Reverse mathematics of first-order theories with finitely many models

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Abstract

We examine the reverse-mathematical strength of several theorems in classical and effective model theory concerning first-order theories and their number of models. We prove that, among these, most are equivalent to one of the familiar systems RCA_0, WKL_0, or ACA_0. We are led to a purely model-theoretic statement that implies WKL_0 but refutes ACA_0 over RCA_0.

1 Introduction

Simpson [22, Ch. II.8 and IV.3] laid the foundation for the study of first-order logic from the point of view of reverse mathematics. There he provided suitable definitions of objects such as theories and models in the language of second-order arithmetic, and proved versions of several important theorems, including the Soundness and Completeness Theorems, in the weak axiom system RCA_0. In [22, Ch. IX.4] he began the study of model theory proper by formalizing and proving the existence theorem for recursively saturated models in the system WKL_0. This work was motivated, however, by its applications to metamathematical conservation theorems. Recently, there has been a surge interest in the reverse mathematics of model theory *per se*, and researchers such as Harris, Hirschfeldt, Lange, Shore, and Slaman have undertaken a systematic study using Simpson's framework.

While much of this work has fallen into the familiar pattern of placing lists of theorems in correspondence with one of several known axiom systems—most often one of the Big Five isolated by Friedman [5, 6]—it has also enriched the field by suggesting totally new axiom systems. For example, Hirschfeldt, Shore, and Slaman [13], in studying the classical existence theorem for atomic models, isolated the new reverse-mathematical principles AMT and $\Pi^0_1\mathsf{G}$. Hirschfeldt, Lange, and Shore [12], drawing on work in effective model theory by Goncharov [8] and Peretyat′kin [19], have studied various versions of the classical existence theorem for homogeneous models, finding further connections with AMT and with induction principles

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such as $\mathsf{B}\Sigma^0_2$ and $\mathsf{I}\Sigma^0_2$, and discovering a new hierarchy of principles $\Pi^0_\mathsf{n}\mathsf{GA}$ between $\mathsf{I}\Sigma^0_\mathsf{n}$ and $\mathsf{I}\Sigma^0_{\mathsf{n}+1}$ but incomparable with $\mathsf{B}\Sigma^0_{\mathsf{n}+1}$.

Given the known connections between reverse and effective mathematics (as described in, for example, Friedman, Simpson, and Smith [7]), it should come as no surprise that the reverse-mathematical approach to model theory also has strong connections with effective model theory. On the one hand, many known results and techniques from the effective setting can be formalized in RCA_0. On the other, the fact that many other results *cannot* be formalized in RCA_0 suggests new questions in effective mathematics.

It has typically been the case in effective model theory that when a particular object is being studied its complexity is tightly controlled, while that of other objects varies freely. An example that comes up frequently is the isomorphism relation: two models are *isomorphic* if there is an isomorphism between them. The Turing degree of the isomorphism is not normally considered, unless it is the main object of interest, as in the study of recursive stability or relative categoricity. Because it is unnatural in reverse mathematics to treat a model or theory differently from an isomorphism—all second-order objects obey the same basic set-existence axioms—our approach here must be more uniform. When interpreted in ω-models, our results over RCA_0 can be viewed as correspondingly uniform results in effective mathematics.

In this paper we address, within various subsystems of second-order arithmetic, the following two questions of basic model theory.

Q1. *Under what conditions is a complete theory $\aleph_0$-categorical?*

Q2. *For what finite values n may we have a complete theory with exactly n models up to isomorphism?*

We assume familiarity with reverse mathematics and with model theory. Subsections §1.1 and §1.2 describe some of our less standard notation, and provide a few useful lemmas in reverse mathematics and in model theory, respectively. Subsections §2.1 and §2.2 summarise our answers to the questions Q1 and Q2, respectively. Most of the proofs are deferred to the remainder of the paper, namely §§3–7. Each section among §§3–7 is built around a particular construction or technique, and is split into four parts: first, a brief description of the construction and its goals; second, a subsection giving the construction itself; third, a 'verification' subsection where basic properties are checked (such as completeness and consistency of a particular theory); and, finally, an 'applications' subsection where the construction is used to prove claims from §2.1 and §2.2.

Suitable machinery is introduced and developed as needed, including a WKL_0 version of the Henkin model construction in §5 and an RCA_0 version of the Fraïssé limit construction in §6. Unless otherwise stated, all reasoning is in RCA_0. A theorem's statement may be tagged with the axiom system in which it is being proved, such as RCA_0, ACA_0, or 'Classical' when reasoning in ZFC.

1.1 Notation for reverse mathematics

Most of our reverse-mathematical notation follows Simpson [22]. We use M and $\mathcal{S}$ to denote the first- and second-order parts, respectively, of a model $(M, \mathcal{S}, +_M, \cdot_M, 0_M, 1_M, <_M)$ of RCA_0. We typically assume, without mention, that we are working inside such a model; when we do mention the model we omit the operation symbols, writing simply $(M, \mathcal{S})$. We say that a set $X \in \mathcal{S}$ is *finite* if it has an upper bound in M. We use the symbol $\{0,1\}^{<M}$ or $2^{<M}$ to denote the set of all finite binary strings in $\mathcal{S}$. We use $\mathsf{I}\Sigma^0_1$ to denote the axiom scheme of induction for Σ^0_1 formulas with parameters from M and $\mathcal{S}$. We also use the following notation.

Definition 1.1. Fix a set $Z \in \mathcal{S}$ in a model $(M, \mathcal{S})$ of RCA_0.

(i) Given a sequence of sets $X_0, \ldots, X_{n-1} \in \mathcal{S}$, where $n \in M$ may be nonstandard, we define the *coded tuple* $\langle X_0, \ldots, X_{n-1} \rangle$ as the predicate:

$$\langle X_0, \ldots, X_{n-1} \rangle(\langle i, k \rangle) \iff k \in X_i.$$

Given a sequence of sets $X_0, X_1, \ldots \in \mathcal{S}$ with indices ranging over all of M, we define the *coded sequence* $\langle X_0, X_1, \ldots \rangle$ similarly:

$$\langle X_0, X_1, \ldots \rangle(\langle i, k \rangle) \iff k \in X_i.$$

We sometimes treat coded tuples and coded sequences as sets, for example by writing $\langle i, k \rangle \in \langle X_0, X_1, \ldots \rangle$. Depending on how the sets X_i are presented, a coded tuple or coded sequence may or may not to be an element of $\mathcal{S}$. In this paper, we usually point out when it is.

(ii) Given a set $Z \in \mathcal{S}$ and a number $s \in M$, let $K_s^Z = \{e < s : \Phi_{e,s}^Z(e) \text{ converges}\}$, where Φ_e is the e-th Turing functional. The *Turing jump enumeration for Z* is the coded sequence $\langle K_0^Z, K_1^Z, \ldots \rangle$. Note that the Turing jump enumeration exists in $\mathcal{S}$ by Δ_1^0 comprehension. We let $K_{at\ s}^Z$ denote the set difference $K_s^Z - K_{s-1}^Z$.

(iii) The *Turing jump of Z*, written K^Z, is the Σ_1^0 predicate

$$K^Z(n) \iff (\exists s)[n \in K_s^Z].$$

We often write $n \in K^Z$ to mean $K^Z(n)$.

The following lemma shows how the Turing jump fits into reverse mathematics.

Lemma 1.2 ($\mathsf{RCA_0}$)**.** *Let $(M, \mathcal{S})$ be a model of $\mathsf{RCA_0}$. Then $(M, \mathcal{S})$ is a model of $\mathsf{ACA_0}$ if and only if K^Z is an element of $\mathcal{S}$ for every $Z \in \mathcal{S}$.*

Proof. See Simpson [22, Ex. VIII.1.12]. □

Lemma 1.2 allows us to obtain reversals from a principle P to $\mathsf{ACA_0}$ by coding $\langle K_0^Z, K_1^Z, \ldots \rangle$ into an object and arguing that, if P holds, then we can use Δ_1^0 comprehension to recover K^Z. We use this method frequently, for example, in the proofs of Proposition 4.5 and Proposition 6.11.

1.2 Background and notation for model theory

All definitions are in the language of second-order arithmetic. Our definitions for basic model-theoretic terms such as *language*, *formula*, *sentence*, *structure*, *model*, *consistent*, and *satisfiable* are mostly as given in Simpson [22, Ch. II.8] and in Hirschfeldt, Lange, and Shore [12]. All structures have countably infinite domain unless otherwise specified. Given a language L, an *L-theory* is any set of L-sentences. A *complete L-theory* is a theory containing either ϕ or $\neg\phi$ for every L-sentence ϕ. Two structures $\mathcal{A}$ and $\mathcal{B}$ are *isomorphic* if there is an isomorphism between them. When we are working in a model $(M, \mathcal{S})$ of $\mathsf{RCA_0}$, the isomorphism must be an element of $\mathcal{S}$. A theory is *$\aleph_0$-categorical* if all of its models are isomorphic.

We shall need the following theorem.

Theorem 1.3 ($\mathsf{RCA_0}$. Weak Completeness Theorem)**.** *Every deductively-closed consistent theory is satisfiable. In particular, every complete consistent theory is satisfiable, and every deductively-closed consistent theory can be extended to a complete consistent theory.*

Originally due to Gödel, the Weak Completeness Theorem 1.3 was formalized in effective mathematics by Morley and translated to reverse mathematics by Simpson [22, Thm II.8.4]. *Weak* is in the name to contrast this with the stronger statement, not provable in RCA_0, which does not include *deductively-closed* as a hypothesis:

Theorem 1.4. *The statement, 'Every consistent theory is satisfiable' is equivalent to* WKL_0 *over* RCA_0.

Proof. See Simpson [22, Thm IV.3.3]. □

One of the Weak Completeness Theorem's immediate consequences is the following theorem of Łos and Vaught.

Theorem 1.5. *(i) (Classical. Łos, Vaught.) If T is an L-theory with only one countable model, then for every L-sentence ϕ, either $T \vdash \phi$ or $T \vdash \neg\phi$.*

(ii) (RCA_0.) *Every deductively-closed theory with exactly one model up to isomorphism is complete.*

(iii) The statement of part (i) is equivalent to WKL_0 *over* RCA_0.

Proof. A proof of part (i) can be found in standard texts such as Marker [14]. Part (ii) and the forward direction of part (iii) are implicit in the proof given in Simpson [22, Ch. II.8] of the Weak Completeness Theorem 1.3.

For the reverse direction of (iii), assume that $\neg\mathsf{WKL}_0$ holds. By Theorem 1.4, there is a language L_0 and a consistent L_0-theory T_0 with no models. We may assume L_0 is a relational language. Let $L_1 = \{\leq\}$ be the language of partial orders, and let T_1 be the theory of dense linear orders without endpoints, which is $\aleph_0$-categorical in RCA_0. Define a new language $L = L_0 \cup L_1 \cup \{R\}$, where R is a new 0-ary relation, and an L-theory T by:

$$T = \{\neg R \to \phi : \phi \in T_0\} \cup \{\neg R \to \text{all relations in } L_1 \text{ are empty}\}$$
$$\cup \{R \to \phi : \phi \in T_1\} \cup \{R \to \text{all relations in } L_0 \text{ are empty}\}$$

This T has exactly one model, but neither proves nor refutes the sentence R. □

Thus, in the system WKL_0, if we wish to show that a theory is complete, it is enough to construct a model and show that it is unique up to isomorphism. This is, in general, not enough in the weaker system RCA_0. Instead, we use a suitably effective notion of quantifier elimination.

Definition 1.6. (i) We say a theory T *has quantifier elimination* if, for every L-formula $\phi(\bar{x})$, there is a quantifier-free L-formula $\psi(\bar{x})$—possibly one of the formal logical symbols Tr or Fa—such that $T \vdash \phi(\bar{x}) \leftrightarrow \psi(\bar{x})$.

(ii) We say a theory T *has effective quantifier elimination* if there is a function which takes as input any L-formula $\phi(\bar{x})$ and returns an L-formula $\psi(\bar{x})$—possibly Tr or Fa—such that $T \vdash \phi(\bar{x}) \leftrightarrow \psi(\bar{x})$.

Any theory with effective quantifier elimination has quantifier elimination, and, in a relational language, any theory with quantifier elimination is complete. The following lemma, used in the work of Hirschfeldt, Shore, and Slaman [13], is our main tool for proving completeness of a theory.

Lemma 1.7 (RCA_0)**.** *Suppose T is a theory and there is a function which takes as input an L-formula $\theta(\bar{x}, y)$ which is a conjunction of literals and returns a quantifier-free L-formula $\psi(\bar{x})$ such that $T \vdash (\exists y)\theta(\bar{x}, y) \leftrightarrow \psi(\bar{x})$. Then T has effective quantifier elimination.*

Proof. Suppose such a function f exists, and fix any L-formula $\phi(\bar{x})$. We show how to produce a ψ such that $T \vdash \phi(\bar{x}) \leftrightarrow \psi(\bar{x})$. Suppose first that $\phi(\bar{x})$ is of the form $(\exists y)\theta(\bar{x}, y)$, where θ is quantifier-free. The usual proof of De Morgan's laws may be carried out in RCA_0, so we may assume that θ is in disjunctive normal form, say $\theta_0(\bar{x}, y) \vee \cdots \vee \theta_{n-1}(\bar{x}, y)$. Since RCA_0 is also strong enough to prove the distributivity of $\exists$ over $\vee$, we have $T \vdash \phi(\bar{x}) \leftrightarrow (\exists y)\theta_0(\bar{x}, y) \vee \cdots \vee (\exists y)\theta_{n-1}(\bar{x}, y)$. We may now use the provided function f to to find quantifier-free formulas $\psi_0(\bar{x}), \ldots, \psi_{n-1}(\bar{x})$ such that $T \vdash (\exists y)\theta_i(\bar{x}, y) \leftrightarrow \psi_i(\bar{x})$ for all $i < n$. Then $T \vdash \phi(\bar{x}) \leftrightarrow \psi_0(\bar{x}) \vee \cdots \vee \psi_{n-1}(\bar{x})$, so $\psi_0 \vee \cdots \vee \psi_{n-1}$ is the desired ψ.

Now suppose that $\phi(\bar{x})$ is a formula of arbitrary quantifier depth $n > 0$. Using the above procedure on the deepest quantifiers of ϕ, we can find a formula which is provably equivalent to ϕ and has quantifier depth $n - 1$. Iterate this procedure using Δ^0_1 recursion to get a quantifier-free ψ such that $T \vdash \phi(\bar{x}) \leftrightarrow \psi(\bar{x})$. □

The following definitions are of central importance to the study of $\aleph_0$-categorical theories.

Definition 1.8. Fix a natural number n, a language L, and a complete, consistent L-theory T.

(i) An *n-type of T* is a set $p(x_0, \ldots, x_{n-1})$ of formulas in variables taken from $\{x_0, \ldots, x_{n-1}\}$ such that $T \subseteq p(x_0, \ldots, x_{n-1})$ and, if $c_0, \ldots, c_{n-1}$ are new constants not in L, then the set

$$\{\phi(c_{i_0}, \ldots, c_{i_{k-1}}) : \phi(x_{i_0}, \ldots, x_{i_{k-1}}) \in p(x_0, \ldots, x_{n-1})\}$$

is a complete, consistent $L \cup \{c_0, \ldots, c_n\}$-theory. We sometimes abbreviate $p(x_0, \ldots, x_{n-1})$ to $p(\bar{x})$, or just p. We often omit n and call $p(\bar{x})$ simply a *type*.

(ii) A type $p(\bar{x})$ of T is *principal* if there is a formula $\phi(\bar{x}) \in p(\bar{x})$ such that $T \vdash \phi(\bar{x}) \rightarrow \psi(\bar{x})$ for all $\psi(\bar{x}) \in p(\bar{x})$. Otherwise, $p(\bar{x})$ is *nonprincipal*.

(iii) Suppose that $\mathcal{A}$ is a model of T and $p(\bar{x})$ is a type. We say that $\mathcal{A}$ *realizes* $p(\bar{x})$ if there is a tuple $\bar{a}$ from its domain such that $\mathcal{A} \models \phi(\bar{a})$ for every $\phi(\bar{x}) \in p(\bar{x})$. Otherwise, we say that $\mathcal{A}$ *omits* $p(\bar{x})$.

An RCA_0 version of the classical Type Omitting Theorem can be proved by an easy Henkin-style construction.

Theorem 1.9 (Classical and RCA_0. Type Omitting Theorem)**.** *Let T be a complete theory and $p(\bar{x})$ a nonprincipal type. There is a model of T that omits $p(\bar{x})$.*

Proof. See Harizanov [9, Theorem 6.1]. □

Much more intricate type-omitting theorems can be found in the work of Millar [17] in effective mathematics. Some of these have been studied in reverse mathematics by Hirschfeldt, Shore, and Slaman [13].

2 Summary of results

The main results of this paper fall into two classes, listed separately in §2.1 and §2.2. Section §2.1 deals with a theorem of Ryll-Nardzewski, Engeler, and Svenonius about $\aleph_0$-categorical theories and their n-types. Section §2.2 deals with theorems about theories, not necessarily $\aleph_0$-categorical, that have only finitely many models.[1]

[1]These are sometimes called *Ehrenfeucht theories*.

2.1 Reverse mathematics and $\aleph_0$-categorical theories

Recall our first question:

Q1. *Under what conditions is a complete theory T $\aleph_0$-categorical?*

In the classical setting, Engeler [4], Ryll-Nardzewski [20], and Svenonius [23] independently discovered a number of properties characterising $\aleph_0$-categorical theories. Many such properties are now known. We focus on the following five:

Theorem 2.1 (Classical. Engeler; Ryll-Nardzewski; Svenonius)**.** *Let T be a complete, consistent theory, and let M denote the true natural numbers ω. The following are equivalent:*

(S1) There is a function $f : M \to M$ such that, for all $n \in M$, T has exactly $f(n)$ distinct n-types.

(S2) There is a function $f : M \to M$ such that, for all $n \in M$, T has no more than $f(n)$ distinct n-types.

(S3) T has only finitely many n-types, for each $n \in M$.

(S4) T is $\aleph_0$-categorical.

(S5) All types of T are principal.

Our approach to the question Q1 is to explore the reverse-mathematical strength of Theorem 2.1, allowing nonstandard M. In other words, we replace Q1 with the more specific question:

Q1′. *What is the strength over* RCA_0 *of each implication* $(\mathsf{S}i \to \mathsf{S}j)$*?*

It is simple to check that the classical proofs of equivalence for principles (S1)–(S5), as found in standard texts such as Marker [14], all work in ACA_0. Over RCA_0, each implication therefore lies somewhere between RCA_0 and ACA_0.

The following table summarizes our results. Each implication $(\mathsf{S}i \to \mathsf{S}j)$ is equivalent to the principle named in the cell in row $(\mathsf{S}i)$ and column $(\mathsf{S}j)$; tautologies of the form $(\mathsf{S}i \to \mathsf{S}i)$ are greyed out; and any other blank cell means 'unknown'. Each of these equivalences is justified in one of Theorem 2.2, Theorem 2.3, and Theorem 2.4 below.

	(S1)	(S2)	(S3)	(S4)	(S5)
(S1)		RCA_0	RCA_0	RCA_0	RCA_0
(S2)	ACA_0		RCA_0	ACA_0	RCA_0
(S3)	ACA_0	ACA_0		ACA_0	RCA_0
(S4)			WKL_0		RCA_0
(S5)	ACA_0	ACA_0	ACA_0	ACA_0	

We begin by isolating, in Theorem 2.2, the implications that require a detailed proof, indicating in each case where in this paper the proof can be found. We then list, in Theorem 2.3, several implications that are easily provable in RCA_0, giving in each case a short argument or reference. All other implications in the table follow by composing implications from Theorems 2.2 and 2.3, as outlined in the proof of Theorem 2.4.

Theorem 2.2. *(i)* $\mathsf{RCA}_0 \vdash (S2 \to S1) \to \mathsf{ACA}_0$. *(Proposition 6.11)*

(ii) $\mathsf{RCA}_0 \vdash (S2 \to S4) \to \mathsf{ACA}_0$. *(Corollary 6.14)*

(iii) $\mathsf{RCA}_0 \vdash (S3 \to S2) \to \mathsf{ACA}_0$. *(Proposition 6.12)*

(iv) $\mathsf{RCA}_0 \vdash (S5 \to S3) \to \mathsf{ACA}_0$. *(Proposition 4.5)*

(v) $\mathsf{RCA}_0 \vdash (S4 \to S3) \leftrightarrow \mathsf{WKL}_0$. *(Propositions 3.5 and 5.6)*

(vi) $\mathsf{RCA}_0 \vdash (\mathit{S5} \to \mathit{S4}) \to \mathsf{ACA}_0$. *(Proposition 4.6)*

□

Theorem 2.3. *(i)* $\mathsf{RCA}_0 \vdash (\mathit{S1} \to \mathit{S2})$

(ii) $\mathsf{RCA}_0 \vdash (\mathit{S2} \to \mathit{S3})$

(iii) $\mathsf{RCA}_0 \vdash (\mathit{S3} \to \mathit{S5})$

(iv) $\mathsf{RCA}_0 \vdash (\mathit{S1} \to \mathit{S4})$

(v) $\mathsf{RCA}_0 \vdash (\mathit{S4} \to \mathit{S5})$

Proof. (i) By definition.

(ii) By definition.

(iii) We prove the contrapositive. Suppose that T has a nonprincipal n-type $p = \{\psi_0(\bar{x}), \psi_1(\bar{x}), \ldots\}$. Then there are infinitely many $m \in M$ such that the formula

$$\theta_m = \bigwedge_{i<m} \psi_i \wedge \neg\psi_m$$

is consistent with T. These θ_m can be extended uniformly to an infinite coded sequence of distinct n-types.

(iv) Suppose that the property (S1) holds of T, and we are given two models $\mathcal{A} \models T$ and $\mathcal{B} \models T$. We can construct an isomorphism $f : \mathcal{A} \to \mathcal{B}$ by an effective version of the usual back-and-forth argument. For an example of an effective back-and-forth argument, see the proof of Lemma 3.5 below.

(v) We prove the contrapositive. Suppose that T has a nonprincipal type p. By the Weak Completeness Theorem 1.3, there is a model $\mathcal{A}$ of T realizing p; and by the Type Omitting Theorem 1.9, there is a model $\mathcal{B}$ that does not realize p. These $\mathcal{A}$ and $\mathcal{B}$ cannot be isomorphic, so T is not $\aleph_0$-categorical.

□

Theorem 2.4. *All equivalences listed in the table are correct.*

Proof sketch. We have already proved many of these equivalences in Theorems 2.2 and 2.3. All others can be deduced from these. For example, we can see that $(\mathsf{S1} \to \mathsf{S5})$ holds in RCA_0 by combining parts (i), (ii), and (iii) of Theorem 2.3:

$$\mathsf{RCA}_0 \vdash (\mathsf{S1} \to \mathsf{S2}) \wedge (\mathsf{S2} \to \mathsf{S3}) \wedge (\mathsf{S3} \to \mathsf{S5})$$

and applying the rules of propositional logic. On the other hand, we can see that $(\mathsf{S5} \to \mathsf{S1})$ implies ACA_0 over RCA_0 by combining parts (i) and (ii) of Theorem 2.3 with part (iv) of Theorem 2.2:

$$\mathsf{RCA}_0 \vdash (\mathsf{S1} \to \mathsf{S2}) \wedge (\mathsf{S2} \to \mathsf{S3}) \wedge ((\mathsf{S5} \to \mathsf{S3}) \to \mathsf{ACA}_0).$$

The remaining directions are similar. □

We can also combine parts of Theorems 2.2 and 2.3 to show that the two remaining directions, $(\mathsf{S4} \to \mathsf{S1})$ and $(\mathsf{S4} \to \mathsf{S2})$, each imply WKL_0 over RCA_0. Hence their strength over RCA_0 lies somewhere between WKL_0 and ACA_0. The question of their precise strength remains open.

Question 2.5. What is the strength over RCA_0 of $(\mathsf{S4} \to \mathsf{S1})$ and $(\mathsf{S4} \to \mathsf{S2})$?

There are other statements besides (S1)–(S5) which are commonly given as pieces of the Ryll-Nardzewski theorem. Here we list a few statements that are provably equivalent, in RCA_0, to one of (S1)–(S5). Some of these will be useful in the work that follows.

(S3′) For each n there is a number k such that any set $\{\phi_0, \ldots, \phi_k\}$ of n-ary formulas contains a pair ϕ_i, ϕ_j, $i \neq j$, such that $T \vdash \phi_i \leftrightarrow \phi_j$.

(S5′) Every model of T is *atomic*, i.e., realizes only principal types.

(S5″) There is an atomic model of T realizing all types of T.

Theorem 2.6. *(i)* RCA_0 *proves that a complete theory T has only finitely many n-types if and only if there is a number k such that any set $\{\phi_0, \ldots, \phi_k\}$ of n-ary formulas contains a pair ϕ_i, ϕ_j, $i \neq j$, such that $T \vdash \phi_i \leftrightarrow \phi_j$. In particular,* $\mathsf{RCA}_0 \vdash (S3 \leftrightarrow S3')$.

(ii) $\mathsf{RCA}_0 \vdash (S5 \leftrightarrow S5')$ *and* $\mathsf{RCA}_0 \vdash (S5 \leftrightarrow S5'')$.

□

2.2 Reverse mathematics and theories with finitely many models

Recall our second question of basic model theory:

Q2. *For what finite values n may we have a complete theory with exactly n models up to isomorphism?*

In the classical setting, this question was settled by work of Ehrenfeucht and work of Vaught. Ehrenfeucht's idea was to add to a linear order a sequence of constant symbols that together give a small number of nonprincipal types, which can either be realized or omitted to give a certain number of nonisomorphic models. This can be carried out in ACA_0.

Theorem 2.7 (Classical and ACA_0. Ehrenfeucht)**.** *For every $n \geq 3$, there is a complete theory T with exactly n models up to isomorphism.*

Proof. See Chang and Keisler [1, Ex. 2.3.16]. □

Vaught's idea was, given a complete theory T which is not $\aleph_0$-categorical, to use the nonprincipal type guaranteed by the Ryll-Nardzewski Theorem 2.1 to show that T has at least three models. This can also be carried out in ACA_0:

Theorem 2.8 (Classical and ACA_0. Vaught)**.** *There is no complete theory with exactly two models up to isomorphism.*

Proof. See Chang and Keisler [1, Thm 2.3.15]. □

Since RCA_0 is enough to prove the Weak Completeness Theorem 1.3 and to prove that some theories are $\aleph_0$-categorical—for instance, the theory of dense linear orders without endpoints—we now have a full answer to Q2 over ACA_0:

Corollary 2.9 (Classical and ACA_0)**.** *Fix $n \geq 1$. There is a complete theory T with exactly n models up to isomorphism if and only if $n = 1$ or $n \geq 3$.*

It is not immediately clear whether Ehrenfeucht's and Vaught's constructions should work in systems weaker than ACA_0. In §7 below, we get a different answer to Q2 in the system $\mathsf{RCA}_0 + \neg\mathsf{WKL}_0$ by adapting a construction of Millar [15] from effective mathematics. Millar's idea was to define a complete decidable theory T with a recursive nonprincipal 1-type $p(x)$ such that there is exactly one decidable model omitting p and exactly $n-1$ decidable models realizing p, both up to classical and up to recursive isomorphism. This construction can be carried out assuming the failure of Weak König's Lemma:

Theorem 2.10 ($\mathsf{RCA_0} + \neg\mathsf{WKL_0}$)**.** *For every $n \geq 1$, there is a complete theory with exactly n models up to isomorphism.*

Proof. See §7.3 below. □

Corollary 2.11. *(i) $\neg\mathsf{WKL_0}$ implies the statement of Ehrenfeucht's Theorem 2.7 over $\mathsf{RCA_0}$.*

(ii) The statement of Vaught's Theorem 2.8 implies $\mathsf{WKL_0}$ over $\mathsf{RCA_0}$.

It remains to answer Q2 in the system $\mathsf{WKL_0} + \neg\mathsf{ACA_0}$. A reasonable first step is to ask whether the proofs of Corollary 2.9 or Theorem 2.10 can be carried out in this system. The work in §5 below gives the following:

Theorem 2.12. *Over $\mathsf{RCA_0}$, the following are equivalent:*

(i) $(\neg\mathsf{WKL_0}) \vee \mathsf{ACA_0}$

(ii) There is a complete theory with a nonprincipal type and only finitely many models up to isomorphism.

(iii) There is a complete theory with infinitely many n-types, for some n, and with only finitely many models up to isomorphism.

Proof. The direction $(i \to ii)$ follows from the use of a nonprincipal type in the proofs of Theorem 2.7 and Theorem 2.10 in the systems $\mathsf{ACA_0}$ and $\mathsf{RCA_0} + \neg\mathsf{WKL_0}$, respectively. The direction $(ii \to iii)$ is immediate. The final direction $(iii \to i)$ follow from Proposition 5.7 below. □

Although Theorem 2.12 is interesting in itself—it is the first example of a natural-seeming statement equivalent to $(\neg\mathsf{WKL_0}) \vee \mathsf{ACA_0}$ or, in its negation, to $\mathsf{WKL_0} + \neg\mathsf{ACA_0}$—it is a serious obstacle if we want a full answer to Q2 over $\mathsf{RCA_0}$. Since the constructions of Ehrenfeucht, Vaught, and Millar each require a nonprincipal type, Theorem 2.12 tells us none of them can be used in the system $\mathsf{WKL_0} + \neg\mathsf{ACA_0}$. Beyond this, we know very little about the case of $\mathsf{WKL_0} + \neg\mathsf{ACA_0}$.

Question 2.13. Fix a model $(M, \mathcal{S})$ of $\mathsf{WKL_0} + \neg\mathsf{ACA_0}$. Is there a complete theory $T \in \mathcal{S}$ with a finite number $n \in M$, $n \geq 2$ of models? If so, what values of n are possible?

3 Coding an extendable binary tree as a theory.

Our first and most straightforward technique is one that has seen heavy use in effective mathematics, and has already been used in reverse mathematics by Hirschfeldt, Shore, and Slaman [13] and by Harris [10]. The earliest published use appears to be Ehrenfeucht [3].

Recall that we are working within a model $(M, \mathcal{S})$ of $\mathsf{RCA_0}$, and that $2^{<M}$ denotes the set of all finite binary strings. We say that a binary tree $\mathcal{T} \subseteq 2^{<M}$ is *extendable* if, for every $\sigma \in \mathcal{T}$, at least one of $\sigma^\frown 0, \sigma^\frown 1$ is in $\mathcal{T}$. (Here the $^\frown$ symbol denotes concatenation.) Fix an extendable binary tree $\mathcal{T}$, and let $L = (U_i)_{i \in M}$ be a relational language with each U_i unary. In §3.1 below we describe a complete L-theory T with the property that, for each $\sigma \in 2^{<M}$,

$$\begin{aligned}\sigma \text{ is in } \mathcal{T} \quad & \text{if and only if } T \vdash (\exists x)\left[\bigwedge_{\substack{i<|\sigma| \\ \sigma(i)=0}} \neg U_i(x) \wedge \bigwedge_{\substack{j<|\sigma| \\ \sigma(j)=1}} U_j(x)\right] \\ & \text{if and only if } T \vdash (\exists^{\geq n} x)\left[\bigwedge_{\substack{i<|\sigma| \\ \sigma(i)=0}} \neg U_i(x) \wedge \bigwedge_{\substack{j<|\sigma| \\ \sigma(j)=1}} U_j(x)\right] \text{ for all } n.\end{aligned}$$

The theory T also has quantifier elimination, so its 1-types are determined entirely by literals of the form $U_i(x)$ and $\neg U_i(x)$. This gives a natural correspondence between the 1-types of T and the paths in $\mathcal{T}$, and between the n-types of T and the coded tuples of paths in $\mathcal{T}$.

We give the full construction in §3.1, some basic verification in §3.2, and a direct application in §3.3. Further applications are obtained in §4, where we examine a specific instance of this construction.

3.1 Construction.

Let $L = (U_i)_{i\in M}$ be a relational language with every U_i unary. Fix an extendable tree $\mathcal{T}$. (*Extendable* is defined at the beginning of this section.) Consider the following axiom schemes:

$$\text{Ax I.}\ (\exists^{\geq n} x)\left[\bigwedge_{\substack{i<|\sigma|\\ \sigma(i)=0}} \neg U_i(x) \wedge \bigwedge_{\substack{j<|\sigma|\\ \sigma(j)=1}} U_j(x)\right] \text{ for every } n \in M \text{ and every } \sigma \in \mathcal{T}.$$

$$\text{Ax II.}\ \neg(\exists x)\left[\bigwedge_{\substack{i<|\sigma|\\ \sigma(i)=0}} \neg U_i(x) \wedge \bigwedge_{\substack{j<|\sigma|\\ \sigma(j)=1}} U_j(x)\right] \text{ for every } \sigma \notin \mathcal{T}.$$

Let T^* be the collection of all sentences in Ax I and II, and let T be the deductive closure of T^*. This completes the construction. Although T^* is clearly in the second-order part of $(M, \mathcal{S})$ by Δ^0_1 comprehension, it is not immediately evident that T is in $\mathcal{S}$. One of our first tasks in the next subsection is to prove that it is.

3.2 Verification.

Here we list some important properties of T, such as its existence, completeness, and consistency. The analogous situation in effective mathematics is described in Harizanov [9, Section 7]. Unfortunately, we cannot rely on the proofs there, since in RCA_0 we do not have access to tools such as strong forms of the Completeness Theorem. Instead we give longer, elementary proofs.

Lemma 3.1 (RCA_0)**.** *T^* has effective quantifier elimination.*

Proof. Fix a quantifier-free L-formula $\phi(\bar{x}, y)$ which is a conjunction of literals. It suffices by Lemma 1.7 to show an effective procedure producing a quantifier-free ψ such that $T \vdash \psi \leftrightarrow (\exists y)\phi(\bar{x}, y)$. By identifying and renaming variables if necessary, we may assume that no conjunct in ϕ is of the form $y = x_i$ or $x_i = y$.

Check whether there is a $\sigma \in \mathcal{T}$ such that $|\sigma| \geq i$ and $\sigma(i) = 0$ whenever $\neg U_i(y)$ is a conjunct in ϕ, and $|\sigma| \geq i$ and $\sigma(i) = 1$ whenever $U_i(y)$ is in ϕ. If there is no such σ, then ϕ contradicts Ax II, so we may let ψ be the formal logical symbol Fa.

Now suppose there is such a σ, and let ψ be the formula obtained from ϕ by replacing each conjunct mentioning y with the propositional symbol Tr. Clearly $T^* \vdash (\exists y)\phi(\bar{x}, y) \to \psi(\bar{x})$. We wish to show the converse. Fix $n = |\bar{x}| + 1$. The following is a version of the Pigeonhole Principle, and is easily seen to be a tautology:

$$\left(\psi(\bar{x}) \wedge \bigwedge_{k<\ell<n} y_k \neq y_\ell\right) \to \left(\psi(\bar{x}) \wedge \bigvee_{k<n} \bigwedge_{i<n-1} y_k \neq x_i\right).$$

As ϕ has no conjunct of the form $y = x_i$ or $x_i = y$, we deduce a second tautology:

$$\left(\psi(\bar{x}) \wedge \bigwedge_{k<n}\left(\bigwedge_{\ell\neq k} y_k \neq y_\ell \wedge \bigwedge_{\substack{i<|\sigma|\\ \sigma(i)=0}} \neg U_i(y_k) \wedge \bigwedge_{\substack{j<|\sigma|\\ \sigma(j)=1}} U_j(y_k)\right)\right) \to \bigvee_{k<n} \phi(\bar{x}, y_k).$$

This statement, together with the instance of Ax I which uses the n and σ specified above, gives $T^* \vdash \psi(\bar{x}) \to (\exists y)\phi(\bar{x}, y)$. □

Proposition 3.2 (RCA_0). *(i) For every L-sentence ϕ, either ϕ is provable from T^*, or $\neg\phi$ is provable from T^*.*

(ii) T is an element of $\mathcal{S}$.

(iii) T is a complete theory. T has quantifier elimination.

Proof. (i) Given an L-sentence ϕ, use the procedure from Lemma 3.1 to produce a quantifier-free ψ such that $T^* \vdash \phi \leftrightarrow \psi$. Since L is relational, ψ is a propositional combination of Tr and Fa, and hence provably equivalent either to Tr or to Fa. If Tr, then ϕ is in T; if Fa, then $\neg\phi$ is in T.

(ii) If T contains a contradiction, that is, a pair of sentences of the form ϕ and $\neg\phi$, then T is the set of all L-sentences, which is certainly in $\mathcal{S}$. Otherwise, by part (i), T contains exactly one of each pair $\{\phi, \neg\phi\}$: we can effectively decide which by searching for the shortest proof of either $T^* \vdash \phi$ or $T^* \vdash \neg\phi$.

(iii) Completeness of T follows from part (i). Quantifier elimination is inherited from T^*. □

Lemma 3.3 (RCA_0)**.** *T is consistent.*

Proof. We build a model $\mathcal{A} \models T$ with domain $\{a_0, a_1, \ldots\}$, beginning with its quantifier-free diagram. For each $i, k \in M$, let $R_k(a_i)$ hold in $\mathcal{A}$ if and only if $\mathsf{left}(\sigma_i)(k) = 1$, where $\mathsf{left}(\sigma_i)$ is the path in $\mathcal{T}$ extending σ which is leftmost with respect to the ordering $0 < 1$. Recursively extend to a full quantifier-free diagram by adding formulas of the form $\neg\phi$ and $\phi \wedge \psi$, in the usual way. It is straightforward to check that this diagram satisfies every axiom in T^*. (Here we are using the usual truth-functional semantics, as given in Simpson [22, Ch. II.8].)

Now we extend to a complete diagram for $\mathcal{A}$. Fix any $\phi(\bar{a})$, where ϕ is a formula and $\bar{a}$ is a tuple of elements. We must decide whether to place $\phi(\bar{a})$ into the diagram of $\mathcal{A}$. By iterating the effective construction of Proposition 3.1, obtain a quantifier-free ψ such that $T^* \vdash \psi \leftrightarrow \phi$. Add $\phi(\bar{a})$ if and only if $\psi(\bar{a})$ is in the quantifier-free diagram. We claim that this process yields a complete, consistent diagram. For a contradiction, suppose that it does not. Then there is a formula $\phi(\bar{a})$ which fails to have one of the following properties:

- If $\phi(\bar{a}) = \neg\theta(\bar{a})$, then ϕ is in the diagram iff $\theta(\bar{a})$ is not in the diagram.
- If $\phi(\bar{a}) = \theta_0(\bar{a}) \wedge \theta_1(\bar{a})$, then $\phi(\bar{a})$ is in the diagram iff both $\theta_0(\bar{a})$ and $\theta_1(\bar{a})$ are in the diagram.
- If $\phi(\bar{a}) = (\forall x)\theta(\bar{a}, x)$, then $\phi(\bar{a})$ is in the diagram iff $\theta(\bar{a}, a_i)$ is in the diagram for every a_i.

But this is impossible by $\mathsf{I}\Sigma^0_1$ and the proof of Proposition 3.1. □

Lemma 3.4 (RCA_0)**.** *(i) The 1-types of T correspond to paths in $\mathcal{T}$ in the following manner. If $p(x)$ is a 1-type of T, define a function $f_p : M \to \{0,1\}$ by $f_p(n) = 1 \iff U_n(x) \in p(x)$. The function f_p is a path in $\mathcal{T}$, and for every path f in $\mathcal{T}$, there is a unique 1-type $p(x)$ such that $f = f_p$.*

(ii) An n-type $p(x_0, \ldots, x_{n-1})$ is uniquely determined by the 1-types induced on its entries. In particular, the correspondence from (i) can be extended to a correspondence between n-types and coded n-tuples $\langle f_0, \ldots, f_{n-1}\rangle$ of paths in $\mathcal{T}$.

Proof. (i) By construction and the fact that T has quantifier elimination.

(ii) By construction, since the language L consists only of unary relations. □

3.3 Applications.

Recall from §2.1 the statements:

(S3) T has only finitely many n-types, for each n.

(S4) T is $\aleph_0$-categorical.

The construction given in §3.1 is enough to show one direction of Theorem 2.2(v):

Proposition 3.5. *Over* RCA_0*, the implication* ($S4 \to S3$) *implies* WKL_0.

Proof. We prove the contrapositive statement that, if WKL_0 fails, there is a theory T satisfying (S4) but not (S3). Let $\mathcal{T}_0$ be an infinite binary tree with no infinite path. Let $\langle \sigma_0, \sigma_1, \ldots \rangle$ be a one-to-one enumeration of all terminal nodes in $\mathcal{T}_0$. Define a second tree $\mathcal{T}$ by

$$\mathcal{T} = \mathcal{T}_0 \cup \{\sigma_i{}^\frown 0^j : i, j \in M\}.$$

Then $\mathcal{T}$ is an extendable tree. (*Extendable* is defined at the beginning of §3.) Let T be the theory obtained from $\mathcal{T}$ using the construction of §3.1. By Lemma 3.4, each path in $\mathcal{T}$ corresponds to a unique 1-type of T. Since $\mathcal{T}$ has infinitely many paths, T has infinitely many distinct 1-types, and so does not satisfy (S3).

On the other hand, each 1-type p of T corresponds to a path f_p in $\mathcal{T}$ of the form $f_p = \sigma_i{}^\frown 0^M$ for some terminal node σ_i of $\mathcal{T}_0$. This σ_i, in turn, is associated with a formula

$$\bigwedge_{\substack{j<|\sigma_i| \\ \sigma_i(j)=0}} \neg U_j(x) \wedge \bigwedge_{\substack{j<|\sigma_i| \\ \sigma_i(j)=1}} U_j(x)$$

which generates p. Hence there is a procedure mapping every 1-type to a formula which generates it. With Lemma 3.4(iii), this gives a procedure for mapping any type of any arity to a formula generating it.

Now suppose that $\mathcal{A}$ and $\mathcal{B}$ are two models of T, with domains $\{a_0, \ldots\}$ and $\{b_0, \ldots\}$, respectively. We now produce an isomorphism from $\mathcal{A}$ to $\mathcal{B}$:

Stage 0. Let f_0 be the empty function.

Odd stages $2s+1$. Suppose that f_{2s} is a finite partial elementary map from $\mathcal{A}$ into $\mathcal{B}$ with domain of size $2s$, enumerated $\langle a_{k_0}, \ldots, a_{k_{2s-1}} \rangle$. Let i be least such that a_i is not in the domain of f_{2s}. Use the procedure outlined above to find a formula $\phi(x_0, \ldots, x_{2s})$ generating $\mathsf{tp}^{\mathcal{A}}(a_{k_0}, \ldots, a_{k_{2s-1}}, a_i)$. Since f_{2s} is a partial elementary map, we know that

$$\mathsf{tp}^{\mathcal{A}}(a_{k_0}, \ldots, a_{k_{2s-1}}) = \mathsf{tp}^{\mathcal{B}}(f_{2s}(a_{k_0}), \ldots, f_{2s}(a_{k_{2s-1}})),$$

and in particular that there exists a b_j not in $\{f_{2s}(a_{k_0}), \ldots, f_{2s}(a_{k_{2s-1}})\}$ and such that $\mathcal{B} \models \phi(f_{2s}(a_{k_0}), \ldots, f_{2s}(a_{k_{2s-1}}), b_j)$. Let j be the least index of such a b_j, and define $f_{2s+1} = f_{2s} \cup \{(a_i, b_j)\}$.

Even stages $2s+2$. Let $\langle a_{k_0}, \ldots, a_{k_{2s}} \rangle$ be an enumeration of the domain of f_{2s+1}. Beginning with the least index j such that b_j is not in the range of f_{2s+1}, perform a procedure similar to the one given for odd stages to find the least index i such that a_i is not in the domain of f_{2s+1} and such that $\mathsf{tp}^{\mathcal{A}}(a_{k_0}, \ldots, a_{k_{2s}}, a_i) = \mathsf{tp}^{\mathcal{B}}(f_{2s+1}(a_{k_0}), \ldots, f_{2s+1}(a_{k_{2s}}), b_j)$. Let $f_{2s+2} = f_{2s+1} \cup \{(a_i, b_j)\}$.

Then Δ^0_1 comprehension allows us to form the limit $f = \bigcup_{s \in M} f_s$. It is straightforward to check that f is an isomorphism. □

The strategy we used to build f in the proof of Proposition 3.5 is called an *effective back-and-forth argument*.

4 A theory with infinitely many 1-types, whose every nonprincipal type computes K^Z.

Recall that we work in a model $(M, \mathcal{S})$ of $\mathsf{RCA_0}$. Fix a set $Z \in \mathcal{S}$. We begin by constructing an infinite ternary tree $\mathcal{T} \subseteq \{0,1,b\}^{<M}$ with infinitely many isolated paths and whose every nonisolated path computes the Turing jump K^Z. We then convert $\mathcal{T}$ into a theory T, and show that T has infinitely many 1-types and that K^Z is Δ^0_1 definable in each nonprincipal type of T. This allows us, in §4.3, to prove some directions of Theorem 2.2. Our construction is similar to some in the literature, for instance, Millar [16].

4.1 Construction.

We define the set $\mathcal{T} \subseteq 2^{<M}$ as follows. Suppose that σ is any string in $\{0,1,b\}^{<M}$ not beginning with b. Then σ can be written uniquely in the form

$$\sigma = i_0{}^\frown b^{t_0}{}^\frown i_1{}^\frown \cdots {}^\frown b^{t_{m-1}}{}^\frown i_m{}^\frown b^{t_*},$$

with $i_k \in \{0,1\}$, $t_k \in M$ for each k, and $t_* \in M$. We let σ be in $\mathcal{T}$ if and only if the following condition holds:

$$\text{For each } k < m,\ t_k \text{ is the least number } \geq k \text{ s.t. } i_0{}^\frown \cdots {}^\frown i_k = K^Z_{t_k} \upharpoonright (k+1). \qquad (1)$$

This completes the construction of $\mathcal{T}$. Before constructing the theory T, we point out that $\mathcal{T}$ is indeed a nonempty extendable tree:

Lemma 4.1 ($\mathsf{RCA_0}$). *(i) The empty string $\emptyset$ is in $\mathcal{T}$.*

(ii) If $\sigma \subseteq \tau$ and $\tau \in \mathcal{T}$, then $\sigma \in \mathcal{T}$.

(iii) If $\sigma \in \mathcal{T}$, then $\sigma^\frown b \in \mathcal{T}$.

Proof. All three claims are immediate. □

Now we code $\mathcal{T}$ as a binary tree $\mathcal{T}_0$ by defining a function $F : \{0,1,b\}^{<M} \to \{0,1\}^{<M}$:

$$F(\emptyset) = \emptyset,\ F(\sigma^\frown 0) = F(\sigma)^\frown 0^\frown 0,\ F(\sigma^\frown 1) = F(\sigma)^\frown 0^\frown 1,\ F(\sigma^\frown b) = F(\sigma)^\frown 1^\frown 0,$$

and letting $\mathcal{T}_0 = \{\tau : \tau \subseteq F(\sigma) \text{ for some } \sigma \in \mathcal{T}\}$. Let T be the theory obtained from $\mathcal{T}_0$ by the method of §3.1. This completes the construction.

4.2 Verification.

We claim that T has infinitely many 1-types, and we claim that K^Z is Δ^0_1 definable in every nonprincipal type of T. By Lemma 3.4, the 1-types of T correspond to paths in $\mathcal{T}_0$, which can be identified naturally with paths in $\mathcal{T}$. We may therefore rephrase the claim that T has infinitely many 1-types as part (ii) of the following lemma.

Lemma 4.2 ($\mathsf{RCA_0}$). *(i) For every $\sigma \in \mathcal{T}$, we have $\sigma^\frown 0 \in \mathcal{T} \iff \sigma^\frown 1 \in \mathcal{T}$.*

(ii) The tree $\mathcal{T}$ has infinitely many paths.

Proof. (i) Immediate from the definition.

(ii) Let $\langle \sigma_0, \sigma_1, \ldots \rangle$ be a one-to-one enumeration of all strings in $\mathcal{T}$ that end in a 1. (There are infinitely many such σ_i.) We know by Lemma 4.1(iii) that $\mathcal{T}$ is extendable, so we may effectively extend every $\sigma \in \mathcal{T}$ to the leftmost path $\mathsf{left}(\sigma) \in \{0,1,b\}^M$ of $\mathcal{T}$ extending σ, using the ordering $0 < 1 < b$. Then the coded sequence $\langle \mathsf{left}(\sigma_0), \mathsf{left}(\sigma_1), \ldots \rangle$ is a sequence of paths through $\mathcal{T}$. Since the mapping from σ_i to $\mathsf{left}(\sigma_i)$ is effective, this coded sequence exists in $\mathcal{S}$ by Δ^0_1 comprehension. It is easy to see that $i \neq j$ implies $\mathsf{left}(\sigma_i) \neq \mathsf{left}(\sigma_j)$, so $\langle \mathsf{left}(\sigma_0), \ldots \rangle$ is a list of infinitely many distinct paths, as desired.

□

It remains to show that K^Z is Δ^0_1 definable in each nonprincipal type of T. This requires a few more facts about $\mathcal{T}$.

Lemma 4.3 (RCA_0). *(i) A path f through $\mathcal{T}$ is isolated if and only if f is of the form $f = \sigma^\frown b^M$ for some finite string σ.*

(ii) K^Z is Δ^0_1 definable in each nonisolated path through $\mathcal{T}$.

(iii) If $\langle f_0, \ldots, f_{n-1} \rangle$ is a tuple of isolated paths through $\mathcal{T}$, then there is a level $\ell \in M$ above which every f_i is isolated.

Proof. (i) For the 'if' direction, suppose that $f = \sigma^\frown b^M$, with $\sigma = i_0{}^\frown b^{t_0}{}^\frown \cdots {}^\frown b^{t_{m-2}}{}^\frown i_{m-1}$. If there is no $t \geq k$ such that $i_0{}^\frown i_1{}^\frown \cdots {}^\frown i_{m-1} = K^Z_t \upharpoonright m$, then f is isolated above σ. If there is such a t, then f is isolated above $\sigma^\frown b^{t+1}$.

For the 'only if' direction, we show the contrapositive. Suppose that f is a path through $\mathcal{T}$ such that $f(m) \in \{0,1\}$ for infinitely many m. By Lemma 4.2(ii), for each such m, the string $\sigma = (f \upharpoonright m)^\frown(1 - f(m))$ is in $\mathcal{T}$, and hence there is a path $g_m \neq f$ with $g_m \upharpoonright (m+1) = (f \upharpoonright m)^\frown(1 - f(m))$. Since these m are cofinal in M, it follows that f is not isolated.

(ii) Suppose that f is an infinite path through $\mathcal{T}$ not ending in a string of b's. Such an f may be written

$$f = i_0{}^\frown b^{t_0}{}^\frown i_1{}^\frown b^{t_1}{}^\frown \cdots,$$

with $i_k \in \{0,1\}$ for every k. For every $s \in M$, the initial segment $\sigma_s \subseteq f$ given by

$$\sigma_s = i_0{}^\frown b^{t_0}{}^\frown \cdots {}^\frown b^{t_{s-1}}{}^\frown i_s$$

is an element of $\mathcal{T}$. It follows from the definition of $\mathcal{T}$ that, for all $m \in M$:

$$(\forall s > t_{m-1}) \left[i_0{}^\frown \cdots {}^\frown i_{m-1} = K^Z_s \upharpoonright m \right].$$

In other words, $i_0{}^\frown \cdots {}^\frown i_{m-1} = K^Z \upharpoonright m$. This gives a Δ^0_1 definition for K^Z.

(iii) Let $\langle f_0, \ldots, f_{n-1} \rangle$ be a coded n-tuple of isolated paths in $\mathcal{T}$. By part (i), each f_j can be written in the form:

$$f_j = i_{j,0}{}^\frown b^{t_{j,0}}{}^\frown \cdots {}^\frown b^{t_{j,m_j-1}}{}^\frown i_{j,m_j-1}{}^\frown b^M.$$

The induction axioms of RCA_0 are not strong enough, at least on their face, to guarantee the existence of the tuple $\langle m_j : j < n \rangle$. This adds to the complexity of our proof.

Every f_j, being isolated, falls into one or more of the following cases:

1. f_j has an initial segment of the form $i_{j,0}{}^\frown b^{t_{j,0}}{}^\frown \cdots {}^\frown i_{j,m}{}^\frown b^{s+1}$ with $s \geq m$ and such that $i_{j,0}{}^\frown \cdots {}^\frown i_{j,m} = K^Z_s \upharpoonright (m+1)$.
2. There is a k such that $i_{j,k} = 0$ while $K^Z(k) = 1$.
3. There is a k such that $i_{j,k} = 1$ while $K^Z(k) = 0$.

Whether f_j falls into case 1 is a Σ^0_1 question, and case 2, also a Σ^0_1 question. Use bounded Σ^0_1 comprehension to partition the indices $j < n$ along these lines:

$$X_1 = \{j < n : f_j \text{ falls into case } 1\},$$

$$X_2 = \{j < n : j \notin X_1 \text{ and } f_j \text{ falls into case } 2\},$$

$$X_3 = \{j < n : j \notin X_1 \cup X_2\}.$$

Then every element of X_3 falls into case 3. It suffices to show that for each $z \in \{1, 2, 3\}$ there is a level ℓ_z above which f_j is isolated for all $j \in X_z$, and take $\ell = \max(\ell_1, \ell_2, \ell_3)$.

First consider $z = 1$. Assign to each $j \in X_1$ a string $\sigma_j \subseteq f_j$ as in the statement of case 1. Then f_j is isolated above the length $|\sigma_j|$. Let ℓ_1 be the maximum of $|\sigma_j|$ as j ranges over X_1.

Now consider $z = 2$. For each $j \in X_2$, the formula $(\exists k \exists s)[i_{j,k} = 0 \text{ and } K^Z_s(k) = 1]$ holds. Use Σ^0_1 bounding to assign to each $j \in X_2$ a pair k_j, s_j witnessing this. Choose any $\sigma_j \subseteq f_j$ of the form

$$\sigma_j = i_{j,0}{}^\frown b^{t_{j,0}}{}^\frown \cdots {}^\frown i_{j,k_j}{}^\frown \tau^\frown b^{s_j+1}$$

where τ is a string. Then f_j is isolated above the length $|\sigma_j|$. Let ℓ_2 be the maximum of $|\sigma_j|$ as j ranges over X_2.

Lastly, consider $z = 3$. Since it is a Π^0_1 question to ask whether two paths are equal, we may assume by bounded Π^0_1 comprehension that the paths f_j are all distinct as j ranges over X_3. Let $j_0, j_1 \in X_3$ be distinct elements, and consider the paths f_{j_0}, f_{j_1}. Let k be least such that $i_{j_0,k} \neq i_{j_1,k}$; we may assume by symmetry that $i_{j_0,k} = 0$ and $i_{j_1,k} = 1$. Then $K^Z(k)$ must equal 0, since otherwise j_0 would be an element of X_2. Let $\sigma_{j_1} = i_{j_1,0}{}^\frown b^{t_{j_1,0}}{}^\frown \cdots {}^\frown i_{j_1,k}$. It follows that f_{j_1} is isolated above $|\sigma_{j_1}|$. Repeat this procedure on pairs from $X_3 - \{j_1\}$, and so on, until there is a σ_j associated to all but one element of X_3, say j'. Let $\sigma_{j'}$ be such that $f_{j'}$ is isolated above $|\sigma_{j'}|$, and let ℓ_3 be the maximum of $|\sigma_j|$ as j ranges over X_3.

Now $\ell = \max(\ell_1, \ell_2, \ell_3)$ is the desired bound.

□

This is enough to verify the last desired property:

Proposition 4.4 (RCA_0)**.** *K^Z is Δ^0_1 definable in each nonprincipal type of T.*

Proof. Let $p(x_0, \ldots, x_{n-1})$ be a nonprincipal n-type for some n. Since the language of T consists only of unary relations, p may be decomposed into 1-types $\langle p_0, \ldots, p_{n-1}\rangle$:

$$p(x_0, \ldots, x_{n-1}) \iff p_0(x_0), \ldots, p_{n-1}(x_{n-1}).$$

The 1-types $\langle p_0, \ldots, p_{n-1}\rangle$ correspond to a tuple $\langle f_0, \ldots, f_{n-1}\rangle$ of paths through $\mathcal{T}$. Since p is nonprincipal, there is an i such that f_i is nonisolated by Lemma 4.3(iii). Therefore K^Z is Δ^0_1 definable from f_i, and hence from p, by Lemma 4.3(ii). □

4.3 Applications.

Recall from §2.1 the statements:

(S3) T has only finitely many n-types, for each n.

(S4) T is $\aleph_0$-categorical.

(S5) All types of T are principal.

We use this section's construction to prove two parts of Theorem 2.2, beginning with part (iv):

Proposition 4.5. *Over* RCA_0*, the implication* ($\mathsf{S5} \to \mathsf{S3}$) *implies* ACA_0*.*

Proof. Suppose that ($\mathsf{S5} \to \mathsf{S3}$) holds, and fix any set $Z \in \mathcal{S}$. Let T be the theory constructed in §4.1. Since T has infinitely many 1-types, T satisfies ($\neg$S3). Then T satisfies ($\neg$S5), i.e., T has a nonprincipal type p. By Proposition 4.4 above, K^Z is Δ^0_1 definable from p, and so K^Z exists by Δ^0_1 comprehension. Since Z was arbitrary, we conclude by Lemma 1.2 that ACA_0 holds. □

Next, we prove Theorem 2.2(vi):

Proposition 4.6. *Over* RCA_0*, the implication* ($\mathsf{S5} \to \mathsf{S4}$) *implies* ACA_0*.*

Proof. Fix any set $Z \in \mathcal{S}$, and let T be the theory constructed in §4.1. It is enough to exhibit two models $\mathcal{A}, \mathcal{B}$ of T such that K^Z is Δ^0_1 definable in any isomorphism $f : \mathcal{A} \to \mathcal{B}$. Let $\langle \sigma_0, \sigma_1, \ldots \rangle$ be a one-to-one enumeration of all strings in the tree $\mathcal{T}_0$. For each σ_i, let $\mathsf{left}(\sigma_i)$ be the leftmost path of $\mathcal{T}_0$ extending σ_i; similarly, let $\mathsf{right}(\sigma_i)$ be the rightmost path extending σ_i. We may form the coded sequences $\langle \mathsf{left}(\sigma_0), \mathsf{left}(\sigma_1), \ldots \rangle$ and $\langle \mathsf{right}(\sigma_0), \mathsf{right}(\sigma_1), \ldots \rangle$ by Δ^0_1 comprehension.

First we build the model $\mathcal{A}$, with domain $\{a_0, a_1, \ldots\}$. For each $i, k \in M$, let $R_k(a_i)$ hold in $\mathcal{A}$ if and only if $\mathsf{left}(\sigma_i)(k) = 1$. It is easy to check that $\mathcal{A}$ satisfies the axioms of §3.1 semantically. Fill in the rest of the diagram as in the proof of Lemma 3.3 so that $\mathcal{A}$ is a model of T. Build a second model $\mathcal{B}$ with domain $\{b_0, b_1, \ldots\}$ by a similar method: for each $i, k \in M$, let $R_k(b_i)$ hold in $\mathcal{B}$ if and only if $\mathsf{right}(\sigma_i)(k) = 1$, and fill in the rest of the diagram.

Now, suppose that $f : \mathcal{A} \to \mathcal{B}$ is an isomorphism. Use f to define a function $g : M \to M$ by $g(i) = j$ whenever $f(a_i) = b_j$. Then $\mathsf{left}(\sigma_i) = \mathsf{right}(\sigma_{g(i)})$ for all $i \in M$. In particular either $\sigma_i \subseteq \sigma_{g(i)}$ or $\sigma_i \supseteq \sigma_{g(i)}$, and the longer of the two, which we denote by $\sigma_i \cup \sigma_{g(i)}$, is isolated in $\mathcal{T}_0$. It follows that σ_i is isolated if and only if there is no string τ such that $\sigma_i \subseteq \tau \subseteq \sigma_i \cup \sigma_{g(i)}$, and such that both $\tau^\frown 0$ and $\tau^\frown 1$ are elements of $\mathcal{T}_0$. This gives a uniform procedure for deciding whether a given σ is isolated, and, in particular, allows us to define a nonisolated path of $\mathcal{T}_0$, and hence a nonisolated path of $\mathcal{T}$. By Lemma 4.3(ii) and Δ^0_1 comprehension, the Turing jump K^Z is an element of $\mathcal{S}$. We conclude by Lemma 1.2 that ACA_0 holds. □

5 Models from a tree of Henkin constructions.

For the following informal discussion, we reason in WKL_0. Fix a set $Z \in \mathcal{S}$, a language L, a complete L-theory T with infinitely many n-types for some n, and a model $\mathcal{A} \models T$ with domain $A = \{a_0, a_1, \ldots\}$. We produce a second model $\mathcal{B} \models T$ with domain $B = \{b_0, b_1, \ldots\}$ such that the Turing jump K^Z is Δ^0_1 definable in any elementary embedding $f : \mathcal{B} \to \mathcal{A}$. We achieve this by making the function $g : M \to M$ defined by $g(m) = n \iff f(b_m) = a_n$ grow roughly as fast as the *modulus function* of K^Z, which is given by $m \mapsto \min\{s > m : K^Z_s \restriction m = K^Z \restriction m\}$. More specifically, we ensure that, if m is an element of $K^Z_{at\ s}$, there is an n-ary formula satisfied in $\mathcal{B}$ by an n-tuple taken from the initial segment $\{b_0, b_1, \ldots, b_{2n(m+1)-1}\}$ of B, but not in $\mathcal{A}$ by any n-tuple from the initial segment $\{a_0, \ldots, a_{s-1}\}$ of A. Then if $f : \mathcal{B} \to \mathcal{A}$ is an elementary embedding, the function given by $m \mapsto \max\limits_{i<2n(m+1)} g(i)$ bounds the modulus function of K^Z.

The model $\mathcal{B}$ itself is obtained by the following method. We construct a binary tree $\mathcal{H}^*$ such that any node $\sigma \in \mathcal{H}^*$ of length s represents the first s-many steps of a Henkin-style construction, and such that the construction along any infinite path of $\mathcal{H}^*$ yields a model $\mathcal{B}$ with the property outlined above. We then show that $\mathcal{H}^*$ is infinite, and apply Weak König's Lemma to obtain $\mathcal{B}$.

5.1 Construction.

We begin with some definitions. Fix a language L and a complete, consistent L-theory T.

Definition 5.1. (i) Let L' be the enriched language $L \cup \{c_0, c_1, \ldots\}$, where each c_i is a constant symbol not in L. Let $\langle \phi_s \rangle_s$ be a one-to-one enumeration of all L'-sentences. First, define a $2^{<M}$-indexed sequence $\langle D_\sigma \rangle_{\sigma \in 2^{<M}}$ of sets of L'-sentences by

$$D_\sigma = \{\phi_s : s < |\sigma| \text{ and } \sigma(s) = 1\} \cup \{\neg\phi_s : s < |\sigma| \text{ and } \sigma(s) = 0\}.$$

Second, define a sequence $\langle W_s \rangle_{s \in M}$ of sets of L'-sentences by recursion:

$$W_0 = \emptyset$$
$$W_{s+1} = \begin{cases} W_s \cup \{\phi_s \to \psi(c_{2k+1})\} & \text{if } \phi_s \text{ is of the form } (\exists\, x)\psi(x), \text{ where } 2k+1 \text{ is the least odd index such that } c_{2k+1} \text{ is not mentioned in } W_s \text{ or in any } D_\sigma \text{ with } |\sigma| \leq s. \\ W_s & \text{if } \phi_s \text{ is not of this form.} \end{cases}$$

Third, define a tree $\mathcal{H} \subseteq 2^{<M}$ by

$$\mathcal{H} = \{\sigma \in 2^{<M} : T \cup D_\sigma \cup W_{|\sigma|} \text{ is consistent}\}.$$

We call $\mathcal{H}$ the *full tree of odd Henkin diagrams*. ('Odd' because we are using only the odd-numbered constants to witness existential sentences.)

(ii) Given an infinite path β in $\mathcal{H}$, let $D_\beta = \bigcup_{s \in M} D_{\beta \restriction s}$. Then D_β is a complete, consistent L'-theory. Define an equivalence relation E on the constants $\{c_0, c_1, \ldots\}$ by $c_i E c_j$ iff $D_\beta \vdash c_i = c_j$. Denote the E-equivalence class of c_i by $[c_i]_E$, and let $\langle b_0, b_1, \ldots \rangle$ be the one-to-one listing of all E-equivalence classes given by

$$b_m = [c_{i_m}]_E, \text{ where } i_m \text{ is least s.t. } c_{i_m} \notin b_k \text{ for all } k < m.$$

Let $\mathcal{B}$ be the L-structure such that, for any L-formula ϕ,

$$\mathcal{B} \models \phi(b_0, \ldots, b_{n-1}) \iff D_\beta \vdash \phi(c_{i_0}, \ldots, c_{i_{m-1}}).$$

Then $\mathcal{B}$ is a model of T. We say that $\mathcal{B}$ is the *Henkin model encoded by* β.

Now fix a model $\mathcal{A}$ of T. We define an infinite subtree $\mathcal{H}^* \subseteq \mathcal{H}$ of the full tree of odd Henkin diagrams such that, if β is an infinite path of $\mathcal{H}^*$ and $\mathcal{B}$ is the Henkin model encoded by β, then K^Z is Δ^0_1 definable in any elementary embedding $f : \mathcal{B} \to \mathcal{A}$. Then $\mathsf{WKL_0}$ ensures that such a path β exists, giving the desired model $\mathcal{B}$.

For each $t \in M$, choose an n-ary L-formula $\theta_t(\bar{x})$ such that $T \vdash (\exists \bar{x})\theta_t(\bar{x})$, and such that θ_t is not satisfied by any tuple taken from $\{a_0, \ldots, a_t\}$ in $\mathcal{A}$. (This is possible by Theorem 2.6(i), since T has infinitely many n-types.) For each $s \in M$, define a finite set D^*_s of L'-sentences:

$$D^*_s = \{\theta_t(c_{2mn}, c_{2mn+2}, \ldots, c_{2(m+1)n-2}) : m, t < s \text{ and } m \in K^Z_{at\ t}\}.$$

Note that $D^*_s \subseteq D^*_{s+1}$ for each s. Define the subtree $\mathcal{H}^*$ of $\mathcal{H}$ by:

$$\mathcal{H}^* = \left\{\sigma \in 2^{<M} : T \cup D_\sigma \cup D^*_{|\sigma|} \cup W_{|\sigma|} \text{ is consistent}\right\}. \quad (2)$$

This completes the construction.

5.2 Verification.

There are two facts to verify: first, that $\mathcal{H}^*$ is infinite, and second, if a model $\mathcal{B}$ is encoded by a path in $\mathcal{H}^*$, then K^Z is Δ^0_1 definable in any elementary embedding of $\mathcal{B}$ into $\mathcal{A}$.

Lemma 5.2 ($\mathsf{RCA_0}$)**.** *The tree $\mathcal{H}^*$ is infinite.*

Proof. Fix any $s \in M$. It suffices to show that $\mathcal{H}^*$ has an element of length s. We may choose a finite tuple $\langle c_i^{\mathcal{A}} : i < N\rangle$ of elements of $\mathcal{A}$ such that $(\mathcal{A}, c_i^{\mathcal{A}} : i < N)$ is a model of $T \cup D^*_s \cup W_s$. In particular, $\langle c_i^{\mathcal{A}} : i < N\rangle$ contains all constants mentioned in $\phi_0, \ldots, \phi_{s-1}$, where $\langle \phi_t\rangle_t$ is the enumeration of all L'-sentences fixed in Definition 5.1(i). Define a string σ of length s by

$$\sigma(t) = \begin{cases} 1 \text{ if } (\mathcal{A}, c_i^{\mathcal{A}} : i < N) \models \phi_t, \\ 0 \text{ otherwise} \end{cases}$$

for all $t < s$. Then $(\mathcal{A}, c_i^{\mathcal{A}} : i < N)$ is a model of $T \cup D_\sigma \cup D^*_s \cup W_s$, so $T \cup D_\sigma \cup D^*_s \cup W_s$ is consistent. Therefore σ is in $\mathcal{H}^*$, as desired. □

Lemma 5.3 ($\mathsf{RCA_0}$)**.** *If $\mathcal{B}$ is the model encoded by an infinite path β in $\mathcal{H}^*$, and $f : \mathcal{B} \to \mathcal{A}$ is an elementary embedding, then K^Z is Δ^0_1 definable from f.*

Proof. Suppose that $\mathcal{B}$ is the model encoded by some path β in $\mathcal{H}^*$, and that $f : \mathcal{B} \to \mathcal{A}$ is an elementary embedding. Define a mapping $h : M \to M$ by

$$h(m) = \text{greatest } j \text{ s.t. } f([c_{2mn+2i}]_E) = a_j \text{ for some } i < n.$$

By the definition of D^*_s, if there is a t such that $m \in K^Z_t$, then $m \in K^Z_{h(m)}$. Hence we have $m \in K^Z \iff m \in K^Z_{h(m)}$, which gives a Δ^0_1 definition for K^Z. □

5.3 Applications.

Recall from §2.1 the statements:

($\mathsf{S3}$) T has only finitely many n-types for each n.

($\mathsf{S4}$) T is $\aleph_0$-categorical.

We say that a model $\mathcal{A}$ of a theory T is *elementary-universal* if, for any model $\mathcal{B}$ of T, there is an elementary embedding from $\mathcal{B}$ into $\mathcal{A}$. The construction in §5.1 above is tailored to give the following result.

Lemma 5.4. $\mathsf{WKL_0} + \neg\mathsf{ACA_0} \vdash$ (*'T has an elementary-universal model'*$\to$ *S3*).

Proof. Suppose that $(M, \mathcal{S})$ is a model of $\mathsf{WKL_0} + \neg\mathsf{ACA_0}$. By Lemma 1.2, we may fix a set $Z \in \mathcal{S}$ whose Turing jump K^Z is not in $\mathcal{S}$. We show that the contrapositive statement ($\neg\mathsf{S3} \to$'T has no elementary-universal model') holds in $(M, \mathcal{S})$.

Fix a complete theory $T \in \mathcal{S}$ with infinitely many n-types, and fix a model $\mathcal{A} \in \mathcal{S}$ of T. Use the construction of §4.1 and Lemma 5.3 to obtain a second model $\mathcal{B} \in \mathcal{S}$ of T such that K^Z is Δ^0_1 definable in every elementary embedding from $\mathcal{B}$ into $\mathcal{A}$. This means, by our choice of Z, that no $f \in \mathcal{S}$ can be an elementary embedding from $\mathcal{B}$ into $\mathcal{A}$. In particular, $\mathcal{A}$ is not elementary-universal. □

Since any model of an $\aleph_0$-categorical theory is elementary-universal, the following is an immediate consequence of Lemma 5.4.

Lemma 5.5. $\mathsf{WKL_0} + \neg\mathsf{ACA_0} \vdash$ (*S4* $\to$ *S3*).

We are ready to prove the remaining direction of Theorem 2.2(v), the other having been proved in Proposition 3.5 above.

Proposition 5.6. $\mathsf{WKL}_0 \vdash$ (*S4* $\rightarrow$ *S3*)

Proof. We know from Lemma 5.5 that $\mathsf{WKL}_0 + \neg\mathsf{ACA}_0 \vdash$ ($\mathsf{S4} \rightarrow \mathsf{S3}$). On the other hand, as noted in §2.1, ACA_0 is sufficiently strong to carry out the usual proof of equivalence of all the principles ($\mathsf{S1}$) through ($\mathsf{S5}$), and in particular $\mathsf{ACA}_0 \vdash$ ($\mathsf{S4} \rightarrow \mathsf{S3}$). Hence we conclude that $\mathsf{WKL}_0 \vdash$ ($\mathsf{S4} \rightarrow \mathsf{S3}$). □

The construction from this section also justifies an assertion in §2.2. The following proposition completes the proof of Theorem 2.12:

Proposition 5.7. *Over* WKL_0*, the following are equivalent:*

(i) ACA_0

(ii) There is a complete theory with a nonprincipal type and only finitely many models.

(iii) There is a complete theory with infinitely many n-types for some n, and only finitely many models.

Proof. Reason in WKL_0. The implication ($i \rightarrow ii$) follows from the use of a nonprincipal type in the proof of Ehrenfeucht's Theorem 2.7 in the system ACA_0. The implication ($ii \rightarrow iii$) is immediate from the definitions.

We prove the final implication ($iii \rightarrow i$) by way of its contrapositive statement ($\neg i \rightarrow \neg iii$). Suppose that $\mathsf{WKL}_0 + \neg\mathsf{ACA}_0$ holds, and let T be a complete theory with infinitely many n-types for some n. Dovetail the proof of Lemma 5.4 to get a coded sequence $\langle \mathcal{A}_0, \mathcal{A}_1, \ldots \rangle$ of models of T such that no $\mathcal{A}_j$ embeds elementarily into any $\mathcal{A}_i$ with $i < j$. (For each triple $\langle i, j, m \rangle$ where $i < j$, if m is in $K^Z_{at\ s}$, use the method of §5.1 to ensure that there is a formula realized by a tuple from among the first $2n(\langle i, j, m \rangle + 1)$-many elements of $\mathcal{A}_j$ but not by any tuple from among the first s-many elements of $\mathcal{A}_i$.) We have produced an infinite list of pairwise nonisomorphic models of T, so (iii) fails, as desired. □

6 Theories with only finitely many n-types for every n.

The *Ryll-Nardzewski function* for a theory T is the Σ^0_2 partial function $\mathrm{RN}_T : M \rightarrow M$ given by:

$$\begin{aligned}\mathrm{RN}_T(n) = m \quad &\Longleftrightarrow \quad T \text{ has exactly } m \text{ different } n\text{-types}\\ &\Longleftrightarrow \quad \text{there exists a sequence } \phi_0, \ldots, \phi_{m-1} \text{ of } n\text{-ary formulas}\\ &\qquad\quad \text{such that } T \vdash \phi_0 \vee \cdots \vee \phi_{m-1} \text{ and } T \nvdash \phi_i \rightarrow \phi_j \text{ for each}\\ &\qquad\quad i \neq j, \text{ and for all } n\text{-ary } \psi \text{ and all } i \text{ s.t. } T \vdash \psi \rightarrow \phi_i \text{ we}\\ &\qquad\quad \text{have } T \vdash \phi_i \rightarrow \psi.\end{aligned}$$

If $\mathrm{RN}_T(n)$ has no value according to the above definition, we treat $\mathrm{RN}_T(n)$ as an infinite number. The properties ($\mathsf{S1}$), ($\mathsf{S2}$), and ($\mathsf{S3}$) from §2.1 can all be phrased in terms of RN_T.

In this section, we prove several directions of Theorem 2.2 by constructing examples of a theory T for which RN_T is finite-valued, but for which RCA_0 cannot prove the existence of RN_T. One of these examples, given in Proposition 6.12, has a RN_T so fast-growing that ACA_0 is needed to prove even that RN_T is dominated by a function in the second-order part of $(M, \mathcal{S})$. A second example, given in the proof of Proposition 6.11 and used again in that of Proposition 6.13, has a RN_T that is slow-growing, but whose existence nonetheless implies ACA_0. Our theories are built using a simple common framework, given in §6.1 below, which takes as a parameter a coded sequence $\langle X_1, X_2, \ldots \rangle$ of sets. By varying this parameter, we control RN_T.

In effective model theory, similar constructions have been done before to control the Turing degree of RN_T for a decidable $\aleph_0$-categorical theory with infinitely many predicates (Palyutin [18] and Venning [24, Ch. 2]) and with a single binary predicate (Herrmann [11], Schmerl [21], and Venning [24, Ch. 3]). Both our construction and our verification are very similar to Palyutin's, when done carefully in second-order arithmetic.[2] Our construction is also similar to Venning's [Ch. 2], but the verification more elementary.

6.1 Construction.

Let L be the language $L = \langle R^n_s \rangle_{s \in M, n \geq 1}$, with each R^n_s an n-ary relation. Let $\langle X_1, X_2, \ldots \rangle$ be a coded sequence of sets. We introduce three axiom schemes:

Ax I. $R^n_s(x_0, \ldots, x_{n-1}) \to x_i \neq x_j$, for each n, s and each pair $i, j < n$ with $i \neq j$.

Ax II. $\neg R^n_s(\bar{x})$, for each n, s such that $s \notin X_n$.

Ax III. $\psi(\bar{x}) \to (\exists y)\phi(\bar{x}, y)$ for every pair ϕ, ψ of formulas with the following properties:

- ϕ and ψ are conjunctions of L'-literals, where $L' = \{R^n_s : n, s < \ell\}$ for some $\ell > |\bar{x}| + 1$;
- For every atomic L'-formula θ with variables in $\bar{x}$, either θ or $\neg\theta$ appears as a conjunct in ψ;
- $\phi(\bar{x}, y)$ is consistent with Ax I and II;
- Every conjunct in ψ is a conjunct in ϕ;

Let T^* denote the collection of all sentences in Ax I–III, and let T be the deductive closure of T^*. This completes the construction. Notice that we have not yet proved the existence either of T^* or of T in the second-order part of $(M, \mathcal{S})$. For T^*, this follows from Lemma 6.2 below, where we prove that the consistency check in Ax III can be performed effectively. For T, existence is proved in Proposition 6.5 using quantifier elimination.

The intuition behind these axioms is as follows. Axiom I is an n-ary version of the irreflexivity property for binary relations: R^n_s holds only of n-tuples whose entries are all distinct. This limits the number of quantifier-free formulas that may hold of an n-tuple. Axiom II relates the parameter $\langle X_1, X_2, \ldots \rangle$ to the number of different quantifier-free formulas that might hold of an n-tuple. Axiom III then binds this number to $\mathrm{RN}_T(n)$ by providing quantifier elimination.

6.2 Verification.

Most of this section is devoted to checking that the T defined in §6.1 is an element of $\mathcal{S}$, is complete, and is consistent. The exception is Lemma 6.10, in which we relate the coded sequence $\langle X_1, X_2, \ldots \rangle$ to the Ryll-Nardzewski function RN_T. The following technical lemma will be useful in this section, and again in §7.

Lemma 6.1 (RCA_0). *Let $L_0 = \langle Q_n \rangle_n$ be a relational language. Let $\Psi = \{\psi_s : s \in M\}$ be L_0-theory where each ψ_s is of the form $(\forall \bar{x}, \bar{y})[\ell_s(\bar{x}) \vee \theta_s(\bar{x}, \bar{y})]$ where θ_s is quantifier-free and ℓ_s is either $Q_n(\bar{x})$ or $\neg Q_n(\bar{x})$, where $n \geq s$ and Q_n is not mentioned in any ψ_t, $t < s$. Then there is a procedure that decides, given a quantifier-free L-formula $\phi(\bar{z})$, whether $\Psi \cup \{(\exists \bar{z})\phi(\bar{z})\}$ is consistent.*

[2] We thank the referee for bringing this paper to our attention.

Proof. Fix a quantifier-free formula $\phi(z_0, \ldots, z_{m-1})$. Let n be the greatest index such that Q_n is mentioned in ϕ, and consider the set $\Psi_n = \{\psi_s : s \leq n\}$. Recall that a theory is *consistent* if does not entail a contradiction. We claim that $\Psi \cup \{(\exists \bar{z})\phi(\bar{x})\}$ is consistent if and only if $\Psi_n \cup \{(\exists \bar{z})\phi(\bar{x})\}$ has an m-element model. We prove this claim by a series of implications:

(a) If $\Psi \cup \{(\exists \bar{z})\phi(\bar{z})\}$ is consistent, then $\Psi_n \cup \{(\exists \bar{z})\phi(\bar{z})\}$ is consistent.

(b) If $\Psi_n \cup \{(\exists \bar{z})\phi(\bar{z})\}$ is consistent, then $\Psi_n \cup \{(\exists \bar{z})\phi(\bar{z})\}$ has an m-element model.

(c) If $\Psi_n \cup \{(\exists \bar{z})\phi(\bar{z})\}$ has an m-element model, then $\Psi \cup \{(\exists \bar{z})\phi(\bar{z})\}$ has an m-element model.

(d) If $\Psi \cup \{(\exists \bar{z})\phi(\bar{z})\}$ has an m-element model, then $\Psi \cup \{(\exists \bar{z})\phi(\bar{z})\}$ is consistent.

Item (a) is immediate. For item (b), notice that it is possible to construct a propositional formula P such that if $\Psi_n \cup \{(\exists \bar{z})\phi(\bar{z})\}$ is consistent then P is consistent, and if P is satisfiable then $\Psi_n \cup \{(\exists \bar{z})\phi(\bar{z})\}$ has an m-element model. (Use one propositional variable to represent the truth value of each relevant ψ_s on each tuple taken from $\bar{z}$.) Item (c) holds because, given an m-element model of $\Psi_n \cup \{(\exists \bar{z})\phi(\bar{z})\}$, we can effectively transform it into a model of $\Psi \cup \{(\exists \bar{z})\phi(\bar{z})\}$ by reassigning the truth values of ℓ_s to satisfy ψ_s for each $s > n$. Item (d) follows from the Soundness Theorem, which is provable in $\mathsf{RCA_0}$—see Simpson [22, Theorem II.8.8].

Our procedure works as follows: Given a formula $\phi(z_0, \ldots, z_{m-1})$, find n as above, and construct the propositional formula P used in (b). Test all truth valuations to see whether P is consistent. If so, $\Psi \cup \{(\exists \bar{z})\phi(\bar{z})\}$ is consistent. If not, $\Psi \cup \{(\exists \bar{z})\phi(\bar{z})\}$ is inconsistent. □

Lemma 6.2 ($\mathsf{RCA_0}$)**.** *There is a procedure to check whether a quantifier-free L-formula ϕ is consistent with Axioms I and II.*

Proof. We may rewrite Axiom I by replacing the $\rightarrow$ with an equivalent $\vee$, and restricting the parameters n, s so as not to conflict with Axiom II:

$$\neg R^n_s(x_0, \ldots, x_{n-1}) \vee x_i \neq x_j, \text{ for each } n, s \in M \text{ such that } s \in X_n \text{ and each pair } i, j < n \text{ such that } i \neq j.$$

Then, after an appropriate reindexing of the relations R^n_s, our axioms meet the hypothesis of Lemma 6.1. The result follows. □

Recall that T^* denotes the collection of all sentences in Ax I–III. We are ready to begin dealing with T^* directly.

Lemma 6.3 ($\mathsf{RCA_0}$)**.** *T^* is an element of $\mathcal{S}$.*

Proof. We can easily tell whether a given formula is in Ax I or Ax II. Lemma 6.2 gives a method for deciding whether or not a formula is in Ax III. □

Lemma 6.4 ($\mathsf{RCA_0}$)**.** *The theory T^* has effective quantifier elimination.*

Proof. By Lemma 1.7, it is enough to give an effective procedure that takes as input any conjunction of literals $\phi(\bar{x}, y)$ and returns a quantifier-free formula $\psi(\bar{x})$ such that $T^* \vdash (\exists y)\phi(\bar{x}, y) \leftrightarrow \psi(\bar{x})$. By performing the appropriate substitutions, we may assume that no literal in ϕ is of the form $(z_0 = z_1)$. First use the effective procedure given by Lemma 6.2 to see whether ϕ is consistent with Axioms I and II. If it is not, we conclude that $T^* \vdash (\exists y)\phi(\bar{x}, y) \leftrightarrow \mathsf{Fa}$.

If it is consistent, let $\psi(\bar{x})$ be the formula produced from ϕ by substituting Tr for each conjunct mentioning the variable y. Let $L' = \{R^n_s : n, s < \ell\}$, where ℓ is a number greater than any n or s such that R^n_s is mentioned in ψ. Use Lemma 6.2 to find all conjunctions $\psi_0, \psi_1, \ldots, \psi_m$ of L'-literals without repetitions such that

- $\psi_i \wedge \phi$ is consistent with Ax I and II.
- Every conjunct of ψ is a conjunct of ψ_i.
- For every atomic L'-formula θ with variables in $\bar{x}$, either θ or $\neg\theta$ appears as a conjunct in ψ_i.

Then $T^* \vdash (\exists y)\phi \to (\psi_0 \vee \cdots \vee \psi_m)$. The converse direction $T^* \vdash (\psi_0 \vee \cdots \vee \psi_m) \to (\exists y)\phi$ follows from Ax III applied to each pair $\phi, \phi \wedge \psi_i$. □

Recall that T denotes the deductive closure of T^*.

Proposition 6.5 (RCA_0). *(i) For every L-sentence ϕ, either ϕ is provable from T^*, or $\neg\phi$ is provable from T^*.*

(ii) T is an element of $\mathcal{S}$.

(iii) T has quantifier elimination. T is a complete theory.

Proof. Similar to the proof of Proposition 3.2. □

Next, we verify that T is consistent. It suffices to show that T has a model. This is achieved in Proposition 6.9 below, using an effective version of the Fraïssé limit construction. This argument is both clean and reusable—we use it again in the proof of Proposition 6.13 and later in §7—but requires some definitions and lemmas. The following definitions are based on those given by Csima, Harizanov, Miller, and Montalbán [2] for Fraïssé limits in recursive mathematics.

Definition 6.6. Fix a language L_0 of relation symbols. Let $\mathbb{K} = \langle \mathcal{A}_0, \mathcal{A}_1, \ldots \rangle$ be a sequence of finite L_0-structures.

(i) We say that $\mathbb{K}$ has the *effective hereditary property (EHP)* if there is a function that, given an index i and a finite set F of elements from $\mathcal{A}_i$, returns an index j and an isomorphism from $\mathcal{A}_j$ to the induced substructure $\mathcal{A}_i \restriction F$.

(ii) We say that $\mathbb{K}$ has the *effective joint embedding property (EJEP)* if there is a function that, given indices $\langle i, j \rangle$, returns an index k and a pair of embeddings $\mathcal{A}_i \hookrightarrow \mathcal{A}_k$ and $\mathcal{A}_j \hookrightarrow \mathcal{A}_k$.

(iii) We say that $\mathbb{K}$ has the *effective amalgamation property (EAP)* if there is a functions that, given indices $\langle i, j, k \rangle$ and injections $f : \mathcal{A}_i \to \mathcal{A}_j$ and $g : \mathcal{A}_i \to \mathcal{A}_k$, returns an index ℓ, an embedding $e : \mathcal{A}_j \hookrightarrow \mathcal{A}_\ell$, and an injection $h : \mathcal{A}_k \to \mathcal{A}_\ell$ such that $h \circ f = e \circ g$ and, if f and g are embeddings, h is an embedding as well.

(iv) Let $\mathcal{A}$ be a countably infinite L_0-structure with domain A. Suppose that there is a pair of functions h_0, h_1 such that h_0 maps finite subsets $F \subseteq A$ surjectively onto the indices $\{0, 1, \ldots\}$ of $\mathbb{K}$, and h_1 maps finite subsets $F \subseteq A$ to isomorphisms from the induced substructure $\mathcal{A} \restriction F$ to $\mathcal{A}_{h_0(F)}$. Suppose further that, for every choice of a finite $F \subseteq A$, a pair of indices $\langle i, j \rangle$, an isomorphism f from $\mathcal{A} \restriction F$ to $\mathcal{A}_i$, and an embedding $g : \mathcal{A}_i \hookrightarrow \mathcal{A}_j$, there is a second finite $G \subseteq A$ containing F and an isomorphism from $\mathcal{A} \restriction G$ to $\mathcal{A}_j$ which agrees with $g \circ f$ on F. Then we say that $\mathcal{A}$ is an *effective Fraïssé limit of* $\mathbb{K}$.

When interpreted in the standard model REC of RCA_0, the definitions of EHP, EJEP, and EAP agree with those of the *computable* hereditary, joint embedding, and amalgamation properties in [2]. Our notion of *effective Fraïssé limit* is essentially the same, except that we require an explicit mapping from finite substructures of $\mathcal{A}$ onto $\mathbb{K}$. (The same effect is achieved in [2] using what they call a *canonical age*.)

Lemma 6.7 (RCA_0)**.** *Let L_0 be a relational language, and let $\mathbb{K} = (\mathcal{A}_i)_{i \in M}$ be a sequence of finite L_0-structures. If $\mathbb{K}$ has the EHP, the EJEP, and the EAP, then $\mathbb{K}$ has an effective Fraïssé limit.*

Proof. Similar to [2, Thm 3.9]. □

Lemma 6.8 (RCA_0)**.** *Let L_0 be a relational language, and let T_0 be an L_0-theory axiomatized by a set T_0' of $\forall\exists$-sentences. Let $\mathbb{K} = \langle \mathcal{A}_0, \mathcal{A}_1, \ldots \rangle$ be a sequence of finite models of the $\forall$ part of T_0 with the EHP, the EJEP, and the EAP. Suppose that, for any $\exists$ L_0-formula $\phi(\bar{x})$ such that $(\forall \bar{x})\phi(\bar{x})$ is in T_0', and any $(\mathcal{A}_i, \bar{b})$ with $\bar{b}$ having the same length as $\bar{x}$, there is an $\mathcal{A}_j$ and an embedding $g : \mathcal{A}_i \hookrightarrow \mathcal{A}_j$ such that $\mathcal{A}_j \models \phi(g(\bar{b}))$. Then any effective Fraïssé limit of $\mathbb{K}$ is a model of T_0.*

Proof. Suppose that $\mathcal{A}$ is an effective Fraïssé limit of $\mathbb{K}$ with domain A. It suffices to show that $\mathcal{A}$ satisfies T_0'. Let ϕ be an n-ary $\exists$ formula such that $(\forall \bar{x})\phi(\bar{x})$ is in T_0'. Fix any n-tuple $\bar{a}$ taken from A, and let $F \subseteq A$ be a finite set containing all entries of $\bar{a}$. Using the functions h_0, h_1 from the definition of effective Fraïssé limit, find an index i and an isomorphism f from the induced substructure $\mathcal{A} \upharpoonright F$ to $\mathcal{A}_i$. By assumption, there is an $\mathcal{A}_j$ and an embedding $g : \mathcal{A}_i \hookrightarrow \mathcal{A}_j$ such that $\mathcal{A}_j \models \phi(g(f(\bar{a})))$. Use the definition of effective Fraïssé limit to get a finite $G \subseteq A$ containing F such that $\mathcal{A} \upharpoonright G$ embeds into $\mathcal{A}_j$ by a mapping agreeing with $g \circ f$ on F. Then $\mathcal{A} \upharpoonright G \models \phi(\bar{a})$, and hence $\mathcal{A} \models \phi(\bar{a})$. Since ϕ and $\bar{a}$ were arbitrary, $\mathcal{A}$ satisfies T_0', as desired. □

We are now ready to verify the consistency of the theory T.

Proposition 6.9 (RCA_0)**.** *T is consistent.*

Proof. Notice that the axioms for T given in §6.1 consist of $\forall\exists$ sentences. To see that T has a model, it is enough to construct a sequence $\mathbb{K} = \langle \mathcal{A}_0, \mathcal{A}_1, \ldots \rangle$ meeting the hypotheses of Lemmas 6.7 and 6.8 with T in place of T_0. We begin by defining $\mathbb{K}$, and then verify that $\mathbb{K}$ has the required properties.

Let Y be the set of all triples $\langle n, s, \sigma \rangle$, where n is a natural number and σ is a function mapping each tuple taken from $\{0, \ldots, n-1\}^{\leq n}$ to a value in $\{0,1\}^{s+1}$, with the property that, if $\bar{y}$ has a repeated entry, we have $\sigma(\bar{y})(t) = 0$ for all $t \leq s$. This Y is an element of $\mathcal{S}$ by Δ^0_1 comprehension. Let G be a surjection $G : M \to Y$. Each $\mathcal{A}_i$ is constructed as follows. Suppose that $G(i) = \langle n, s, \sigma \rangle$. Let $\mathcal{A}_i$ be the L-structure with domain $\{a_0, \ldots, a_{n-1}\}$ such that, for all $1 \leq k \leq n$, all $t \leq s$, and all k-tuples $\langle j_0, \ldots, j_{k-1} \rangle$ taken from $\{0, \ldots, n-1\}$, we have

$$\mathcal{A}_i \models R^k_t(a_{j_0}, \ldots, a_{j_{k-1}}) \iff (t \in X_k \text{ and } \sigma(\langle j_0, \ldots, j_{k-1} \rangle)(t) = 1)\,,$$

and $\mathcal{A}_i \models \neg R^k_t(\bar{a})$ for all other $t, k, \bar{a}$.

It is clear from the definition that $\mathbb{K}$ has the EHP, the EJEP, and the EAP, and hence by Lemma 6.7 has an effective Fraïssé limit $\mathcal{A}$. It can be checked that $\mathbb{K}$ satisfies the hypothesis of Lemma 6.8, and hence $\mathcal{A}$ is a model of T. □

We now show how the coded sequence $\langle X_1, X_2, \ldots \rangle$ relates to $\mathrm{RN}_T(n)$.

Lemma 6.10 (RCA_0)**.** *Define a function F on tuples $\bar{a} = \langle a_1, \ldots, a_n \rangle \in M^{<M}$ by:*

$$\mathsf{F}(\emptyset) = 1,$$

$$\mathsf{F}(\bar{a}) = \sum_{m=1}^{n} S(n,m) \prod_{k=1}^{m} 2^{\left(\frac{m!}{(m-k)!}\right) a_k}, \text{ whenever } |\bar{a}| \geq 1,$$

where $S(n,m)$ is the number of ways to partition an n-element set into m nonempty subsets.[3] *The following statements hold.*

(i) *If $\bar{a} = \langle a_1, \ldots, a_n\rangle$ and $\bar{b} = \langle b_1, \ldots, b_n\rangle$ are n-tuples such that $\mathsf{F}(a_1, \ldots, a_k) = \mathsf{F}(b_1, \ldots, b_k)$ for all $k \leq n$, then $\bar{a} = \bar{b}$.*

(ii) *If the tuple $\langle |X_1|, \ldots, |X_n|\rangle$ exists in M, then $\mathrm{RN}_T(n) = \mathsf{F}(|X_1|, \ldots, |X_n|)$.*

(iii) *If $\mathrm{RN}_T(n)$ is finite, then $\langle |X_1|, \ldots, |X_n|\rangle$ exists in $\mathcal{S}$.*

(iv) *The Ryll-Nardzewski function RN_T exists in $\mathcal{S}$ if and only if the function $n \mapsto |X_n|$ exists in $\mathcal{S}$.*

Proof. (i) This is immediate when $n = 0$. If $\langle a_0, \ldots, a_k\rangle = \langle b_0, \ldots, b_k\rangle$ and $\mathsf{F}(a_0, \ldots, a_{k+1}) = \mathsf{F}(b_0, \ldots, b_{k+1})$, then it is clear from the definition of F that $a_{k+1} = b_{k+1}$. The result now follows by Δ^0_1 induction.

(ii) If $n = 0$, then there is exactly one 0-type, namely T itself, so $\mathrm{RN}_T(0) = 1 = \mathsf{F}(\emptyset)$. The case when $n \geq 1$ follows by a straightforward induction.

(iii) It is clear that, for all $k \leq n$, we have $|X_k| \leq \mathrm{RN}_T(n)$. Using bounded Σ^0_1 comprehension we may form the set $\{\langle k, i\rangle : |X_k| \geq i \text{ and } k \leq n\}$, from which $\langle |X_1|, \ldots, |X_n|\rangle$ is Δ^0_1 definable.

(iv) The 'if' direction is immediate from part (ii). For the 'only if' direction, suppose RN_T is in $\mathcal{S}$, and fix n. We know by parts (i), (ii), and (iii) that $\langle |X_1|, \ldots, |X_n|\rangle$ is in $\mathcal{S}$, and is the unique n-tuple satisfying that $\mathrm{RN}_T(k) = \mathsf{F}(|X_1|, \ldots, |X_k|)$ for every $k \leq n$. Thus we can find $|X_n|$ by testing each n-tuple for this property.

□

6.3 Applications

Recall from §2.1 the statements:

(S1) There is a function f such that, for all n, T has exactly $f(n)$ distinct n-types.

(S2) There is a function f such that, for all n, T has no more than $f(n)$ distinct n-types.

(S3) T has only finitely many n-types, for each n.

We now use the construction of §6.1 to prove Theorem 2.2(i):

Proposition 6.11. *Over RCA_0, the implication (S2 → S1) implies ACA_0.*

Proof. Suppose that (S2 → S1) holds. Let Z be any set, and recall from Definition 1.1 the *Turing jump* K^Z and its enumeration $\langle K^Z_0, K^Z_1, \ldots\rangle$. Define sets $X_1, X_2, \ldots$ by, for each s, n,

$$s \in X_{n+1} \iff n \in K^Z_{at\ s}.$$

The coded sequence $\langle X_1, \ldots\rangle$ exists by Δ^0_1 comprehension. Let T be the theory constructed by the method of §6.1 using $\langle X_1, \ldots\rangle$ as its parameter. Since each X_n has size ≤ 1, we can see by Lemma 6.10(ii) that RN_T is dominated by the function $f(n) = \mathsf{F}(\underbrace{1, 1, \ldots, 1}_{n \text{ times}})$. Hence T satisfies (S2). Since (S2 → S1) holds, T satisfies (S1) as well, that is, RN_T is an element of $\mathcal{S}$. By Lemma 6.10(iv), the function $n \mapsto |X_{n+1}|$ is in $\mathcal{S}$ as well. But this is the characteristic function of K^Z. We conclude by Lemma 1.2 that ACA_0 holds. □

Next, we verify Theorem 2.2(iii):

[3] These $S(n,m)$ are called *Stirling numbers of the second kind.*

Proposition 6.12. *Over* RCA_0*, the implication* ($\mathsf{S3} \to \mathsf{S2}$) *implies* ACA_0*.*

Proof. Suppose that ($\mathsf{S3} \to \mathsf{S2}$) holds. Fix any set Z. Define sets $X_1, X_2, \ldots$ by, for each $s, n \in M$,

$$s \in X_{n+1} \iff (\exists t)[t \leq s < 2t \land n \in K^Z_{at\ t}].$$

If $n \in K^Z_{at\ t}$ for some t, then $|X_{n+1}| = t$; if there is no such t, then $|X_{n+1}| = 0$. The coded sequence $\langle X_1, \ldots \rangle$ exists by Δ^0_1 comprehension. Let T be the theory constructed by the method of §6.1 using $\langle X_1, \ldots \rangle$ as its parameter.

For each $n \geq 1$, $K \upharpoonright n$ exists by bounded Σ^0_1 comprehension, so way may form the tuple $\langle |X_1|, \ldots, |X_n| \rangle$. It follows by Lemma 6.10(ii) that $\mathrm{RN}_T(n)$ is a finite number, and $K^Z \upharpoonright n = K^Z_{\mathrm{RN}_T(n)} \upharpoonright n$. Thus T satisfies ($\mathsf{S3}$). Since ($\mathsf{S3} \to \mathsf{S2}$) holds, T satisfies ($\mathsf{S2}$) as well. Let f be a function such that $f(n) \geq \mathrm{RN}_T(n)$ for all n. Then we have $K^Z \upharpoonright n = K^Z_{f(n)} \upharpoonright n$ for all n, so K^Z is in $\mathcal{S}$ by Δ^0_1 comprehension. We conclude by Lemma 1.2 that ACA_0 holds. □

Finally, we prove Theorem 2.2(ii). In fact, we prove a stronger result.

Proposition 6.13. *Over* RCA_0*, the implication* ($\mathsf{S2} \to$ 'T has a prime model'*) implies* ACA_0*.*

Proof. Fix any set Z. Define a coded sequence of sets $\langle X_1, \ldots \rangle$ and a theory T as in the proof of Proposition 6.11 above. As we have seen, T satisfies ($\mathsf{S2}$). We construct two models $\mathcal{A}, \mathcal{B}$ of T such that, if $\mathcal{C}$ is a third model, and $e_0 : \mathcal{C} \hookrightarrow \mathcal{A}$, $e_1 : \mathcal{C} \hookrightarrow \mathcal{B}$ are embeddings, then K^Z is computable from e_0 and e_1. The models $\mathcal{A}, \mathcal{B}$ will be the effective Fraïssé limits of sequences $\mathbb{K}_0$ and $\mathbb{K}_1$, respectively.

Let Y be the set of all pairs $\langle n, \sigma \rangle$ such that n is a natural number, and $\sigma : \{0, \ldots, n-1\}^{\leq n} \to \{0, 1\}$ is a function such that $\sigma(\bar{x}) = 0$ whenever $\bar{x}$ has a repeated entry. This Y is a recursive set. Let $G : M \to Y$ be an infinite-to-one surjection. We use G to define sequences $\mathbb{K}_0 = \langle \mathcal{A}_0, \mathcal{A}_1, \ldots \rangle$ and $\mathbb{K}_1 = \langle \mathcal{B}_0, \mathcal{B}_1, \ldots \rangle$ of finite structures. If $G(i) = \langle n, \sigma \rangle$, then $\mathcal{A}_i$ has domain $\{a_0, \ldots, a_{n-1}\}$ and, for all s and all tuples $\langle j_0, \ldots, j_{k-1} \rangle \in \{0, \ldots, n-1\}^{\leq n}$ of length $k \geq 1$,

$$\mathcal{A}_i \models R^k_s(a_{j_0}, \ldots, a_{j_{k-1}}) \iff (s \in X_k \text{ and } \sigma(j_0, \ldots, j_{k-1}) = 1 \text{ and } i > s),$$

and, for all other $s, k, \bar{a}$, we have $\mathcal{A}_i \models \neg R^k_s(\bar{a})$. The structure $\mathcal{B}_i$ has domain $\{b_0, \ldots, b_{n-1}\}$ and, for all s and all tuples $\langle j_0, \ldots, j_{k-1} \rangle \in \{0, \ldots, n-1\}^{\leq n}$,

$$\mathcal{B}_i \models R^k_s(b_{j_0}, \ldots, b_{j_{k-1}}) \iff (s \in X_k \text{ and } (\sigma(j_0, \ldots, j_{k-1}) = 1 \text{ or } i \leq s)),$$

and, for all other $s, k, \bar{b}$, we have $\mathcal{B}_i \models \neg R^k_s(\bar{b})$. The coded sequences $\mathbb{K}_0, \mathbb{K}_1$ exist by Δ^0_1 comprehension. It can be checked that $\mathbb{K}_0$ and $\mathbb{K}_1$ each have the EHP, the EJEP, and the EAP, and satisfy the hypotheses of Proposition 6.8. Hence, by Propositions 6.7 and 6.8, $\mathbb{K}_0$ has an effective Fraïssé limit $\mathcal{A} \models T$ and $\mathbb{K}_1$ has an effective Fraïssé limit $\mathcal{B} \models T$.

Now suppose that $\mathcal{C}$ is a model of T with domain C, and $e_0 : \mathcal{C} \hookrightarrow \mathcal{A}$, $e_1 : \mathcal{C} \hookrightarrow \mathcal{B}$ are embeddings. Given a finite $F \subseteq C$, we may use e_0 and the fact that $\mathcal{A}$ is an effective Fraïssé limit to find an index i and an isomorphism from the induced substructure $\mathcal{C} \upharpoonright F$ to $\mathcal{A}_i$. Likewise, we may use e_1 to find an index j and an isomorphism from $\mathcal{C} \upharpoonright F$ to $\mathcal{B}_j$, giving an isomorphism from $\mathcal{A}_i$ to $\mathcal{B}_j$.

Fix enumerations $\bar{a}$ of the elements of $\mathcal{A}_i$ and $\bar{b}$ of the elements of $\mathcal{B}_j$ such that $(\mathcal{A}_i, \bar{a}) \cong (\mathcal{B}_j, \bar{b})$. Let n be the cardinality of F, and suppose that $n \in K^Z$. Then there is an s such that $n \in K^Z_s$ and $X_{n+1} = \{s\}$. We claim that $s \leq \max(i, j)$. To see this, assume that $j < s$, so that $\mathcal{B}_j \models R^{n+1}_s(\bar{b})$ by construction of $\mathcal{B}_j$. Then $\mathcal{A}_i \models R^{n+1}_s(\bar{a})$ as well, which implies by construction of $\mathcal{A}_i$ that $i \geq s$. Our claim now proven, we deduce that n is in K^Z if and only if n is in $K^Z_{\max(i,j)}$. Hence K^Z exists by Δ^0_1 comprehension. We conclude by Lemma 1.2 that ACA_0 holds. □

Corollary 6.14. *Over* RCA_0*, the implication* ($\mathsf{S2} \to \mathsf{S4}$) *implies* ACA_0*.*

7 Theories with finitely many models

In this section, we present a construction due to Millar [15]. Given any $n \geq 2$, it builds a complete, decidable theory T with exactly n decidable models, both up to classical isomorphism and up to recursive isomorphism. We use this construction largely unchanged in the system $\mathsf{RCA}_0 + \neg\mathsf{WKL}_0$ to prove Theorem 2.10. The construction itself is given in §7.1 below. We begin with some definitions and an overview of our goals.

Definition 7.1. A *disjoint* Σ^0_1 *pair* is a coded sequence $\langle U_s, V_s\rangle_{s\in M}$ of pairs $U_s, V_s \subseteq M$ with the following properties:

- Each U_s and V_s is finite, with $\max(U_s \cup V_s) < s$.
- $U_s \cap V_s = \emptyset$ for every s.
- $U_s \subseteq U_{s+1}$ and $V_s \subseteq V_{s+1}$ for every s.

Given a disjoint Σ^0_1 pair $\langle U_s, V_s\rangle_s$, a set $C \subseteq M$ is called a *separating set for* $\langle U_s, V_s\rangle_s$ if, for every s, we have $U_s \subseteq C \subseteq (M - V_s)$. If no such C exists, then $\langle U_s, V_s\rangle_s$ is called an *inseparable* Σ^0_1 *pair*. The Σ^0_1 *separation principle* is the statement: There is no inseparable Σ^0_1 pair.

In the standard model REC of RCA_0, a disjoint Σ^0_1 pair $\langle U_s, V_s\rangle_s$ can be written as a pair of recursive approximations $\langle U_s\rangle_s$, $\langle V_s\rangle_s$ to disjoint r.e. sets $U = \lim_s U_s$ and $V = \lim_s V_s$. If $\langle U_s, V_s\rangle_s$ is an inseparable Σ^0_1 pair in REC, then the limits U and V are *recursively inseparable* in the sense of recursion theory.

We are interested in these pairs, first, because they figure in Millar's construction, and second, because of the following result of Friedman, Simpson, and Smith [7] pinpointing the reverse-mathematical complexity of the Σ^0_1 separation principle.

Lemma 7.2. $\mathsf{RCA}_0 \vdash \mathsf{WKL}_0 \leftrightarrow (\Sigma^0_1 \text{ separation})$

Proof. See Simpson [22, Lemma IV.4.4]. □

Fix a natural number $n \geq 2$ and a disjoint Σ^0_1 pair $\langle U_s, V_s\rangle_s$. Our construction in §7.1 is of a complete, decidable theory T with the following properties:

1. T has exactly one nonprincipal 1-type $p(x)$.
2. For every $k < n$, T has a decidable model $\mathcal{A}$ with exactly k distinct elements realizing p.
3. For every $k \in M$, if $\mathcal{A}, \mathcal{B}$ are models of T each with exactly k distinct elements realizing p, then there is an isomorphism $f : \mathcal{A} \cong \mathcal{B}$ which is Δ^0_1 definable in $\mathcal{A} \oplus \mathcal{B}$.
4. If $\mathcal{A}$ is a model of T with at least n distinct elements realizing p, then there is a separating set C for $\langle U_s, V_s\rangle_s$ which is Δ^0_1 definable in $\mathcal{A}$.

If we are working within a model of $\mathsf{RCA}_0 + \neg\mathsf{WKL}_0$ and $\langle U_s, V_s\rangle_s$ is an inseparable Σ^0_1 pair as given by Lemma 7.2, then the properties above imply that T has exactly n nonisomorphic models. (This is proved in §7.3 below.)

7.1 Construction

Fix a natural number $n \geq 2$ and a disjoint Σ^0_1 pair $\langle U_s, V_s\rangle_s$. Let $L = \langle P_s, R_s\rangle_{s\in M}$ be a language with every P_s unary and every R_s n-ary. Consider the following axiom schemes:

Ax I. $P_s(x) \to P_t(x)$, whenever $t \leq s$.

Ax II. $R_k(x_0, \ldots, x_{n-1}) \to \bigwedge_{i<j<n}(P_k(x_i) \wedge x_i \neq x_j)$

Ax III. $\left(\bigwedge_{i<j<n}(P_s(x_i) \wedge x_i \neq x_j)\right) \to R_k(x_0, \ldots, x_{n-1})$, whenever $k \in U_s$.

Ax IV. $\left(\bigwedge_{i<j<n}(P_s(x_i) \wedge x_i \neq x_j)\right) \to \neg R_k(x_0, \ldots, x_{n-1})$, whenever $k \in V_s$.

Ax V. $\psi(\bar{x}) \to (\exists y)\phi(\bar{x}, y)$ for every pair ϕ, ψ of formulas with the following properties:

- ϕ and ψ are conjunctions of L'-literals, where $L' = \{P_s, R_s : s < \ell\}$ for some ℓ;
- For every atomic L'-formula θ with variables in $\bar{x}, y$, either θ or $\neg\theta$ appears as a conjunct in ψ;
- $\phi(\bar{x}, y)$ is consistent with Ax I–IV;
- Every conjunct in ψ is a conjunct in ϕ;

Let T^* be the collection of all sentences in Ax I–V, and let T be the deductive closure of T^*. This completes the construction. Notice that we have not yet established that either T^* or T is in $\mathcal{S}$. The existence of T^* is a consequence of Lemma 7.3 below, while that of T is part of Proposition 7.5.

The intuition behind these axioms is as follows. Given an element a of a model and an index s, the statement $P_s(a)$ is read as, 'a is turned on at stage s'. Axiom I says that the stages at which an element is turned on form an initial segment of M—possibly $\emptyset$ or all of M. Axiom II says that R_k can hold of a tuple $\bar{a}$ only if the entries of $\bar{a}$ are all distinct and are all turned on at stage k. Axioms III and IV together say that if $\bar{a}$ is a tuple of distinct elements, all turned on at stage s, then $U_s \subseteq \{k : R_k(\bar{a}) \text{ holds}\} \subseteq M - V_s$. As with the similar axiom in §6.1 above, Axiom V gives the theory effective quantifier elimination.

7.2 Verification

Lemma 7.3 (RCA_0). *There is a procedure to decide whether a given L-formula ϕ is consistent with Axioms I–IV.*

Proof. Assume that $k \in U_s \cup V_s$ implies $k < s$. Combine Axioms I, III, and IV into a single equivalent scheme of the form:

$$P_s(x_0) \to \left(\bigwedge_{t \leq s} P_t(x_0) \wedge \left(\bigwedge_{i<j<n} P_s(x_i) \wedge x_i \neq x_j\right) \to \left(\bigwedge_{k \in U_s} R_k(x_0, \ldots, x_{n-1}) \wedge \bigwedge_{k \in V_s} \neg R_k(x_0, \ldots, x_{n-1})\right)\right).$$

As in the proof of Lemma 6.2, we may replace the initial $\to$ with $\vee$ in both this scheme and Axiom II and perform an appropriate reindexing of the relations to get a sequence of sentences satisfying the hypothesis of Lemma 6.1 above. The result follows. □

It follows that T^* is in $\mathcal{S}$.

Lemma 7.4 (RCA_0). *The theory T^* has quantifer elimination.*

Proof. Similar to the proof of Lemma 6.4. □

Proposition 7.5. *T is in $\mathcal{S}$, is complete, and has quantifier elimination.*

Proof. Similar to the proof of Proposition 3.2. □

Lemma 7.6 (RCA_0). *The theory T is consistent.*

Proof. It suffices by the Soundness Theorem to show that T has a model. Suppose that $\mathcal{A}$ is a finite L-structure, and suppose that there is an s_0 such that, for every $s \geq s_0$, every n-tuple $\bar{a}$ of elements of $\mathcal{A}$, and every entry a_i of $\bar{a}$, we have $\mathcal{A} \models \neg P_s(a_i)$ and $\mathcal{A} \models \neg R_s(\bar{a})$. Then there is a recursive procedure to check whether $\mathcal{A}$ is a model of Axioms I–IV. Let $\mathbb{K}$ be an infinite-to-one enumeration of all finite L-structures which have such an s_0 and which are consistent with Axioms I–IV. This $\mathbb{K}$ satisfies the hypotheses of Lemma 6.8, and hence, by Lemmas 6.7 and 6.8, has an effective Fraïssé limit which is a model of T. □

Lemma 7.7 ($\mathsf{RCA_0}$)**.** *T has exactly one nonprincipal 1-type $p(x)$. Furthermore, $P_s(x)$ is in $p(x)$ for every s, and if $q(x)$ is a 1-type of T not equal to $p(x)$, then there is an s such that $\neg P_s(x) \in q$.*

Proof. As in Harizanov [9, Lemma 10.7]. □

Lemma 7.8 ($\mathsf{RCA_0}$)**.** *For every $k < n$, T has a decidable model $\mathcal{A}$ with exactly k distinct elements realizing p.*

Proof. Use a Fraïssé construction similar to that in the proof of Lemma 7.6, except, instead of just one, allow up to k distinct elements to realize p. □

Lemma 7.9 ($\mathsf{RCA_0}$)**.** *Fix a number $k < n$ and models $\mathcal{A}, \mathcal{B}$ of T. If $\mathcal{A}$ and $\mathcal{B}$ each have exactly k distinct elements realizing p, then $\mathcal{A} \cong \mathcal{B}$.*

Proof. An effective back-and-forth argument. □

Lemma 7.10 ($\mathsf{RCA_0}$)**.** *If $\mathcal{A}$ is a model of T with at least n distinct elements realizing p, then there is a separating set C for $\langle U_s, V_s \rangle_s$. In particular, $\langle U_s, V_s \rangle_s$ is not an inseparable Σ^0_1 pair.*

Proof. Suppose $\mathcal{A}$ is such a model, and let $\bar{a}$ be a tuple of distinct elements all realizing p. Define $C = \{k : \mathcal{A} \models R_k(\bar{a})\}$. Then Ax III ensures that $U_s \subseteq C$ for all s, and Ax IV ensures $V_s \subseteq M - C$ for all s. Therefore, C is a separating set for $\langle U_s, V_s \rangle_s$. □

7.3 Application

We now prove the remaining theorem from §2.2.

Proof of Theorem 2.10. Assume $\mathsf{WKL_0}$ fails. When $n = 1$, use the $\aleph_0$-categorical theory constructed in the proof of Proposition 3.5. (Alternatively, we could use an effectively $\aleph_0$-categorical theory such as the theory of dense linear orders without endpoints.) Now suppose $n \geq 2$. Lemma 7.2 tells us that there is an inseparable Σ^0_1 pair $\langle U_s, V_s \rangle_s$. Let T be the theory constructed by the method of §7.1 using $\langle U_s, V_s \rangle_s$ and the given n. Lemmas 7.8, 7.9, and 7.10 together imply that T has exactly n models up to isomorphism. □

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