that the corresponding difference operator can be defined as:

$$\Delta'^{\alpha} = \Delta_{x_1}^{\alpha_1}\Delta_{x_2}^{\alpha_2}\Delta_{x_3}^{\alpha_3}\Delta_{x_4}^{\alpha_4}\,.$$

Finally for the vector field X we write X^α to denote the following composition of vector fields:

$$X_1^{\alpha_1}X_2^{\alpha_2}X_3^{\alpha_3}X_4^{\alpha_4}\,.$$

Following [FR16] we define the symbol classes $S^m_{\rho,\delta}(\mathcal{B}_4)$, where $0 \leq \delta < \rho \leq 1$ and $m \in \mathbb{R}$, as the set of symbols σ for which the following quantities are finite:

$$\|\sigma\|_{S^m_{\rho,\delta},a,b,c} := \sup_{\lambda\in\mathbb{R}\setminus\{0\},\mu\in\mathbb{R},x\in\mathcal{B}_4} \|\sigma(x,\lambda,\mu)\|_{S^m_{\rho,\delta},a,b,c}\,, \quad a,b,c\in\mathbb{N}_0\,,$$

where

$$\|\sigma(x,\lambda,\mu)\|_{S^m_{\rho,\delta},a,b,c} := \sup_{\substack{[a]\leq a\\ [\beta]\leq b,|\gamma|\leq c}} \|\pi_{\lambda,\mu}(I-\mathcal{L})^{\frac{\rho[\alpha]-m-\delta[\beta]+\gamma}{2}} X^\beta \Delta'^{\alpha}\sigma(x,\lambda,\mu)\pi_{\lambda,\mu}(I-\mathcal{L})^{-\frac{\gamma}{2}}\|_{op}\,.$$

There is a natural quantization on any type-I Lie group introduced by [Tayl84] that can be served as the analogue of the Kohn-Nirenberg quantization on $\mathbb{R}^n$. In particular, the *quantization*, i.e., the mapping $\sigma \mapsto Op(\sigma)$ produces operators associated with a symbol σ (for example in the class of symbols $S^m_{\rho,\delta}(\mathcal{B}_4)$) on $\mathcal{S}(\mathcal{B}_4)$ given by:

$$Op(\sigma)\phi(x) = 2^{-3}\pi^{-4}\int_{\lambda\neq 0}\int_{\mu\in\mathbb{R}} Tr\left(\pi_{\lambda,\mu}(x)\sigma(x,\lambda,\mu)\pi_{\lambda,\mu}(\phi)\right)\,d\mu\,d\lambda\,. \tag{5}$$

Here we have used our notation for the description of the dual, as well as for the symbol and the Plancherel measure, see (4).

Let us note that by (2), we see that for the symbol σ quantized as:

$$\sigma(x,\lambda,\mu) = Op(a_{\kappa_x,\lambda,\mu})\,,$$

then its symbol that is given by

$$a_{\kappa_x,\lambda,\mu}(v,\xi) = (2\pi)^2\mathcal{F}_{\mathbb{R}^4}(\kappa_x)(\xi,\frac{\lambda}{2}v^2-\frac{\mu}{2\lambda},-\lambda v,\lambda)\,,$$

shall be called the (λ,μ)-symbol, where $\{\kappa_x(y)\}$ is the kernel of the symbol $\{\sigma(x,\lambda,\mu)\}$, i.e.,

$$\sigma(x,\lambda,\mu) = \pi_{\lambda,\mu}(\kappa_x)\,.$$

The above, together with the property of the Fourier transform

$$\hat{\phi}(\pi_{\lambda,\mu})\pi_{\lambda,\mu}(x) = \mathcal{F}_{\mathcal{B}_4}(\phi(x\cdot))(\pi_{\lambda,\mu})\,,$$

and the properties of the trace yield the following alternative formula for the quantization given in (5):

$$Op(\sigma)(\phi)(x) = 2^{-3}\pi^{-4}\int_{\lambda\neq 0}\int_{\mu\in\mathbb{R}} Tr\left(Op(a_{\kappa_x,\lambda,\mu})Op(a_{\phi(x\cdot),\lambda,\mu})\right)\,d\mu\,d\lambda\,.$$

The last formula shows that the quantization formula (5) can be expressed in terms of composition of quantization of symbols in the Euclidean space.

Similarly, for the case of the Heisenberg group $\mathbb{H}_n$, (3) implies that the operator $Op(\sigma)$ on $\mathcal{S}(\mathbb{H}_n)$ involves 'Euclidean objects', and in particular:

$$Op(\sigma)\phi(x) = c_n \int_{\lambda \neq 0} Tr\left(Op^W(a_{x,\lambda})Op^W[\mathcal{F}_{\mathbb{R}^{2n+1}}(\phi(x\cdot))(\sqrt{|\lambda|}\cdot, \sqrt{\lambda}\cdot, \lambda)]\right) |\lambda|^n \, d\lambda \,,$$

where the symbol $a_{x,\lambda}$ (also called the λ-symbol) given by:

$$a_{x,\lambda}(\xi, u) = \mathcal{F}_{\mathbb{R}^{2n+1}}(\kappa_x^{'})(\sqrt{|\lambda|}\xi, \sqrt{\lambda}u, \lambda) \,,$$

where $\{\kappa_x^{'}(y)\}$ is the kernel of the symbol σ, is such that

$$\sigma(x, \lambda) := \sigma(x, \pi_\lambda) = Op^W(a_{x,\lambda}) \,.$$

For our notation, especially for the Plancherel measure $c_n|\lambda|^n$ on $\mathbb{H}_n$, see [FR16, Chapter 6].

In contrast with the case of the Engel group $\mathcal{B}_4$, in the setting of the Heisenberg group $\mathbb{H}_n$, one can renormalise $a_{x,\lambda}$ as

$$a_{x,\lambda}(\xi, u) := \tilde{a}_{x,\lambda}(\sqrt{|\lambda|}\xi, \sqrt{\lambda}u) \,,$$

and therefore, one can characterise the symbol classes $S^m_{\rho,\delta}(\mathbb{H}_n)$ by the property that these λ-symbols belong to some Shubin spaces, called λ-type version of the usual Shubin classes, leading to sufficient criteria for ellipticity and hypoellipticity of operators on $\mathbb{H}_n$ in terms of the invertibility properties of their λ-symbols, see [FR16, Chapter 6].

Acknowledgement

I would like to thank Professor Michael Ruzhansky for introducing me to this topic and for comments leading to improvements of the current work.

References

[BFKG12] H. Bahouri, C. Fermanian-Kammerer and I. Gallagher. *Phase-space analysis and pseudo-differential calculus on the Heisenberg group*, Astérisque, **342**, 2012. See also revised version of March 2013 of arXiv:0904.4746.

[BGR10] U. Boscain, J. P. Gauthier and F. Rossi. Hypoelliptic heat kernel over 3-step nilpotent Lie groups. *J. Math. Sci.*,Vol.199, No.6, 2014.

[Dix57] J. Dixmier. Sur les représentations unitaires des groupes de Lie nilpotents, volume III of *Canad. J. Math.10*, pages 321-348, 1957.

[FR14] V. Fischer and M. Ruzhansky. A pseudo-differential calculus on the Heisenberg group. *C. R. Math. Acad. Sci. Paris*, 352(3):197–204, 2014.

[FR16] V. Fischer and M. Ruzhansky. *Quantization on nilpotent Lie groups*, volume 314 of *Progress in Mathematics*. Birkhäuser/Springer, [Open access book], 2016.

[FS82] G.B. Folland and E. Stein. *Hardy spaces on Homogeneous groups*. Mathematical Notes **28**. Princeton University Press, 1982.

[RT10] M. Ruzhansky and V. Turuner. *Pseudo-differential operators and symmetries: Background analysis and advanced topics.* Pseudo-Differential Operators: Theory and Applications,**2**, Birkhäuser, Verlag, 2010.

[Tayl84] M.E. Taylor. *Noncommutative microlocal analysis. I. Mem.* Amer. Math. Soc. **52**, 1984.

Marianna Chatzakou:
Department of Mathematics
Imperial College London
180 Queen's Gate, London SW7 2AZ
United Kingdom
E-mail address m.chatzakou16@imperial.ac.uk