Stopbands in fast ion-acoustic soliton propagation revisited

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ABSTRACT

It was recently discovered that, in a multispecies plasma with two positive ion species (one cold and one warm adiabatic) and Boltzmann-distributed electrons, a “stopband” could occur, i.e., there was a range of velocities where no fast ion-acoustic solitons could propagate between two ranges where propagation was possible. Several extensions were subsequently investigated, including the effects of the cool ions having finite temperature and of nonthermal electron distributions. Efforts were made to estimate existence domains in plasma parameter space, often by *ad hoc* arguments, but the illustrations were invariably restricted to the same specific set of parameter values or their neighborhoods. In contrast, here, a systematic and structured study is given: physical arguments determine a range of compositional parameters so that a Sagdeev pseudopotential analysis establishes in a consistent way the various curves limiting the existence domains in parameter space. This is done for four models, namely, Boltzmann, nonthermal Cairns, superthermal kappa, and nonextensive Tsallis electron distributions, and for each, existence domains are plotted in the space of ion charge-to-mass ratio and inverse electron temperature, and detailed examples are presented. Contrary to reports in the literature, stopbands are shown to exist for large deviations from isothermality. However, their range in parameter space is shifted from that obtained for Boltzmann electrons. This establishes that the stopband phenomenon is robust and governed primarily by the cold and warm ion properties and the electron temperature, the form of the electron distribution having only a quantitative effect.

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I. INTRODUCTION

Solitary waves are aperiodic nonlinear structures that propagate through a medium with an unchanging shape and are, thus, stationary in a co-moving reference frame. Their occurrence and behavior in plasmas have often been studied through a reductive perturbation expansion of the underlying fluid equations,^{1,2} leading to a Korteweg–de Vries³ type of equation. This approach was pioneered by Zabusky and Kruskal⁴ in studying ion-acoustic waves, who found a whole string of particle-like interacting properties and coined the name “soliton.” This name is often used in an improper way, as a shorthand to denote just a solitary wave, a practice that we will follow, even if their collisional interactions cannot be ascertained.

Contemporaneously with that work, Sagdeev⁵ introduced a fully nonlinear pseudopotential analysis. Integration of the Poisson equation for electrostatic modes in multispecies plasmas leads to an energy-like integral, in terms of a Sagdeev pseudopotential, provided that the species (charge) densities can be expressed as functions of the

electrostatic potential ϕ . The pseudo-energy integral can then be analyzed for solitary structures, following an analogy with the motion of a particle in a potential field, as in classical mechanics.

Whereas reductive perturbation theory permits the study of interactions between solitons and of their stability, but only for relatively small amplitudes, Sagdeev pseudopotential theory admits *a priori* unbounded amplitudes, but one cannot study their collisional interactions nor stability. Both methods come together for the one-soliton solution, provided that the pseudopotential results are sufficiently limited in amplitude.

Although the Sagdeev technique is in principle valid for arbitrarily large solutions, there are obviously physical constraints and limits that arise from a breakdown of the model assumptions, leading to a restriction of the amplitude.

One typical constraint relates to the neglect of inertia for hot plasma species, on the grounds that their temperature or pressure effects dominate. This is tantamount to having their thermal velocity

much larger than the wave or soliton velocities. Unfortunately, this constraint is sometimes violated, and invariably, no attempt is made to take it into consideration. The avoidance of Landau damping is also a model limitation, when the wave velocity is too close to one of the thermal velocities of the plasma species. Being essentially a kinetic property, it is more difficult to quantify in fluid theory.

Other limits on the soliton velocities and amplitudes are connected with the description of the inertial plasma species, such as whether their densities can reach a sonic point or an infinite compression.^{6,7} All these limitations typically lead to an end point of a range of solitons. For some models, soliton existence may also be limited by the forming of double layers, but that is not relevant for the model we are discussing. To reiterate, a Sagdeev pseudopotential framework can handle solitons with large amplitudes, but there are always physical arguments to limit how large these amplitudes can be.

Invariably, the observation was made that soliton existence domains were limited by single-valued boundaries. It was, thus, a surprise when Nsengiyumva *et al.*⁸ discovered that in a plasma with two positive ion species (one cold and one warm adiabatic) and Boltzmann electrons, under certain modalities, a “stopband” occurred, a range in velocities where no fast ion-acoustic solitons could propagate and the warm ion sonic point limit was double-valued in soliton velocities for a given charge density ratio. They found that this could occur only for a plasma in which the cold ion charge-to-mass ratio $\mu > 1$, i.e., one in which the cold ions are effectively the lighter species. They discussed in detail a plasma with $\mu = 2$ and the normalized inverse electron temperature $\tau = 0.36$. This value of μ can, for instance, model a plasma of cold protons, together with hot alpha particles.^{9–12} Reference 8 also reported that stopbands were observed in the approximate ranges $1.3 \leq \mu \leq 3$ and $0.1 \leq \tau \leq 1$. An attempt was made to quantify approximately the existence range in the important $\{\mu, \tau\}$ parameter space, based on some physical and some *ad hoc* assumptions.

Maharaj and Bharuthram then wrote a series of three papers,^{13–15} which offered various extensions of the paper of Nsengiyumva *et al.*⁸

Their first paper¹³ shows that a nonthermal electron distribution, such as a Cairns^{16–18} and a kappa^{19–22} or a two-temperature Boltzmann electron distribution, may also support stopbands in principle. *Inter alia*, they found that the Cairns- and kappa-distributed electrons yield results that deviate in opposing directions from those of the Boltzmann description. The studies, however, centered mainly on the same key compositional parameter values, $\mu = 2$ and $\tau = 0.36$, studied in Ref. 8, and thus, stopbands were found only for relatively small deviations from isothermality.

The next paper¹⁴ includes an adiabatic description of the cool ions, rather than the strict cold ion assumption used earlier,^{8,13} and the electrons being Boltzmann, Cairns, or kappa, as the case may be. In that paper, it was found that finite temperature cool ions led to a reduction in stopband existence domains. Again, however, all figures are for the same $\mu = 2$ and $\tau = 0.36$. Here, there is a subtle change in the definition of μ , now a pure mass ratio of the two ions rather than the ratio of the mass-per-charges. This severely restricts the applicability of the results because for $\mu = 2$, the heavier ion has to be deuterium, if the cold ions are protons.

In a further paper, Maharaj and Bharuthram¹⁵ include Boltzmann, Cairns, or kappa electron distributions, with cold and warm adiabatic ions as before,^{8,13} but now add a third ion species,

which is hot and Boltzmann-distributed and therefore inertialess like the electrons. It is intended to apply to dusty plasmas, as there are two massive positively charged dust species, in the presence of hot electrons and protons. This is an unusual dusty plasma model, as negative dust particles are much more common in the literature.

This paper also discusses what ranges in μ and τ can be admitted, but it is unfortunately based on a rather restricted approach. First, for $\mu = 2$, they investigate only the range $\tau = 0.27$ to 0.45, much as was discussed by Nsengiyumva *et al.*,⁸ now paying particular attention to when the warm ion limit curve splits into two branches. Then, for $\tau = 0.36$, only the range $\mu = 1.85$ to 2.15 is covered. Unfortunately, this limited set again leads to the false conclusion that only small deviations from isothermal electrons can support stopbands. In fact, our present study will show that strong superthermality clearly admits stopbands, but for values of μ and τ that differ from those considered in Refs. 13 and 15.

To summarize, in Refs. 8 and 13–15 dealing with stopbands, there are efforts to delineate existence ranges in the space of compositional parameters but not in a consistent way, that is, without a really systematic approach. Moreover, the illustrative examples were almost all done for a single pair of values for two of the main compositional parameters, namely, $\mu = 2$ and $\tau = 0.36$, or their neighborhood.

This has motivated us to give a fully systematic analysis of this unusual stopband phenomenon, where the admissible limits in μ are determined on physical grounds, before the possible ranges in τ and in the ratio f of the cold ion to warm adiabatic ion charge densities are considered. Thus, the existence ranges of the stopbands in parameter space are determined, in terms of normalized soliton velocities M , which can be translated into soliton amplitudes. This step-by-step analysis for the case when the electrons are Boltzmann-distributed can, then, be adapted for three commonly considered nonthermal electron distributions, viz., for Cairns nonthermal, kappa superthermal, and Tsallis^{23–27} nonextensive electrons. In particular, it is shown that all these nonthermal distributions support stopbands, even for strong deviations from isothermality. However, there is an associated shift in existence ranges in the $\{\mu, \tau\}$ space and f values from the ranges found for Boltzmann electrons.

The complexities in parameter space when cool ion temperature and/or an additional hot ion species are introduced will be detrimental to the thoroughness of our targeted analysis. They are, therefore, not addressed in this paper, but the reader is referred to the pertinent results given by Maharaj and Bharuthram.^{13–15}

This paper is structured as follows: After this Introduction, we give the complete analysis of the Boltzmann case in Sec. II. This is, then, briefly extended to the Cairns nonthermal (Sec. III), kappa superthermal (Sec. IV), and Tsallis nonextensive (Sec. V) models. Section VI includes an overview and comparison of different electron descriptions, and Sec. VII shows our conclusions.

II. BASIC ANALYSIS AND BOLTZMANN ELECTRONS

We follow first the model investigated by Nsengiyumva *et al.*⁸ for a plasma with Boltzmann electrons and cold and warm adiabatic positive ions, repeat some of their expressions for clarity, arrange the analysis and discussion in a more coherent and systematic scheme, and extend their results. At a later stage, we will generalize this for various nonthermal electron distributions, while keeping the two ion species as before.

The charge densities in normalized variables are

$$n_e = (1 + f) \exp(\tau\varphi), \quad (1)$$

$$n_c = \frac{f}{\sqrt{1 - \frac{2\mu\varphi}{M^2}}}, \quad (2)$$

and

$$n_w = \frac{1}{2} \left[\sqrt{(M+1)^2 - 2\varphi} - \sqrt{(M-1)^2 - 2\varphi} \right] \quad (3)$$

for the Boltzmann electrons, cold and warm adiabatic positive ions, respectively. The way n_w is written supposes that we are specifically considering only the fast ion-acoustic mode, for which $M > 1$ in the normalization used. Here, $f = Z_c n_{c0} / Z_w n_{w0}$ is the cold ion charge density in equilibrium, $\tau = T_w / Z_w T_e$ is the inverse electron temperature, $\mu = Z_c m_w / Z_w m_c$ is an effective warm to cold ion mass ratio, and the warm ion thermal speed $v_{tw} = (T_w / m_w)^{1/2}$ has been used as the yardstick for all velocities. Other variables have their usual meaning. The normalization is inspired by an earlier study of acoustic solitary waves in dusty and/or multi-ion plasmas with cold, adiabatic, and hot constituents.²⁸

The ion charge densities are real until upper limits in positive φ are reached,

$$\varphi \leq \varphi_{lc} = \frac{M^2}{2\mu} \quad (4)$$

for the cold ions and

$$\varphi \leq \varphi_{lw} = \frac{1}{2} (M-1)^2 \quad (5)$$

for the warm ions. These limits correspond to an infinite compression for the cold ions and to a sonic point for the warm ions.^{6,7} Clearly, the model breaks down when the lower of these two limiting potentials is reached.⁸ It is noteworthy that of the parameters specifying the actual plasma composition (μ , τ , f), only μ is involved in φ_{lc} , while φ_{lw} is independent of all plasma parameters.⁸ The normalized soliton speed M is yet to be determined.

From a possible equality of $\varphi_{lc} = \varphi_{lw}$, we determine a crossover Mach number, given by

$$M_{co} = \frac{\sqrt{\mu}}{\sqrt{\mu} - 1}. \quad (6)$$

We remind the reader that Ref. 8 found empirically that M_{co} plays a key role in the forming of stopbands.

We have seen in the earlier stopband papers^{8,13–15} that such a crossover is necessary to permit stopbands to occur, so suitable plasma compositions require two inertial species of the same sign but with different temperatures. The simplest plasma model is, thus, cold and warm adiabatic positive ions, with the hot electrons closing the description. Mathematically, a translation to two negative ions, with similar thermal characteristics, in the presence of hot Boltzmann protons, would yield an equivalent description, with all signs reversed. Such an unusual model has been discussed in dusty plasma physics, where the heavy dust is negatively charged, and all electrons have been accreted onto the dust grains.²⁹ Extensions to different, nonthermal

electron distributions have been shown not to lead to qualitative but only to quantitative changes. The overall picture is, thus, robust.

Supposing for a moment that such a crossover exists, which requires $\mu > 1$,⁸ one can deduce the following equivalences:

$$\varphi_{lw} \leq \varphi_{lc} \iff (M-1)^2 \leq \frac{M^2}{\mu} \iff M \leq M_{co}. \quad (7)$$

This will help to structure the discussion at a later stage, but it is only of interest provided $M \geq M_s$, the true acoustic speed in the specific plasma under consideration; otherwise, there are no solitons.^{30,31} M_{co} decreases with increasing μ but always such that $M_{co} \geq 1$. Physically, we shall also require $M_{co} > M_s$. From (6), the limiting cases are of course that for $\mu \rightarrow 1$, one has $M_{co} \rightarrow \infty$, and at the other end, $\mu \rightarrow \infty$ means $M_{co} \rightarrow 1$. Both these limits are clearly not physically relevant. In passing, we recall⁸ that the range $\mu > 1$ relates to a plasma in which the heavier ion species is hotter than the lighter one, a situation that has been observed, for instance, in a number of solar wind measurements.^{9–12}

In Fig. 1, it is shown how the limiting curves for φ_{lw} and φ_{lc} vary in $\{M, \varphi\}$ space and where there are crossovers. Importantly, this diagram is independent of how the electrons are described, and thus, it applies also to Secs. III–V, which deal with other electron distributions.

Let us return briefly to the case $\mu \leq 1$. There is, then, no crossover, and from $M > 1$, it follows that $\mu(M-1)^2 < \mu M^2 \leq M^2$; hence, $\varphi_{lw} < \varphi_{lc}$ so that soliton amplitudes will always be limited by the warm ion sonic point, as that is reached first when M increases from M_s over the whole range of compositional parameters.

To proceed, we close the set of equations by substituting (1), (2), and (3) into Poisson's equation,

$$\frac{d^2\varphi}{dx^2} = n_e - n_c - n_w. \quad (8)$$

After integration, this yields the usual energy integral,

$$\frac{1}{2} \left(\frac{d\varphi}{dx} \right)^2 + S(\varphi, M) = 0, \quad (9)$$

in which the Sagdeev pseudopotential⁵ is given by⁸

FIG. 1. Curves for φ_{lw} (full) and φ_{lc} (dashed), with the latter for several values of μ . Note that φ_{lw} is independent of μ .

$$S(\varphi, M) = \frac{1+f}{\tau} [1 - \exp(\tau\varphi)] + \frac{fM^2}{\mu} \left(1 - \sqrt{1 - \frac{2\mu\varphi}{M^2}} \right) + \frac{1}{6} \left\{ 2 + 6M^2 - [(M+1)^2 - 2\varphi]^{3/2} + [(M-1)^2 - 2\varphi]^{3/2} \right\}. \quad (10)$$

For the fast ion-acoustic solitary waves, the minimum velocity is the acoustic speed $M = M_s$, defined through the larger of the two roots of

$$S''(\varphi, M)|_{\varphi=0} = 0, \quad (11)$$

where dashes refer to partial derivatives of the Sagdeev pseudopotential with respect to φ .

From (10) and (11), one obtains that

$$-(1+f)\tau + \frac{1}{M_s^2 - 1} + \frac{\mu f}{M_s^2} = 0, \quad (12)$$

and hence, for the fast ion-acoustic mode,

$$M_s^2 = \frac{1 + \mu f + \tau(1+f) + \sqrt{\Delta}}{2\tau(1+f)}. \quad (13)$$

The discriminant Δ in (13) can be written in two ways,

$$\begin{aligned} \Delta &= [1 + \mu f + \tau(1+f)]^2 - 4\mu\tau f(1+f) \\ &= [1 - \mu f + \tau(1+f)]^2 + 4\mu f, \end{aligned} \quad (14)$$

with the first form being the traditional one and the second easily permitting one to prove that $\Delta > 0$ and $M_s > 1$, as required for the fast mode. Coupled with the condition for soliton existence, $M \geq M_s$,^{30,31} this leads to the condition $M \geq M_s > 1$. Analogous forms will be used in dealing with other electron distributions.

For completeness, we stress that the warm ion parameters were used to define the normalization; hence, the limits $\tau \rightarrow 0$ and/or $f \rightarrow 0$ are not admissible on physical grounds, as $\tau = 0$ means that the warm ions are cold and $f = 0$ that there are no cold ions, in which case the model is not interesting in our search for stopbands in M . Thus, one can start from $\tau = 0.1$ and $f = 0.1$ to be physically correct, even though the mathematical limits at zero can be computed and will help in delineating the existence ranges. Moreover, τ too small invalidates the chosen normalization, because the yardstick for the velocities would be so small that all normalized velocities would become very large.

For $f \rightarrow \infty$, the model also breaks down. Indeed, because of the normalization with respect to the warm ion parameters, the contribution of the warm ions in $S(\varphi, M)$ is independent of variations in f , μ , and τ . As one considers increasing values of f , one is essentially increasing the cold ion component of the plasma at the expense of the warm ions. Thus, the cold ion and electron contributions tend to balance each other more and more, when a soliton root is sought, and the relative importance of the warm ions diminishes accordingly.

Using the Boltzmann assumption to describe the electrons is mathematically equivalent to the limit $m_e \rightarrow 0$, as electron inertia can be neglected in comparison to their thermal properties. In other words, their thermal velocity must be the highest velocity in the system. Clearly, large τ would bring the warm ion and electron thermal

velocities too close to each other, leading to a breakdown of the model. In addition, it is important to avoid possible Landau damping, by ensuring that the thermal velocities are not too close to the wave or soliton speeds. These factors lead to the (implicitly assumed but seldom mentioned in full) ordering that $v_{tw} \ll v_{soliton} \ll v_{te}$ in the original physical units. The first strong inequality is necessary to minimize ion Landau damping, while the second strong inequality translates the (inertialless) Boltzmann assumption. We emphasize that, in Secs. III–V, when nonthermal electron distributions are considered, these, too, are inertialless, and the same arguments and ordering hold.

Given that we use the warm ion thermal velocity as the standard in the normalization, the ordering becomes

$$1 \ll M \ll \sqrt{T_e m_w / (T_w m_e)}. \quad (15)$$

In the domain where stopbands can be encountered, it will turn out that we find M values of less than 4, so that the second strong inequality, concerning inertialless electrons, is easily fulfilled for the usual mass ranges, provided that the ions are not positrons. Therefore, we will limit the discussion to the range $\tau \geq 0.1$.

Next, we will define a physically acceptable range for μ . As μ determines M_{co} , we see for $\mu = 4$ that $M_{co} = 2$, already a low value. Higher values for μ and consequently lower values for M_{co} do not permit an acceptable range in M , which upholds the ordering (15) and avoids warm ion Landau damping. On the lower end, since $\mu > 1$ is needed for the existence of a crossover regime and thus stopband existence, we will be on the safe side from large M_{co} and take $\mu = 1.5$ as the lower limit. We will see below a further argument in τ for that choice and thus, for the analysis in this paper, consider the range $1.5 \leq \mu \leq 4$.

For further use, we define two auxiliary functions, analogous to those of Ref. 8,

$$\begin{aligned} A(\varphi, M) &= \frac{1}{\tau} [1 - \exp(\tau\varphi)] + \frac{M^2}{\mu} \left(1 - \sqrt{1 - \frac{2\mu\varphi}{M^2}} \right), \\ B(\varphi, M) &= \frac{1}{\tau} [1 - \exp(\tau\varphi)] + \frac{1}{6} \left\{ 2 + 6M^2 - [(M+1)^2 - 2\varphi]^{3/2} + [(M-1)^2 - 2\varphi]^{3/2} \right\}, \end{aligned} \quad (16)$$

so that $S(\varphi, M) = A(\varphi, M)f + B(\varphi, M)$. Using this in the two limits $\varphi_{\ell c}$ and $\varphi_{\ell w}$, we denote, in short,

$$\begin{aligned} A_c(M) &= A(\varphi_{\ell c}, M), & B_c(M) &= B(\varphi_{\ell c}, M), \\ A_w(M) &= A(\varphi_{\ell w}, M), & B_w(M) &= B(\varphi_{\ell w}, M). \end{aligned} \quad (17)$$

Because $\varphi_{\ell c}$ and $\varphi_{\ell w}$ signal the end of ranges of possible solutions,

$$\begin{aligned} S(\varphi_{\ell c}, M) &= A_c(M)f + B_c(M) = 0, \\ S(\varphi_{\ell w}, M) &= A_w(M)f + B_w(M) = 0 \end{aligned} \quad (18)$$

represent curves in $\{f, M\}$ space, for given μ and τ .

From the earlier analyses,^{8,13–15} we note that stopbands exist, provided that the warm ion sonic limit curve has a range in f where it is double valued in M , i.e., it doubles on itself, as it were, between (f_{co}, M_{co}) and (f_{tp}, M_{tp}) . The logical ordering is, thus, $f_{co} < f_{tp}$ and $M_{tp} < M_{co}$. Typical limiting curves are shown in Fig. 2 for the choice $\mu = 3$ and $\tau = 0.95$. How we obtain these values is discussed in a systematic way below. It will quickly become clear that many cases can similarly

FIG. 2. Bounding curves for $\phi_{\ell w}$ (full red) and $\phi_{\ell c}$ (dashed blue) in $\{f, M\}$ space, with M_s represented by the dotted black curve, for $\mu = 3$ and $\tau = 0.95$. The special values delimiting the stopband region are $f_{co} = 1.158$, $f_{ip} = 4.174$, $M_{co} = 2.366$, and $M_{ip} = 2.176$, for Boltzmann electrons.

be treated for other parameters and electron models and that these will give rise to analogous figures.

As illustrated in Fig. 2, the two special points must guide the logical way of determining the existence domains in compositional parameter space. First, M_{co} is a function of μ only, and the crossover between warm and cold ion limits occurs at $\phi_{\ell w} = \phi_{\ell c}$. As $S(\phi_{\ell w}, M_{co}) = 0$ is linear in f , the analytical determination of f_{co} is fairly straightforward, numerically. For $M = M_{co}$, one has $\phi_{\ell w} = \phi_{\ell c}$ and $S(\phi_{\ell w}, M_{co}) = S(\phi_{\ell c}, M_{co}) = 0$, whence (18) yields

$$f_{co} = -\frac{B_w(M_{co})}{A_w(M_{co})} = -\frac{B_c(M_{co})}{A_c(M_{co})}. \quad (19)$$

Given that M_{co} is a function of μ only, f_{co} becomes a function of the compositional parameters μ and τ . The parts of $A(\phi, M)$ and $B(\phi, M)$ containing τ in (16) and subsequent expressions are equal so that we can from (19) derive that f_{co} changes with τ as

$$\frac{\partial f_{co}}{\partial \tau} = \frac{B_w(M_{co}) - A_w(M_{co})}{A_w^2(M_{co})} \cdot \frac{\partial A_w(M_{co})}{\partial \tau}. \quad (20)$$

One can prove that

$$\frac{\partial A_w(M_{co})}{\partial \tau} = -\frac{1}{\tau^2} [1 - \exp(\tau\phi) + (\tau\phi) \exp(\tau\phi)] < 0 \quad (21)$$

because the expression between square brackets is always positive for $\tau\phi > 0$. Parenthetically, we note that this proof depends explicitly on the Boltzmann form of the electron distribution. Furthermore,

$$\begin{aligned} B_w(M_{co}) - A_w(M_{co}) &= \frac{1}{3} (1 + 3M_{co}^2 - 4M_{co}^{3/2}) - (M_{co} - 1)^2 \\ &= -\frac{2}{3} (M_{co}^{1/2} - 1)^2 (2M_{co}^{1/2} + 1) < 0. \end{aligned} \quad (22)$$

The last result is independent of the hot electron description and will also hold for other, nonthermal electron distributions. Combining (21) and (22), the r.h.s. of (20) is positive and f_{co} increases as τ increases. This result proves the correctness of the assertion in Ref. 8, regarding the effect of increasing τ on f_{co} , and will also be observed in

the graphs below. Unfortunately, as the elegant mathematical proof of (21) is valid specifically for Boltzmann electrons, it does not apply to the other electron distributions studied below, so the fact that f_{co} increases with τ will be borne out by the graphs in those cases.

The mathematical limits $f_{co} = 0$ and $f_{co} \rightarrow \infty$ can be evaluated numerically, even though they are physically not relevant, as being outside the acceptable ranges. The result is two limit values of τ at given μ , namely, τ_{min} from $f_{co} = 0 \iff B_w(M_{co}) = 0$ and τ_{maxB} from $f_{co} \rightarrow \infty \iff A_w(M_{co}) = 0$. This much was already suggested in the original paper⁸ but only evaluated numerically at $\mu = 2$.

On the other hand, the turning point at f_{ip} occurs when the warm ion limit curve has a vertical tangent, in other words when $dM/df \rightarrow \infty$ or equivalently, when $df/dM = 0$. Solving the lower equation in (18) for f and taking the derivative with respect to M yield

$$\frac{df}{dM} = \frac{d}{dM} \left(-\frac{B_w(M)}{A_w(M)} \right) = 0, \quad (23)$$

an expression in M , μ , and τ . Therefore, Eq. (23) defines $M = M_{ip}$ as an implicit function of μ and τ through

$$A_w(M) \frac{dB_w(M)}{dM} = B_w(M) \frac{dA_w(M)}{dM}. \quad (24)$$

Once a value $M = M_{ip}$ is obtained numerically, for chosen μ and τ , substitution in the expression between brackets in (23) yields

$$f_{ip} = -\frac{B_w(M_{ip})}{A_w(M_{ip})}. \quad (25)$$

It is this essential part of the reasoning which eluded the authors of the earlier papers.^{8,13–15}

Bearing in mind the definition of f as the ratio of the cold ion to the warm ion charge density, one may view the above results in the following way: for the lower end of τ , the range in f over which a stopband in M can be found runs over a finite range in increasing normalized cold ion density from f_{co} to finite f_{ip} . As one considers the increasing values of τ , both these limits in f increase until the upper limit reaches infinity, i.e., the range then stretches from a finite cold ion fraction until one reaches the model limit with the vanishing of the warm ion density. For higher values of τ , the lower limit on the cold ion density increases until it, too, reaches the model limit.

Given the obvious ordering between the two special points, the extreme value $f_{ip} \rightarrow \infty$ is reached at τ_{maxA} , but obviously, f_{co} is still finite. Thus, $\tau_{maxA} < \tau_{maxB}$, and for τ inside this range, the warm ion limit curve is split into two branches, an upper one connected to the point (f_{co}, M_{co}) and a lower one, starting from low f , as observed in Ref. 15. In this τ range, all values $f \geq f_{co}$ will yield stopbands, but, of course, f too large will not be physically relevant.

We recognize that the above approach has similarities in concept to an earlier attempt to obtain an estimate of acceptable ranges in τ and μ , the relative charge-to-mass ratio of the ion species.⁸ However, a part of that work was based on the acoustic speed, M_s , the lower limit of soliton speeds, as an approximate value for M_{co} at the lower and upper limits in f .

Having decided upon a physically acceptable range for the compositional parameters that might support stopbands, we can use (19), (24), and (25) to guide the discussion. The minimum limit $f_{co} = 0$ comes from $B_w(M_{co}) = 0$ and gives a curve for τ_{min} as a function of μ ,

whereas $f_{co} \rightarrow \infty$ means that $A_w(M_{co}) = 0$ gives the curve for τ_{maxB} . When $f_{ip} \rightarrow \infty$, this determines the curve for τ_{maxA} , with the help of (24). Figure 3 shows the curves for τ_{min} , τ_{maxA} , and τ_{maxB} as functions of μ and thus delimits the existence ranges in $\{\mu, \tau\}$ space. For all three of these limiting curves, the normalized electron temperature increases with μ . This confirms and expands the observation first made in Ref. 8 that the values of τ over which stopbands were found rose with increasing μ .

There are stopbands for all μ and τ between the curves for τ_{min} and τ_{maxA} for $f_{co} < f < f_{ip} < \infty$. For μ and τ between the curves τ_{maxA} and τ_{maxB} , there are stopbands having $f_{co} < f < \infty$ so that the warm ion limit curve is split into two, given that $f_{ip} \rightarrow \infty$. Mathematically, the curves would tend to $\tau \rightarrow 0$ for $\mu \rightarrow 1$, but this is not physically realistic, as discussed above. Even for $\mu = 1.5$, the values for the corresponding τ are already close to or below 0.1, thus justifying our decision to consider only $\mu \geq 1.5$.

Another way to visualize the existence domains is given in Fig. 4, showing the curves f_{co} and f_{ip} for the case of Boltzmann electrons, for chosen $\mu = 2, 3$, and 4, respectively. In each pair of curves with the same coding, the lower one represents f_{co} and the upper one f_{ip} . The coding is blue for $\mu = 2$, red dashes for $\mu = 3$, and black dots for $\mu = 4$. All curves are limited in τ to where f_{ip} and/or f_{co} becomes infinite but usually at physically unacceptably high values of f . Therefore, the ordinates have been cut off above $f = 10$ for graphical clarity but extend to infinity in f .

The interpretation of Fig. 4 is that specific values $\mu = 2, 3, 4$ have been chosen to generate graphs, which are indicative of the trends when the compositional parameters vary. Nevertheless, for reasons of continuity, it is clear that any μ between 1.5 and 4 (not just integers) will give a physically acceptable stopband in M , as already indicated in Fig. 3. Now, f_{co} increases with τ , as proved already by (20)–(22). After that, it is straightforward to plot the bounding curves, the corresponding Sagdeev pseudopotentials, and the soliton profiles, if wanted.

As seen from Figs. 3 and 4, for increasing μ , stopbands will occur for increasing values of τ . Figure 2 clearly shows that for a given value of f , a first soliton range is encountered between M_s and the lower of the two values on the curve representing ϕ_{lc} , whereas the second range starts at the higher value on the red curve and the curve

FIG. 3. Graphs of τ_{min} (blue), τ_{maxA} (black dotted), and τ_{maxB} (red dashed) as functions of μ .

FIG. 4. Curves for f_{co} (lower curve) and f_{ip} (upper curve), for $\mu = 2$ (blue), $\mu = 3$ (red dashes), and $\mu = 4$ (black dots).

representing ϕ_{lc} . To show that all these arguments and graphs indeed indicate a range of stopbands in M , velocities at which no solitons exist, sandwiched between two ranges that allow solitons, we have plotted in Fig. 5 the corresponding Sagdeev pseudopotentials. Sagdeev pseudopotentials indicated by the three thicker red curves are for the limiting Mach numbers, in ascending values $M = 2.008, 2.301$, and 2.593. In between, the thin blue curves represent possible soliton pseudopotentials, whereas the dashed thin black curves are pseudopotentials without solitons, in the stopband range or beyond ϕ_{lc} .

In the case where $f_{ip} \rightarrow \infty$, there is a regime beyond τ_{maxA} , until τ_{maxB} is reached when $f_{co} \rightarrow \infty$. The limiting curve representing ϕ_{lc} , then, splits into two branches,¹⁵ and there are stopbands between these two, for all f . However, these f values quickly get very large and are, thus, of limited physical interest.

An example of the limiting curves for the choice $\mu = 2$ and $\tau = 0.375$ is shown in Fig. 6. We note that then $\tau = 0.375 > \tau_{maxA} = 0.371$, and therefore, the curve representing ϕ_{lc} has been split into two branches. Figure 6(a) clearly shows that for a given value of f , a first soliton range is encountered between M_s and the lower of the two

FIG. 5. Sagdeev pseudopotentials for the model with Boltzmann electrons and $\mu = 3$, $\tau = 0.95$, and $f = 3$. The three thicker red curves are for the limiting Mach numbers, in ascending values $M = 2.008, 2.301, 2.593$. In between, the thin blue curves represent possible soliton pseudopotentials, whereas the dashed thin black curves are pseudopotentials without solitons, in the stopband range and beyond ϕ_{lc} .

FIG. 6. (a) Bounding curves for ϕ_{tw} (full red) and ϕ_{tc} (dashed blue) in $\{f, M\}$ space, with M_s represented by the dotted black curve, for $\mu=2$ and $\tau=0.375$. The special values delimiting the stopband region are $f_{co}=3.157$ and $M_{co}=3.414$, with $f_p \rightarrow \infty$, for Boltzmann electrons. (b) Sagdeev pseudopotentials for the model with Boltzmann electrons and $\mu=2$, $\tau=0.375$, and $f=8$, with curve coding as in Fig. 5. The three thicker red curves are for the limiting Mach numbers, in ascending values $M=2.870$, 3.393 , and 3.551 .

branches representing ϕ_{tw} , whereas the second range starts at the higher red curve and is limited by ϕ_{tc} .

To show that these stopbands in M correspond to velocities at which no solitons exist, sandwiched between two ranges that allow solitons, we have plotted the corresponding Sagdeev pseudopotentials in Fig. 6(b), for $f=8$. Sagdeev pseudopotentials indicated by the three thicker red curves are for the limiting Mach numbers, in ascending values $M=2.870$, 3.393 , and 3.551 . In between, the thin blue curves represent possible soliton pseudopotentials, whereas the dashed thin black curves are examples of pseudopotentials without solitons, in the stopband range, or beyond ϕ_{tc} . We note that, although $\tau_{maxB}=0.431$, $f=10$ is already reached for $\tau=0.411$. Further typical examples, but for questionably high f , can be found elsewhere.¹⁵

After this rather extensive discussion of the Boltzmann electron case, we will apply, in brief, the same methodological framework, *mutatis mutandis*, when, instead of Boltzmann electrons, we incorporate Cairns nonthermal, kappa superthermal, or Tsallis nonextensive electrons. This framework consists in first determining the relevant limits in compositional parameter space and then determining the equivalent expressions (19) and (25), to illustrate the existence domains as in Fig. 3. This allows one to choose a convenient set μ, τ

plus the relevant nonthermality parameter and draw the limiting curves as in Fig. 2. Finally, a choice for f and the corresponding M yields the Sagdeev pseudopotentials, proving the existence of the stopband, as shown in Fig. 5. This will be addressed below.

III. CAIRNS NONTHERMAL ELECTRONS

When the electrons are no longer in thermal equilibrium, described by Boltzmann distributions (1), various possibilities are encountered in the literature. All of these models reduce in the limit to Boltzmann distributions.

First, we will discuss the Cairns nonthermal distributions,¹⁶ with (1) replaced by^{16,18}

$$n_e = (1+f)(1-\beta\tau\phi + \beta\tau^2\phi^2) \exp(\tau\phi). \quad (26)$$

It is obvious that the Cairns distribution includes the Boltzmann case for $\beta=0$, while an increase in β represents an increase in nonthermality. Because of the peculiar way in which the Cairns distribution was constructed, to explain soliton polarity changes observed in space plasma observations,¹⁶ there is an upper limit at $\beta=4/7 \simeq 0.57$, beyond which the underlying Cairns phase space distribution induces beam unstable configurations.¹⁷ Of course, once the Cairns phase space distribution has been integrated to obtain the species' charge density (26), this limit on β is not visible at the fluid description level but needs to be kept in mind. Purely mathematically, there is no difficulty in going to higher β .

The changes in the Sagdeev pseudopotential, due to Cairns electrons,^{16,18,32} and its building blocks yield

$$\begin{aligned} A(\phi, M) &= \frac{1}{\tau} [1 + 3\beta - (1 + 3\beta - 3\beta\tau\phi + \beta\tau^2\phi^2) \exp(\tau\phi)] \\ &\quad + \frac{M^2}{\mu} \left(1 - \sqrt{1 - \frac{2\mu\phi}{M^2}} \right), \\ B(\phi, M) &= \frac{1}{\tau} [1 + 3\beta - (1 + 3\beta - 3\beta\tau\phi + \beta\tau^2\phi^2) \exp(\tau\phi)] \\ &\quad + \frac{1}{6} \left\{ 2 + 6M^2 - [(M+1)^2 - 2\phi]^{3/2} \right. \\ &\quad \left. + [(M-1)^2 - 2\phi]^{3/2} \right\}. \end{aligned} \quad (27)$$

From the modified Sagdeev pseudopotential elements (27), one finds that the acoustic speed satisfies

$$-(1+f)(1-\beta)\tau + \frac{1}{M_s^2 - 1} + \frac{\mu f}{M_s^2} = 0, \quad (28)$$

and hence, in a completely analogous way, for the fast ion-acoustic mode,

$$M_s^2 = \frac{1 + \mu f + (1+f)(1-\beta)\tau + \sqrt{\Delta}}{2(1+f)(1-\beta)\tau}. \quad (29)$$

The discriminant Δ in (29) can be written as

$$\Delta = [1 - \mu f + (1+f)(1-\beta)\tau]^2 + 4\mu f. \quad (30)$$

From here on, the discussion leading to (19)–(25) can be followed, except (21), with the inclusion of the additional nonthermality parameter β .

Since $\beta = 0$ reduces the Cairns to the Boltzmann case, we have chosen $\beta = 0.5$ for the illustrations in Fig. 7, as this is near the maximum permitted and will best show the differences between the Cairns and the Boltzmann cases. So, in Fig. 7(a), the curves for $\tau_{\min}$, $\tau_{\max A}$, and $\tau_{\max B}$ have been plotted as functions of μ , at $\beta = 0.5$, with color coding as in Fig. 3. To enable one to evaluate the effects of nonthermal electrons on the stopband existence domains, the equivalent curves for Boltzmann electrons are also shown in light gray. It is seen that at equal μ , the curves for τ increase with β .

An example of the limiting curves in $\{f, M\}$ space for the choice $\mu = 2$, $\tau = 0.4$, and $\beta = 0.5$ is shown in Fig. 7(b). The corresponding Sagdeev pseudopotentials for $f = 5$ are shown in Fig. 7(c). The limiting Mach numbers are $M = 3.237$, 3.397, and 3.487. The color coding is the same as in Figs. 2 and 5.

It is clear that, for appropriate μ and τ , even strongly nonthermal Cairns distributions support stopbands and that the existence range in τ increases with μ , much as for the isothermal case.

IV. KAPPA SUPERHERMAL ELECTRONS

The second model for nonthermal electrons is the kappa distribution,^{20–22} which after integration over the phase space distribution yields the electron density as

$$n_e = (1+f) \left(1 - \frac{\tau\varphi}{\kappa - \frac{3}{2}} \right)^{\frac{1}{2}-\kappa}. \quad (31)$$

Here, the Boltzmann distribution is obtained for $\kappa \rightarrow \infty$; hence, decreasing κ represents increasing deviation from the isothermal, and $\kappa > \frac{3}{2}$ is needed to be able to define energy and temperature, when integrating the phase space distribution over all microscopic velocities. The restriction $\kappa > \frac{3}{2}$ is clearly visible in the integrated charge density (31).

The changes in the Sagdeev pseudopotential building blocks, due to the kappa electrons,^{32–34} are shown as follows:

$$\begin{aligned} A(\varphi, M) &= \frac{1}{\tau} \left[1 - \left(1 - \frac{\tau\varphi}{\kappa - \frac{3}{2}} \right)^{\frac{1}{2}-\kappa} \right] + \frac{M^2}{\mu} \left(1 - \sqrt{1 - \frac{2\mu\varphi}{M^2}} \right), \\ B(\varphi, M) &= \frac{1}{\tau} \left[1 - \left(1 - \frac{\tau\varphi}{\kappa - \frac{3}{2}} \right)^{\frac{1}{2}-\kappa} \right] + \frac{1}{6} \left\{ 2 + 6M^2 \right. \\ &\quad \left. - [(M+1)^2 - 2\varphi]^{3/2} + [(M-1)^2 - 2\varphi]^{3/2} \right\}. \end{aligned} \quad (32)$$

FIG. 7. Graphs corresponding to Figs. 3, 2, and 5, respectively, for the case of Cairns nonthermal electrons with $\beta = 0.5$. In (b), $\mu = 2$ and $\tau = 0.4$, and in (c), $f = 5$. The Mach numbers for the limiting curves of the Sagdeev pseudopotentials in (c) are $M = 3.237$, 3.397, and 3.487.

From (32), one obtains for the acoustic speed that

$$-\frac{(2\kappa-1)(1+f)\tau}{2\kappa-3} + \frac{1}{M_s^2-1} + \frac{\mu f}{M_s^2} = 0, \quad (33)$$

leading to

$$M_s^2 = \frac{(2\kappa-3)(1+\mu f) + (2\kappa-1)(1+f)\tau + \sqrt{\Delta}}{2(2\kappa-1)(1+f)\tau}, \quad (34)$$

where the discriminant Δ can be written as

$$\Delta = [(2\kappa-3)(1-\mu f) + (2\kappa-1)(1+f)\tau]^2 + 4(2\kappa-3)^2\mu f. \quad (35)$$

From here, the discussion leading to (19) and (25) can be followed, with the addition of the extra compositional parameter κ . The higher the value of κ , the closer the distribution approaches the Boltzmann profile in the limit $\kappa \rightarrow \infty$; thus, $\kappa = 3$ characterizes strong deviations from the Boltzmann distribution.

Following the same pattern as for the Cairns electron case in Sec. III, we show in Fig. 8 the graphs corresponding to Figs. 3, 2, and 5, respectively, for the case of kappa superthermal electrons with $\kappa = 3$. In (b), $\mu = 3$ and $\tau = 0.5$, and in (c), $f = 2.2$. The Mach numbers for the limiting curves of the Sagdeev pseudopotentials in (c) are $M = 2.062$, 2.331, and 2.512, and the curve coding is the same as before.

Comparing the results with the included (light gray) Boltzmann curves in Fig. 8(a), it is immediately clear that the kappa electrons cause an opposite trend, contrary to what the Cairns electrons yield. Now, higher μ corresponds to rather strong decreases in acceptable τ , as compared to the Boltzmann case.

V. TSALLIS NONEXTENSIVE ELECTRONS

A third model is the so-called Tsallis nonextensive distribution,^{23–27} which is a more recent addition to the nonlinear wave literature and not yet as widely used as the others we have dealt with in the preceding sections (Secs. II–IV). After integration over the phase space distribution, this Tsallis distribution yields the electron density as^{27,32}

$$n_e = (1+f)[1 + (q-1)\tau\varphi]^{\frac{q+1}{2(q-1)}}. \quad (36)$$

There is a new parameter q characterizing this distribution, and one can prove that the Boltzmann distribution is obtained for $q = 1$. To have finite density and energy when the Tsallis phase space distribution is integrated over all microscopic velocities, one needs to keep

FIG. 8. Graphs corresponding to Figs. 3, 2, and 5, respectively, for the case of kappa superthermal electrons with $\kappa = 3$. In (b), $\mu = 3$ and $\tau = 0.5$, and in (c), $f = 2.2$. The Mach numbers for the limiting curves of the Sagdeev pseudopotentials in (c) are $M = 2.062, 2.331$, and 2.512 .

to the range $1/3 < q \leq 1$.^{26,27} This restriction is not obvious when inspecting expressions like (36) and is often overlooked in soliton theory applications where the erroneous limit $-1 < q \leq 1$ is used when computing the number or charge densities, disregarding the fact that there is a temperature used in the normalization, which cannot be defined for the range $-1 < q \leq 1/3$.²⁷ For $q > 1$, the microscopic velocities are restricted to a finite interval, which prevents this range of the Tsallis model from adequately describing the high-energy wings that differentiate superthermal distributions from the Boltzmann profiles.

The Sagdeev pseudopotential building blocks, modified for Tsallis electrons,³² are now

$$\begin{aligned} A(\varphi, M) &= \frac{2}{\tau(3q-1)} \left\{ 1 - [1 + (q-1)\tau\varphi]^{\frac{3q-1}{2(q-1)}} \right\} \\ &\quad + \frac{M^2}{\mu} \left(1 - \sqrt{1 - \frac{2\mu\varphi}{M^2}} \right), \\ B(\varphi, M) &= \frac{2}{\tau(3q-1)} \left\{ 1 - [1 + (q-1)\tau\varphi]^{\frac{3q-1}{2(q-1)}} \right\} \\ &\quad + \frac{1}{6} \left\{ 2 + 6M^2 - [(M+1)^2 - 2\varphi]^{3/2} \right. \\ &\quad \left. + [(M-1)^2 - 2\varphi]^{3/2} \right\}. \end{aligned} \quad (37)$$

From the modified Sagdeev pseudopotential, one obtains from the defining equation for the acoustic speed,

$$-\frac{1}{2}(1+f)(q+1)\tau + \frac{1}{M_s^2 - 1} + \frac{\mu f}{M_s^2} = 0, \quad (38)$$

that

$$M_s^2 = \frac{2(1+f\mu) + (1+f)(q+1)\tau + \sqrt{\Delta}}{2(1+f)(q+1)\tau}, \quad (39)$$

with a modified discriminant

$$\Delta = [2(1 - \mu f) + (1+f)(q+1)\tau]^2 + 16\mu f. \quad (40)$$

The discussion leading to (19) and (25) can be followed, with the addition of the extra compositional parameter q . When $q \rightarrow 1$ for $1/3 < q \leq 1$, the nonextensive distribution approaches the Boltzmann profile; thus, $q = 0.4$ characterizes strong deviations from the Boltzmann distribution.

Examples of the curves corresponding to Figs. 3, 2, and 5, respectively, are shown in Fig. 9 for the case of Tsallis nonextensive electrons with $q = 0.4$. In (b) we have typical values like $\mu = 2$ and $\tau = 0.35$, and in (c) we consider $f = 2$. The Mach numbers for the limiting curves of the Sagdeev pseudopotentials in (c) are $M = 3.216, 3.393$, and 3.492 . It is seen that the Tsallis curves of τ_{maxB} lie below the corresponding Boltzmann curve, whereas those of τ_{min} and τ_{maxA} are above the Boltzmann curves.

Another point of note is that when Figs. 7(a) and 9(a) are compared, the differences are not very large. In one way, this is not immediately expected because the Cairns model is an *ad hoc* artifact constructed to explain certain polarity changes in satellite observations of nonlinear ion-acoustic waves in the upper ionosphere.¹⁶ On the other hand, the Tsallis model is based on thermodynamical and statistical arguments²³ so that it might have given results rather different from the Cairns model. Also, a correspondence or equivalence is sometimes claimed between the kappa and Tsallis descriptions.^{25,35} In that case, the trends between Figs. 8(a) and 9(a), with both τ_{min} and τ_{maxB} varying from the common Boltzmann limits in opposite ways, are rather surprising.

VI. COMPARISON OF ELECTRON DESCRIPTIONS

Here, in Table I, we have gathered the τ limits for some of the cases we discussed or illustrated above. For the nonthermal distributions, key results are given for both the cases of very strong deviations from the isothermal case illustrated in Secs. III–V and some of intermediate nonthermality. We shall consider these data in conjunction with our earlier figures.

In passing, we note that for the Boltzmann electron distribution, the values of τ_{min} and τ_{maxB} agree with the τ limits found earlier by Nsengiyumva *et al.*⁸

For each of the nonthermal distributions, at a fixed value of β , κ , or q , we see that all 3 limiting τ values increase monotonically with μ , as is the case for the Boltzmann distribution. Thus, for each electron distribution considered, the range of τ that supports stopbands shifts to higher values of τ for increasing μ .

Since the Cairns, kappa, and Tsallis distributions include the Boltzmann distribution for characteristic parameters $\beta = 0$, $\kappa \rightarrow \infty$, and $q = 1$, respectively, we can order the discussion by groups.

Reading the Boltzmann and Cairns cases together, we see that, additionally, an increase in nonthermality (β) leads to an increase in each of the limiting τ values. However, we recall that stopbands will be found in the range of τ values given by $\tau_{maxB} - \tau_{min}$. Closer

FIG. 9. Graphs corresponding to Figs. 3, 2, and 5, respectively, for the case of Tsallis nonextensive electrons with $q = 0.4$. In (b), $\mu = 2$ and $\tau = 0.35$, and in (c), $f = 2$. The Mach numbers for the limiting curves of the Sagdeev pseudopotentials in (c) are $M = 3.216, 3.393$, and 3.492 .

examination now shows that an increase in β from zero to 0.5 actually corresponds to a small decrease in the range of appropriate τ . For instance, for $\mu = 3$, the stopband range drops from 1.041 for $\beta = 0$, through 0.991 ($\beta = 0.25$) to 0.905 for $\beta = 0.5$.

Next, comparing the Boltzmann and Tsallis cases, we notice analogous trends and, at first sight, similar results to those for Cairns electrons. One noticeable difference is that while τ_{min} and τ_{maxA} increase with q , τ_{maxB} actually decreases for a higher deviation from isothermality. Interestingly, the stopband range of τ is very similar to that for

TABLE I. Typical μ and τ values for different electron models. Curves in $\{\tau, f\}$ space start at τ_{min} where $f_{co} = 0$ and then τ_{maxA} where $f_{ip} \rightarrow \infty$ and τ_{maxB} where $f_{co} \rightarrow \infty$.

Electrons	μ	τ_{min}	τ_{maxA}	τ_{maxB}
Boltzmann	2	0.137	0.371	0.431
	3	0.305	1.067	1.346
	4	0.460	1.880	2.512
Cairns, $\beta = 0.25$	2	0.165	0.399	0.438
	3	0.378	1.185	1.369
	4	0.578	2.134	2.555
Cairns, $\beta = 0.5$	2	0.200	0.417	0.443
	3	0.480	1.265	1.385
	4	0.753	2.314	2.584
Kappa, $\kappa = 10$	2	0.121	0.321	0.367
	3	0.270	0.929	1.148
	4	0.408	1.640	2.142
Kappa, $\kappa = 3$	2	0.078	0.195	0.215
	3	0.176	0.575	0.673
	4	0.268	1.029	1.255
Tsallis, $q = 0.7$	2	0.153	0.388	0.431
	3	0.346	1.142	1.348
	4	0.526	2.042	2.515
Tsallis, $q = 0.4$	2	0.174	0.394	0.416
	3	0.400	1.192	1.301
	4	0.614	2.171	2.429

Cairns electrons, with the range for $\mu = 3$ running from 1.041, through 1.002 (for $q = 0.7$) to 0.901 for $q = 0.4$.

However, comparing finally the Boltzmann and kappa distributions, one finds a very different picture. First, there is an opposite trend from the Cairns case in that an increase in superthermality is reflected in a decrease in all three τ limits at given μ . Furthermore, the τ range for stopbands is significantly narrower than for the other two nonthermal distributions. For instance, for $\mu = 3$, it changes from 1.041, through 0.878 ($\kappa = 10$) to 0.497 for $\kappa = 3$, less than half the τ range for Boltzmann electrons. Unfortunately, we have no ready explanation for this, other than to comment that there have been other cases where the κ distribution was found to yield soliton behavior that differs from, say, that of the Cairns distribution, for instance, Ref. 33.

Finally, we are now in a position to attempt to estimate the width in M of stopbands. In principle, they lie, for given $f \geq f_{co}$, between the lower and upper branches of the warm ion limit curves for ϕ_{lw} . Moreover, the width decreases as f increases beyond f_{co} . Looking at the maximum M , this is M_{co} , for f_{co} , and only depends on μ and decreases when μ increases. As we have delimited the range in μ as $1.5 \leq \mu \leq 4$, this yields $5.44 \geq M_{co} \geq 2.0$ for all models.

The minimum value of M for a stopband occurs where the corresponding f_{co} crosses the lower branch of the warm ion limit curve. The problem here is that this depends for Boltzmann electrons not only on μ but also on τ and, for the nonthermal extension, also on β , κ , or q . It has not been possible to consider all these cases in numerical detail, given the ranges in τ visualized in Figs. 3, 7(a), 8(a), and 9(a). Hence, the stopband range in M is at most of the order of 4, decreasing to 1 or even less when μ reaches 4.

VII. CONCLUSIONS

We have revisited in this paper the problem that under certain circumstances, a soliton range can end at a warm ion sonic point limit, and a new range opens at another point on the warm ion sonic point limit, in cases where for certain f , the warm ion limit curve is double valued in M . The second propagation range then ends when the cold ion infinite compression limit is encountered. Sandwiched between these two ranges, there is a band in M where no solitons can propagate. This property was first reported by Nsengiyumva *et al.*⁸ and designated a “stopband” in M and later studied further in a series of papers by Maharaj and Bharuthram.^{13–15}

Unfortunately, these studies were limited in many examples to a specific set of parameter values only, $\mu = 2$ and $\tau = 0.36$, or their immediate neighborhood, and efforts to give a rigorous, systematic discussion of the possible existence domains in terms of the

compositional parameters μ , τ , and f were lacking. Estimates of where the ends of these ranges lie were based on a mixture of physical and *ad hoc* assumptions. We, therefore, set up a coherent and logical framework that works for different electron distributions, both Boltzmann and nonthermal, and finds the existence ranges in terms of physical arguments and reasoned limits.

1. In the Boltzmann electron case, there are three parameters that define the plasma composition, μ , τ and f , in normalized variables. For the nonthermal electron distributions, additional dimensionless parameters are introduced to measure the deviation from the isothermal case. Because $\tau=0$ means that the warm ions are cold and $f=0$ that there are no cold ions, we have chosen to start our calculations from $\tau=0.1$ and $f=0.1$. However, the mathematical limits at zero can be computed and will help in delineating the existence ranges in τ .
2. The crossover between the warm and cold ion limit curves at M_{co} plays a key role in the forming of stopbands. It leads to $\mu > 1$ being one of the necessary conditions to find a stopband range.⁸ To both justify the neglect of hot electron inertia in the use of the Boltzmann and the common nonthermal distributions and to avoid Landau damping, one is led to the ordering $1 \ll M \ll \sqrt{T_e m_w / (T_w m_e)}$. This is obeyed for the normal plasma of, say, electrons, protons, and higher mass ions, provided that τ remains of order unity, leading ultimately to the inverse electron temperature range $0.1 \leq \tau < 3$. As μ too close to 1 leads to large M_{co} at very small τ , we start from $\mu = 1.5$, which yields $M_{co} = 5.44$. On the other hand, $\mu = 4$ gives $M_{co} = 2$ and endangers the window needed to avoid Landau damping so that the physically acceptable range in effective inverse cold ion mass is $1.5 \leq \mu \leq 4$. This kind of *a priori* analysis of the physical limits on the compositional parameters is, unfortunately, lacking in the earlier papers.
3. The analysis indicates two special points to determine in a logical way the existence domains in compositional parameter space, namely, (f_{co}, M_{co}) and (f_{tp}, M_{tp}) . First, the crossover between warm and cold ion limits occurs at $\phi_{\ell w} = \phi_{\ell c}$. As $S(\phi_{\ell w}, M_{co}) = 0$ is linear in f , the analytical determination of f_{co} is straightforward, as a function of μ and τ . The mathematical limits $f_{co} = 0$ and $f_{co} \rightarrow \infty$ can be evaluated numerically, even though they are physically not relevant, but lead to two limit values τ_{min} and τ_{maxB} , at given μ . This much was suggested in the original paper⁸ but only evaluated numerically for $\mu = 2$.
4. The other special point occurs when the double-valued warm ion curve has a vertical tangent, leading to M_{tp} as an implicit function of μ and τ , to be determined numerically. Then, $S(\phi_{\ell w}, M_{tp}) = 0$ yields f_{tp} in terms of the same two parameters. This essential part of the reasoning eluded the authors of the earlier papers.^{8,13–15} The obvious ordering $f_{co} < f_{tp}$ gives for $f_{tp} \rightarrow \infty$ another maximum, τ_{maxA} , where f_{co} is still finite. In the range $\tau_{maxA} < \tau < \tau_{maxB}$, the warm ion limit curve is split into two branches,¹⁵ an upper one connected to the point (f_{co}, M_{co}) and a lower one, starting from low f . Here, all values $f \geq f_{co}$ will yield stopbands, but, of course, f too large will not be physically relevant.
5. The above reasoning is applicable to all the hot electron distributions considered, viz., the isothermal Boltzmann case as in the original paper⁸ and the nonthermal Cairns and superthermal

kappa distributions discussed in more recent contributions.^{13–15}

We have revisited these in our paper, in the light of our strictly systematic approach, and added a more recent addition to the plasma soliton literature, the nonextensive Tsallis distributions. To illustrate the non-Boltzmann cases, the values of the additional parameters, β , κ , and q , respectively, have been chosen to yield large deviations from the Boltzmann distributions, as clearly seen in the figures.

6. Rather than being (mostly) restricted to the “standard” values of $\mu = 2$ and $\tau = 0.36$ or their neighborhood, we have constructed existence diagrams showing the curves for τ_{min} , τ_{maxA} , and τ_{maxB} as functions of μ . These are illustrated in Figs. 3 and 7(a), 8(a) and 9(a), for some typical nonthermal parameters. It is worth stressing that all values of μ and τ between the curves τ_{min} and τ_{maxB} will yield Sagdeev pseudopotentials having a stopband range, and this has been checked and shown in some explicit graphs, in parts (b) and (c) of the figures quoted. It is also clear that for all four electron models, the τ limit curves increase with μ . Hence, for instance, searching for stopbands by trial and error at fixed τ for different values of μ can lead to erroneous deductions.
7. Comparing the stopband existence ranges for the nonthermal distributions with those of the Boltzmann distribution, one finds that for both the Cairns and Tsallis distributions, there is an increase in τ_{min} and τ_{maxA} for increased nonthermality, through increased β or decreased q , respectively, whereas the values for τ_{maxB} show only a small increase. In contrast, the kappa distribution shows a marked decrease, almost halved, compared to the Boltzmann values.
8. Some of these trends are surprising because the apparent similarity hides the fact that the Cairns distribution is an artifact, advanced to explain some of the polarity changes in ionospheric wave observations, whereas the Tsallis distribution has been grounded in thermodynamics and statistics. Moreover, there have been attempts to link the kappa and Tsallis distributions,^{25,35} which renders the resolutely opposite behavior in τ rather enigmatic and unexpected.

However, the broad conclusion at this stage is that the overall picture of the stopband phenomenon is robust and relies most on the behavior or influence of the cold and warm ion properties and on the electron temperature, with the actual form of the electron distribution playing only a quantitative role.

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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