REGULAR PSEUDO-HYPEROVALS AND REGULAR PSEUDO-OVALS IN EVEN CHARACTERISTIC

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ABSTRACT. S. Rottey and G. Van de Voorde characterized regular pseudo-ovals of \( PG(3n-1,q), q = 2^h, h > 1 \) and \( n \) prime. Here an alternative proof is given and slightly stronger results are obtained.

1. INTRODUCTION

Pseudo-ovals and pseudo-hyperovals were introduced in [10]; see also [12]. These objects play a key role in the theory of translation generalized quadrangles [6, 12]. Pseudo-hyperovals only exist in even characteristic. A characterization of regular pseudo-ovals in odd characteristic was given in [2]; see also [12]. In [8] a characterization of regular pseudo-ovals and regular pseudo-hyperovals in \( PG(3n-1,q), q \) even, \( q \neq 2 \) and \( n \) prime, is obtained. Here a shorter proof is given and slightly stronger results are obtained.

2. OVALS AND HYPEROVALS

A \( k \)-arc in \( PG(2,q) \) is a set of \( k \) points, \( k \geq 3 \), no three of which are collinear. Any non-singular conic of \( PG(2,q) \) is a \( (q+1) \)-arc. If \( K \) is any \( k \)-arc of \( PG(2,q) \), then \( k \leq q + 2 \). For \( q \) odd \( k \leq q + 1 \) and for \( q \) even a \( (q+1) \)-arc extends to a \( (q+2) \)-arc; see [3]. A \( (q+1) \)-arc is an oval; a \( (q+2) \)-arc, \( q \) even, is a complete oval or hyperoval.

A famous theorem of B. Segre [9] tells us that for \( q \) odd every oval of \( PG(2,q) \) is a non-singular conic. For \( q \) even, there are many ovals that are not conics [3]; also, there are many hyperovals that do not contain a conic [3].

3. GENERALIZED OVALS AND HYPEROVALS

Arcs, ovals and hyperovals can be generalized by replacing their points with \( m \)-dimensional subspaces to obtain generalized \( k \)-arcs, generalized ovals and generalized hyperovals. These have strong connections to generalized quadrangles, projective planes, circle geometries, flocks and other structures. See [6, 12, 10, 11, 2, 7]. Below, some basic definitions and results are formulated; for an extensive study, many applications and open problems, see [12].

A generalized \( k \)-arc of \( PG(3n-1,q), n \geq 1 \), is a set of \( k \) \( (n-1) \)-dimensional subspaces of \( PG(3n-1,q) \) every three of which generate \( PG(3n-1,q) \). If \( q \) is odd then \( k \leq q^n + 1 \), if \( q \) is even then \( k \leq q^n + 2 \). Every generalized \( (q^n+1) \)-arc of \( PG(3n-1,q), q \) even, can be extended to a generalized \( (q^n+2) \)-arc.
If $\mathcal{O}$ is a generalized $(q^n + 1)$-arc in $\PG(3n - 1, q)$, then it is a pseudo-oval or generalized oval or $[n-1]$-oval of $\PG(3n - 1, q)$. For $n = 1$, a $[0]$-oval is just an oval of $\PG(2, q)$. If $\mathcal{O}$ is a generalized $(q^n + 1)$-arc in $\PG(3n - 1, q)$, $q$ even, then it is a pseudo-hyperoval or generalized hyperoval or $[n-1]$-hyperoval of $\PG(3n - 1, q)$. For $n = 1$, a $[0]$-hyperoval is just a hyperoval of $\PG(2, q)$.

If $\mathcal{O} = \{\pi_0, \pi_1, \ldots, \pi_{q^n}\}$ is a pseudo-oval of $\PG(3n - 1, q)$, then $\pi_i$ is contained in exactly one $(2n - 1)$-dimensional subspace $\tau_i$ of $\PG(3n - 1, q)$ which has no point in common with $(\pi_0 \cup \pi_1 \cup \cdots \cup \pi_{q^n}) \setminus \pi_i$, with $i = 0, 1, \ldots, q^n$; the space $\tau_i$ is the tangent space of $\mathcal{O}$ at $\pi_i$. For $q$ even the $q^n + 1$ tangent spaces of $\mathcal{O}$ contain a common $(n - 1)$-dimensional space $\pi_{q^n + 1}$, the nucleus of $\mathcal{O}$; also, $\mathcal{O} \cup \{\pi_{q^n + 1}\}$ is a pseudo-hyperoval of $\PG(3n - 1, q)$. For $q$ odd, the tangent spaces of a pseudo-oval $\mathcal{O}$ are the elements of a pseudo-oval $\mathcal{O}^*$ in the dual space of $\PG(3n - 1, q)$.

4. Regular pseudo-ovals and pseudo-hyperovals

In the extension $\PG(3n-1, q^n)$ of $\PG(3n-1, q)$, consider $n$ planes $\xi_i$, $i = 1, 2, \ldots, n$, that are conjugate in the extension $\mathbb{F}_{q^n}$ of $\mathbb{F}_q$ and which span $\PG(3n - 1, q^n)$. This means that they form an orbit of the Galois group corresponding to this extension and span $\PG(3n - 1, q^n)$.

In $\xi_1$ consider an oval $\mathcal{O}_1 = \{x_0^{(1)}, x_1^{(1)}, \ldots, x_{q^n}^{(1)}\}$. Further, let $x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{(n)}$, with $i = 0, 1, \ldots, q^n$, be conjugate in $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$. The points $x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{(n)}$ define an $(n - 1)$-dimensional subspace $\tau_i$ over $\mathbb{F}_q$ for $i = 0, 1, \ldots, q^n$. Then, $\mathcal{O} = \{\pi_0, \pi_1, \ldots, \pi_{q^n}\}$ is a generalized oval of $\PG(3n - 1, q)$. These objects are the regular or elementary pseudo-ovals. If $\mathcal{O}_1$ is replaced by a hyperoval, and so $q$ is even, then the corresponding $\mathcal{O}$ is a regular or elementary pseudo-hyperoval.

All known pseudo-ovals and pseudo-hyperovals are regular.

5. Characterizations

Let $\mathcal{O} = \{\pi_0, \pi_1, \ldots, \pi_{q^n}\}$ be a pseudo-oval in $\PG(3n - 1, q)$. The tangent space of $\mathcal{O}$ at $\pi_i$ will be denoted by $\tau_i$, with $i = 0, 1, \ldots, q^n$. Choose $\pi_i$, $i \in \{0, 1, \ldots, q^n\}$, and let $\PG(2n - 1, q) \subseteq \PG(3n - 1, q)$ be skew to $\pi_i$. Further, let $\tau_i \cap \PG(2n - 1, q) = \eta_i$ and $\langle \pi_i, \tau_i \rangle \cap \PG(2n - 1, q) = \eta_i$, with $j \neq i$. Then $\{\eta_0, \eta_1, \ldots, \eta_{q^n}\} = \Delta^*$ is an $(n - 1)$-spread of $\PG(2n - 1, q)$.

Now, let $q$ be even and let $\pi$ be the nucleus of $\mathcal{O}$. Let $\PG(2n - 1, q) \subseteq \PG(3n - 1, q)$ be skew to $\pi$. If $\zeta_j = \PG(2n - 1, q) \cap \langle \pi, \pi_j \rangle$, then $\{\zeta_0, \zeta_1, \ldots, \zeta_{q^n}\} = \Delta$ is an $(n - 1)$-spread of $\PG(2n - 1, q)$.

Next, let $q$ be odd. Choose $\pi_i$, with $i \in \{0, 1, \ldots, q^n\}$. If $\pi_i \cap \tau_j = \delta_j$, with $j \neq i$, then $\{\delta_0, \delta_1, \ldots, \delta_{q^n} \} = \Delta^*$ is an $(n - 1)$-spread of $\pi_i$.

Finally, let $q$ be even and let $\mathcal{O} = \{\pi_0, \pi_1, \ldots, \pi_{q^n+1}\}$ be a pseudo-hyperoval in $\PG(3n - 1, q)$. Choose $\pi_i$, with $i \in \{0, 1, \ldots, q^n + 1\}$, and let $\PG(2n - 1, q) \subseteq \PG(3n - 1, q)$ be skew to $\pi_i$. Let $\langle \pi_i, \tau_i \rangle \cap \PG(2n - 1, q) = \eta_j$, with $j \neq i$. Then $\{\eta_0, \eta_1, \ldots, \eta_{q^n+1}\} = \Delta^*$ is an $(n - 1)$-spread of $\PG(2n - 1, q)$.

**Theorem 5.1** (Casse, Thas and Wild [2]). Consider a pseudo-oval $\mathcal{O}$ with $q$ odd. Then at least one of the $(n - 1)$-spreads $\Delta_0, \Delta_1, \ldots, \Delta_{q^n}, \Delta^*_0, \Delta^*_1, \ldots, \Delta^*_{q^n}$ is regular.
if and only if they all are regular if and only if the pseudo-oval $O$ is regular. In such a case $O$ is essentially a conic over $\mathbb{F}_q^n$.

**Theorem 5.2** (Rottey and Van de Voorde [8]). Consider a pseudo-oval $O$ in $\text{PG}(3n-1, q)$ with $q = 2^h$, $h > 1$, $n$ prime. Then $O$ is regular if and only if all $(n-1)$-spreads $\Delta_0, \Delta_1, \ldots, \Delta_{q^n}$ are regular.

6. ALTERNATIVE PROOF AND IMPROVEMENTS

**Theorem 6.1.** Consider a pseudo-hyperoval $O$ in $\text{PG}(3n-1, q)$, $q = 2^h$, $h > 1$ and $n$ prime. Then $O$ is regular if and only if all $(n-1)$-spreads $\Delta_i$, with $i = 0, 1, \ldots, q^n + 1$, are regular.

**Proof.** If $O$ is regular, then clearly all $(n-1)$-spreads $\Delta_i$, with $i = 0, 1, \ldots, q^n + 1$, are regular.

Conversely, assume that the $(n-1)$-spreads $\Delta_0, \Delta_1, \ldots, \Delta_{q^n+1}$ are regular. Let $O = \{\pi_0, \pi_1, \ldots, \pi_{q^n+1}\}$ and let $O = \{\beta_0, \beta_1, \ldots, \beta_{q^n+1}\}$ be the dual of $O$, with $\beta_i$ being the dual of $\pi_i$.

Choose $\beta_i, i \in \{0, 1, \ldots, q^n + 1\}$, and let $\beta_i \cap \beta_j = \alpha_{ij}, j \neq i$. Then

$$\{\alpha_{i0}, \alpha_{i1}, \ldots, \alpha_{i,i-1}, \alpha_{i,i+1}, \ldots, \alpha_{i,q^n+1}\} = \Gamma_i$$

is an $(n-1)$-spread of $\beta_i$.

Now consider $\beta_i, \beta_j, \Gamma_i, \Gamma_j, \alpha_{ij}, j \neq i$. In $\Gamma_j$ we next consider a $(n-1)$-regulus $\gamma_j$ containing $\alpha_{ij}$. The $(n-1)$-regulus $\gamma_j$ is a set of maximal spaces of a Segre variety $S_{1:n-1}^1$; see Section 4.5 in [4]. The $(n-1)$-regulus $\gamma_j$ and the $(n-1)$-spread $\Gamma_i$ of $\beta_i$ generate a regular $(n-1)$-spread $\Sigma(\gamma_j, \Gamma_i)$ of $\text{PG}(3n-1, q)$. This can be seen as follows. The elements of $\Gamma_i$ intersect $n$ lines $U_1, U_2, \ldots, U_n$ which are conjugate in $\mathbb{F}_q^n$ over $\mathbb{F}_q$, that is, they form an orbit of the Galois group corresponding to this extension. Let $\alpha_{ij} \cap U_l = \{w_l\}$, with $l = 1, 2, \ldots, n$. Now consider the transversals $T_1, T_2, \ldots, T_n$ of the elements of $\gamma_j$, with $T_l$ containing $w_l$. The $n$ planes $T_l U_l = \theta_i$ intersect all elements of $\gamma_j$ and $\Gamma_i$. The $(n-1)$-dimensional subspaces of $\text{PG}(3n-1, q)$ intersecting $\theta_1, \theta_2, \ldots, \theta_n$ are the elements of the regular $(n-1)$-spread $\Sigma(\gamma_j, \Gamma_i)$. The elements of this spread correspond to the points of a plane $\text{PG}(2, q^n)$, with its lines corresponding to the $(2n-1)$-dimensional spaces containing at least two (and then $q^n + 1$) elements of the spread. Hence the $q + 2$ elements of $\Theta$ containing an element of $\gamma_j$, say $\beta_i = \beta_{i1}, \beta_{i2}, \ldots, \beta_{i,q^n+2} = \beta_j$, correspond to lines of $\text{PG}(2, q^n)$. Dualizing, the elements $\pi_{i1}, \pi_{i2}, \ldots, \pi_{i,q^n+2}$ correspond to points of $\text{PG}(2, q^n)$.

Now consider $\beta_{i1}$ and $\gamma_j$, and repeat the argument. Then there arise $n$ planes $\theta_i$ intersecting all elements of $\gamma_j$ and $\Gamma_i$. The $(n-1)$-dimensional subspaces of $\text{PG}(3n-1, q)$ intersecting $\theta_1, \theta_2, \ldots, \theta_n$ are the elements of the regular $(n-1)$-spread $\Sigma(\gamma_j, \Gamma_i)$. The elements of this spread correspond to the points of a plane $\text{PG}'(2, q^n)$, and the lines of this plane correspond to the $(2n-1)$-dimensional spaces containing $q^n + 1$ elements of the spread. Hence $\beta_{i1}, \beta_{i2}, \ldots, \beta_{i,q^n+2}$ correspond to lines of $\text{PG}'(2, q^n)$. Dualizing, the elements $\pi_{i1}, \pi_{i2}, \ldots, \pi_{i,q^n+2}$ correspond to points of $\text{PG}'(2, q^n)$.

First, assume that $\{\theta_1, \theta_2, \ldots, \theta_n\} \cap \{\theta'_1, \theta'_2, \ldots, \theta'_n\} = \emptyset$. Consider $\pi_{i1}, \pi_{i2}, \pi_{i3}, \pi_{i4}$. The planes of $\text{PG}(3n-1, q^n)$ intersecting these four spaces constitute a set $\mathcal{N}$ of
maximal spaces of a Segre variety $S_{2n-1}$ [1]. The planes $\theta_1, \theta_2, \cdots, \theta_n, \theta'_1, \theta'_2, \cdots, \theta'_n$ are elements of $M$. It follows that $(\theta_1 \cup \theta_2 \cup \cdots \cup \theta_n) \cap (\theta'_1 \cup \theta'_2 \cup \cdots \cup \theta'_n) = \emptyset$.

Consider any $(n-1)$-dimensional subspace $\pi \in \{\pi_{i_3}, \pi_{i_4}, \cdots, \pi_{i_{q+2}}\}$ of $\text{PG}(3n-1, q)$. We will show that $\pi$ is a maximal subspace of $S_{2n-1}$. Let $\pi_i \cap \pi_j = \{t_{ij}\}$, $\pi'_i \cap \pi_j = \{t'_{ij}\}$, $i = 1, 2, \cdots, n, j = i_1, i_2, \cdots, i_{q+2}$. If $t_{ij}, t'_{ij}, t_{ij}, t'_{ij}$ distinct, then $v_1, v_2, \cdots, v_n$ are conjugate and similarly $v_1', v_2', \cdots, v_n'$ are conjugate. Hence $\langle v_1, v_2, \cdots, v_n \rangle = \langle v_1', v_2', \cdots, v_n' \rangle$ defines a $(n-1)$-dimensional space over $F_q$ which intersects $\theta_1, \theta_2, \cdots, \theta'_n$ (over $F_{q^n}$). The points $t_{ij}$, with $j = i_1, i_2, \cdots, i_{q+2}$, generate a subspace of $\theta_i$, and the points $t'_{ij}$, with $j = i_1, i_2, \cdots, i_{q+2}$, generate a subspace of $\theta'_i$ with $i = 1, 2, \cdots, n$. Let $q = 2^h$ and let $F_2^v$ be the subfield of $F_{q^n} = F_{2^h}$ over which these subplanes are defined; so $v|hn$. Then $v < hn$ as otherwise the spreads of $\text{PG}(3n-1, q)$ defined by $\theta_1, \theta_2, \cdots, \theta_n$ and $\theta'_1, \theta'_2, \cdots, \theta'_n$ coincide, clearly not possible. The $(n-1)$-regulus $\gamma_j$ implies that the subplanes contain a line over $F_q$, so $h|v$. As $n$ is prime we have $v = h$, so $2^n = q$. Hence the $2n$ subplanes are defined over $F_q$. It follows that the $q+2$ elements $\pi_{i_1}, \pi_{i_2}, \cdots, \pi_{i_{q+2}}$ are maximal subspaces of the Segre variety $S_{2n-1}$. Hence $\pi$ is a maximal subspace of $S_{2n-1}$. It follows that $\pi_{i_1}, \pi_{i_2}, \cdots, \pi_{i_{q+2}}$ are maximal subspaces of $S_{2n-1}$.

Now consider a $\text{PG}(2, q)$ which intersects $\pi_{i_1}, \pi_{i_2}, \cdots, \pi_{i_{q+2}}$. The $(n-1)$-dimensional spaces $\pi_{i_1}, \pi_{i_2}, \cdots, \pi_{i_{q+2}}$ are maximal spaces of $S_{2n-1}$ which intersect $\text{PG}(2, q)$; they are maximal spaces of the Segre variety $S_{2n-1} \cap \text{PG}(3n-1, q)$ of $\text{PG}(3n-1, q)$.

Consider $\pi_{i_1}$ and also a $\text{PG}(2n-1, q)$ skew to $\pi_{i_1}$. If we project $\pi_{i_2}, \pi_{i_3}, \cdots, \pi_{i_{q+2}}$ from $\pi_{i_1}$ onto $\text{PG}(2n-1, q)$, then by the foregoing paragraph the $q+1$ projections constitute a $(n-1)$-regulus of $\text{PG}(2n-1, q)$. Similarly, we can project from $\pi_{i_s}$, $s$ any element of $\{1, 2, \cdots, q+2\}$. Equivalently, if $s \in \{1, 2, \cdots, q+2\}$ then the spaces $\beta_s$, with $t = 1, 2, \cdots, s-1, s+1, \cdots, q+2$, form a $(n-1)$-regulus of $\beta_s$.

Now assume that the condition $\{\theta_1, \theta_2, \cdots, \theta_n\} \cap \{\theta'_1, \theta'_2, \cdots, \theta'_n\} = \emptyset$ is satisfied for any choice of $\beta_i, \beta_j, \gamma_j, \beta_{ij}$ in such a case every $(n-1)$-regulus contained in a spread $\Gamma_s$ defines a Segre variety $S_{2n-1}$ over $F_q$. Let us define the following design $\mathcal{D}$. The points of $\mathcal{D}$ are the elements of $\emptyset$, a block of $\mathcal{D}$ is a set of $q+2$ elements of $\emptyset$, containing at least one space of a $(n-1)$-regulus contained in some regular spread $\Gamma_s$, and incidence is containment. Then $\mathcal{D}$ is a $4-(q^n+2, q+2, 1)$ design. By Kantor [5] this implies that $q = 2$, a contradiction.

Consequently, we may assume that for at least one quadruple $\beta_i, \beta_j, \gamma_j, \beta_{ij}$ we have

$$\{\theta_1, \theta_2, \cdots, \theta_n\} = \{\theta'_1, \theta'_2, \cdots, \theta'_n\}.$$  
In such a case the $q^n+2$ elements of $\mathcal{O}$ correspond to lines of the plane $\text{PG}(2, q^n)$. It follows that $\mathcal{O}$ is regular.

**Theorem 6.2.** Consider a pseudo-oval $\mathcal{O}$ in $\text{PG}(3n-1, q)$, with $q = 2^h$, $h > 1$ and $n$ prime. Then $\mathcal{O}$ is regular if and only if all $(n-1)$-spreads $\Delta_0, \Delta_1, \cdots, \Delta_{q^n}$ are regular.

**Proof.** If $\mathcal{O}$ is regular, then clearly all $(n-1)$-spreads $\Delta_0, \Delta_1, \cdots, \Delta_{q^n}$ are regular. Conversely, assume that the $(n-1)$-spreads $\Delta_0, \Delta_1, \cdots, \Delta_{q^n}$ are regular. Let $\mathcal{O} = \{\pi_0, \pi_1, \cdots, \pi_{q^n}\}$, $\pi_{q^n+1}$ be the nucleus of $\mathcal{O}$, let $\tilde{\mathcal{O}} = \mathcal{O} \cup \{\pi_{q^n+1}\}$, let $\tilde{\mathcal{O}}$ be the dual of $\mathcal{O}$, let $\bar{\mathcal{O}}$ be the dual of $\tilde{\mathcal{O}}$, and let $\beta_i$ be the dual of $\pi_i$. 

Choose $\beta_i, i \in \{0, 1, \cdots, q^n + 1\}$, and let $\beta_i \cap \beta_j = \alpha_{ij}, j \neq i$. Then
\begin{equation}
\{\alpha_{i0}, \alpha_{i1}, \cdots, \alpha_{i,i-1}, \alpha_{i,i+1}, \cdots, \alpha_{i,q^n+1}\} = \Gamma_i
\end{equation}
is an $(n-1)$-spread of $\beta_i$.

Now consider $\beta_i, \beta_j, \Gamma_i, \Gamma_j, \alpha_{ij}$, with $j \neq i$ and $i, j \in \{0, 1, \cdots, q^n\}$. In $\Gamma$, we next consider a $(n-1)$-regulus $\gamma_j$ containing $\alpha_{ij}$ and $\alpha_{j,q^n+1}$. The $(n-1)$-regulus $\gamma_j$ is a set of maximal spaces of a Segre variety $S_{1,q^{n-1}}$. The $(n-1)$-regulus $\gamma_j$ and the $(n-1)$-spread $\Gamma_i$ of $\beta_i$ generate a regular $(n-1)$-spread $\Sigma(\gamma_j, \Gamma_i)$ of $PG(3n-1,q)$. Such as in the proof of Theorem 6.1 we introduce the elements $U_i, u_i, T_i, \theta_i, l = 1, 2, \cdots, n$, and the plane $PG(2, q^n)$. The $q + 2$ elements of $\hat{O}$ containing an element of $\gamma_j$, say $\beta_i = \beta_i, \beta_{i_2}, \cdots, \beta_{i_3}, \beta_j = \beta_{q^n+1}, \beta_{q^n+1} + 1$, correspond to lines of $PG(2, q^n)$. Dualizing, the elements $\pi_{i_1}, \pi_{i_2}, \cdots, \pi_{i_{q^n+1}}$ correspond to points of $PG(2, q^n)$.

Now consider $\beta_{i_2}$ and $\gamma_{j}$, and repeat the argument. Then there arise $n$ planes $\theta'_i$ of $PG(3n-1,q^n)$ intersecting all elements of $\gamma_j$ and $\Gamma_i$, and a $(n-1)$-spread $\Sigma(\gamma_j, \Gamma_i)$ of $PG(3n-1,q^n)$. The elements of this spread correspond to the points of a plane $PG'(2, q^n)$. The spaces $\beta_{i_1}, \beta_{i_2}, \cdots, \beta_{i_{q^n+1}}, \beta_{q^n+1}$ correspond to lines of $PG'(2, q^n)$. Dualizing, the elements $\pi_{i_1}, \pi_{i_2}, \cdots, \pi_{i_{q^n+1}}, \pi_{i_{q^n+1}}$ correspond to points of $PG'(2, q^n)$.

First, assume that $\{\theta_1, \theta_2, \cdots, \theta_n\} \cap \{\theta'_1, \theta'_2, \cdots, \theta'_n\} = \emptyset$. Consider $\pi_{i_1}, \pi_{i_2}, \pi_{i_3}, \pi_{i_4}$. The planes of $PG(3n-1,q^n)$ intersecting these four spaces constitute a set $M$ of maximal spaces of a Segre variety $S_{2n-1}$. The planes $\theta_1, \theta_2, \cdots, \theta_n, \theta'_1, \theta'_2, \cdots, \theta'_n$ are elements of $M$. It follows that $\{\theta_1 \cup \theta_2 \cup \cdots \cup \theta_n\} \cap \{\theta'_1 \cup \theta'_2 \cup \cdots \cup \theta'_n\} = \emptyset$. Let $\pi \in \{\pi_{i_1}, \pi_{i_2}, \cdots, \pi_{i_{q^n+1}}\}$. As in the proof of Theorem 6.1 one shows that $\pi$ is a maximal subspace of $S_{2n-1}$. It follows that $\pi_{i_1}, \pi_{i_2}, \cdots, \pi_{i_{q^n+1}}, \pi_{i_{q^n+1}}$ are maximal subspaces of $S_{2n-1}$.

Next consider a $PG(2, q)$ which intersects $\pi_{i_1}, \pi_{i_2}, \pi_{i_3}, \pi_{i_4}$. The $(n-1)$-dimensional spaces $\pi_{i_1}, \pi_{i_2}, \cdots, \pi_{i_{q^n+1}}, \pi_{i_{q^n+1}}$ are maximal spaces of $S_{2n-1}$ which intersect the plane $PG(2, q)$; they are maximal spaces of the Segre variety $S_{2n-1} \cap PG(3n-1,q^n)$. As in the proof of Theorem 6.1 it follows that the spaces $\beta_{q^n+1} \cap \beta_{i},$ with $t = 1, 2, \cdots, q + 1$, form a $(n-1)$-regulus of $\beta_{q^n+1}$.

Now assume that the condition $\{\theta_1, \theta_2, \cdots, \theta_n\} \cap \{\theta'_1, \theta'_2, \cdots, \theta'_n\} = \emptyset$ is satisfied for any choice of $\beta_1, \beta_2, \gamma_j, \beta_{i_2}, j \neq i$ and $i, j \in \{0, 1, \cdots, q^n\}$. Let $\alpha_1, \alpha_2, \alpha_3$ be distinct elements of $\Gamma_{q^n+1}$. Then $\beta_1, \beta_2, \gamma_j, \beta_{i_2}$ can be chosen in such a way that $\alpha_1 \in \beta_1, \alpha_2 \in \beta_2, \alpha_3 \in \gamma_j, \beta_{i_2} \cap \gamma_j$ with $\alpha_3 \in \beta_{i_2}$. Hence the $(n-1)$-regulus in $\beta_{q^n+1}$ defined by $\alpha_1, \alpha_2, \alpha_3$ is subset of $\Gamma_{q^n+1}$. From [4], Theorem 4.23, now follows that the $(n-1)$-spread $\Gamma_{q^n+1}$ of $\beta_{q^n+1}$ is regular. By Theorem 6.1 the pseudo-hyperoval $\hat{O}$ is regular, and so $\hat{O}$ is regular. But in such a case the condition $\{\theta_1, \theta_2, \cdots, \theta_n\} \cap \{\theta'_1, \theta'_2, \cdots, \theta'_n\} = \emptyset$ is never satisfied, a contradiction.

Consequently, we may assume that for at least one quadruple $\beta_1, \beta_2, \gamma_j, \beta_{i_2}$ we have $\{\theta_1, \theta_2, \cdots, \theta_n\} = \{\theta'_1, \theta'_2, \cdots, \theta'_n\}$. In such a case the $q^n + 2$ elements of $\hat{O}$ correspond to lines of the plane $PG(2, q^n)$. It follows that $\hat{O}$, and hence also $\hat{O}$, is regular.

**Theorem 6.3.** Consider a pseudo-hyperoval $\hat{O}$ in $PG(3n-1,q), q = 2^h, h > 1$ and $n$ prime. Then $\hat{O}$ is regular if and only if at least $q^n - 1$ elements of $\{\Delta_0, \Delta_1, \cdots, \Delta_{q^n+1}\}$ are regular.
Proof. If \( \mathcal{O} \) is regular, then clearly all \((n-1)\)-spreads \( \Delta_i \), with \( i = 0, 1, \ldots, q^n + 1 \), are regular.

Conversely, assume that \( \rho \), with \( \rho \geq q^n - 1 \), elements of \( \{ \Delta_0, \Delta_1, \ldots, \Delta_{q^n + 1} \} \) are regular.

If \( \rho = q^n + 2 \), then \( \mathcal{O} \) is regular by Theorem 6.1; if \( \rho = q^n + 1 \), then \( \mathcal{O} \) is regular by Theorem 6.2.

Now assume that \( \rho = q^n \) and that \( \Delta_2, \Delta_3, \ldots, \Delta_{q^n + 1} \) are regular. We have to prove that \( \Delta_0 \) is regular. We use the arguments in the proof of Theorem 6.2. If one of the elements \( \alpha_1, \alpha_2, \alpha_3 \), say \( \alpha_1 \), in the proof of Theorem 6.2 is \( \beta_0 \cap \beta_1 \), then let \( \gamma_j \) contain \( \beta_j \cap \beta_1, \beta_j \cap \beta_0, \beta_j \cap \beta_1 \) and let \( \beta_{ij} \neq \beta_1, \beta_0 \), with \( i, j \in \{2, 3, \ldots, q^n + 1 \} \). Now see the proof of the preceding theorem.

Finally, assume that \( \rho = q^n - 1 \) and that \( \Delta_3, \Delta_4, \ldots, \Delta_{q^n + 1} \) are regular. We have to prove that \( \Delta_0 \) is regular. We use the arguments in the proof of Theorem 6.2. If exactly one of the elements \( \alpha_1, \alpha_2, \alpha_3 \), say \( \alpha_1 \), in the proof of Theorem 6.2 is \( \beta_0 \cap \beta_1 \) or \( \beta_0 \cap \beta_2 \), then proceed as in the preceding paragraph with \( \beta_{ij} \neq \beta_1, \beta_2 \). Now assume that two of the elements \( \alpha_1, \alpha_2, \alpha_3 \), say \( \alpha_1 \) and \( \alpha_2 \), are \( \beta_0 \cap \beta_1 \) and \( \beta_0 \cap \beta_2 \). Now consider all \((n-1)\)-reguli in \( \Delta_0 \) containing \( \alpha_1 \) and \( \alpha_3 \), and assume, by way of contradiction, that no one of these \((n-1)\)-reguli contains \( \alpha_2 \). The number of these \((n-1)\)-reguli is \( q^{n-2} - \frac{q^n}{q-1} \), and so \( q = 2 \), a contradiction. It follows that the \((n-1)\)-regulus in \( \beta_0 \) defined by \( \alpha_1, \alpha_2, \alpha_3 \) is contained in \( \Delta_0 \). Now we proceed as in the proof of Theorem 6.2.

7. Final remarks

7.1. The cases \( q = 2 \) and \( n \) not prime

For \( q = 2 \) or \( n \) not prime other arguments have to be developed.

7.2. Improvement of Theorem 6.3

Let \( \mathcal{D} = (P, B, \in) \) be an incidence structure satisfying the following conditions.

(i) \( |P| = q^n + 1 \), \( q \) even, \( q \neq 2 \);

(ii) the elements of \( B \) are subsets of size \( q + 1 \) of \( P \) and every three distinct elements of \( P \) are contained in at most one element of \( B \);

(iii) \( Q \) is a subset of size \( \delta \) of \( P \) such that any triple of elements in \( P \) with at most one element in \( Q \), is contained in exactly one element of \( B \).

Assumption: Any such \( \mathcal{D} \) is a \( 3 - (q^n + 1, q + 1, 1) \) design whenever \( \delta \leq \delta_0 \) with \( \delta_0 \leq q - 2 \).

Theorem 7.1. Consider a pseudo-hyperoval \( \mathcal{O} \) in \( \text{PG}(3n-1, q) \), \( q = 2^h, h > 1 \) and \( n \) prime. Then \( \mathcal{O} \) is regular if and only if at least \( q^n + 1 - \delta_0 \) elements of \( \{ \Delta_0, \Delta_1, \ldots, \Delta_{q^n + 1} \} \) are regular.

Proof. Similar to the proof of Theorem 6.3.

7.3. Acknowledgement

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