Adaptive Fregean Set Theory

Diderik Batens*

Centre for Logic and Philosophy of Science
Universiteit Gent, Belgium

Diderik.Batens@ugent.be

Abstract

This paper defines provably non-trivial theories that characterize Frege's notion of a set, taking into account that the notion is inconsistent. By choosing an adaptive underlying logic, consistent sets behave classically notwithstanding the presence of inconsistent sets. Some of the theories have a full-blown presumably consistent set theory T as a subtheory, provided T is indeed consistent. An unexpected feature is the presence of classical negation within the language.

Keywords: Fregean set theories, adaptive logics, content guidance, paraconsistency.

1 Aim Of This Paper

This paper concerns non-trivial theories that characterize Frege's notion of a set. There are several such theories and they have remarkable properties.

While the first proposed paraconsistent set theory [11] proceeded in the wake of Quine's **NF**, a lot of attention was paid to paraconsistent versions of Frege's set theory [10, 17, 27]. The topic is fascinating for several reasons. First, Frege's notion of a set is generally considered natural, as is witnessed by the many mathematics curricula that offer a course on naive set theory. Next, it is important that we try to understand the way in which mathematicians moved from Frege's inconsistent set theory to the presumably consistent set theories that succeeded it.

The third reason requires a few more words. While many believe that the restrictions imposed by the **ZF**-axioms are sufficient to avoid the antinomies, doubts were raised, from the outset, on the issue whether those restrictions are necessary. Meanwhile paraconsistent logics came into being and gained recognition. As a result, a more general question may be raised: Is it possible

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to devise an inconsistent but non-trivial set theory that is based on Frege's notion but nevertheless serves the purposes a decent set theory should serve?

Some specifications are required. First, I do not aim at a set theory that Frege would have liked; as I understand Frege, he would have considered an inconsistent set theory unpalatable. I am after a set theory “based on Frege's notion”, viz. with ‘the same’ axioms as Frege's. There is a complication because classical logic **CL** is unavoidably replaced by a paraconsistent logic and because Frege's axioms, even some instances of the abstraction axiom schema Abs by themselves, are **CL**-trivial. Here is how to get around that. Two strings $\sigma_1\sigma_2\dots\sigma_n$ and $\tau_1\tau_2\dots\tau_n$ are *symbol-wise equivalent in CL* iff (i) $\sigma_i = \tau_i$ if σ_i is not a logical symbol and (ii) σ_i and τ_i have the same meaning within **CL** if σ_i is a logical symbol.¹ *Desideratum 1*: every axiom of the paraconsistent set theory is symbol-wise equivalent to one of Frege's axioms and *vice versa*.

It seems desirable to accompany the formal *Desideratum 1* with some substantive demands. First, sets should be truly extensional. *Desideratum 2*: two sets are identical just in case they have the same members as well as the same non-members. Next, Abstraction should be taken serious. *Desideratum 3*: to every property corresponds a set such that (i) every object that has (does not have) the property is a member (non-member) of the set and (ii) no object is a member (non-member) of a set without a reason.

While inconsistencies are unavoidable, we want the paraconsistent set theory to be consistent where it can be consistent. We should tread carefully here. If it is unavoidable that one of three sets is inconsistent and no further information is available, then there is no *systematic formal reason* to select any of the three as consistent. This leads to *Desideratum 4*: if a set (or sentence) can be selected as consistent in view of a systematic formal reason, then the set (or sentence) is consistent.

To serve the purposes a decent set theory should serve, it seems to me that the inconsistent set theory should include a consistent set theory. In view of the generality of the Abs schema, that does not even sound difficult. The consistent theory may require a restricted membership relation—the universal set is not a member of any set in **ZF**. As I shall be unable to say much on the topic in this paper, I phrase a rather vague *Desideratum 5*: if there are consistent set theories, they are subtheories of the paraconsistent Fregean set theory.

The main aim of this paper is to devise set theories that fulfil the five *desiderata* as well as obvious *desiderata*, for example that I am after a non-trivial theory in the set theoretic language.

The aforementioned publications on paraconsistent set theory were all impressive at the time of their appearance. They are, however, not satisfactory with respect to the five *desiderata*. Set theories that have no detachable implication fail with respect to *desiderata 2, 3, 4 and 5*. Even $\emptyset \in \{\emptyset\}$ and $\{x \mid x \notin x\} \notin \{\emptyset\}$ are not provable if the theories meet *Desideratum 1*; actually nearly nothing is.² Set theories phrased in terms of a relevant implication turn identity into an intensional operator; all those known to me fail *Desideratum 2*.

¹The paraconsistent logic **CLuNs**, presented in Section 2, has, for example, implications that have different meanings within **CLuNs** but the same meaning within **CL**.

²More is provable if one defines $t_1 = t_2 =_{df} \forall x(x \in t_1 \Leftrightarrow x \in t_2)$, in which $A \Leftrightarrow B =_{df} (\neg A \vee B) \wedge (A \vee \neg B)$. Yet, the defined identity is non-transitive: in some models M there is a set t_1 such that $M \models \forall x(x \in t_1 \wedge x \notin t_1)$ as well as at least two consistent sets, t_2 and t_3 , that differ from each other. So $M \models \forall x x = t_1$ and $M \not\models t_2 = t_3$.

That $\{x \mid A(x)\}$ and $\{x \mid B(x)\}$ have the same members *as well as* the same non-members is insufficient for proving $\{x \mid A(x)\} = \{x \mid B(x)\}$; this requires a proof of $\forall x(A(x) \leftrightarrow B(x))$ for $\leftrightarrow$ the relevant equivalence of the set theory.³

Where R names the Russell set, $\{x \mid \neg x \in x\}$, the paraconsistent Fregean set theory we are after will unavoidably have $R \in R \wedge \neg R \in R$ as a theorem. So there are inconsistent sets, sets of which some members are also non-members, next to consistent sets; and the negation $\neg$ has to be paraconsistent. Yet, we want to interpret the inconsistent theory as consistently as possible. As $\neg R \in R$ and $R \in R \vee A$ hold for all A , to derive A from these formulas by Disjunctive Syllogism is simply an inferential mistake. Yet if t_1 is a consistent set, and even if t_2 is a non-member of t_1 and not also a member of t_1 , A is true whenever $\neg t_2 \in t_1$ and $t_2 \in t_1 \vee A$ are true. Capitalizing on consistent membership is the more justified as Frege (mistakenly) believed and (for good reasons) intended his theory to be consistent.

An inconsistency-adaptive approach seems appropriate. Details follow in Section 2, but here are some suggestions. Inconsistency-adaptive logics allow for inconsistencies and possibly for other abnormalities, but localize and minimize them. Parts of the theory that do not involve an abnormality behave as they would in terms of **CL**. Actually, the type of problem under consideration is the main intended application context of inconsistency-adaptive logics: to obtain a minimally inconsistent ‘interpretation’ of theories that were intended as consistent but turned out inconsistent. However, the original intention was to use the minimally inconsistent ‘interpretation’ as a starting point to devise a consistent theory. Only much later came the idea to devise theories the underlying logic of which is adaptive. Such theories are not semi-recursive sets of formulas, but are up to Π_1^1 -complex. In view of *Desideratum 4*, no simpler theory will do for theories that explicate the Fregean notion of a set.

Joint work with Peter Verdée led to a lecture at the 2006 Studia Logica Conference in Torun. Verdée continued to work on the topic. This resulted in several lectures and papers [24, 25] about some four Fregean set theories. For all straightforward choices of adaptive logics AL, Verdée found that the presence of Curry sets caused a major problem. The gain expected from the adaptive approach was not obtained. So the adaptive set-theories failed *Desideratum 4*. Verdée was able to devise adaptive set theories that, by present lights, fulfil *Desideratum 4*. Unfortunately, each of these fails *Desideratum 1*; they either add an axiom or impose a restriction on an axiom. Other proposals are ad hoc in requiring an adaptive logic that is not in standard format.

This first paper on the topic proceeds in terms of Abstraction rather than (full) Comprehension—see Section 4. The paper may be rewritten in terms of Comprehension, but I avoid connected difficulties for now.

One of the extremely impressive properties of the adaptive Fregean set theories AFS is that the *classical* negation is locally definable. This will obviously be explained, but what it comes to is that a classical negation may be introduced from the very beginning, thus impressively raising the expressive power of the AFS. For example, notwithstanding the presence of a non-triviality proof, the

³Zach Weber [26] studied so-called Doppelgänger, sets that have the same members without being identical. He shows, among other things, that the considered examples should not be identical because they differ with respect to non-membership. Yet, he does not even mention sets for which $\forall x(A(x) \leftrightarrow B(x))$ is unprovable even if $\{x \mid A(x)\}$ and $\{x \mid B(x)\}$ have the same members as well as the same non-members.

‘classical’ Russell set exists—a drastic difference with all paraconsistent set theories known to me. Other impressive features are saved for Section 9 because stating them here would require too lengthy an explanation.

In view of space limitations, the present paper does not contain all available results. It describes an approach that leads to adaptive Fregean set theories that fulfil the *desiderata*. The results show the approach sensible and promising. Needless to say, the results do not circumvent the limitative theorems or produce other fake miracles. That they offer perspectives that differ from the traditional ones is fascinating enough.

2 Logical and Linguistic Preliminaries

In this paper, the metalanguage will be classical and “false” will function as the classical negation of “true”. So nothing will be said to be both true and false, but some formulas will be said to be true together with their negation. I shall need set-theoretic statements in the metalanguage and these will be classical, say **ZFC**-ish. Confusion is excluded in that all metalinguistic statements will contain symbols that do not belong to the object language.

A metalinguistic expression of the form $A(\alpha)$ will denote a formula in which at most the variable α occurs free. Let $\exists A$, respectively $\forall A$, denote A preceded by an existential, respectively universal, quantifier over every variable free in A ; let QA denote A preceded by a quantifier over every variable free in A .

Let $\mathcal{L}_s$ be the standard predicative language schema with the following logical symbols $\{\neg, \wedge, \vee, \supset, \equiv, \forall, \exists, =\}$. The sets of sentential letters, individual constants, individual variables, and predicates of rank r will be named $\mathcal{S}$, $\mathcal{C}$, $\mathcal{V}$, and $\mathcal{P}^r$ respectively. Let $\mathcal{F}_s$ be the set of formulas of $\mathcal{L}_s$ and $\mathcal{W}_s$ be the set of closed formulas of $\mathcal{L}_s$.

The popular paraconsistent logic **CLuNs** is known under many names. I refer to a surveying study [8] for further references. The version of **CLuNs** that I shall need consists of full positive **CL** including Replacement of Identicals (RoI), $\forall x \forall y (x = y \supset (A(x) \supset A(y)))$, Excluded Middle, $A \vee \neg A$, plus the standard axioms to drive negation inwards: $\neg \neg A \equiv A$, $\neg(A \wedge B) \equiv \neg A \vee \neg B$, $\dots$, $\neg \exists x A \equiv \forall x \neg A$.

The handiest **CLuNs**-semantics for present purposes is the three-valued one, with values T (consistently true), I (inconsistent), and F (false). By $\mathcal{L}_\mathcal{O}$ I shall mean a pseudo-language schema that is exactly like the standard predicative language schema $\mathcal{L}_s$, except that the role of the countable set $\mathcal{C}$ of individual constants is played by $\mathcal{C} \cup \mathcal{O}$, with $\mathcal{O}$ a possibly uncountable set of pseudo-constants.⁴ Let $\mathcal{W}_\mathcal{O}$ be the set of closed pseudo-formulas of $\mathcal{L}_\mathcal{O}$. In the **CLuNs**-semantics, each model $M = \langle D, v \rangle$ is described in terms of an $\mathcal{L}_\mathcal{O}$ in which $\mathcal{C} \cup \mathcal{O}$ has at least the cardinality of D .⁵

Where $M = \langle D, v \rangle$ is a **CLuNs**-model, D is a set and v an assignment function defined as follows. (i) Sentential letters are assigned a truth value, $v: \mathcal{S} \rightarrow \{T, I, F\}$. (ii) Constants and pseudo-constants are mapped to the

⁴The standard view requires (languages and) language schemes to have countably many symbols. The prefix “pseudo-” indicates that countability may fail.

⁵Phrasing the semantics in terms of $\mathcal{L}_\mathcal{O}$ is just a neater way to phrase the semantics as if “the elements of the domain D were names for themselves” (which strictly speaking is nonsense). Doing as if one has a name for every element of D is fictional (for uncountable D), but allows for a simpler handling of the quantifiers than less fictional descriptions.

domain, $v: \mathcal{C} \cup \mathcal{O} \rightarrow \mathbf{D}$, in such a way that $\{v(\alpha) \mid \alpha \in \mathcal{C} \cup \mathcal{O}\} = \mathbf{D}$. (iii) Predicates of rank r are assigned a triple of r -tuples of $\mathbf{D}$, $\langle \Sigma_1, \Sigma_2, \Sigma_3 \rangle$, such that $\Sigma_1 \cup \Sigma_2 \cup \Sigma_3 = \wp(\mathbf{D}^r)$ and $\Sigma_1 \cap \Sigma_2 = \Sigma_1 \cap \Sigma_3 = \Sigma_2 \cap \Sigma_3 = \emptyset$. To simplify the notation, consider v as composed in this case of the three functions v^T , v^I , and v^F , with $v^T(\pi^r) = \Sigma_1$, $v^I(\pi^r) = \Sigma_2$, and $v^F(\pi^r) = \Sigma_3$. Identity is handled as a binary predicate with the special restriction that $v^T(=) \cup v^I(=) = \{\langle o, o \rangle \mid o \in \mathbf{D}\}$.

The valuation determined by M , $v_M: \mathcal{W}_{\mathcal{O}} \rightarrow \{T, I, F\}$, handles atomic formulas as follows:⁶

$$\begin{aligned} \text{where } A \in \mathcal{S}, v_M(A) &= v(A) \\ v_M(\pi^r \alpha_1 \dots \alpha_r) &= T \text{ iff } \langle v(\alpha_1), \dots, v(\alpha_r) \rangle \in v^T(\pi^r) \\ v_M(\pi^r \alpha_1 \dots \alpha_r) &= I \text{ iff } \langle v(\alpha_1), \dots, v(\alpha_r) \rangle \in v^I(\pi^r) \\ v_M(\pi^r \alpha_1 \dots \alpha_r) &= F \text{ iff } \langle v(\alpha_1), \dots, v(\alpha_r) \rangle \in v^F(\pi^r) \end{aligned}$$

The meaning of three connectives is defined by the matrices

	$\neg$		$\supset$	T	I	F		$\wedge$	T	I	F
T	F	T	T	T	I	F	T	T	T	I	F
I	I	I	T	T	I	F	I	I	I	I	F
F	T	F	T	T	T	T	F	F	F	F	F

and the value of universally quantified formulas is determined by

$$\begin{aligned} v_M(\forall \alpha A(\alpha)) &= T \text{ iff } v_M(A(\beta)) = T \text{ for all } \beta \in \mathcal{C} \cup \mathcal{O} \\ v_M(\forall \alpha A(\alpha)) &= F \text{ iff } v_M(A(\beta)) = F \text{ for at least one } \beta \in \mathcal{C} \cup \mathcal{O} \\ v_M(\forall \alpha A(\alpha)) &= I \text{ otherwise.}^7 \end{aligned}$$

The remaining logical symbols are defined explicitly:

$$\begin{aligned} A \vee B &=_{df} \neg(\neg A \wedge \neg B) \\ A \equiv B &=_{df} (A \supset B) \wedge (B \supset A) \\ \exists \alpha A(\alpha) &=_{df} \neg \forall \alpha \neg A(\alpha) \end{aligned}$$

$M \models A$ (M verifies A) iff $v_M(A) \in \{T, I\}$. M is a *model* of Γ iff $M \models A$ for all $A \in \Gamma$. $\Gamma \models A$ (A is a *semantic consequence* of Γ) iff all models of Γ verify A . $\Gamma \vDash A$ (A is *valid*) iff all models verify A .⁸ Where $T = \langle \Gamma, \mathbf{L} \rangle$ is a theory, with Γ a set of non-logical axioms and $\mathbf{L}$ a logic, $\vdash_{\mathbf{T}} A$ abbreviates $\Gamma \vdash_{\mathbf{L}} A$.

The following statements give a feel of detachment in **CLuNs**:

$$\begin{aligned} A \supset B, A &\vdash_{\mathbf{CLuNs}} B \\ A \supset B, \neg B &\not\vdash_{\mathbf{CLuNs}} \neg A \quad \text{but} \quad A \supset B, \neg B \vdash_{\mathbf{CLuNs}} \neg A \vee (B \wedge \neg B) \\ A \vee B, \neg B &\not\vdash_{\mathbf{CLuNs}} A \quad \text{but} \quad A \vee B, \neg B \vdash_{\mathbf{CLuNs}} A \vee (B \wedge \neg B) \end{aligned}$$

Many further logical symbols are definable within **CLuNs**. Thus $A \sqsupset B =_{df} \neg A \vee B$ is contraposable but not detachable, $A \rightarrow B =_{df} (A \supset B) \wedge (\neg B \supset \neg A)$ is contraposable and detachable. Apart from the earlier introduced $\Leftrightarrow$, two definable equivalences will play a role in the sequel:

$$\begin{aligned} A \leftrightarrow B &=_{df} (A \equiv B) \wedge (\neg A \equiv \neg B) \\ A \Leftrightarrow B &=_{df} (B \supset A) \wedge (\neg B \supset \neg A) \end{aligned}$$

⁶The aim is a semantics for $\mathcal{L}_s$ and $\mathcal{W}_s \subseteq \mathcal{W}_{\mathcal{O}}$.

⁷Viz. iff $v_M(A(\beta)) \in \{T, I\}$ for all $\beta \in \mathcal{C} \cup \mathcal{O}$ and $v_M(A(\beta)) = I$ for at least one $\beta \in \mathcal{C} \cup \mathcal{O}$.

⁸Propositional **CLuNs** is decidable. Drive negations inward and next proceed as for propositional **CL**, except that both A and $\neg A$ may be true. In view of this, I shall leave it to the reader to spell out propositional steps (as well as obvious steps from positive logic).

Both $(A \leftrightarrow B) \leftrightarrow (B \leftrightarrow A)$ and $(A \leftrightarrow B) \leftrightarrow (\neg B \leftrightarrow \neg A)$ are **CLuNs-theorems**, $\leftrightarrow$ is detachable and Replacement of Equivalents holds for it. The asymmetric equivalence $\Rightarrow$ has a strong right-left direction and a weak left-right direction: $A \Rightarrow B$ **CLuNs**-entails $B \supset A$ and $\neg B \supset \neg A$, but only $A \supset B$ and $\neg A \supset \neg B$. While $(A \Rightarrow B) \leftrightarrow (\neg A \Rightarrow \neg B)$ is a **CLuNs**-theorem, neither $(A \Rightarrow B) \leftrightarrow (B \Rightarrow A)$ nor $(A \Rightarrow B) \leftrightarrow (\neg B \Rightarrow \neg A)$ is.

Frege's axioms Ext and Abs will be introduced in Section 4. The theory defined from these axioms by **CLuNs** will be called **PFS** (Paraconsistent Fregean Set theory). **PFS** is not the theory I am after but only a preparatory step towards Adaptive Fregean Set theories AFS. As there is a manifold of them and many are interesting, AFS does not name a system but the family of systems. All members of the family are defined from the two **PFS**-axioms by different adaptive logics 'based on' **CLuNs**.

Several introductions to (inconsistency-)adaptive logics are available, including recent ones [3, 7, 6, 20]. So I shall stick to the bare essentials.

An adaptive logic is a triple composed of (i) a lower limit logic, which is a logic that has static proofs [4]—roughly a Tarski logic—(ii) a *set of abnormalities* Ω , which is a subset of $\mathcal{W}_s$ characterized by a possibly restricted logical form, and (iii) an adaptive strategy. In the present paper, we shall only consider adaptive logics that have **CLuNs** as their lower limit and Minimal Abnormality as their strategy. By way of example, let $\Omega = \{Q(A \wedge \neg A) \mid A \in \mathcal{F}^a\}$, in which $\mathcal{F}^a$ is the set of atomic formulas (those that contain no logical symbols other than identity).⁹ The logic determined by these choices, will be called **CLuNs^m**.¹⁰ The underlying logic of the AFS that will be introduced later will be different from **CLuNs^m**. Yet, the latter is a good starting point to explain the idea.

Like all adaptive logics that have Minimal Abnormality as their strategy, **CLuNs^m** is most easily understood in semantic terms. Where M is a **CLuNs**-model, $Ab(M) = \{A \in \Omega \mid M \Vdash A\}$. M is a *minimally abnormal CLuNs-model* of Γ iff M is a **CLuNs**-model of Γ and no **CLuNs**-model M' of Γ is such that $Ab(M') \subset Ab(M)$. $\Gamma \models_{\mathbf{CLuNs}^m} A$ iff every minimally abnormal **CLuNs**-model of Γ verifies A .

The most fascinating aspect of adaptive logics are the dynamic proofs. These offer a syntactic definition of **CLuNs^m**-consequence sets and at the same time explicate the way in which we estimate membership of such a consequence set by finitary means.

Lines of annotated dynamic proofs consist of four elements: a number, a formula, a justification, and a *condition*, which is a finite subset of Ω . Where (i) Γ is the premise set, (ii) $Dab(\Delta)$ denotes a disjunction of the members of a finite $\Delta \subseteq \Omega$, and (iii)

$$A \quad \Delta$$

abbreviates a line at which A occurs on the condition Δ , the rules of inference can be phrased as three generic rules:

⁹Extending the lower limit logic **CLuNs** with an axiom that abnormalities entail triviality— $(A \wedge \neg A) \supset B$ will do—delivers the *upper limit logic*, in the present case **CL**.

¹⁰In many papers, **CLuNs^m** names a variant with $\Omega = \{\exists(A \wedge \neg A) \mid A \in \mathcal{F}^a\}$.

Prem	If $A \in \Gamma$:	$\frac{\dots \quad \dots}{A \quad \emptyset}$
RU	If $A_1, \dots, A_n \vdash_{\mathbf{CLuNs}} B$:	$\frac{\begin{array}{c} A_1 \quad \Delta_1 \\ \dots \quad \dots \\ A_n \quad \Delta_n \end{array}}{B \quad \Delta_1 \cup \dots \cup \Delta_n}$
RC	If $A_1, \dots, A_n \vdash_{\mathbf{CLuNs}} B \vee Dab(\Theta)$:	$\frac{\begin{array}{c} A_1 \quad \Delta_1 \\ \dots \quad \dots \\ A_n \quad \Delta_n \end{array}}{B \quad \Delta_1 \cup \dots \cup \Delta_n \cup \Theta}$

Only RC *introduces* new non-empty conditions. Prem introduces empty conditions and RU merely adds up conditions into a union.¹¹ Note that **CLuNs** is a compact logic and that the above generic rules are finitary, viz. have a finite number of local premises. A proof stage is a sequence of lines. A proof is a chain of stages. An application of a rule brings the proof to the next stage.

At each stage s , the marking definition determines for every line i whether i is unmarked or marked—this means that it is respectively IN or OUT—in view of (i) the condition of i and (ii) the minimal *Dab*-formulas of stage s . I shall only consider the marking definition for Minimal Abnormality. A *choice set* of $\Sigma = \{\Delta_1, \Delta_2, \dots\}$ is a set that contains an element out of each member of Σ . A *minimal choice set* of Σ is a choice set of Σ of which no proper subset is a choice set of Σ . Where $Dab(\Delta_1), \dots, Dab(\Delta_n)$ are the *Dab*-formulas that occur on the empty condition in stage s , $\Phi_s(\Gamma)$ is the set of the minimal choice sets of $\{\Delta_1, \dots, \Delta_n\}$. Let $Dab(\Delta_1), Dab(\Delta_2), \dots$ be the *Dab*-formulas that are **CLuNs**-derivable from Γ and let $\Phi(\Gamma)$ be the set of minimal choice sets of $\Sigma = \{\Delta_1, \Delta_2, \dots\}$. It was proved [3, Lemma 4] that $\varphi \in \Phi(\Gamma)$ iff there is a Minimal Abnormal **CLuNs**-model M of Γ such that $\varphi = Ab(M)$. $\Phi_s(\Gamma)$ may be seen as the best estimate, at stage s , of $\Phi(\Gamma)$.

Definition 1 *Marking for Minimal Abnormality: where A is the formula and Δ is the condition of line i , line i is marked at stage s iff (i) there is no $\varphi \in \Phi_s(\Gamma)$ such that $\varphi \cap \Delta = \emptyset$, or (ii) for some $\varphi \in \Phi_s(\Gamma)$, there is no line at which A is derived on a condition Θ for which $\varphi \cap \Theta = \emptyset$.*

Definition 2 *A is finally derived from Γ at line i of a finite **CLuNs**^m-proof stage s iff (i) A is the second element of line i , (ii) line i is not marked at stage s , and (iii) every extension of the proof in which line i is marked may be further extended in such a way that line i is unmarked.*

Definition 3 $\Gamma \vdash_{\mathbf{CLuNs}^m} A$ (A is finally **CLuNs**^m-derivable from Γ) iff A is finally derived at a line of a **CLuNs**^m-proof stage from Γ .

Marks at stage s may be seen as the estimate of final derivability in view of insights available at s . Establishing final **CLuNs**^m-derivability requires (i) a finite proof stage and (ii) a metatheoretic reasoning about extensions of the stage

¹¹Provably [3, Lemma 1] a line at which A occurs on the condition Δ can be obtained in a **CLuNs**^m-proof from Γ iff $\Gamma \vdash_{\mathbf{CLuNs}} A \vee Dab(\Delta)$; the latter obtains iff, in a **CLuNs**-proof from Γ , a line can be obtained at which $A \vee Dab(\Delta)$ is derived on the condition $\emptyset$.

and extensions of these. I shall consider other adaptive logics, all of which will have **CLuNs** as their lower limit logic. Semantic consequence and derivability for those logics are as for **CLuNs**^m—only Ω and the name of the adaptive logic need to be replaced. For all of them holds $\Gamma \vdash A$ iff $\Gamma \models A$ [3, Theorem 9].

In **AFS**, abnormalities are allowed but minimized. In semantic terms, the models of an **AFS** are a selection of the **PFS**-models. So **AFS** have all **PFS**-theorems as theorems. Interesting **AFS** have further theorems; these form the *adaptive gain*.

The reason to choose **CLuNs** rather than another paraconsistent logic as underlying **PFS** is mainly a matter of experience. In comparison to empirical theories, mathematical theories are conceptually simple, they are not blurred by empirical data, and their conceptual difficulties relate to properties of their most atomic concepts. **CLuNs** connects inconsistencies in that removing some of them comes to removing infinitely many of them at once.¹² In mathematical theories, disconnecting those inconsistencies by moving to a weaker logic does not open any interesting roads for removing involved inconsistencies.

The choice for **CLuNs**^m is provisional but, at this point, natural. We shall see that **PFS** requires some atomic contradictions to be true, for example $R \in R \wedge R \notin R$, in which R is the Russell set. The natural reaction is to allow for contradictions but to minimize them. Minimizing contradictions in **CLuNs** has to be done by minimizing *atomic* contradictions—otherwise a flip-flop results, as was shown a long time ago [2, p. 50].¹³

3 The Language of Inconsistent Set Theory

The language of set theory, $\mathcal{L}_S$, takes part of its signature from $\mathcal{L}_s$, viz. $\{\neg, \wedge, \vee, \supset, \equiv, \forall, \exists, =\} \cup \mathcal{V}$. Apart from these, there is the binary predicate $\in$ and there are *abstracting terms* $\{\alpha \mid A(\alpha)\}$, in which $A(\alpha)$ is called the *touchstone* of the abstracting term—I shall sometimes write “set” instead of “abstracting term”. We stipulate that α is bound in $\{\alpha \mid A(\alpha)\}$. The only terms are abstracting terms and members of $\mathcal{V}$; all individual constants will be abbreviations of abstracting terms. Let $\mathcal{T}$ be the set of *terms* and $\mathcal{T}^c$ the set of *constant terms*. Let $\mathcal{F}_S$ be the set of formulas of $\mathcal{L}_S$ and $\mathcal{W}_S$ be the set of sentences (closed formulas) of $\mathcal{L}_S$.

$$\begin{aligned} !(A) &=_{df} A \wedge \neg A \\ t_1 \neq t_2 &=_{df} \neg t_1 = t_2 \\ t_1 \notin t_2 &=_{df} \neg t_1 \in t_2 \\ \perp &=_{df} \forall x \forall y (x = y \wedge x \neq y \wedge x \in y \wedge x \notin y) \\ \top &=_{df} \perp \supset \perp \\ \sim A &=_{df} A \supset \perp \end{aligned}$$

¹²Where M is a **CLuNs**-model, (i) $M \models A \wedge \neg A$ iff $M \models (A \wedge \neg A) \wedge \neg (A \wedge \neg A)$, (ii) $M \models A \wedge \neg A$ iff $M \models (A \wedge B) \wedge \neg (A \wedge B)$ whenever $M \models B$, and so on.

¹³A flip-flop delivers the upper limit consequence set of normal premise sets, as do all adaptive logics, but delivers only the lower limit consequences of inconsistent premise sets. So there is no adaptive gain if the premise set requires some abnormalities to be true.

Clearly $\top$ is a **CLuNs**-theorem and in that sense a ‘top constant’; $\perp$ is a *local* ‘bottom constant’:¹⁴ it entails every sentence of $\mathcal{L}_S$, but not every sentence of some extensions of $\mathcal{L}_S$.¹⁵ Similarly, $\sim$ functions *locally* as classical negation. It is easily seen that, for every domain D , there is a trivial model $M = \langle D, v \rangle$, a model such that $v_M(A) = I$ for all $A \in \mathcal{W}_S$. The following tables obtain in non-trivial models—all $\{\neg, \sim\}$ -combinations are equivalent to one of these:

	$\neg$	$\sim$	$\sim\sim$	$\neg\sim$	$\sim\neg$	$\sim\sim\neg$		
T	F	F	T	T	T	F	$\sim\neg A$	A is true and consistent
I	I	F	T	T	F	T	$A \wedge \neg A$	A is inconsistent
F	T	T	F	F	F	T	$\sim A$	A is false

So M is inconsistent iff $v_M(\neg A) = v_M(A)$ for some $A \in \mathcal{W}_S$; M is trivial iff $v_M(\sim A) = v_M(A)$ for some $A \in \mathcal{W}_S$, viz. iff $v_M(A) = I$ for all $A \in \mathcal{W}_S$.

Both $A \vdash_{\mathbf{CLuNs}} \sim\neg A \vee (A \wedge \neg A)$ and $\neg A \vdash_{\mathbf{CLuNs}} \sim A \vee (A \wedge \neg A)$ hold; in words: if A is true, A is either consistently true or inconsistent; if $\neg A$ is true, A is either false or inconsistent. Also $A \supset \top$, $\sim\perp$, and $\sim\neg A \vee (A \wedge \neg A) \vee \sim A$ are **CLuNs**-theorems. However, $\perp \supset A$ and $\sim A \supset \neg A$ are only local **CLuNs**-theorems; they hold in $\mathcal{L}_S$ but not in all extensions of $\mathcal{L}_S$. The same holds for $(\sim\neg A \wedge (A \wedge \neg A)) \supset B$, $(\sim A \wedge (A \wedge \neg A)) \supset B$, $(\sim\neg A \wedge \sim A) \supset B$, $\sim(\sim\neg A \wedge (A \wedge \neg A))$, $\sim(\sim A \wedge (A \wedge \neg A))$ and $\sim(\sim\neg A \wedge \sim A)$. So the members of $\{\sim\neg A, A \wedge \neg A, \sim A, \sim A\}$ are jointly exhaustive and, locally, are mutually exclusive whenever $A \in \mathcal{W}_S$. As this paper concerns the AFS as set theories, local **CLuNs**-theorems will henceforth be called **CLuNs**-theorems.

Some sets will be given a special name.¹⁶

$$\begin{aligned}
\emptyset &=_{df} \{x \mid \perp\} \\
U &=_{df} \{x \mid \top\} \\
G &=_{df} \{x \mid \exists x \exists y (y \in x \wedge y \notin x)\} \\
R &=_{df} \{x \mid x \notin x\} \\
Z &=_{df} \{x \mid x \in x\} \\
Q_n &=_{df} \{x \mid \neg \exists y_1 \dots \exists y_n (x \in y_1 \wedge y_1 \in y_2 \wedge \dots \wedge y_n \in x)\} \quad (1 \leq n) \\
D &=_{df} \{x \mid x \neq x\} \\
\emptyset^\circ &=_{df} \{x \mid \forall y (x = y \wedge x \neq y \wedge x \in y \wedge x \notin y)\} \\
\emptyset^\Delta &=_{df} \{x \mid \forall y x \in y\} \\
U^\Delta &=_{df} \{x \mid \exists y x \in y\} \\
D^* &=_{df} \{x \mid \sim x = x\} \\
R^* &=_{df} \{x \mid \sim x \in x\} \\
Q_n^* &=_{df} \{x \mid \sim \exists y_1 \dots \exists y_n (x \in y_1 \wedge y_1 \in y_2 \wedge \dots \wedge y_n \in x)\} \quad (1 \leq n) \\
N_0 &=_{df} \{x \mid \sim \exists y y \in x\} \\
N_1 &=_{df} \{x \mid \exists y (y \in x \wedge \forall z (z \in x \supset z = y))\}
\end{aligned}$$

¹⁴It was shown [5, p. 25 *sub* (C)] that a local bottom constant is also definable within several interesting paraconsistent arithmetic theories.

¹⁵This refers to the logic. In the context of **PFS** and **AFS** the situation is more complicated and studied elsewhere.

¹⁶Sets only exist by virtue of the axiom schema Abs—see the next section; the definition sign merely indicates renaming or abbreviation.

$$\begin{aligned}
N_n &=_{df} \{x \mid \exists y_1 \dots \exists y_n (\sim y_1 = y_2 \wedge \dots \wedge \sim y_1 = y_n \wedge \dots \wedge \sim y_{n-1} = y_n \wedge \\
&\quad y_1 \in x \wedge \dots \wedge y_n \in x \wedge \\
&\quad \forall z (z \in x \supset (z = y_1 \vee \dots \vee z = y_n))\} \quad (n \geq 2) \\
C &=_{df} \{x \mid \forall y \sim(y \in x \wedge y \notin x)\}
\end{aligned}$$

Set G will turn out remarkable; D , $\emptyset^\circ$ and $\emptyset^\Delta$ are variants of the empty set $\emptyset$; U^Δ is a variant of the universal set U . The presence of $\sim$ warrants the existence of the variant D^* and of the ‘classical’ Russell set R^* . Remarkably, R^* does not cause triviality—see Theorem 6 in Section 5. As $\vdash_{\mathbf{CLuNs}} \forall x \forall y \neg(y \in x \wedge y \notin x)$, the touchstone of C , the set of consistent sets, requires classical negation.

If a **CLuNs**-theory in the language $\mathcal{L}_S$ is inconsistent, some of its theorems are disjunctions of formulas of the form $Q(t_1 = t_2 \wedge t_1 \neq t_2)$ or $Q(t_1 \in t_2 \wedge t_1 \notin t_2)$. So the theory requires the existence of sets that are different from themselves—because $\alpha = \beta, \alpha \neq \beta \vdash_{\mathbf{CLuNs}} \alpha \neq \alpha$ —or the existence of sets of which some entities are both members and non-members. The **CLuNs**-theory **PFS**, defined in the next Section, has no theorems of the form $Q \alpha \neq \beta$; see Theorem 4, but **PFS** has inconsistent sets— t is an *inconsistent set* iff $\exists x(x \in t \wedge x \notin t)$ holds.

Consider some definable operators, first some subset relations.

$$\begin{aligned}
t_1 \subseteq t_2 &=_{df} \forall z(z \in t_1 \rightarrow z \in t_2) \\
t_1 \subsetneq t_2 &=_{df} \forall z((z \in t_1 \rightarrow z \in t_2) \wedge t_1 \neq t_2) \\
t_1 \subset t_2 &=_{df} \forall z((z \in t_1 \rightarrow z \in t_2) \wedge \sim t_1 = t_2)
\end{aligned}$$

Replacing $\rightarrow$ by $\supset$, one obtains weaker relations which have rather uninteresting properties. If an **AFS** has no models that verify $\exists x x \neq x$, then it has $\forall(t_1 \subsetneq t_2 \leftrightarrow t_1 \subset t_2)$ as a theorem.

$$\begin{aligned}
\bar{t} &=_{df} \{x \mid x \notin t\} \\
\tilde{t} &=_{df} \{x \mid \sim x \in t\} \\
t_1 - t_2 &=_{df} \{x \mid x \in t_1 \wedge x \notin t_2\} \\
t_1 \setminus t_2 &=_{df} \{x \mid x \in t_1 \wedge \sim x \in t_2\} \\
\{t_1, \dots, t_n\} &=_{df} \{x \mid x = t_1 \vee \dots \vee x = t_n\} \quad (n \geq 1) \\
\bigcap(t) &=_{df} \{x \mid \forall y(y \in t \supset x \in y)\} \\
\bigcup(t) &=_{df} \{x \mid \exists y(y \in t \wedge x \in y)\}^{17} \\
\wp(t) &=_{df} \{x \mid x \subseteq t\} \\
i(t) &=_{df} \{x \mid x \in t\} \\
\oplus(t) &=_{df} \{x \mid \sim \sim x \in t\} \\
\ominus(t) &=_{df} \{x \mid \sim \sim x \notin t\} \\
\boxplus\{x \mid A(x)\} &=_{df} \{x \mid \sim \sim(x \in \{x \mid A(x)\} \wedge A(x))\} \\
\boxminus\{x \mid A(x)\} &=_{df} \{x \mid \sim \sim(x \notin \{x \mid A(x)\} \wedge \neg A(x))\}
\end{aligned}$$

These operations seem self-explanatory but their precise meaning obviously depends on the Abstraction axiom, by virtue of which sets exist.¹⁸

¹⁷Where $t = \{t_1, \dots, t_n\}$, I shall also write $t_1 \cap \dots \cap t_n$ and $t_1 \cup \dots \cup t_n$.

¹⁸Ordered pairs and sequences are defined as usual in terms of sets, but will not be given any special attention in this paper.

4 The Axioms and Elementary Considerations

Desideratum 1 forces us to make some choices. To the detachable and contraposable material implication of **CL** correspond three different implications in **CLuNs**, all introduced before: $A \sqsupset B$, $A \supset B$ and $A \rightarrow B$; their meanings as specified within **CLuNs** coincide in **CL**. Similarly, the material equivalence of **CL** corresponds to a manifold of different equivalences in **CLuNs**. A perspicuous view on the matter sees an equivalence as the conjunction of four implications: $A \rightsquigarrow_1 B$, $\neg A \rightsquigarrow_2 \neg B$, $B \rightsquigarrow_3 A$ and $\neg B \rightsquigarrow_4 \neg A$, each of the four squiggly arrows being either the non-detachable $\sqsupset$ or the detachable $\supset$. So Frege's axioms may provisionally be rendered as

$$\forall x \forall y (\forall z (z \in x \rightsquigarrow_1 z \in y) \rightsquigarrow_5 x = y) \quad (1)$$

$$\forall \beta (\beta \in \{\alpha \mid A(\alpha)\} \rightsquigarrow_2 A(\beta)) \quad (2)$$

with (2) the conjunction of (3)–(6):

$$\forall \beta (\beta \in \{\alpha \mid A(\alpha)\} \rightsquigarrow_1 A(\beta)) \quad (3)$$

$$\forall \beta (\beta \notin \{\alpha \mid A(\alpha)\} \rightsquigarrow_2 \neg A(\beta)) \quad (4)$$

$$\forall \beta (A(\beta) \rightsquigarrow_3 \beta \in \{\alpha \mid A(\alpha)\}) \quad (5)$$

$$\forall \beta (\neg A(\beta) \rightsquigarrow_4 \beta \notin \{\alpha \mid A(\alpha)\}) \quad (6)$$

The following heuristic journey leads to the choices.

(A) There are inconsistent sets in **PFS**. Consider the R -instance of (2) for R : $R \in R \rightsquigarrow_2 R \notin R$. For both choices of $\rightsquigarrow_1$ and $\rightsquigarrow_2$, the corresponding instance of (3) entails $R \notin R$ and the corresponding instance of (4) entails $R \in R$.

(B) The ‘collection’ of non-members of an inconsistent set is not a function of the ‘collection’ of members. So, in the presence of inconsistent sets, one needs both collections in order to characterize a set.

(C) By a reasoning as in (A), both $R^* \in R^*$ and $R^* \notin R^*$ are **PFS**-theorems. So $\rightsquigarrow_1$ cannot be $\supset$; if it were, the relevant instance of (3) would entail $\sim R^* \in R^*$, causing **PFS** to be trivial.

(D) Let X abbreviate $\{x \mid \sim \sim x \notin x\}$. By a reasoning as in (A), both $X \notin X$ and $X \in X$ are **PFS**-theorems. So $\rightsquigarrow_2$ cannot be $\supset$; if it were, the relevant instance of (4) would entail $\sim X \notin X$, causing **PFS** to be trivial.

(E) That $\rightsquigarrow_3$ and $\rightsquigarrow_4$ are detachable does not cause triviality—see Theorem 6. So they should be detachable. Indeed, it is Frege's basic idea that $A(t)$ is sufficient for $t \in \{x \mid A(x)\}$ and that $\neg A(t)$ is sufficient for $t \notin \{x \mid A(x)\}$ —but see Section 9 on $\text{Abs}^{\leftrightarrow}$.

(F) In view of *Desideratum 2*, $\rightsquigarrow_1$ should be the contraposable equivalence $\leftrightarrow$ rather than $\equiv$ ¹⁹ and $\rightsquigarrow_5$ should be detachable.

So we have settled for the following axiom and schema:

$$\text{Ext } \forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \supset x = y)$$

$$\text{Abs } \forall \beta (\beta \in \{\alpha \mid A(\alpha)\} \Leftrightarrow A(\beta))$$

¹⁹Any other choice is ridiculous; e.g., we want neither $\vdash_{\text{PFS}} G = U$ nor $\vdash_{\text{AFS}} \{x \mid A(x)\} = \{x \mid A(x) \wedge \exists y (y \in y \wedge y \notin y)\}$.

PFS-models $M = \langle D, v \rangle$ have the following properties. In view of Ext: if o_1 and o_2 are distinct elements of D , then there is an $o_3 \in D$ such that, for at least two $X \in \{T, I, F\}$, if $\langle o_3, o_1 \rangle \in v^X(\in)$ then $\langle o_3, o_2 \rangle \notin v^X(\in)$. In view of Abs: for all $\alpha \in \mathcal{O}$, (i) if $M \Vdash A(\alpha)$ then $\langle v(\alpha), v(\{\beta \mid A(\beta)\}) \rangle \in v^T(\in) \cup v^I(\in)$ and (ii) if $M \Vdash \neg A(\alpha)$ then $\langle v(\alpha), v(\{\beta \mid A(\beta)\}) \rangle \in v^I(\in) \cup v^F(\in)$.

Define $\{\alpha \mid A(\alpha)\}$ as *clean* in **PFS**-model M iff $M \Vdash \forall \beta (\beta \in \{\alpha \mid A(\alpha)\} \leftrightarrow A(\beta))$; $\{\alpha \mid A(\alpha)\}$ is *unclean with respect to membership* in M iff $M \Vdash \exists \beta (\beta \in \{\alpha \mid A(\alpha)\} \wedge \sim A(\beta))$; it is *unclean with respect to non-membership* in M iff $M \Vdash \exists \beta (\beta \notin \{\alpha \mid A(\alpha)\} \wedge \sim \neg A(\beta))$. The set $\{\alpha \mid A(\alpha)\}$ is *unclean with respect to membership* in set theory **S** iff $\vdash_{\mathbf{S}} \exists \beta (\beta \in \{\alpha \mid A(\alpha)\} \wedge \sim A(\beta))$; similarly for non-membership iff $\vdash_{\mathbf{S}} \exists \beta (\beta \notin \{\alpha \mid A(\alpha)\} \wedge \sim \neg A(\beta))$. In view of (C), R^* is unclean with respect to membership in **PFS**: $\vdash_{\mathbf{PFS}} R^* \in R^* \wedge \sim R^* \in R^*$.²⁰ In (D), $\{x \mid \sim \sim x \notin x\}$ is shown unclean with respect to non-membership in **PFS**. With the exception of abstracting terms that define sets identical to G , all abstracting terms are unclean in some non-trivial **PFS**-models. By going adaptive one may eliminate **PFS**-models in which terms are needlessly unclean, for example models verifying $\emptyset \in \emptyset$, or in which unclean terms are more unclean than required, for example models verifying $R \in R^*$. One may call t_1 a clean or unclean (non-)member of t_2 ; thus R is an unclean member of $\emptyset$ if $R \in \emptyset$ and is an unclean non-member of U if $R \notin U$.

Let us have a closer look at **PFS**, keeping in mind that the ultimate aim is an adequate **AFS**.

In view of RoI, replacing the implication in Ext by the standard equivalence $\equiv$ results in a **PFS**-theorem, but replacing it by the contraposable $\leftrightarrow$ does not. The latter replacement would cause inconsistent sets to be different from all sets, including themselves. The feature, which holds in Zach Weber's dialethic set theory [26, 27, 28], comes to an idle spreading of inconsistency in **PFS**.²¹

As abstracting terms are unclean in some **PFS**-models, several elements of the domain may function as the assignment value of the same abstracting term. So, where $t \in \mathcal{T}^c$, some models differ only in $v(t)$. Going adaptive, one may eliminate **PFS**-models in which terms are more unclean than required.

The **PFS**-theorems $\forall x x \notin \emptyset$ and $\forall x x \in U$ are **CLuNs**-equivalent to $\neg \exists x x \in \emptyset$ and $\neg \exists x x \notin U$ respectively. We obviously want $\sim \exists x x \in \emptyset$ and $\sim \exists x x \notin U$ as **AFS**-theorems; only these warrant that $\emptyset$ and U are clean.

The maximally inconsistent set G is remarkable. The abstracting terms that are **PFS**-provably identical to G ,²² are the only terms that are clean in all non-trivial models as well as, contingent on D , invariant over all models. As an effect, $\vdash_{\mathbf{PFS}} \forall x (\forall y (y \in x \wedge y \notin x) \equiv x = G)$.

As $R \in R$ and $R \notin R$ are **PFS**-theorems, so are $R \in Z$ and $R \notin Z$. None of the following four are **PFS**-theorems: $Z \in Z$, $Z \notin R$, $Z \notin Z$, and $Z \in R$. Moreover, we do not want any of them as **AFS**-theorems because there apparently are no reasons to prefer either the former two or the latter two. So

²⁰On the one hand, R^* is defined within **PFS** in the same way as R is defined in Frege's original set theory and the negation in R^* is classical within $\mathcal{L}_S$. On the other hand, R^* has some members for which the touchstone does not hold; if it did, **PFS** would be trivial. So whether, within **PFS**, the Russell set R^* is called classical is partly a matter of convention.

²¹Yet, the complicated identity is definable in **PFS**, for example, by $t_1 \simeq t_2 =_{df} \forall x (x \in t_1 \leftrightarrow x \in t_2)$. This gives $\vdash_{\mathbf{PFS}} t_1 \simeq t_2 \equiv t_1 = t_2$ and $\vdash_{\mathbf{PFS}} \neg t_1 \simeq t_2 \equiv (t_1 \neq t_2 \vee \exists x (x \in t_1 \wedge x \notin t_1))$ —*sic*. In view of Theorem 4 below in the text, it is easy enough to ensure that an **AFS** has all formulas $\sim Q(t_1 = t_2 \wedge t_1 \neq t_2)$ as theorems—see **AFS**₂ in Section 8.

²²If $\not\vdash_{\mathbf{PFS}} t = G$, then $t = G$ is not a theorem of any **AFS**.

Z is a set that we want to be underdetermined by the theory, a set that should be a member of itself in some models and a non-member of itself in others.

The sets Q_n ($n \geq 1$) are Quine's [18] variants to the Russell set. The following **CLuNs**-proof for Q_1 is easily generalized to all Q_n .

1	$\forall x(x \in Q_1 \Leftrightarrow \neg \exists y(x \in y \wedge y \in x))$	Abs
2	$Q_1 \in Q_1 \Leftrightarrow \neg \exists y(Q_1 \in y \wedge y \in Q_1)$	1
3	$Q_1 \in Q_1 \Leftrightarrow \neg(Q_1 \in Q_1 \wedge Q_1 \in Q_1)$	2
4	$Q_1 \in Q_1 \Leftrightarrow Q_1 \notin Q_1$	3
5	$Q_1 \notin Q_1 \supset Q_1 \in Q_1$	4
6	$Q_1 \notin Q_1 \vee Q_1 \notin Q_1$	4
7	$Q_1 \in Q_1$	5
8	$Q_1 \notin Q_1$	6

We obviously want $D, D^*, \emptyset^\circ, \emptyset^\Delta$ and $\emptyset$ to be identical in **AFS**, but they are not provably identical in **PFS**. Indeed, there are models in which some of the terms are unclean and others clean, or more unclean or less unclean.

Each of the following are **PFS**-theorems: $R^* \in R, R^* \notin R$; for all $i \geq 1$, $Q_i^* \in Q_i^*, Q_i^* \notin Q_i^*, Q_i \in R, Q_i \notin R, Q_i^* \in R$ and $Q_i^* \notin R$; $R \notin R^*$ and, for all $i \geq 1$, $Q_i \notin R^*$ and $Q_i^* \notin R^*$. The following non-**PFS**-theorems are desirable as **AFS**-theorems: $\sim R \in R^*$ and, for all $i \geq 1$, $\sim Q_i \in R^*$ and $\sim Q_i^* \in R^*$.

The N_i are the sets of sets with i members: N_0 the set of empty sets, N_1 the set of singletons, etc. The occurrence of the classical negation is essential: $\vdash_{\mathbf{PFS}} G \in \{x \mid \neg \exists y y \in x\}$ but we want $\sim G \in N_0$ in all **AFS**.

While $\bar{t}$ depends on the members of t as well as on the non-members of t , $\tilde{t}$ depends only on the members of t . A similar comment applies to $t_1 - t_2$ and $t_1 \setminus t_2$. An extreme case is $i(t)$: whether t is clean or not, every member (non-member) of t is a member (non-member) of $i(t)$, but $i(t)$ may have further members as well as further non-members in **PFS**-models in which it is unclean. Sets $\oplus(t), \ominus(t), \boxplus\{x \mid A(x)\}$ and $\boxminus\{x \mid A(x)\}$ are consistent iff clean—note that $\not\vdash_{\mathbf{PFS}} R \notin \oplus(R)$ and $\not\vdash_{\mathbf{PFS}} R^* \in \boxplus(R^*)$.

Sets are extremely ‘malleable’. Expressions such as $\{x \mid A(x) \vee (x = t \wedge \exists x!(x \in x))\}$, $\{x \mid x \in \{x \mid A(x)\} \vee (x = t \wedge \exists x!(x \in x))\}$ and $\{x \mid A(x)\} \cup (\{t\} \cap G)$ enable one to define a set exactly like $\{x \mid A(x)\}$ but with t a member of it. Other expressions enable one to define a set exactly like $\{x \mid A(x)\}$ but with t a non-member of it. Where $t_1 \in t_2 \wedge t_1 \notin t_2$, one may define a t_3 which, in **AFS**, is exactly like t_2 except that $\sim t_1 \in t_3$ or that $\sim t_1 \notin t_3$ as one prefers. Instead of defining a set by ‘adding’ or ‘removing’ t_1 as a member or non-member, one may, by similar means, add or remove the *members* of t_1 as members or non-members.

5 Some Models

Consider first M_1^1 , a trivial **PFS**-model. Its singleton domain and assignment function may, with notational abuse, be summarized as follows.

$$M_1^1 \quad D = \{m\}$$

$$\text{for all } t \in \mathcal{T}^c \cup \mathcal{O}, v(t) = m \quad \frac{m}{m} \Big| \begin{array}{c} m \\ =, \neq \end{array} \quad \frac{m}{m} \Big| \begin{array}{c} m \\ \in, \notin \end{array}$$

Spelling out the predicates: $v^T(\in) = v^F(\in) = v^T(=) = v^F(=) = \emptyset$ and $v^I(\in) = v^I(=) = \{\langle m, m \rangle\}$. As all abstracting terms are mapped to m , G is clean and terms t for which $\not\vdash_{\mathbf{PFS}} t = G$ are maximally unclear in M_1^1 . The model verifies all sentences of $\mathcal{L}_S$. Moreover, it is a *collapsed* model of Fregean set theory. The term is taken from collapsed models of arithmetic [5, 14, 15, 16, 21, 22]. However, arithmetic is *presumed* to be consistently given by the Peano axioms, whereas the adaptive Fregean set theory is inconsistent as well as under construction.

Let model M_2^1 be exactly like M_1^1 except in that $v^T(=) = \{\langle m, m \rangle\}$ whereas $v^I(=) = v^F(=) = \emptyset$. Clearly M_2^1 verifies $\forall x \forall y x = y$, $\forall x \forall y x \in y$ and $\forall x \forall y x \notin y$. So M_2^1 verifies Ext as well as all instances of Abs.²³

Theorem 4 *No PFS-theorem has the form $Q \alpha \neq \beta$.*

Proof. M_2^1 is a **PFS**-model and $M_2^1 \not\models \exists x \exists y x \neq y$. ■

Theorem 5 *No PFS-theorem has the form $Q \sim \alpha \in \beta$.*

Proof. M_2^1 is a **PFS**-model, $M_2^1 \not\models \perp$ and $M_2^1 \models \forall x \forall y x \in y$. ■

Theorem 6 **PFS** *is non-trivial.*

Proof. M_2^1 is a **PFS**-model and $M_2^1 \not\models \perp$. ■

Proofs of Strong Reassurance occur in several places [3, Theorem 4]. Applied to AFS, this comes to: if **PFS** is non-trivial, no AFS is trivial.²⁴

Theorem 7 *All AFS are non-trivial.*

Let the domain and assignment of **PFS**-model M_1^3 be as follows.

M_1^3	$D = \{l, m, u\}$							
$v(\emptyset) = l$		l	m	u		l	m	u
$v(U) = u$	l	$=$	$\neq$	$\neq$	l	$\notin$	$\in, \notin$	$\in$
$v(t) = m$ otherwise ²⁵	m	$\neq$	$=, \neq$	$\neq$	m	$\notin$	$\in, \notin$	$\in$
	u	$\neq$	$\neq$	$=$	u	$\notin$	$\in, \notin$	$\in$

The tables summarize: $v^T(\in) = \{\langle l, u \rangle, \langle m, u \rangle, \langle u, u \rangle\}$, $v^I(\in) = \{\langle l, m \rangle, \langle m, m \rangle, \langle u, m \rangle\}$, $v^F(\in) = \{\langle l, l \rangle, \langle m, l \rangle, \langle u, l \rangle\}$, $v^T(=) = \{\langle l, l \rangle, \langle u, u \rangle\}$, $v^I(=) = \{\langle m, m \rangle\}$, and $v^F(=)$ comprises the six other couples.

To see that Ext is verified, note that l , m , and u are different from each other with respect to the combination of their ‘positive extension’ (members) and ‘negative extension’ (non-members):

	$\in$	$\notin$
l		l, m, u
m	l, m, u	l, m, u
u	l, m, u	

²³These are, for every $A(x)$: $\forall y(A(y) \supset y \in \{x \mid A(x)\})$, $\forall y(\neg A(y) \supset y \notin \{x \mid A(x)\})$, $\forall y(y \notin \{x \mid A(x)\} \vee A(y))$ and $\forall y(y \in \{x \mid A(x)\} \vee \neg A(y))$.

²⁴In semantic terms, Strong Reassurance (also called Stopperedness or Smoothness) comes to this: for all adaptive logics in standard format, if a premise set has a lower limit model M that is not minimally abnormal, then the premise set has a minimally abnormal lower limit model M' such that $Ab(M') \subset Ab(M)$. So as **PFS** has non-trivial models (by Theorem 6), every AFS has non-trivial models. The essential step in the proof is the absence of infinitely descending chains of abnormal models.

²⁵“Otherwise” abbreviates “for all $t \in \mathcal{O} \cup \mathcal{T}^c - \{\emptyset, U\}$ ”; similarly for other models below in the text. Equivalent touchstones do not warrant identical abstracting terms.

Abs is also verified. The following table lists, in the first two rows, the couples $\langle S_1, S_2 \rangle$ for which $S_1 \cup S_2 = D$. The third row mentions the elements of D that have all of S_1 as members and all of S_2 as non-members. Please check that the assignment to abstracting terms agrees with this.

$\in t$	—	l	l	m	...	m, u	l, m, u	...	l, m, u	l, m, u
$\notin t$	l, m, u	m, u	l, m, u	l, u	...	l	l, m, u	...	u	—
$v(t)$	l/m	m	m	m	...	m	m	...	m	m/u

In M_1^3 , the sets $\emptyset$ and U are clean and so are all t such that $\vdash_{\mathbf{PFS}} t_1 = G$; all other abstracting terms are unclean.

Let M_2^3 be exactly like M_1^3 except that identity is consistent and $\langle m, m \rangle \in v^T(=)$. This model is just as much a **PFS**-model as M_1^3 and showing that it proceeds as for M_1^3 .

I mention some more **PFS**-models; some will be referred to in the sequel.

M_1^2	$D = \{l, m\}$																		
$v(\emptyset) = l$ $v(t) = m$ otherwise	<table> <tr> <td></td> <td>l</td> <td>m</td> </tr> <tr> <td>l</td> <td>$\notin$</td> <td>$\in, \notin$</td> </tr> <tr> <td>m</td> <td>$\notin$</td> <td>$\in, \notin$</td> </tr> </table> <table> <tr> <td></td> <td>l</td> <td>m</td> </tr> <tr> <td>l</td> <td>$=$</td> <td>$\neq$</td> </tr> <tr> <td>m</td> <td>$\neq$</td> <td>$=$</td> </tr> </table>		l	m	l	$\notin$	$\in, \notin$	m	$\notin$	$\in, \notin$		l	m	l	$=$	$\neq$	m	$\neq$	$=$
	l	m																	
l	$\notin$	$\in, \notin$																	
m	$\notin$	$\in, \notin$																	
	l	m																	
l	$=$	$\neq$																	
m	$\neq$	$=$																	

M_2^2	$D = \{m, u\}$																		
$v(U) = u$ $v(t) = m$ otherwise	<table> <tr> <td></td> <td>m</td> <td>u</td> </tr> <tr> <td>m</td> <td>$\in, \notin$</td> <td>$\in$</td> </tr> <tr> <td>u</td> <td>$\in, \notin$</td> <td>$\in$</td> </tr> </table> <table> <tr> <td></td> <td>m</td> <td>u</td> </tr> <tr> <td>m</td> <td>$=$</td> <td>$\neq$</td> </tr> <tr> <td>u</td> <td>$\neq$</td> <td>$=$</td> </tr> </table>		m	u	m	$\in, \notin$	$\in$	u	$\in, \notin$	$\in$		m	u	m	$=$	$\neq$	u	$\neq$	$=$
	m	u																	
m	$\in, \notin$	$\in$																	
u	$\in, \notin$	$\in$																	
	m	u																	
m	$=$	$\neq$																	
u	$\neq$	$=$																	

M_1^4	$D = \{l, m, r, u\}$									
		l	m	r	u		l	m	r	u
$v(R) = r$	l	$\notin$	$\in, \notin$	$\in$	$\in$	l	$=$	$\neq$	$\neq$	$\neq$
$v(U) = u$	m	$\notin$	$\in, \notin$	$\in, \notin$	$\in$	m	$\neq$	$=$	$\neq$	$\neq$
$v(t) = m$ otherwise	r	$\notin$	$\in, \notin$	$\in, \notin$	$\in$	r	$\neq$	$\neq$	$=$	$\neq$
	u	$\notin$	$\in, \notin$	$\notin$	$\in$	u	$\neq$	$\neq$	$\neq$	$=$

M_2^4	$D = \{l, m, r, u\}$									
		l	m	r	u		l	m	r	u
$v(R) = r$	l	$\notin$	$\in, \notin$	$\in, \notin$	$\in$	l	$=$	$\neq$	$\neq$	$\neq$
$v(U) = u$	m	$\notin$	$\in, \notin$	$\in, \notin$	$\in$	m	$\neq$	$=$	$\neq$	$\neq$
$v(t) = m$ otherwise	r	$\notin$	$\in, \notin$	$\in, \notin$	$\in$	r	$\neq$	$\neq$	$=$	$\neq$
	u	$\notin$	$\in, \notin$	$\notin$	$\in$	u	$\neq$	$\neq$	$\neq$	$=$

Checking that M_1^4 and M_2^4 are **PFS**-models proceeds as for M_1^3 and is just as easy. Model M_2^4 verifies $\exists x(x \notin R \wedge \sim x \in x)$ whereas M_1^4 falsifies it—actually R is clean in M_1^4 . As no abstracting term names l in either model, there is no abstracting term t such that $M_2^4 \Vdash t \notin R \wedge \sim t \in t$ and $M_1^4 \nVdash t \notin R \wedge \sim t \in t$.

6 Abnormalities Revisited

A set that is unclean in a model is inconsistent in that model. Yet, a set may be inconsistent as well as clean in a model: $M_1^4 \Vdash \exists x(x \in R \wedge x \notin R)$

but $M_1^4 \Vdash \forall x(x \in R \leftrightarrow x \notin x)$. In M_2^4 , to the contrary, R is inconsistent and unclean, $M_2^4 \Vdash \exists x(x \notin R \wedge x \in x)$. This seems a good reason to prefer M_1^4 over M_2^4 ,²⁶ but **CLuNs**^m does not make such preference: $Ab(M_2^4) = Ab(M_1^4)$ —the models verify the same non-quantified formulas because no constant term names l ; they verify the same quantified formulas because the effect of $\langle l, r \rangle \in v^I(\in)$ is that $M_2^4 \Vdash \exists x(x \in R \wedge x \notin R)$, but M_1^4 and M_1^4 verify the same formula in view of $\langle r, r \rangle \in v^I(\in)$. In order that M_1^4 be preferred over M_2^4 , formulas of the forms (7) and (8) should count as abnormalities.²⁷ Call these *new abnormalities* as opposed to the *old* ones. Some examples: $\emptyset \in Z \wedge \sim \emptyset \in \emptyset$, $\exists x(x \in \emptyset \wedge \sim \perp)$, $\emptyset \notin R \wedge \sim \emptyset \notin \emptyset$ and $\emptyset \notin \{\emptyset\} \wedge \sim \emptyset \neq \emptyset$.

$$Q(t \in \{y \mid A(y)\} \wedge \sim A(t)) \quad (7)$$

$$Q(t \notin \{y \mid A(y)\} \wedge \sim \neg A(t)) \quad (8)$$

Model M_1^4 is a far cry of a minimal abnormal **CLuNs**^m-model of **PFS**, so let me strengthen the argument. By present insights R is clean in some minimal abnormal **CLuNs**^m-models of **PFS**; let M_1 be a non-denumerable such model, whence most members of the domain D are not denoted by abstracting terms. Let M_2 be exactly like M_1 except in that a member of D that is not named by an abstracting term is not a member of itself but is a member as well as a non-member of R . In terms of the old abnormalities, $Ab(M_2) = Ab(M_1)$, but R is clean in M_1 and unclean in M_2 .

A different way to present the new abnormalities is by pointing out that the transition from a non-detachable implication to a detachable one may be justified by the falsehood of the connected contradiction, as suggested in the left schema, which is (locally) equivalent to the middle one, but also by the falsehood of the specific ‘abnormality’, as in the right schema.²⁸

$$\frac{A \supset B}{(A \supset B) \vee (A \wedge \neg A)} \quad \frac{\neg A \vee B}{(\sim A \vee B) \vee (A \wedge \neg A)} \quad \frac{A \supset B}{(A \supset B) \vee (A \wedge \sim B)}$$

As explained in Section 4, we want to retain **PFS**-models verifying $Z \in Z$ and **PFS**-models verifying $Z \notin Z$ as **AFS**-models, but not models verifying both, which are unnecessarily inconsistent. As $M \Vdash Z \in Z \wedge Z \notin Z$ does not cause uncleanness in M , the old abnormalities should be retained with the new ones. So the set of abnormalities will from now on read:

$$\begin{aligned} \Omega = & \{Q(A \wedge \neg A) \mid A \in \mathcal{F}^a\} \cup \\ & \{Q(t \in \{y \mid A(y)\} \wedge \sim A(t)) \mid A \in \mathcal{F}_S\} \cup \\ & \{Q(t \notin \{y \mid A(y)\} \wedge \sim \neg A(t)) \mid A \in \mathcal{F}_S\}. \end{aligned} \quad (9)$$

We are not yet completely home yet, but all abnormalities considered later on will belong to this Ω . Let **CLuNs**_d^m be exactly like **CLuNs**^m except that the set of abnormalities is the Ω defined in (9).

²⁶By present lights, *Desideratum 3* rules out every uncleanness that is not provable in **PFS**. See Section 4 for $\vdash_{\mathbf{PFS}} R^* \in R^* \wedge \sim R^* \in R^*$.

²⁷Strictly speaking, the form of (7) is $Q((t \in \{y \mid A(y)\} \supset A(t)) \wedge (t \in \{y \mid A(y)\} \wedge \sim A(t)))$ and similarly for (8). However, the first conjunct is an obvious consequence of Abs.

²⁸Obviously $((A \supset B) \wedge (A \wedge \sim B))$ will also do for the right disjunct of the right schema, but the complication is useless for (7) and (8).

7 The Adaptive Gain Vanishing

Consider a stage of a **CLuNs**_d^m-proof from Ext and Abs. Let Z abbreviate the abnormality $\exists z(z \in \{\emptyset\} \wedge \sim z = \emptyset)$. *Dab*-consequences and minimal *Dab*-consequences of $\{\text{Ext}, \text{Abs}\}$ will be called *Dab*-theorems and minimal *Dab*-theorems (of **PFS**).

1	$\forall z(z \in \{\emptyset\} \iff z = \emptyset)$	RU	$\emptyset$
2	$\emptyset \in \{\emptyset\} \iff \emptyset = \emptyset$	1; RU	$\emptyset$
3	$\emptyset = \emptyset$	RU	$\emptyset$
4	$\emptyset \in \{\emptyset\}$	3, 2; RU	$\emptyset$
5	$\forall z(z \in \{\emptyset\} \supset z = \emptyset)$	1; RC	$\{\exists z!(z \in \{\emptyset\})\}$
6	$\emptyset \in \{\emptyset\} \wedge \forall z(z \in \{\emptyset\} \supset z = \emptyset)$	4, 5; RU	$\{\exists z!(z \in \{\emptyset\})\}$
7	$\exists y(y \in \{\emptyset\} \wedge \forall z(z \in \{\emptyset\} \supset z = y))$	6; RU	$\{\exists z!(z \in \{\emptyset\})\}$
8	$\{\emptyset\} \in N_1 \iff \exists y(y \in \{\emptyset\} \wedge \forall z(z \in \{\emptyset\} \supset z = y))$	RU	$\emptyset$
9	$\{\emptyset\} \in N_1$	8, 7; RU	$\{\exists z!(z \in \{\emptyset\})\}$
10	$\{\emptyset\} \in N_1 \vee \exists z!(z \in \{\emptyset\})$	9; RD	$\emptyset$
11	$\forall z(z \in \{\emptyset\} \supset z = \emptyset) \vee \exists z(z \in \{\emptyset\} \wedge \sim z = \emptyset)$	RU	$\emptyset$
12	$\forall z(z \in \{\emptyset\} \supset z = \emptyset)$	11; RC	$\{Z\}$
13	$\emptyset \in \{\emptyset\} \wedge \forall z(z \in \{\emptyset\} \supset z = \emptyset)$	4, 12; RU	$\{Z\}$
14	$\exists y(y \in \{\emptyset\} \wedge \forall z(z \in \{\emptyset\} \supset z = y))$	13; RU	$\{Z\}$
15	$\{\emptyset\} \in N_1$	8, 14; RU	$\{Z\}$
16	$\{\emptyset\} \in N_1 \vee \exists z(z \in \{\emptyset\} \wedge \sim z = \emptyset)$	15; RD	$\emptyset$

The formula of line 1 is an Abs axiom with $\{y \mid y = \emptyset\}$ replaced by $\{\emptyset\}$. The formulas of 2 and 8 are instances of Abs axioms and the formulas of 3 and 11 **CLuNs**-theorems. RD is a derived rule, which relies on the fact that A is derivable from Γ on condition Δ just in case $A \vee \text{Dab}(\Delta)$ is derivable from Γ on condition $\emptyset$ —see footnote 11. Lines 11–15 show that, in the presence of the new abnormalities, $\{\emptyset\} \in N_1$ is not only derivable on the condition $\{\exists z(z \in \{\emptyset\} \wedge z \notin \{\emptyset\})\}$ but also on the condition $\{\exists z(z \in \{\emptyset\} \wedge \sim z = \emptyset)\}$. According to the insights provided by the present proof stage, $\{\emptyset\} \in N_1$ belongs to the adaptive gain: it is conditionally derived at unmarked lines 9 and 15.

A basic insight, due to Peter Verdée [24, 25], is that formulas only derivable on a non-empty condition from the **PFS**-axioms are not finally **CLuNs**^m-derivable from these axioms. The proof is related to the Curry paradox. Verdée's proof easily generalizes to **CLuNs**_d^m and turns out to have more variants than the influenza virus. Let $C_{[A]}$ abbreviate $\{x \mid x \in x \supset A\}$ and let $K_{[A]}$ abbreviate $\{x \mid x \notin x \vee A\}$, where for the time being *no variables are free in A*.

The following four formulas are **PFS**-theorem schemas.

$$C_{[A]} \in C_{[A]} \quad (10)$$

$$A \vee !(C_{[A]} \in C_{[A]}) \quad (11)$$

$$K_{[A]} \in K_{[A]} \quad (12)$$

$$A \vee !(K_{[A]} \in K_{[A]}) \quad (13)$$

I only list a proof of (12) and (13); see lines 3 and 5:

1	$\forall x(x \in K_{[A]} \iff (x \notin x \vee A))$	Abs	$\emptyset$
2	$K_{[A]} \in K_{[A]} \iff (K_{[A]} \notin K_{[A]} \vee A)$	1; RU	$\emptyset$
3	$K_{[A]} \in K_{[A]}$	2; RU	$\emptyset$

4	$A \vee K_{[A]} \notin K_{[A]}$	2; RU	$\emptyset$
5	$A \vee !(K_{[A]} \in K_{[A]})$	3, 4; RU	$\emptyset$

Consider the case where a formula A is derived on the condition Δ , whence $A \vee Dab(\Delta)$ is derivable on the empty condition—see footnote 11. There are then several ways, all relying on (11), to derive a *Dab*-formula that has $Dab(\Delta)$ as a disjunct. I shall call them *roads* and number them for future reference. For the *first road*, I rely on the obvious fact that, if $A \notin \mathcal{F}^a$, then there are $A_1, \dots, A_n \in \mathcal{F}^a$ (for a $n \geq 1$) such that $A \wedge \neg A \vdash_{\mathbf{CLuNs}} (A_1 \wedge \neg A_1) \vee \dots \vee (A_n \wedge \neg A_n)$.

i+1	$A \vee Dab(\Delta)$	$\dots$	$\emptyset$
i+2	$\neg A \vee !(C_{[\neg A]} \in C_{[\neg A]})$	(11)	$\emptyset$
i+3	$(A \wedge \neg A) \vee Dab(\Delta) \vee !(C_{[\neg A]} \in C_{[\neg A]})$	i+1, i+2; RU	$\emptyset$
i+4	$!(A_1) \vee \dots \vee !(A_n) \vee Dab(\Delta) \vee !(C_{[\neg A]} \in C_{[\neg A]})$	i+3; RU	$\emptyset$

The presence of the local classical negation provides the *second road*:

i+1	$A \vee Dab(\Delta)$	$\dots$	$\emptyset$
i+2	$\sim A \vee !(C_{[\sim A]} \in C_{[\sim A]})$	(11)	$\emptyset$
i+3	$Dab(\Delta) \vee !(C_{[\sim A]} \in C_{[\sim A]})$	i+1, i+2; RU	$\emptyset$

And here is a *third road*:

i+1	$Dab(\Delta) \vee !(C_{[Dab(\Delta)]} \in C_{[Dab(\Delta)]})$	(11)	$\emptyset$
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This may be applied more generally. If $B \in \Delta$ then, in view of (11),

$$B \vee !(C_{[B]} \in C_{[B]})$$

is a *Dab*-theorem. Similar proofs give one the *K*-variants of these *roads*.

So the proof stage listed at the outset of this section may be extended, taking the formula of line 10 for $A \vee Dab(\Delta)$, with a derivation of any of the following three formulas on the condition $\emptyset$.

$$!(\{\emptyset\} \in N_1) \vee \exists z!(z \in \{\emptyset\}) \vee !(C_{[\neg\{\emptyset\} \in N_1]} \in C_{[\neg\{\emptyset\} \in N_1]}) \quad (14)$$

$$\exists z!(z \in \{\emptyset\}) \vee !(C_{[\sim\{\emptyset\} \in N_1]} \in C_{[\sim\{\emptyset\} \in N_1]}) \quad (15)$$

$$\exists z!(z \in \{\emptyset\}) \vee !(C_{[\exists z!(z \in \{\emptyset\})]} \in C_{[\exists z!(z \in \{\emptyset\})]}) \quad (16)$$

Similarly for the three *K*-variants.

Any such extension of the proof stage at the outset of this section causes 9 to be marked and to remain marked in all further extensions—the *Dab*-theorems are minimal and will remain minimal. One may proceed in the same way with respect to line 15, identifying $A \vee Dab(\Delta)$ with the formula of 16. In this case the formulas are the following three plus their *K*-variants.

$$!(\{\emptyset\} \in N_1) \vee \exists z(z \in \{\emptyset\} \wedge \sim z = \emptyset) \vee !(C_{[\neg\{\emptyset\} \in N_1]} \in C_{[\neg\{\emptyset\} \in N_1]}) \quad (17)$$

$$\exists z(z \in \{\emptyset\} \wedge \sim z = \emptyset) \vee !(C_{[\sim\{\emptyset\} \in N_1]} \in C_{[\sim\{\emptyset\} \in N_1]}) \quad (18)$$

$$\exists z(z \in \{\emptyset\} \wedge \sim z = \emptyset) \vee !(C_{[\exists z(z \in \{\emptyset\} \wedge \sim z = \emptyset)]} \in C_{[\exists z(z \in \{\emptyset\} \wedge \sim z = \emptyset)]}) \quad (19)$$

So 15 will be marked in the extensions and will remain marked if these are further extended.

An adaptive gain is also blocked by the new abnormalities.

$$\begin{array}{llll}
i+1 & A \vee Dab(\Delta) & \dots & \emptyset \\
i+2 & C_{[\sim A]} \in C_{[\sim A]} & (10) & \emptyset \\
i+3 & Dab(\Delta) \vee (C_{[\sim A]} \in C_{[\sim A]} \wedge \sim(C_{[\sim A]} \in C_{[\sim A]} \supset \sim A)) & i+1, i+2; RU & \emptyset
\end{array}$$

So this was the *fourth road* and here is the *fifth road*:

$$\begin{array}{llll}
i+1 & Dab(\Delta) \vee \sim Dab(\Delta) & RU & \emptyset \\
i+2 & C_{[Dab(\Delta)]} \in C_{[Dab(\Delta)]} & (10) & \emptyset \\
i+3 & Dab(\Delta) \vee (C_{[Dab(\Delta)]} \in C_{[Dab(\Delta)]} \wedge \sim(C_{[Dab(\Delta)]} \in C_{[Dab(\Delta)]} \supset Dab(\Delta)) & i+1, i+2; RU & \emptyset
\end{array}$$

The proofs of the *K*-variants proceed similarly.

So Curry eats the adaptive gain in a most gluttonous way. There are many variants: Curry formulas with x free in the touchstone, Curry formulas based on Quine sets, unions like $R \cup \{x \mid A(x)\}$, all kinds of transformations of such formulas by De Morgan-like properties and the like.

Some gluttons are unrelated to the ones discovered by Verdée. If, for example, $\exists x!(x \in t)$ is verified by some models and falsified by others, then

$$\exists x!(x \in t) \vee !(t \in \{x \mid \sim \exists y!(y \in x) \wedge (x \neq t \vee x \in G)\}) \quad (20)$$

is a minimal *Dab*-theorem. Similarly, if $\vdash_{\mathbf{PFS}} A(t)$ and t is an unclean non-member of $\{x \mid A(x)\}$ in some but not all **PFS**-models, then

$$\begin{aligned}
& (t \notin \{x \mid A(x)\} \wedge \sim \neg A(t)) \vee !(\{x \mid A(x)\} \in \{x \mid \\
& (\forall y(y \in x \Leftrightarrow A(y)) \wedge (t \in x \leftrightarrow A(t))) \wedge (x \neq \{x \mid A(x)\} \vee x \in G)\}) \quad (21)
\end{aligned}$$

is a minimal *Dab*-theorem. This *sixth* and *seventh road* are obtained by the malleability discussed in Section 4. Deciphering (20): either t is an inconsistent set or t is a member of the set of consistent sets of which t is a non-member. Deciphering (21) is left to the reader. The role played by Curry-like sets in the previous paragraphs is played here by (what I shall call) *funny normality sets*; they comprise normal versions of a certain set as a member, but are inconsistent otherwise. No self-reference or circular reference or similar occurs in (20) or (21). So the vanishing of the adaptive gain is not an effect of odd forms of reference.

The overall conclusion is that **CLuNs^m** and **CLuNs_>^m** do not define interesting AFS, but AFS that reduce to **PFS**. This is not a deadlock, however. Ways out have been known for a long time.

8 Some Adaptive Fregean Set Theories

Define a sequence $\Omega_1, \Omega_2, \dots$ such that (i) $\Omega_i \subset \Omega_{i+1}$ for all i and (ii) $\Omega_1 \cup \Omega_2 \cup \dots = \Omega$, where Ω is as in (9). Let the adaptive logics **CLuNs_{>1}^m**, **CLuNs_{>2}^m**, $\dots$ be exactly like **CLuNs_>^m** except that the set of abnormalities of **CLuNs_{>i}^m** is Ω_i . The AFS is the sequential superposition:

$$\dots \text{Cn}_{\mathbf{CLuNs}_{>3}^m}(\text{Cn}_{\mathbf{CLuNs}_{>2}^m}(\text{Cn}_{\mathbf{CLuNs}_{>1}^m}(\{\text{Abs}, \text{Ext}\}))) \dots \quad (22)$$

There are several straightforward ways to write proof stages of a so defined AFS as sequences rather than as sets of sequences. Below, I consider a few AFS, illustrating some possibilities offered by sequential superpositions.

As many sets of abnormalities are around, I shall use such expressions as Dab_{Ω} -theorems, Dab_{Ω_1} -theorems, ...; a *minimal* Dab_{Ω_i} -theorem will be one that is minimal among the Dab_{Ω_i} -theorems. Note that, for example, Dab_{Ω_3} -theorems are **CLuNs**-derivable from $Cn_{\mathbf{CLuNs}_{\mathcal{P}_2}^m}(Cn_{\mathbf{CLuNs}_{\mathcal{P}_1}^m}(\{Abs, Ext\}))$. Such expressions and statements will have a contextual meaning, dependent on the specific sequential superposition of adaptive logics under discussion.

A formula *in PNF* (prenex normal form) starts with a possibly empty sequence of quantifiers followed by a $\wedge$ - $\vee$ -function of elements of the form $\nu t_1 \xi t_2$ in which ν is nothing or a string from $\{\neg, \sim, \sim\neg, \sim\sim, \sim\sim\neg\}$ ²⁹ and $\xi \in \{=, \in\}$. D is a PNF of A iff D is in PNF and $\vdash_{\mathbf{CLuNs}} \forall(D \leftrightarrow A)$.

A partition $\langle \mathcal{T}_0, \mathcal{T}_1, \dots \rangle$ of $\mathcal{T}$ will be defined. If $t \in \mathcal{T}_i$, then i is said to be the *degree* of t . First, $\mathcal{V} \subseteq \mathcal{T}_0$. For abstracting terms, define $\{x \mid A\} \in \mathcal{T}_n$ iff n is the smallest number such that, for a PNF D of A , n is the sum of (i) the number of occurrences of $\vee$ and $\wedge$ within D and (ii) the maximum of the degrees of the abstracting terms occurring in D .³⁰ Some examples: $x \in \mathcal{T}_0$, $\emptyset, U \in \mathcal{T}_0$ (because $\vdash_{\mathbf{CLuNs}} \perp \leftrightarrow \exists y \sim y = y$ and $\vdash_{\mathbf{CLuNs}} \top \leftrightarrow \forall y \sim \sim y = y$), $\{\emptyset\}, \{\{\emptyset\}\}, \dots \in \mathcal{T}_0$, $R, \emptyset, C_{[\perp]}, C_{[\top]} \in \mathcal{T}_0$, $R \cup \emptyset \in \mathcal{T}_1$ (because there is one $\vee$ and no $\wedge$ in $x \in R \vee x \in \emptyset$ and $R, \emptyset \in \mathcal{T}_0$), $\{\emptyset\} \cup \{\{\emptyset\}\} \in \mathcal{T}_1$, $G \in \mathcal{T}_1$, $C_{[x \in R]} \in \mathcal{T}_1$, $C_{[x \in R]} \cup R \in \mathcal{T}_2$ (because there is one $\vee$ and no $\wedge$ in $x \in C_{[x \in R]} \vee x \in R$ and 1 is the maximal degree of $C_{[x \in R]}$ and R), $C_{[x \in G]} \in \mathcal{T}_2$, $\{x \mid \sim \exists y!(y \in x) \wedge (x \neq \emptyset \vee x \in G)\} \in \mathcal{T}_4$, $\{x \mid (\forall y(y \in x \iff A(y))) \wedge (t \in x \leftrightarrow A(t)) \wedge (x \neq \{x \mid A(x)\} \vee x \in G)\} \in \mathcal{T}_d$ where d is either 10 or the degree of $\{x \mid A(x)\}$ plus 9, whatever is larger, etc.

Define a sequence $\langle \Xi_0, \Xi_1, \dots \rangle$ of subsets of Ω in terms of the partition of $\mathcal{T}$ —compare to (9) to see the relation between Ω and the Ξ_i .³¹

$$\begin{aligned} \Xi_i = & \{Q!(t_1 = t_2) \mid t_1, t_2 \in \mathcal{T}_0 \cup \dots \cup \mathcal{T}_i\} \cup \\ & \{Q!(t_1 \in t_2) \mid t_2 \in \mathcal{T}_0 \cup \dots \cup \mathcal{T}_i\} \cup \\ & \{Q(t \in \{y \mid A(y)\} \wedge \sim A(t)) \mid \{y \mid A(y)\} \in \mathcal{T}_0 \cup \dots \cup \mathcal{T}_i\} \cup \\ & \{Q(t \notin \{y \mid A(y)\} \wedge \sim \neg A(t)) \mid \{y \mid A(y)\} \in \mathcal{T}_0 \cup \dots \cup \mathcal{T}_i\} \quad (23) \end{aligned}$$

Finally, define the sequence $\langle \Omega_1, \Omega_2, \dots \rangle$ by positing $\Omega_{i+1} = \Xi_i$. Let **AFS**₁ be the sequential superposition (22), each $\mathbf{CLuNs}_{\mathcal{P}_i}^m$ being defined by the lower limit logic **CLuNs**, the set of these Ω_i and the Minimal Abnormality strategy.

The proof that there is an adaptive gain in **AFS**₁ may have the same complexity as a proof that **ZFC** is consistent. So this is beyond our reach. Yet, one may establish positive properties of **AFS**₁, for example that it is non-trivial by Theorem 7 and that no know mechanism prevents an adaptive gain, see Theorem 11.

Claim 8 By present lights, *there are* $A \in \Omega_1$ *such that* $A \notin \Delta$ *whenever* $Dab_{\Omega_1}(\Delta)$ *is a minimal* Dab_{Ω_1} -*theorem of* **PFS**.

The expression *by present lights* refers to the state of the art and invites the reader to show the author wrong. As for classical theories, relying on similar

²⁹The table in Section 3 clarifies the matter. Every formula has one or more PNF. Obtaining them is straightforward in view of the **CLuNs**-semantics; the only unusual step is that $A \supset B$ is eliminated in favour of $\sim A \vee B$, which contains a defined symbol.

³⁰That the touchstones of two abstracting terms have the same PNF entails that the terms have the same degree, not that they are identical in all **PFS**-models.

³¹For expressions of the form $t_1 \in t_2$ only the degree of t_2 is relevant for determining the Ξ_i to which the corresponding abnormality belongs.

claims is unavoidable; *cf.* by present lights **PA** is consistent. There are arguments in support of the claim. (i) Sets that behave abnormally in **PFS** are proven to behave so by pretty simple proofs: that R and Z are inconsistent, that R^* is unclean with respect to membership, etc. (ii) Nothing suggests that $\emptyset$ or U need be inconsistent and nothing suggests that those sets can only be consistent if other sets are inconsistent; similarly for $\{t\}$, $\hat{t}$, $\oplus t$ and many other set forming operations. (iii) Nothing suggests that R would be unclean or that $\emptyset$ would be an unclean member of R^* or that cleanness for those cases can only be obtained at the expense of having more uncleanness in other sets; similarly for $t_1 - t_2$ or $t_1 \cup t_2$ and other set forming operations. (iv) Nothing suggests that sets that are consistent in the small models from Section 5 would become inconsistent when the domain is extended with consistent sets. (v) If the aforementioned sets would be inconsistent, respectively unclean, one wonders why the sets and set forming operations that also occur in **ZF** would behave classically there. That the **AFS**₁-counterpart t of a **ZF**-set has inconsistent and unclean members or non-members does not cause t to be inconsistent or unclean.

Claim 9 By present lights, $\exists x \exists y!(x = y)$ is not a disjunct of a minimal Dab_{Ω_1} -theorem of **PFS**.

That is: in minimally abnormal **CLuNs**₁^m-models of $\{\text{Abs}, \text{Ext}\}$ no two sets are identical as well as different. Apart from specifications of reasons for Claim 8, a particular reason to believe Claim 9 is that all models mentioned in Section 5 are consistent with respect to identity or have a variant that only differs from the original model in that identity is consistent in it. Note that the theoremhood of the third formula in Lemma 10 states that identity is classical in **AFS**₁. This drastically strengthens the way in which *Desideratum 2* is fulfilled.

Lemma 10 If Claim 9 holds true, then $\forall x \forall y \sim(x = y \wedge x \neq y)$, $\forall x \sim x \neq x$ and $\forall x \forall y(\neg x = y \leftrightarrow \sim x = y)$ are **AFS**₁-theorems.

Proof. The classical negation of $\exists x \exists y!(x = y)$ is **CLuNs**-equivalent to $\forall x \forall y \sim(x = y \wedge x \neq y)$; from this together with the **PFS**-theorem $\forall x x = x$ follows $\forall x \sim x \neq x$; two conjuncts of $\forall x \forall y(\neg x = y \leftrightarrow \sim x = y)$ are **CLuNs**-theorems, the two other are derivable on the condition $\exists x \exists y!(x = y)$. ■

Theorem 11 If Claim 9 is true, the seven roads from Section 7 do not establish that every Ω_1 -abnormality is a disjunct of a minimal Dab_{Ω_1} -theorem.

Proof outline. Suppose that Claim 9 is true and consider line $i+1$, where $A \vee Dab_{\Omega_1}(\Delta)$ is derived, of each of the seven roads. I shall show that, for some A , none of the seven roads establishes that their last line, $i+3$ or $i+4$, is a minimal Dab_{Ω_1} -formula. This is obvious for roads 3 and 5–7: the constructed sets are not in $\mathcal{T}_0$.

Let A of line $i+1$ be one of the three formulas mentioned in Lemma 10 or an instance of them or another weakening of them and let $Dab(\Delta)$ be $\exists x \exists y!(x = y)$. For each of these A , neither $C_{[\neg A]}$ nor $C_{[\sim A]}$ belongs to $\mathcal{T}_0$; even $\sim x \in x \vee \sim(t \neq t)$ belongs, for all t , to a $\mathcal{T}_i$ with $i \geq 1$. So roads 1, 2 and 4 do not establish that their last line, $i+3$ or $i+4$, is a minimal Dab_{Ω_1} -formula.

The proof for the $K_{[\neg A]}$ and $K_{[\sim A]}$ variants proceeds similarly. ■

It may also be shown that the seven roads do not prevent normality results for sets and for the membership relation. This claim depends even more on *present lights* and requires a lengthy proof. I shall retain the results for a subsequent paper that studies theorems of one or more **AFS**. Salient normality statements include: (a) t is consistent: $\sim\exists x!(x \in t)$; (b) t_1 is a consistent member or non-member of t_2 : $\sim!(t_1 \in t_2)$; (c) $\{x \mid A(x)\}$ is clean with respect to membership, respectively non-membership: $\forall x(x \in \{x \mid A(x)\} \supset A(x))$, respectively $\forall x(x \notin \{x \mid A(x)\} \supset \neg A(x))$; (d) t is either not a member or a clean member, respectively not a non-member or a clean non-member, of $\{x \mid A(x)\}$: $t \in \{x \mid A(x)\} \supset A(t)$, respectively $t \notin \{x \mid A(x)\} \supset \neg A(t)$; (e) variants in quantifiers of (a) and (c)—replace $\exists$ by $\forall$ and vice versa. Each of these formulas B is derivable on a condition $\{D\}$ such that $\vdash_{\mathbf{CLuNs}} \sim B \leftrightarrow D$ (as well as on some other conditions).

By *present lights* **AFS**₁ is an adaptive Fregean set theory that heavily extends **PFS** and may be expected to offer a sensible analysis of the Fregean notion of a set. The theory **AFS**₁ is provably non-trivial, handles the known inconsistent sets in a decent way—as consistently as possible, among other things—and, for all we presently know, handles as consistent sets that may be consistent. The latter includes the collections of members and the collections of non-members, $\oplus(t)$ and $\ominus(t)$, of even inconsistent t —there are some exceptions as is discussed below. Also, sets that may be clean are handled as clean by **AFS**₁. This includes the collections of clean members and clean non-members, $\boxplus(t)$ and $\boxminus(t)$, of necessarily unclean t —here too some exceptions are discussed below.

Slightly reordering the abnormalities offers new perspectives. Let **AFS**₂ be like **AFS**₁ except that the sequence of Ω_i is defined as follows:

$$\begin{aligned}\Omega_1 &= \{Q!(t_1 = t_2) \mid t_1, t_2 \in \mathcal{T}\} \\ \Omega_2 &= \Omega_1 \cup \{Q!(t_{n+1} \in \{t_1, \dots, t_n\}) \mid n \geq 1; t_1, \dots, t_{n+1} \in \mathcal{T}\} \\ \Omega_3 &= \Omega_2 \cup \{Q!(t_1 \in \tilde{t}_2), Q!(t_1 \in \oplus(t_2)), Q!(t_1 \in \ominus(t_2)), \mid t_1, t_2 \in \mathcal{T}\} \\ \Omega_{i+4} &= \Omega_{i+3} \cup \Xi_i \quad (i \in \mathbb{N})\end{aligned}$$

Three sets of abnormalities were inserted before Ξ_1 . All differences I found count in favour of **AFS**₂.

In the present **CLuNs**₁^m-proof, the one with the present choice for Ω_1 , $\forall x \forall y (\sim x = y \vee \sim x \neq y)$ is derivable on the condition $\{\exists x \exists y!(x = y)\}$. The line at which it is so derived will be unmarked in all extensions of the proof because there are **PFS**-models in which $v^I(=) = \emptyset$. Several small such models were mentioned in Section 5, but it is not difficult to show that, if M is a **PFS**-model and the assignment of M' is exactly like that of M except in that the members of $v^I(=)$ in M are members of $v^T(=)$ in M' , then M' is a **PFS**-model—see also Theorem 4. The difference with **AFS**₁ is that the proof of Lemma 10 requires Claim 9, which relies on *present lights*, whereas no such reliance is needed in **AFS**₂.³²

The **CLuNs**₂^m-consequence set also offers simple and dependable new theorems: $\forall x \sim!(x \in \{t_1, \dots, t_n\})$ is derivable on the condition $\exists x!(x = t_1) \vee \dots \vee \exists x!(x = t_n)$ and neither disjunct can occur in a minimal Dab_{Ω_2} -theorem because, for all t , $\sim\exists x!(x = t)$ is a member of the **CLuNs**₁^m-consequence set.³³

³²If the reliance were mistaken in **AFS**₁, the effect in **AFS**₂ might be that the membership relation would be less normal than expected.

³³In view of the sequence of Ξ_i , it is very likely that $\sim A$ is an **AFS**₁-theorem for any A

Minimizing the abnormalities of $\Omega_3 - \Omega_2$ at once minimizes uncleanness. The effect of the present Ω_3 is that the sets defined by those operators are consistent as well as clean for *nearly* all t_2 —the exceptions will be commented upon. The other involved sets may be consistent and clean and whether they are is not influenced by minimizing abnormalities in Ω_2 .

What is the effect of the seven roads from Section 7 on **AFS**₂ given its ordering of the Ω_i ? In view of the present Ω_3 , the theorems of **AFS**₂ differ from those of **AFS**₁, but the adaptive gain is increased rather than diminished by **AFS**₂. The sensitive point is whether $\tilde{t} \in t$, $\oplus(t) \in t$ and $\ominus(t) \in t$. For many t , the sets $\tilde{t}$, $\oplus(t)$ and $\ominus(t)$ are as we hoped them to be in both **AFS**₁ and **AFS**₂. For example, for most t , it holds that $\forall x(x \in \oplus(t) \leftrightarrow x \in t)$. This holds even for $Z \in Z$, which is verified by some **PFS**-models and not by others, and similarly for $\oplus(Z) \in Z$ and $\oplus(Z) \in \oplus(Z)$.

Yet, consider $\oplus(R^*)$. Two relevant instances of Abs give one $\sim \oplus(R^*) \in \oplus(R^*) \supset \oplus(R^*) \in \oplus(R^*)$. So $\vdash_{\mathbf{PFS}} \oplus(R^*) \in \oplus(R^*)$; but then $\vdash_{\mathbf{PFS}} \oplus(R^*) \notin R^*$ by Abs. The intended meaning of $\oplus(R^*)$ is just as contradictory as that of R^* : in order for $\oplus(R^*)$ to be a clean member of R^* , we need $\sim \oplus(R^*) \in \oplus(R^*)$. So $\oplus(R^*)$ is definitely not a member of the consistent set of members of R^* . But then contradiction should and does result: Abs gives one $\vdash_{\mathbf{PFS}} \oplus(R^*) \in R^* \vee \oplus(R^*) \notin \oplus(R^*)$, whence follows the *Dab* _{Ω} -theorem

$$\vdash_{\mathbf{PFS}} !(\oplus(R^*) \in R^*) \vee !(\oplus(R^*) \in \oplus(R^*)). \quad (24)$$

As $R^*, \oplus(R^*) \in \mathcal{T}_0$, the **CLuNs** _{$\mathbf{p}_1$} ^{*m*}-consequence set of **AFS**₁ will not choose between the disjuncts of (24); neither $\oplus(R^*) \in R^*$ nor $\oplus(R^*) \notin \oplus(R^*)$ will be derivable.³⁴ The **CLuNs** _{$\mathbf{p}_3$} ^{*m*}-consequence set of **AFS**₂ will definitely make a choice. In view of (24), $!(\oplus(R^*) \in R^*)$ will be derived on the condition $!(\oplus(R^*) \in \oplus(R^*))$ because, unlike the former, the latter is a member of the Ω_3 of **AFS**₂. It seems highly plausible that the derivation is final: in which way might $!(\oplus(R^*) \in \oplus(R^*))$ be a disjunct of a minimal *Dab* _{Ω_2} -formula of **AFS**₂? Incidentally, $\sim!(\oplus(R^*) \in \oplus(R^*))$ is also derivable on the condition $!(\oplus(R^*) \in \oplus(R^*))$. Similar results may be obtained for $\oplus(R)$, $\ominus(Z)$, $\widetilde{R^*}$ and the like.

So **AFS**₂ is preferable over **AFS**₁. As R^* has unclean members anyway, including itself, it seems sensible to promote $\oplus(R^*)$, which has a similarly contradictory touchstone, to an unclean member of R^* . Doing so (i) agrees with the **PFS**-theorem $\oplus(R^*) \in \oplus(R^*)$ and (ii) arguably turns $\oplus(R^*)$ into a clean and consistent set.

Incidentally, the situation of $\boxplus$ is quite different from that of the operators mentioned in the present Ω_3 . Both $!(\boxplus(R^*) \in \boxplus(R^*))$ and $\boxplus(R^*) \notin R^*$ are **PFS**-theorems, whence $\boxplus(R^*)$ is both inconsistent and unclean (and an unclean member of itself). Of course, $\boxplus(t)$ behaves fine for less paradoxical t ; even $\boxplus(R^*)$ behaves as it should for ‘most’ of its members—including R^* : both $R^* \notin \boxplus(R^*)$ and $R^* \in \boxplus(R^*)$ are **PFS**-theorems. Also, we can state and prove that $\boxplus(R^*)$ is not a clean member of R^* : $\sim(\boxplus(R^*) \in R^* \wedge \sim \boxplus(R^*) \in \boxplus(R^*))$. Yet $\boxplus(R^*)$ is not the clean and consistent set comprising the clean members of R^* ; it is bound to have itself as an unclean member.

that is a member of the Ω_2 of **AFS**₂. Yet, **AFS**₂ offers certainty.

³⁴It is possible that a ‘later’ **CLuNs** _{$\mathbf{p}_i$} ^{*m*}-consequence set of **AFS**₁ introduces further *Dab*-formulas, which eventually lead to a choice. I did not find any.

Consider a third example, **AFS**₃. For some presumed consistent set theories **CS**, the set of **CS**-sets, call it S_{CS} , (and the set of **CS**-classes, call it C_{CS}) is definable within **AFS** and the sentences of **CS** may be translated into $\mathcal{L}_S$ sentences. Further $\mathcal{L}_S$ -sets may be needed to define S_{CS} and C_{CS} ; let $\mathfrak{H}$ be the *metatheoretic* set comprising S_{CS} , C_{CS} as well as those other $\mathcal{L}_S$ -sets.

The set of abnormalities of **CS**-sets is defined as follows—drop and adjust references to classes as required by the specific **CS**.

$$\begin{aligned}\Omega_{\text{CS}} = & \{Q!(t_1 = t_2) \mid t_1, t_2 \in S_{\text{CS}} \vee t_1, t_2 \in C_{\text{CS}}\} \cup \\ & \{Q!(t_1 \in t_2) \mid t_1 \in S_{\text{CS}}; t_2 \in S_{\text{CS}} \cup C_{\text{CS}}\} \cup \\ & \{Q(t \in \{x \mid A(x)\} \wedge \sim A(t)) \mid t \in S_{\text{CS}}; \{x \mid A(x)\} \in S_{\text{CS}} \cup C_{\text{CS}}\} \cup \\ & \{Q(t \notin \{x \mid A(x)\} \wedge \sim \neg A(t)) \mid t \in S_{\text{CS}}; \{x \mid A(x)\} \in S_{\text{CS}} \cup C_{\text{CS}}\}\end{aligned}$$

The abnormalities of $\mathfrak{H}$ are defined as follows—adjust for the specific **CS**.

$$\begin{aligned}\Omega_{\mathfrak{H}} = & \{Q!(t_1 = t_2) \mid t_1, t_2 \in \mathfrak{H}\} \cup \\ & \{Q!(t_1 \in t_2) \mid t_1, t_2 \in \mathfrak{H}\} \cup \\ & \{Q(t \in \{x \mid A(x)\} \wedge \sim A(t)) \mid t, \{x \mid A(x)\} \in \mathfrak{H}\} \cup \\ & \{Q(t \notin \{x \mid A(x)\} \wedge \sim \neg A(t)) \mid t, \{x \mid A(x)\} \in \mathfrak{H}\}\end{aligned}$$

Let **AFS**₃ be like **AFS**₁ except that the Ω_i are defined as follows:

$$\begin{aligned}\Omega_1 &= \{Q!(t_1 = t_2) \mid t_1, t_2 \in \mathcal{T}\} \\ \Omega_2 &= \Omega_1 \cup \Omega_{\mathfrak{H}} \\ \Omega_3 &= \Omega_2 \cup \Omega_{\text{CS}} \\ \Omega_{i+4} &= \Omega_{i+3} \cup \Xi_i \quad (i \in \mathbb{N})\end{aligned}$$

The first set in the union of the definiens of Ω_{CS} and the first set in the union of the definiens of $\Omega_{\mathfrak{H}}$ may obviously be dropped in view of the present Ω_1 . While Ω_2 warrants that the members of $\mathfrak{H}$ are consistent and clean, Ω_3 warrants that the members of those members are consistent and clean. The reasons to believe that Ω_2 and Ω_3 provide those warrants coincide with the reasons to believe that **CS** is a consistent set theory.

Fact 12 *If **CS**-sets (and **CS**-classes) form a **AFS**₃-set, then if **CS** is indeed consistent, then **CS** is a consistent and clean subtheory of **AFS**₃.*

What I mean by a subtheory is that any **CS**-theorem A corresponds to an **AFS**₃-theorem $t(A)$; in the simplest case $t(A) = B \supset A$ in which B states that all sets mentioned in A are **CS**-sets.

All **AFS** require study. One approach reorders the Ω_i to establish final provability as illustrated by **AFS**₂. A second approach aims at showing that many operators (unions, intersections, ...) define sets the cleanness of which has no effect on the normality of other sets except for artificial constructions (Curry-like sets, funny normality sets, and presumably more). The operators function as desired if they define sets with a lower degree than the artificial constructions. A third approach capitalizes on presumed consistent set theories **CS** in view of (i) if $\Gamma \vdash_{\text{CLuNs}} A$, then $\Gamma \vdash_{\text{CL}} A$, (ii) if $\Gamma \vdash_{\text{CL}} A$, then there is a finite $\Delta \subset \Omega$ such that $\Gamma \vdash_{\text{CLuNs}} A \vee Dab(\Delta)$, and (iii) if $\Gamma \vdash_{\text{CL}} A$, **CS** is consistent

and certain (not too exotic) conditions are fulfilled by the **AFS**, then $\vdash_{\text{AFS}} A$ whenever $\vdash_{\text{AFS}} B$ for all $B \in \Gamma$. This approach need not be restricted to a single **CS**. A fourth approach introduces one or more S_{CS} as illustrated by **AFS**₃. New problems may arise, especially concerning the translation functions. Different ways for incorporating such theories may have to be devised. If some presumably consistent first order set theories would resist incorporation, this might reveal a potentially fatal restriction on adaptive Fregean set theories. A fifth approach concerns the study of the set of consistent **AFS**-sets. This may be interpreted naively, but one may also delineate a consistent set theory within an **AFS**_{*i*} by devising an as large as possible set S the members of which are consistent and have only members of S as members. This provides ways to devise alternative consistent set theories. That it happens within an **AFS** will definitely have heuristic advantages, but possibly also technical advantages. A sixth approach handles lots of sensible work that goes in the opposite direction. Implicitly, **PFS**-models were described within a classic set theory—clearly **ZFC** will do. Each adaptive logic in the sequential superposition (22) selects a set of **PFS**-models of its premisses and its consequence set comprises the $\mathcal{L}_S$ -formulas true in all selected models. The dynamic proofs of the adaptive logics and of their sequential superposition can also be described in terms of **ZFC**. All this suggests that any **AFS** can be embedded within **ZFC** and other ‘classic’ set theories.

Incidentally, if a $\text{CLuNs}_{\triangleright_i}^m$ of a certain **AFS**_{*j*} would destroy the adaptive gain, it reduces to **CLuNs**. But that is harmless: all $\text{CLuNs}_{\triangleright_i}^m$ in (22) deliver a consequence set that is closed under **CLuNs** as well as transitive.

9 In Conclusion

The main purpose of the present paper was to offer examples of theories that characterize Frege’s notion of a set, taking into account that the notion requires inconsistent sets, and to spell out the method for devising such theories. The five *desiderata* from Section 1 are all fulfilled. *Desideratum* 4 will be better understood at this point in the paper in terms of *Dab*-formulas. A specific problem for adaptive set theories pertained to the adaptive gain and was adequately solved.

In subsequent paragraphs, I shall first comment on adaptive Fregean set theories and next on two implications for the methodology of adaptive logics.

The idea behind my approach is double. (1) Frege’s idea was that $A(t)$ is the *support* of $t \in \{x \mid A(x)\}$ and $\neg A(t)$ the support of $t \notin \{x \mid A(x)\}$. Yet $R \in R$ and $R \notin R$ are proved, in Frege’s original theory and **PFS** alike, without support, viz. from $R \notin R \supset R \in R$ and $\neg R \notin R \supset R \notin R$. This causes inconsistency. More significant are the separate steps: if only one of them applied, one would obtain set membership without support whence an unclean set. That both apply in the case of R hides the lack of support and blurs the insight. The fundamental problem is uncleanness; inconsistency is a side effect. In this respect R^* is significant: R^* is *not* $\sim$ -inconsistent in **PFS** and hence unclean. (2) The second idea pursues the first: the adaptive gain is eaten by abstracting terms that have unsupported members or non-members; such terms cause supported set membership to be involved in *Dab*-formulas of the $\text{CLuNs}_{\triangleright_i}^m$ -theory. The insight led to a complicated ordering of subsets of the Ω from (9). As only the *relation* between touchstones matters, the present

Ξ -sequence arguably offers a drastic but satisfactory simplification.

Inconsistent sets may be thought of as couples of consistent sets. This may be expressed within **AFS**₂ where both members of $\langle \oplus t, \ominus t \rangle$ are arguably consistent sets. In return, for *any* two consistent sets t_1 and t_2 , the couple $\langle t_1, t_2 \rangle$ defines a Fregean set provided $\forall x(x \in t_1 \vee x \in t_2)$ holds. The phrase may be read as a statement on **PFS**-models described in terms of any decent set theory, including an **AFS**, in which case the quantifiers are interpreted with respect to a domain D . It may also be read as an absolute statement in terms of a decent set theory, provided the latter allows for consistent sets t_1 and t_2 for which the condition holds.

Some collections of sets were sometimes said too large to be consistent sets. Yet some **AFS** enable one to handle many of those collections precisely so: U , $\tilde{t}$, $\oplus(R)$, $\ominus(R)$, $\oplus(R^*)$, and so on. Some such sets, for example $\oplus(Z)$, are contingent on the model but nevertheless consistent in all **AFS**-models.

The local classical negation plays a significant role within **PFS** and within the **AFS**. First, many functions that generate consistent sets in the **AFS** do so in view of the presence of the classical negation—Section 3 offers many examples of sets that cannot be defined without a classical negation. Next, the classical negation offers a simple means to express that there is a gain: some $t_1 \neq t_2$ have the force of $\sim t_1 = t_2$ and some $t_1 \notin t_2$ the force of $\sim t_1 \in t_2$.³⁵ Third, the classical negation allows one to express that a set t is consistent: $\sim \exists x(x \in t \wedge x \notin t)$.³⁶ Fourth, the presence of this classical negation does not cause triviality.³⁷

As I extensively explained in Section 4, the language $\mathcal{L}_S$ is such that **CLuNs** distinguishes between several implications that coincide according to **CL** and the same holds even more for equivalences; so one has to make a choice anyway. Yet, some might think to have ‘arguments’ for the claim that the equivalence in **Abs** should be symmetric. They are easily satisfied by moving the asymmetry from the connective to the superposed adaptive logics. Take any of the **AFS** defined in Section 8. Phrase **Abs** in terms of the **LP**-equivalence, $A \Leftrightarrow B$ in the terminology of the present paper, and call the result **Abs** ^{$\Leftrightarrow$} . Renumber the Ω_i to Ω_{i+1} and define Ω_1 as follows.

$$\{Q(A(t) \wedge \sim t \in \{y \mid A(y)\}) \mid A \in \mathcal{F}_S\} \cup \{Q(\neg A(t) \wedge \sim t \notin \{y \mid A(y)\}) \mid A \in \mathcal{F}_S\}$$

The new **AFS** _{i} delivers exactly the same consequence set as the old one because, as the old **PFS** was non-trivial anyway, the new **CLuNs** _{$\triangleright_1^m$} eliminates *all* new Ω_1 -abnormalities. In other words, the new **CLuNs** _{$\triangleright_1^m$} -consequence set of $\{\text{Abs}^{\Leftrightarrow}, \text{Ext}\}$ is identical to the **CLuNs**-closure of $\{\text{Abs}, \text{Ext}\}$ and it is provable that the consequence set assigned by an adaptive logic **AL** to a premise set Γ is identical to the consequence set assigned by **AL** to the closure of Γ by the lower limit logic of **AL** [3, Theorem 15,(3)]. In the present case the **AL** is the old **CLuNs** _{$\triangleright_1^m$} , which is the new **CLuNs** _{$\triangleright_2^m$} . Let us stop the frivolity here and

³⁵There is no *need* for a classical negation in order to have the gain: if, for example, $A \wedge \neg A$ can be upheld to be false, B can be derived from $\neg A$ and $A \vee B$.

³⁶While $\exists x(x \in t \wedge x \notin t)$ definitely expresses that t is inconsistent, $\neg \exists x(x \in t \wedge x \notin t)$, being a **CLuNs**-theorem, offers a slightly too easy way to prove the consistency of t .

³⁷This is realized by having the equivalence of **Abs** non-detachable in the left-right direction. But why would that decrease the merits of the **AFS**? Graham Priest [17] propagates, as the foundational tool for building **LP**-models, a set theory which lacks connectives that are detachable in *any* respect. Also, I have argued, *in tempore non suspecto*, that minimally abnormal interpretations of inconsistent classical theories may be obtained by weakening other logical symbols than the classical negation [1, 7].

let **PFS** and the **AFS** again refer to what they referred to before the present paragraph.³⁸

An important point concerns the methodology of adaptive logics in general. When a classical theory T , which obviously was intended as consistent, turns out inconsistent, no general purpose trick turns T into an adequate minimally abnormal theory. A suitable adaptive approach is only reached by study and analysis of the relevant domain. The present paper illustrates that clearly: the choice of the implications and equivalences in the axioms, the introduction of abnormalities relating to cleanness, and the need for a sequential superposition and more precisely for one that saves the adaptive gain. Handling inconsistency as here discussed is a methodological matter and, solving standard problems aside, interesting methods are content driven or content guided [19].

The present paper clearly suggests that *abnormalities* do not concern the structure of models—the domain D and the sets defined over it—but concern the relation between the language and the models. Abnormalities pertain to the assignment, to the way in which the abstracting terms are mapped on the domain—so to the models in the plain sense: $M = \langle D, v \rangle$. The **CLuNs^m**-models of Γ are not those in which certain inconsistent sets are absent. Suppose that $D = \{l, m, r, u\}$ and that we represent (paraconsistent) sets by the couple of the collection of their members and the collection of their non-members, as in $\langle \{l, m\}, \{r, u\} \rangle$. Then what might it even mean that $\langle \{l, m\}, \{l, r, u\} \rangle$ does not exist in a model? The **CLuNs^m**-models of Γ are the **CLuNs**-models of Γ in which the linguistic entities are mapped to sets in such a way that verified inconsistencies are minimized.

Concluding the paper, I hope to have shown that the adaptive study of Frege’s notion of a set is not only sensible but also promising. It is even promising for studying the variety of consistent set theories. It is correct that **AFS** are complex theories [9, 12, 13, 23] whence establishing the final derivability of certain formulas may be difficult and tentative. In return, there is a non-triviality proof—so a proof of $\sim$ -consistency. So even if obtaining a warrant of theoremhood may be demanding for some formulas and may seem hopeless for some others, one knows that the theory one is spelling out cannot possibly turn out a chimera.

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³⁸Similarly, one may modify **Ext** to $\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \supset x = y)$ and modify Ω_1 to $\{Q(A(t) \wedge \sim t \in \{y \mid A(y)\}) \mid A \in \mathcal{F}_S\} \cup \{Q(\neg A(t) \wedge \sim t \notin \{y \mid A(y)\}) \mid A \in \mathcal{F}_S\} \cup \{Q(\forall z (z \in x \leftrightarrow z \in y) \wedge \sim x = y)\}$. Again the new **CLuNs^m_{s1}**-consequence set of the so modified axioms is identical to the **CLuNs**-closure of $\{\text{Abs}, \text{Ext}\}$. However, $\leftrightarrow$ cannot be replaced by $\Leftrightarrow$ in **Ext**; no adaptive logic can repair the too weak sufficient condition.

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