Accepted Manuscript

International Journal of Computational Methods

Article Title: Equal-Order Polygonal Analysis for Fluid Computation in Curved Domain

Author(s): T. Vu-Huu, C. Le-Thanh, H. Nguyen-Xuan, M. Abdel Wahab

DOI: 10.1142/S0219876220400034

Received: 07 July 2019

Accepted: 22 January 2020

To be cited as: T. Vu-Huu *et al.*, Equal-Order Polygonal Analysis for Fluid Computation in Curved Domain, *International Journal of Computational Methods*, doi: 10.1142/S0219876220400034

Link to final version: <https://doi.org/10.1142/S0219876220400034>

This is an unedited version of the accepted manuscript scheduled for publication. It has been uploaded in advance for the benefit of our customers. The manuscript will be copyedited, typeset and proofread before it is released in the final form. As a result, the published copy may differ from the unedited version. Readers should obtain the final version from the above link when it is published. The authors are responsible for the content of this Accepted Article.

EQUAL-ORDER POLYGONAL ANALYSIS FOR FLUID COMPUTATION IN CURVED DOMAIN

T. VU-HUU

Faculty of Civil Engineering, Vietnam Maritime University

*Department of Electrical energy, metals, mechanical constructions and systems, Faculty of
Engineering and Architecture, Ghent University*

Email address: truongvh.ctt@vimaru.edu.vn, huttruong.vu@ugent.be

C. LE-THANH

Faculty of Civil Engineering and Electricity, Ho Chi Minh City Open University, Vietnam

Email address: cuong.lt@ou.edu.vn,

H. NGUYEN-XUAN

*CIRTech Institute, Ho Chi Minh City University of Technology (HUTECH), Ho Chi Minh City,
Vietnam*

*Department of Architectural Engineering, Sejong University, 209 Neungdong-ro, Gwangjin-gu,
Seoul 05006, Republic of Korea*

Email address: magd.abdelwahab@ugent.be

M. ABDEL WAHAB*

*Division of Computational Mechanics, Ton Duc Thang University, Ho Chi Minh City, Vietnam
Faculty of Civil Engineering, Ton Duc Thang University, Ho Chi Minh City, Vietnam*

**Corresponding author: magd.abdelwahab@tdt.edu.vn*

Abstract: Our research introduces a novel advanced method for the fluid computation on the complicated domain. It is an advanced polygonal finite element applied equal-order scheme, within each element, both velocity and pressure fields are represented by the polygonal basis function system. In detail, this technique is based on a combination of the equal-order mixed scheme method, the polygon finite element method (PFEM) and the adapted local projection method that is used to stabilize the pressure result. A numerical test, in this research, of the incompressible fluid flow governing by Stokes system in the curved domain is performed to present the applicability of our developed polygon element. Besides, through the numerical test in this research, we will show the high flexibility of PFEM for curved domain. In addition, in order to simulate the test, MATLAB is utilized for coding and programming.

Keywords: Stokes equation, fluid computation, polygonal finite element method, mixed scheme.

1. Introduction

PFEM, one of the most advanced finite element methods in the recent years, is a powerful technique that contains many desirable characteristics, e.g. the

flexibility, the advantages of Voronoi diagrams' properties as well as the benefits of mesh generation algorithms (Sukumar and Tabarraei 2004, Tabarraei and Sukumar 2006, Chau, Chau et al. 2017). In addition, PFEM produces the calculation progress of higher accuracy for applying in the quadrilateral or triangular discretizations with the lower demand of the mesh size (Nguyen-Xuan 2017). However, the fact is that the application of PFEM in fluid problems still has limitations. Specially, there only exists a research of PFEM for fluid computation that is presented by Talischi, et.al. (Talischi, Pereira et al. 2014). This research introduced a low-order mixed scheme of polygonal finite element that has crucial obstacles. Especially, it does not work for quadrilateral and triangular meshes. Thus, this paper introduces the applying of PFEM into the fluid problem. This application is based on the polygonal equal-order mixed scheme, named Pe_1Pe_1 , that both pressure and velocity fields are constructed by the set of Wachspress polygonal functions. And the other applications using numerical methods were also conducted in Ref. (Xiao, Fang et al. 2013).

It is well-known that an equal-order mixed scheme method causes the instability problem, the poor or non-convergence (Brezzi and Douglas 1988). In other words, it should be noted that our proposed element, Pe_1Pe_1 , does not naturally fulfilled the discrete inf-sup condition governs the stability and convergence of mixed finite approximation in fluid flow computation (Chapelle and Bathe 1993, Bathe 2001, Elman, Silvester et al. 2014). In this case, the inconsistency elimination of approximations is necessary. This is done by adding a special term that accomplishes the convergence and the stability of the approximation progress, e.g. Refs. (Chapelle and Bathe 1993, Bathe 2001, Barth, Bochev et al. 2004). It is called the stabilization method. Furthermore, as known, it is necessary for equal-order mixed method that completely produces the instability (Elman, Silvester et al. 2014). Our proposed method is handled by adding an adopted term of pressure projection to generates the stabilization. Then, in order to take the advantages of PFEM, the coordinate of Wachspress that generates the system of basis shape functions (Wachspress 1975, Floater, Gillette et al. 2014) are adapted for Pe_1Pe_1 element. In addition, to get the quality polygonal mesh, the advanced methods of polygonal meshing developed in Ref. (Talischi, Paulino et al. 2012) are applied to solve the significant bottleneck of generation of polygonal mesh.

This paper's content contains: the introduction is presented in Section 1. And then, the presentations of the incompressible Stokes equation and discretization

system is given in Section 2. Section 3, is to demonstrates the finite mixed method as well as the polygonal shape functions. Next, a numerical test is produced in Section 4. Finally, Section 5, will put forward the conclusions.

2. Stokes governing equation and discretization system

In general case of an incompressible Newtonian fluid on the domain Ω , the Stokes equations take the strong form as:

$$-\mu \nabla^2 \mathbf{u} + \nabla p = \mathbf{f}, \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (2)$$

where μ is dynamic viscosity constant; ∇p gradient of pressure; $\mathbf{u}$ velocity of fluid and $\mathbf{f}$ is body force vector. Then, the approximation of Eq. (1) and (2) by a pair of test function $\{\mathbf{v} \in \mathbf{H}^1(\Omega)^2 \mid \mathbf{v} = 0 \text{ on } \Gamma\Omega^D, q \in L^2(\Omega) \mid \int_{\Omega} q d\Omega = 0\}$ generates the weak form as:

$$\int_{\Omega} (\nabla p - \mu \nabla^2 \mathbf{u}) \cdot \mathbf{v} d\Omega = \mathbf{f}, \quad (3)$$

$$\int_{\Omega} (\nabla \cdot \mathbf{u}) q d\Omega = 0. \quad (4)$$

Utilizing theorem of divergence and integration by parts into Eq. (3) gives:

$$\begin{aligned} \int_{\Omega} \nabla \mathbf{v} : \nabla \mathbf{u} d\Omega - \int_{\Omega} (\nabla p - \mu \nabla^2 \mathbf{u}) \cdot \mathbf{v} d\Omega \\ = \int_{\Omega} (\nabla \mathbf{v}) p d\Omega + \int_{\Gamma\Omega} \mathbf{v} \cdot \left(-p \mathbf{n} + \frac{\partial \mathbf{u}}{\partial n} \right) d\Gamma\Omega. \end{aligned} \quad (5)$$

Then, the result of the above replacing produces:

$$\mu \int_{\Omega} \nabla \mathbf{v} : \nabla \mathbf{u} d\Omega - \int_{\Omega} (\nabla \cdot \mathbf{v}) p d\Omega = \int_{\Gamma\Omega^N} \mathbf{f} \cdot \mathbf{v} d\Gamma\Omega, \quad (6)$$

$$\int_{\Omega} (\nabla \cdot \mathbf{u}) q d\Omega = 0, \quad (7)$$

in which, $\int_{\Omega} (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \mathbf{v} d\Omega$ stands for the diffusion term and the convection one is indicated by $\nu \int_{\Omega} \nabla \mathbf{v} : \nabla \mathbf{u} d\Omega$.

In case of using the Pe_1Pe_1 element and the Eqs. (1) and (2) system discretized by the set basis functions $\{\psi_k\}$ and $\{\phi_j\}$ for pressure and velocity, respectively:

$$\mathbf{u} = \sum_{j=n_u+1}^{n_u+n_\Gamma} \boldsymbol{\phi}_j \mathbf{u}_j + \sum_{j=1}^{n_u} \boldsymbol{\phi}_j \mathbf{u}_j, \quad (8)$$

$$p = \sum_{k=1}^{n_p} \psi_k \mathbf{p}_k, \quad (9)$$

with $\sum_{j=1}^{n_u} \boldsymbol{\phi}_j \mathbf{u}_j \in \mathbf{X}_0^h, j = n_u + 1, \dots, n_u + n_\Gamma$ and $k = 1, \dots, n_p$ is the fixed coefficients of degree of freedoms.

Then, the discrete system can be expressed linearly as follow:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B}^T \\ \mathbf{B} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix}, \quad (10)$$

with, $\mathbf{A}$ and $\mathbf{B}$ respectively is the vector-Laplacian and the matrix of divergence:

$$\mathbf{A} = \left[\int_{\Omega} \nabla \boldsymbol{\phi}_j : \nabla \boldsymbol{\phi}_i d\Omega \right], \quad (11)$$

$$\mathbf{B} = \left[- \int_{\Omega} (\nabla \cdot \boldsymbol{\phi}_j) \psi_k d\Omega \right]. \quad (12)$$

Besides, the right-hand side terms are:

$$\mathbf{f} = \int_{\Gamma \Omega^N} \mathbf{f} \cdot \boldsymbol{\phi}_i d\Gamma - \sum_{j=n_u+1}^{n_u+n_\Gamma} \mathbf{u}_j \int_{\Omega} \nabla \boldsymbol{\phi}_j : \nabla \boldsymbol{\phi}_i d\Omega, \quad (13)$$

$$\mathbf{g} = \sum_{j=n_u+1}^{n_u+n_\Gamma} \mathbf{u}_j \int_{\Omega} (\nabla \cdot \boldsymbol{\phi}_j) \psi_k d\Omega. \quad (14)$$

As mentioned before, the mixed scheme of equal-order pair completely does not meet the inf-sup condition. It means that $\text{Pe}_1 \text{Pe}_1$ leads to unstable approximated results and has inherent inconsistency. Therefore, the discrete system of Eq. (10) need a special treatment eliminating the inconsistency of approximation progress. The added term used in this research is the adopted stabilization local projection pressure that is written as follows:

$$\mathbf{C} = \left[c_{ij} = \sum_{i,j=1}^{n_{ne}} \int_{\Omega_e} \left(\psi_j - \frac{1}{n_{ne}} \right) \left(\psi_i - \frac{1}{n_{ne}} \right) d\Omega \right] \quad (15)$$

where c_{ij} the local pressure projection and the indices e and n_e is the e^{th} element and the total number of the element in the $\mathfrak{I}_h$, respectively. As a result, the stabilized discrete system of Stokes now becomes:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B}^T \\ \mathbf{B} & -\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix}. \quad (16)$$

3. Equal-order mixed scheme of polygonal finite element

This section is to demonstrate the Pe_1Pe_1 element as shown in Fig. 1. Within each element, both velocity and pressure fields are represented by the similar polygonal basis functions.

Fig. 1. Pe_1Pe_1 element.

For a convex polygon of a domain $\mathfrak{V}$ with n_{ne} vertices, $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{n_{ne}}$, ($n_{ne} \geq 3$), the polygonal shape functions of Wachspress coordinators for a point $\mathbf{v} \in \Omega^e$ are defined by:

$$\psi_i^e = \frac{\psi_i}{\sum_{j=1}^{n_{ne}} \psi_j} = \frac{\psi_i}{\psi} \quad \text{with} \quad \psi_i = \frac{S(\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1})}{S(\mathbf{v}, \mathbf{x}_{i-1}, \mathbf{x}_i)S(\mathbf{v}, \mathbf{x}_i, \mathbf{x}_{i+1})} \quad (17)$$

where $S(\mathbf{x}_a, \mathbf{x}_b, \mathbf{x}_c)$ is the area of the triangle $[\mathbf{x}_a, \mathbf{x}_b, \mathbf{x}_c]$, see in Fig. 2 (a). And as the perpendicular coordinates, see in Fig. 2 (c). For $h_i(\mathbf{x})$ stands for the distance from the sides $\mathbf{e}_i$ to $\mathbf{v}$.

(a) Triangular coordinates.

(b) 3D sample.

(c) Perpendicular coordinates.

Fig. 2. Wachspress shape functions for a hexagon.

Then, the Wachspress polygonal shape functions are defined as:

$$\psi_i^e = \frac{\psi_i}{\psi} = \frac{\tilde{\psi}_i}{\sum_{j=1}^{n_{ne}} \tilde{\psi}_j} \text{ with } \tilde{\psi}_i = \det(\mathbf{p}_{i-1}, \mathbf{p}_i), \quad (18)$$

in which, $\mathbf{p}_i(\mathbf{x})$ is defined by the ratio $\frac{\mathbf{n}_i}{h_i(\mathbf{x})}$, where $\mathbf{n}_i$ outward normal vector of $\mathbf{e}_i$. And the gradient is clarified by:

$$\nabla \psi_i^e = \psi_i^e \left[\mathbf{v}_i - \sum_{i=1}^{n_{ne}} \psi_i^e \mathbf{v}_i \right] \text{ with } \mathbf{v}_i = \mathbf{p}_{i-1} + \mathbf{p}_i. \quad (19)$$

4. Numerical example

In this section, we introduce a numerical test for 2-D incompressible Stokes flow under the utilizing of the Pe_1Pe_1 element. For this test, a domain of annular $\Omega = \{(x, y) \in \mathbb{R}^2: 0.5^2 < r_\Omega^2 = x^2 + y^2 < 2^2\}$ is generated as in Fig. 3, and for this domain, r_Ω is the radius of domain Ω . The following conditions are

- The constant of kinematic viscosity is one for all domain;
- The source term $\mathbf{f}$ and boundary data on $\Gamma\Omega$ are given so that the exact solution is:

$$\begin{aligned}
u_x &= \sin(y)\sin(x), \\
u_y &= \cos(x)\cos(y), \\
p &= \sin(y)\sin(x).
\end{aligned}
\tag{20}$$

From the polygonal mesh of this test, see Fig. 3, it can be seen that the good regular elements for the curved domain ideally created. Furthermore, from the result of this test presented in the Fig. 4, it can be seen that Pe_1Pe_1 element completely generated good result for both pressure and velocity of fluid computation. Then, the results of fluid pressure and velocity at a particular node (1,1), shown in Fig. 3, are presented to consider the convergence and accuracy of Pe_1Pe_1 element for the curved domain.

Fig. 3. The annual domain.

(a)

Fig. 4. The result of fluid velocity (a) and pressure (b) in the annular domain.

It is clear from Fig. 4 that fluid velocity and pressure results are visualized smoothly. In addition, the velocity and pressure at Node (1,1) are presented in Table 1 and Fig. 5 shows the convergence rate for the fluid pressure and velocity.

Table 1. The fluid pressure and velocity at the tested point (1,1).

No.	Total DOF of mesh	Velocity (m/s)				Pressure (kPa)	
		v_x	Error (%)	v_y	Error (%)	p	Error (%)
1	606	0.7152	1.0131	0.2934	0.4917	0.7180	1.3987
2	1794	0.7118	0.5221	0.2926	0.2450	0.7120	0.5542
3	3597	0.7105	0.3405	0.2919	0.0062	0.7102	0.3033
4	5997	0.7086	0.0776	0.2918	0.0481	0.7097	0.2247
5	7197	0.7090	0.1317	0.2918	0.0497	0.7083	0.0374
6	8997	0.7088	0.1016	0.2918	0.0318	0.7083	0.0261
7	14997	0.7083	0.0380	0.2918	0.0267	0.7083	0.0254
8	29997	0.7082	0.0319	0.2919	0.0015	0.7082	0.0051
Exact solutions		0.7081		0.2919		0.7081

(a)

(b)

(c)

Fig. 5. The result of fluid velocity (a), (b) and pressure (c) at Node (1,1) of the domain.

5. Conclusion

Our developed Pe_1Pe_1 element satisfied the computation of the fluid flow problem in the complicated domain. In addition, from the results of the numerical test, the applicability of PFEM to fluid problem is very satisfactory. In addition, the high flexibility of PFEM for curved domain is proved through the numerical test in this research. However, in this research we only present the development of our proposed method for Stokes fluid problem. Thus, a further task of our research is to study to solve the stationary or transient incompressible Navier-Stokes problems and then the fluid-structure interaction ones. Besides, as known, solving Navier-Stokes equations asks the solutions for nonlinear term that causes the high computation cost. The demand of reduction of computation cost is one of the most crucial desire for our research. Thus, non-linear model reduction for the Navier–Stokes equations using residual DEIM method will be considered (Xiao, Fang et al. 2013).

Acknowledgement

The authors are grateful to the VLIR-UOS TEAM Project, VN2017TEA454A103, ‘An innovative solution to protect Vietnamese coastal riverbanks from floods and erosion’ financially supported by Flemish Government.

References

- Barth, T., P. Bochev, M. Gunzburger and J. Shadid (2004). "A taxonomy of consistently stabilized finite element methods for the Stokes problem." SIAM Journal on Scientific Computing **25**(5): 1585-1607.
- Bathe, K.-J. (2001). "The inf-sup condition and its evaluation for mixed finite element methods." Computers & structures **79**(2): 243-252.
- Brezzi, F. and J. Douglas (1988). "Stabilized mixed methods for the Stokes problem." Numerische Mathematik **53**(1-2): 225-235.
- Chapelle, D. and K.-J. Bathe (1993). "The inf-sup test." Computers & structures **47**(4-5): 537-545.

Chau, K. N., K. N. Chau, T. Ngo, K. Hackl and H. Nguyen-Xuan (2017). "A polytree-based adaptive polygonal finite element method for multi-material topology optimization." Computer Methods in Applied Mechanics and Engineering.

Elman, H. C., D. J. Silvester and A. J. Wathen (2014). Finite elements and fast iterative solvers: with applications in incompressible fluid dynamics, Numerical Mathematics and Scientific Computation.

Floater, M., A. Gillette and N. Sukumar (2014). "Gradient bounds for Wachspress coordinates on polytopes." SIAM Journal on Numerical Analysis **52**(1): 515-532.

Nguyen-Xuan, H. (2017). "A polytree-based adaptive polygonal finite element method for topology optimization." International Journal for Numerical Methods in Engineering **110**(10): 972-1000.

Sukumar, N. and A. Tabarraei (2004). "Conforming polygonal finite elements." International Journal for Numerical Methods in Engineering **61**(12): 2045-2066.

Tabarraei, A. and N. Sukumar (2006). "Application of polygonal finite elements in linear elasticity." International Journal of Computational Methods **3**(04): 503-520.

Talischi, C., G. H. Paulino, A. Pereira and I. F. Menezes (2012). "PolyMesher: a general-purpose mesh generator for polygonal elements written in Matlab." Structural and Multidisciplinary Optimization **45**(3): 309-328.

Talischi, C., A. Pereira, G. H. Paulino, I. F. Menezes and M. S. Carvalho (2014). "Polygonal finite elements for incompressible fluid flow." International Journal for Numerical Methods in Fluids **74**(2): 134-151.

Wachspress, E. L. (1975). A rational finite element basis, Elsevier.

Xiao, D., F. Fang, J. Du, C. Pain, I. Navon, A. Buchan, A. H. Elsheikh and G. Hu (2013). "Non-linear Petrov–Galerkin methods for reduced order modelling of the Navier–Stokes equations using a mixed finite element pair." Computer Methods In Applied Mechanics and Engineering **255**: 147-157.