

The $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded general linear Lie superalgebra

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Abstract

We present a novel realisation of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded Lie superalgebra $\mathfrak{gl}(m_1, m_2 | n_1, n_2)$ inside an algebraic extension of the enveloping algebra of the \mathbb{Z}_2 -graded Lie superalgebra $\mathfrak{gl}(m | n)$, with $m = m_1 + m_2$ and $n = n_1 + n_2$. A consequence of this realisation is that the representations of $\mathfrak{gl}(m | n)$ “lift up” to representations of $\mathfrak{gl}(m_1, m_2 | n_1, n_2)$, with matrix elements differing only by a sign, which we are able to characterise concisely.

1 Introduction

Colour algebras or colour superalgebras are a generalization of Lie superalgebras introduced by Rittenberg and Wyler [1, 2]. Such algebras are characterized by a grading via an abelian group Γ , and the simplest case not coinciding with a Lie superalgebra is for $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$. Such $\mathbb{Z}_2 \times \mathbb{Z}_2$ colour superalgebras have lately raised interest again, and in recent papers they are usually referred to as $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded Lie superalgebras. We shall follow this terminology, although it is a bit misleading (as these algebras are not Lie superalgebras).

The notion of considering gradings beyond \mathbb{Z}_2 and their corresponding colour Lie (super)algebras goes back to the 1970’s and 1980’s. Most of the attention on these structures has been on various mathematical aspects [3–7]. As remarked in [8], physical applications of $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded Lie superalgebras are very limited compared with the fundamental importance of Lie algebras and Lie superalgebras in theoretical and mathematical physics. For some of the renewed physical interest, we refer to [8] and references therein. Very recent work consists of the importance of $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded Lie superalgebras in the analysis of the Lévy-Leblond equations [9, 10], and various potential applications in the setting of supersymmetric and superconformal quantum mechanics [11–15].

The algebra studied here is the $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded general linear Lie superalgebra $\mathfrak{gl}(m_1, m_2 | n_1, n_2)$. This algebra plays a special role among all finite dimensional $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded Lie superalgebras, because any other finite dimensional $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded Lie superalgebra (of appropriate dimension) can be realised as a subalgebra of $\mathfrak{gl}(m_1, m_2 | n_1, n_2)$ [16, 17]. As an example, we refer to the embedding of the orthosymplectic $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded Lie superal-

gebra $\mathfrak{osp}(2m_1 + 1, 2m_2|2n, 0)$ in $\mathfrak{gl}(2m_1 + 1, 2m_2|2n, 0)$ used in [17], where it is identified as one of the so-called parastatistics algebras (see also [18]).

In view of this, we considered it worthwhile to study $\mathfrak{gl}(m_1, m_2|n_1, n_2)$ and its representations. Our main result is a rather straightforward realisation of $\mathfrak{gl}(m_1, m_2|n_1, n_2)$ inside an algebraic extension of the enveloping algebra of the ordinary Lie superalgebra $\mathfrak{gl}(m|n)$, with $m = m_1 + m_2$ and $n = n_1 + n_2$.

Our result should also be seen in the right context. Already in [16], Scheunert showed that there is a bijection between colour Lie (super)algebras (referred to as ϵ Lie algebras in [16]) and ordinary Lie (super)algebras (provided the grading group Γ is finitely generated). This led to the perception that there is nothing beyond Lie algebras and Lie superalgebras. Finding appropriate realisations of such bijections is, however, not trivial.

Many years after Scheunert's result, McAnally and Bracken [19] developed a technique to construct the Lie superalgebra $\mathfrak{gl}(p|q)$ out of the colour Lie algebra $\mathfrak{gl}(n)$ for $p + q = n$. This ‘‘colour $\mathfrak{gl}(n)$ ’’ is a colour Lie algebra whose n^2 basis elements E_{ij} are graded according to a general grading group Γ , with a bracket depending on a phase function for Γ . The technique of [19] is inspired by the so-called Klein operators discussed by Kleeman [20, 21]. In [19], the authors give a realisation of the $\mathfrak{gl}(p|q)$ generators as elements of the enveloping algebra of colour $\mathfrak{gl}(n)$, thus establishing the mentioned bijection. Although this result is remarkable, the construction itself is rather complicated, even if one tries to apply this to a simple case like $\mathfrak{gl}(m_1, m_2|n_1, n_2)$.

The present contribution is closely related to the above construction, but we follow a path in the opposite direction. Our starting point is the Lie superalgebra $\mathfrak{gl}(m|n)$, and inside an algebraic extension of its enveloping algebra we construct operators that realise $\mathfrak{gl}(m_1, m_2|n_1, n_2)$. Our construction is so straightforward that we need in the end only two additional multiplicative operators in order to get the $\mathfrak{gl}(m_1, m_2|n_1, n_2)$ generators out of those from $\mathfrak{gl}(m|n)$.

Having established the realisation of $\mathfrak{gl}(m_1, m_2|n_1, n_2)$ inside an algebraic extension of the enveloping algebra of $\mathfrak{gl}(m_1 + m_2|n_1 + n_2)$, we demonstrate how the result impacts the representation theory. Specifically, we show how our algebraic results allow us to lift up irreducible covariant representations of $\mathfrak{gl}(m|n)$ to representations of the colour algebra $\mathfrak{gl}(m_1, m_2|n_1, n_2)$. The addition of the afore-mentioned multiplicative operators allow us to easily and explicitly understand how the matrix elements are affected only by a sign.

2 Definitions and algebraic structure

It is instructive to briefly recall the definition of a \mathbb{Z}_2 -graded Lie superalgebra [22, 23]. Let \mathfrak{g} be a \mathbb{Z}_2 -graded vector space over the complex field \mathbb{C} , in the sense that it decomposes as a direct sum

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1.$$

For a homogeneous element $x \in \mathfrak{g}_0$ or \mathfrak{g}_1 , it is also useful to introduce the *degree* as

$$d(x) = \begin{cases} 0; & x \in \mathfrak{g}_0, \\ 1; & x \in \mathfrak{g}_1. \end{cases}$$

Then \mathfrak{g} is a Lie superalgebra if it is endowed with a bilinear product $[[\cdot, \cdot]] : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ that preserves grading, i.e. if $x \in \mathfrak{g}_{d(x)}$, $y \in \mathfrak{g}_{d(y)}$, then $[[x, y]] \in \mathfrak{g}_{d(x)+d(y)}$, with the sum $d(x) + d(y)$ taken modulo 2, and products involving nonhomogeneous elements are determined by extending the Lie product $[[\cdot, \cdot]]$ through linearity. The Lie product must also satisfy

$$[[x, y]] = -(-1)^{d(x) \cdot d(y)} [[y, x]], \quad (\text{graded antisymmetry}) \quad (1)$$

$$[[[x, y], z]] = [[x, [y, z]]] - (-1)^{d(x) \cdot d(y)} [[y, [x, z]]], \quad (\text{graded Jacobi identity}). \quad (2)$$

An important example for our purposes is the Lie superalgebra $\mathfrak{gl}(m|n)$. The standard basis of $\mathfrak{gl}(m|n)$ is given by the (homogeneous) elementary matrices E_{ij} (with 1 in the entry of row i , column j and 0 elsewhere) with grading conveniently characterised via the indices. To this end, set

$$d_i = \begin{cases} 0; & i = 1, \dots, m, \\ 1; & i = m + 1, \dots, m + n. \end{cases}$$

We then define $d_{ij} := d(E_{ij}) = d_i + d_j$ modulo 2.

The \mathbb{Z}_2 -graded Lie product is then given by

$$[[E_{ij}, E_{k\ell}]] = \delta_{jk} E_{i\ell} - (-1)^{d_{ij} \cdot d_{k\ell}} \delta_{i\ell} E_{kj}. \quad (3)$$

The universal enveloping algebra $U(\mathfrak{gl}(m|n))$ is then the unital associative algebra with the free product of generators E_{ij} subject to the algebraic relations

$$E_{ij} E_{k\ell} - (-1)^{d_{ij} \cdot d_{k\ell}} E_{k\ell} E_{ij} = \delta_{jk} E_{i\ell} - (-1)^{d_{ij} \cdot d_{k\ell}} \delta_{i\ell} E_{kj}. \quad (4)$$

Having introduced $\mathfrak{gl}(m|n)$ in this standard way, the definition of its various extensions to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded Lie superalgebras $\mathfrak{gl}(m_1, m_2|n_1, n_2)$ generalises naturally. Following Rittenberg and Wyler [1, 2] and Tolstoy [17], let $\tilde{\mathfrak{g}}$ be a $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded vector space over the complex field \mathbb{C} , that decomposes as

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_{(0,0)} \oplus \tilde{\mathfrak{g}}_{(1,1)} \oplus \tilde{\mathfrak{g}}_{(1,0)} \oplus \tilde{\mathfrak{g}}_{(0,1)}.$$

For a homogeneous element $x_a \in \tilde{\mathfrak{g}}_a$, $a = (0, 0)$, $(1, 1)$, $(1, 0)$, or $(0, 1)$, define the *degree* as $\tilde{d}(x_a) = a$. Then $\tilde{\mathfrak{g}}$ is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded Lie superalgebra if it is endowed with a bilinear product $[[\cdot, \cdot]] : \tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}} \rightarrow \tilde{\mathfrak{g}}$ that preserves this grading, i.e. if $x \in \tilde{\mathfrak{g}}_{\tilde{d}(x)}$, $y \in \tilde{\mathfrak{g}}_{\tilde{d}(y)}$, then $[[x, y]] \in \tilde{\mathfrak{g}}_{\tilde{d}(x)+\tilde{d}(y)}$. In this case, the sum is componentwise addition modulo 2. That is, if $\tilde{d}(x) = (x_1, x_2)$ and $\tilde{d}(y) = (y_1, y_2)$ then $\tilde{d}(x) + \tilde{d}(y) = (x_1 + y_1, x_2 + y_2)$ with the sum in each component being taken modulo 2. As before, products involving nonhomogeneous elements are determined by extending $[[\cdot, \cdot]]$ through linearity.

The product must also satisfy

$$[[x, y]] = -(-1)^{\tilde{d}(x) \cdot \tilde{d}(y)} [[y, x]], \quad (\text{graded antisymmetry}) \quad (5)$$

$$[[[x, y], z]] = [[x, [y, z]]] - (-1)^{\tilde{d}(x) \cdot \tilde{d}(y)} [[y, [x, z]]], \quad (\text{graded Jacobi identity}). \quad (6)$$

In the case that $\tilde{d}(x) = (x_1, x_2)$ and $\tilde{d}(y) = (y_1, y_2)$ then the grading factors appearing in the above expressions are determined by the “dot product” $\tilde{d}(x) \cdot \tilde{d}(y) = x_1y_1 + x_2y_2$ modulo 2.

The main object of our study in this paper is the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded general linear superalgebra $\mathfrak{gl}(m_1, m_2 | n_1, n_2)$, where the labels m_1, m_2, n_1 and n_2 are non-negative integers. As in the \mathbb{Z}_2 -graded case, a convenient basis is the standard one comprising (homogeneous) elementary matrices \tilde{E}_{ij} with 1 in the entry of row i , column j and 0 elsewhere. To describe the grading of these basis elements, we use the graded index notation introduced in [17], whereby

$$\tilde{d}_i = \begin{cases} (0, 0); & i = 1, \dots, m_1 \\ (1, 1); & i = m_1 + 1, \dots, m_1 + m_2 \\ (1, 0); & i = m_1 + m_2 + 1, \dots, m_1 + m_2 + n_1 \\ (0, 1); & i = m_1 + m_2 + n_1 + 1, \dots, m_1 + m_2 + n_1 + n_2, \end{cases}$$

where we adopt the convention that in a situation where one of the labels m_1, m_2, n_1 or n_2 is zero, the case $i = j, \dots, j-1$ does not exist.¹ We then define $\tilde{d}_{ij} := \tilde{d}(\tilde{E}_{ij}) = \tilde{d}_i + \tilde{d}_j$, where the sum is taken componentwise, modulo 2, as described earlier. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded Lie product then takes on the same form as (3), namely

$$[[\tilde{E}_{ij}, \tilde{E}_{k\ell}]] = \delta_{jk} \tilde{E}_{i\ell} - (-1)^{\tilde{d}_{ij} \cdot \tilde{d}_{k\ell}} \delta_{i\ell} \tilde{E}_{kj}. \quad (7)$$

The universal enveloping algebra $U(\mathfrak{gl}(m_1, m_2 | n_1, n_2))$ is then the unital associative algebra with the free product of generators \tilde{E}_{ij} subject to the algebraic relations

$$\tilde{E}_{ij} \tilde{E}_{k\ell} - (-1)^{\tilde{d}_{ij} \cdot \tilde{d}_{k\ell}} \tilde{E}_{k\ell} \tilde{E}_{ij} = \delta_{jk} \tilde{E}_{i\ell} - (-1)^{\tilde{d}_{ij} \cdot \tilde{d}_{k\ell}} \delta_{i\ell} \tilde{E}_{kj}. \quad (8)$$

It is worth remarking that the universal enveloping algebras of both the \mathbb{Z}_2 and $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded superalgebras described above enjoy relations with both commutators and anticommutators. A worthwhile exercise is to compare the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded $\mathfrak{gl}(m_1, m_2 | n_1, n_2)$ with the \mathbb{Z}_2 -graded $\mathfrak{gl}(m_1 + m_2 | n_1 + n_2)$.

For readers unfamiliar with the topic, it should once again be emphasized that a $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded Lie superalgebra is not a Lie superalgebra nor a Lie algebra, but a different structure. For example, comparing $\mathfrak{gl}(1, 1 | 1, 1)$ and $\mathfrak{gl}(2 | 2)$ commutators can become anticommutators and vice versa. E.g. in $U(\mathfrak{gl}(2 | 2))$ the bracket of E_{13} and E_{41} is an anticommutator $\{E_{13}, E_{41}\} = E_{43}$, whereas in $U(\mathfrak{gl}(1, 1 | 1, 1))$ it is a commutator: $[\tilde{E}_{13}, \tilde{E}_{41}] = -\tilde{E}_{43}$. Similarly, the commutator $[E_{21}, E_{13}] = E_{23}$ in $\mathfrak{gl}(2 | 2)$ becomes an anticommutator $\{\tilde{E}_{21}, \tilde{E}_{13}\} = \tilde{E}_{23}$ in $\mathfrak{gl}(1, 1 | 1, 1)$.

3 Algebraic extension of $U(\mathfrak{gl}(m | n))$

The goal of this section is to give a realisation of $U(\mathfrak{gl}(m_1, m_2 | n_1, n_2))$ in terms of an algebraic extension of $U(\mathfrak{gl}(m | n))$. This ultimately allows us to establish an injection mapping irreducible $\mathfrak{gl}(m | n)$ -modules into irreducible $\mathfrak{gl}(m_1, m_2 | n_1, n_2)$ -modules.

¹Note that in the case where two or more of these labels are zero, the algebra reduces to a Lie superalgebra or Lie algebra.

First of all, define the following elements H_k from $U(\mathfrak{gl}(m|n))$, which are just sums of E_{ii} 's:

$$H_k = E_{11} + E_{22} + \cdots + E_{kk}. \quad (9)$$

Then the following relations hold in $U(\mathfrak{gl}(m|n))$

$$\begin{aligned} [H_k, E_{ij}] &= 0, \quad \text{for } k < i < j; \quad i < j \leq k; \quad i > j > k; \quad k \geq i > j, \\ [H_k, E_{ij}] &= E_{ij}, \quad \text{for } i \leq k < j; \\ [H_k, E_{ij}] &= -E_{ij}, \quad \text{for } i > k \geq j. \end{aligned} \quad (10)$$

The operators we actually need are the following elements:

$$B_k = (-1)^{H_k}; \quad B_k^{-1} = (-1)^{-H_k}. \quad (11)$$

Such elements can be viewed as belonging to an extension or closure of the universal enveloping algebra of $\mathfrak{gl}(m|n)$, using

$$(-1)^A = \exp(i\pi A) = \sum_j \frac{(i\pi A)^j}{j!}.$$

We denote this algebraic extension of $U(\mathfrak{gl}(m|n))$ that includes such elements by $\tilde{U}(m, n)$. Of course, if one considers finite-dimensional $\mathfrak{gl}(m|n)$ representations in which the action of the Cartan generators is diagonal, the interpretation of elements such as $(-1)^{H_k}$ in $\tilde{U}(m, n)$ is uncomplicated.

Utilising the commutation relations (10), it is rather straightforward to deduce the following relations between the B_k 's and the E_{ij} 's:

$$\begin{aligned} B_k E_{ij} &= E_{ij} B_k, \quad \text{for } k < i < j; \quad i < j \leq k; \quad i > j > k; \quad k \geq i > j, \\ B_k E_{ij} &= -E_{ij} B_k, \quad \text{for } i \leq k < j; \quad i > k \geq j, \end{aligned} \quad (12)$$

$$\begin{aligned} B_k^{-1} E_{ij} &= E_{ij} B_k^{-1}, \quad \text{for } k < i < j; \quad i < j \leq k; \quad i > j > k; \quad k \geq i > j, \\ B_k^{-1} E_{ij} &= -E_{ij} B_k^{-1}, \quad \text{for } i \leq k < j; \quad i > k \geq j. \end{aligned} \quad (13)$$

In other words, the plus or minus sign when switching B_k with E_{ij} depends on whether k sits inside or outside the range determined by i and j .

We are now in a position to describe the main result of the paper, namely a realisation of $\mathfrak{gl}(m_1, m_2|n_1, n_2)$ in $\tilde{U}(m|n)$, for $m = m_1 + m_2$ and $n = n_1 + n_2$. For this, all we need is to multiply certain $\mathfrak{gl}(m|n)$ generators E_{ij} by B_{m_1} and/or B_m .

- (a) $\tilde{E}_{ij} = E_{ij}$, for $i, j = 1, \dots, m$; $i, j = m + 1, \dots, m + n_1$;
and $i, j = m + n_1 + 1, \dots, m + n$,
- (b) $\tilde{E}_{ij} = E_{ij} B_{m_1}$, for $i = 1, \dots, m$; $j = m + 1, \dots, m + n_1$,
- (c) $\tilde{E}_{ij} = E_{ij} B_m$, for $i = m + 1, \dots, m + n_1$; $j = m + n_1 + 1, \dots, m + n$,
- (d) $\tilde{E}_{ij} = E_{ij} B_{m_1} B_m$, for $i = 1, \dots, m$; $j = m + n_1 + 1, \dots, m + n$,
- (e) $\tilde{E}_{ij} = B_{m_1}^{-1} E_{ij}$, for $i = m + 1, \dots, m + n_1$; $j = 1, \dots, m$,
- (f) $\tilde{E}_{ij} = B_m^{-1} E_{ij}$, for $i = m + n_1 + 1, \dots, m + n$; $j = m + 1, \dots, m + n_1 + 1$,
- (g) $\tilde{E}_{ij} = B_{m_1}^{-1} B_m^{-1} E_{ij}$, for $i = m + n_1 + 1, \dots, m + n$; $j = 1, \dots, m$. (14)

So simply said, the index set $[1, m+n] \times [1, m+n]$ is split into 7 regions: region (a) where there is no difference between E_{ij} and \tilde{E}_{ij} ; regions (b), (c) and (d) where a multiplication to the right takes place by B_{m_1} , B_m or their product; and regions (e), (f) and (g) where a multiplication to the left takes place by $B_{m_1}^{-1}$, B_m^{-1} or their product. It should be noted that in the case $m_1 = 0$, then $B_{m_1} = I$.

It is straightforward to check that equations (14) give a realisation of $U(\mathfrak{gl}(m_1, m_2|n_1, n_2))$ inside $\tilde{U}(m, n)$. For this purpose we must check that relations (8) hold. It is quite a task to check all possible cases from all possible index regions, and it would be too tedious to write down all of these. We restrict ourselves to giving just two representative cases, where moreover some sign change takes place. As a first example, let $i = 1, \dots, m_1$, $j = m+1, \dots, m+n_1$, $k = m+1, \dots, m+n_1$, $l = m_1+1, \dots, m$. Then $\tilde{E}_{ij} \in \mathfrak{gl}(m_1, m_2|n_1, n_2)_{(1,0)}$, $\tilde{E}_{kl} \in \mathfrak{gl}(m_1, m_2|n_1, n_2)_{(0,1)}$ and $\tilde{d}_{ij} \cdot \tilde{d}_{kl} = 0$. So

$$\begin{aligned} \tilde{E}_{ij}\tilde{E}_{kl} - \tilde{E}_{kl}\tilde{E}_{ij} &= E_{ij}B_{m_1}B_{m_1}^{-1}E_{kl} - B_{m_1}^{-1}E_{kl}E_{ij}B_{m_1} \\ &= E_{ij}E_{kl} + E_{kl}E_{ij} = \delta_{jk}E_{il} = \delta_{jk}\tilde{E}_{il}. \end{aligned} \quad (15)$$

The sign change is because for the chosen index sets $E_{ij}B_{m_1} = -B_{m_1}E_{ij}$, and is necessary because E_{ij} and E_{kl} belong to $\mathfrak{gl}(m|n)_1$. As a second example, let $i = 1, \dots, m_1$, $j = m+1, \dots, m+n_1$, $k = m+n_1+1, \dots, m+n$, $l = 1, \dots, m_1$. Then again $\tilde{E}_{ij} \in \mathfrak{gl}(m_1, m_2|n_1, n_2)_{(1,0)}$, $\tilde{E}_{kl} \in \mathfrak{gl}(m_1, m_2|n_1, n_2)_{(0,1)}$ and $\tilde{d}_{ij} \cdot \tilde{d}_{kl} = 0$. So

$$\begin{aligned} \tilde{E}_{ij}\tilde{E}_{kl} - \tilde{E}_{kl}\tilde{E}_{ij} &= E_{ij}B_{m_1}B_{m_1}^{-1}B_m^{-1}E_{kl} - B_{m_1}^{-1}B_m^{-1}E_{kl}E_{ij}B_{m_1} \\ &= -B_m^{-1}E_{ij}E_{kl} - B_m^{-1}E_{kl}E_{ij} = -B_m^{-1}(E_{ij}E_{kl} + E_{kl}E_{ij}) \\ &= -B_m^{-1}\delta_{il}E_{kj} = -\delta_{il}\tilde{E}_{kj}. \end{aligned} \quad (16)$$

It is worth pointing out that these results are also valid for the cases where one of the labels m_1, m_2, n_1, n_2 is zero.

Perhaps more interesting than the actual proof is how we found this solution. It relies on comparing the brackets for \tilde{E}_{ij} ($1 \leq i, j \leq k$) with those of E_{ij} ($1 \leq i, j \leq k$), for $k = 1, 2, \dots$, and gradually increase k . For $k \leq m_1 + m_2 = m$, the commutators are the same, and nothing should change. For $k = m+1$, one sees that for the first time certain commutators between E_{ik} and E_{jl} ($j, l < k$) should become anticommutators for the corresponding elements \tilde{E}_{ik} and \tilde{E}_{jl} . A close inspection shows that it is sufficient to multiply the elements $E_{i, m+1}$ by B_{m_1} , as $m_1 + 1$ is the first index for which the elements \tilde{E}_{jl} have grading $(1, 1)$. One can continue this process with a similar outcome, until $k = m+n_1$. For $k = m+n_1+1$ a change from commutator of $E_{m+n_1, m+n_1+1}$ and $E_{m, m+n_1}$ to anticommutator should take place, and careful inspection leads to a multiplication by B_m to achieve this. Then one continues increasing k , always examining the mutual relations, until one reaches the value $m+n$.

4 Characterisation of representations

To see how easy it is to lift the $\mathfrak{gl}(m_1, m_2|n_1, n_2)$ realisation to representations, let us restrict to the important class of covariant representations of $\mathfrak{gl}(m|n)$. Given the universal

nature of our construction, it would be possible to apply it to the class of irreducible unitary representations [24], but for the purposes of this article, it is sufficient to demonstrate our point on the covariant representations. Such representations are labelled by a partition λ satisfying the (m, n) -hook condition, and have a highest weight with coordinates $(\mu_{1r}, \mu_{2r}, \dots, \mu_{rr})$ (where $r = m + n$) in the common weight basis $(\epsilon_1, \dots, \epsilon_m, \delta_1, \dots, \delta_n)$ of $\mathfrak{gl}(m|n)$ (see [25] for a detailed description). The basis vectors of such a representation are labelled by particular triangular Gelfand-Zetlin patterns, of the following form:

$$|\mu\rangle = \begin{pmatrix} \mu_{1r} & \cdots & \mu_{m-1,r} & \mu_{mr} & \mu_{m+1,r} & \cdots & \mu_{r-1,r} & \mu_{rr} \\ \mu_{1,r-1} & \cdots & \mu_{m-1,r-1} & \mu_{m,r-1} & \mu_{m+1,r-1} & \cdots & \mu_{r-1,r-1} & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & \\ \mu_{1,m+1} & \cdots & \mu_{m-1,m+1} & \mu_{m,m+1} & \mu_{m+1,m+1} & & & \\ \mu_{1m} & \cdots & \mu_{m-1,m} & \mu_{mm} & & & & \\ \mu_{1,m-1} & \cdots & \mu_{m-1,m-1} & & & & & \\ \vdots & \ddots & & & & & & \\ \mu_{11} & & & & & & & \end{pmatrix}, \quad (17)$$

where all entries μ_{ij} are positive integers. For a certain representation, the top row is fixed and corresponds to the highest weight. The other labels μ_{ij} should satisfy a number of conditions, described in [25, Proposition 6].

For convenience, let us also introduce a notation for the sum of all entries on row i of (17):

$$|\mu_i| = \mu_{1,i} + \mu_{2,i} + \cdots + \mu_{i,i}.$$

From [25, Theorem 7], one gets the following action of the generators of the Cartan subalgebra of $\mathfrak{gl}(m|n)$:

$$E_{ii}|\mu\rangle = (|\mu_i| - |\mu_{i-1}|) |\mu\rangle, \quad (18)$$

where $|\mu_0| = 0$. As a consequence, the operators B_k act as

$$B_k|\mu\rangle = (-1)^{|\mu_k|} |\mu\rangle. \quad (19)$$

The main result of [25] was the explicit computation of all matrix elements of the $gl(m|n)$ Chevalley generators in such a covariant representation, i.e. of

$$(\mu' | E_{i,i+1} | \mu) \text{ and } (\mu' | E_{i+1,i} | \mu) \quad i = 1, 2, \dots, r-1.$$

Following the construction of the previous section, the covariant representations described here augment to irreducible representations of $\mathfrak{gl}(m_1, m_2 | n_1, n_2)$, with the following diagonal actions

$$\tilde{E}_{ii}|\mu\rangle = (|\mu_i| - |\mu_{i-1}|) |\mu\rangle, \quad (20)$$

and with

$$\begin{aligned}
(\mu'|\tilde{E}_{i,i+1}|\mu) &= (\mu'|E_{i,i+1}|\mu) \quad i \neq m, m+n_1, \\
(\mu'|\tilde{E}_{m,m+1}|\mu) &= (-1)^{|\mu_{m_1}|}(\mu'|E_{m,m+1}|\mu), \\
(\mu'|\tilde{E}_{m+n_1,m+n_1+1}|\mu) &= (-1)^{|\mu_m|}(\mu'|E_{m+n_1,m+n_1+1}|\mu), \\
(\mu'|\tilde{E}_{i+1,i}|\mu) &= (\mu'|E_{i+1,i}|\mu) \quad i \neq m, m+n_1, \\
(\mu'|\tilde{E}_{m+1,m}|\mu) &= (-1)^{|\mu_{m_1}|}(\mu'|E_{m+1,m}|\mu), \\
(\mu'|\tilde{E}_{m+n_1+1,m+n_1}|\mu) &= (-1)^{|\mu_m|}(\mu'|E_{m+n_1+1,m+n_1}|\mu).
\end{aligned}$$

Matrix elements of other basis elements \tilde{E}_{ij} follow, as usual, by taking appropriate $\mathfrak{gl}(m_1, m_2|n_1, n_2)$ brackets.

It should not be too surprising that the matrix elements of these covariant representations of $\mathfrak{gl}(m_1, m_2|n_1, n_2)$ essentially differ to the matrix elements of the Lie superalgebra $\mathfrak{gl}(m_1 + m_2|n_1 + n_2)$ by a sign in certain entries. Firstly, it is easy to see that the quadratic Casimir of $\mathfrak{gl}(m_1, m_2|n_1, n_2)$ has the same form as that of $\mathfrak{gl}(m|n)$:

$$I_2 = \sum_{i,j} (-1)^{\tilde{d}_{ij} \cdot \tilde{d}_{ij}} \tilde{E}_{ij} \tilde{E}_{ji}.$$

Then, on a highest weight representation with highest weight $(\mu_{1r}, \dots, \mu_{mr}, \mu_{m+1,r}, \dots, \mu_{rr})$, the eigenvalues also have the same form:

$$\sum_{i=1}^m (\mu_{ir}(\mu_{ir} + m - n - 2i + 1)) - \sum_{k=1}^n (\mu_{m+k,r}(\mu_{m+k,r} + m + n - 2k + 1)).$$

Using characteristic identity techniques [26–28], the resulting formulae for the square of the matrix elements of the generators for $\mathfrak{gl}(m_1, m_2|n_1, n_2)$ and $\mathfrak{gl}(m_1 + m_2|n_1 + n_2)$ are the same, and therefore the matrix entries on such a representation can only differ by a sign.

5 Conclusion and outlook

We have provided a straightforward and explicit construction of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded colour Lie superalgebra $\mathfrak{gl}(m_1, m_2|n_1, n_2)$ inside an algebraic extension of the enveloping algebra of the Lie superalgebra $\mathfrak{gl}(m_1 + m_2|n_1 + n_2)$. The motivation for studying the general linear $\mathfrak{gl}(m_1, m_2|n_1, n_2)$ is the fact that Ado's Theorem is known to also hold for the colour case [16, 17], so some knowledge of explicit results should be worthwhile to develop. The simplicity of our construction allows us to lift up representations of $\mathfrak{gl}(m|n)$ to representations of $\mathfrak{gl}(m_1, m_2|n_1, n_2)$, where $m_1 + m_2 = m$ and $n_1 + n_2 = n$. We furthermore demonstrated this process on the class of irreducible covariant tensor representations.

There are various immediate consequences of the correspondence presented in this article. For instance, using equations (14), it is possible to write down Serre relations [29] satisfied by the Chevalley generators $\tilde{E}_{i,i+1}$ and $\tilde{E}_{i+1,i}$ of $\mathfrak{gl}(m_1, m_2|n_1, n_2)$. In some cases, the relative signs of the coefficients in the relations may change as a result of equations

(12) and (13), which is particularly evident in the cases where one of the labels m_1 , m_2 , n_1 or n_2 is zero. Such a presentation would allow, for example, one to develop the quantum analogue and express the relations of the associated quantum group along the lines of [30–32].

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