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Types of triangle in plane Hamiltonian triangulations and applications to domination and k-walks

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Abstract

We investigate the minimum number $t_0(G)$ of faces in a Hamiltonian triangulation G so that any Hamiltonian cycle C of G has at least $t_0(G)$ faces that do not contain an edge of C. We prove upper and lower bounds on the maximum of these numbers for all triangulations with a fixed number of facial triangles. Such triangles play an important role when Hamiltonian cycles in triangulations with 3-cuts are constructed from smaller Hamiltonian cycles of 4-connected subgraphs. We also present results linking the number of these triangles to the length of 3-walks in a class of triangulation and to the domination number.

Keywords: graph, Hamiltonian cycle, domination, 3-walk

Math. Subj. Class.: 05C45, 05C10, 05C38

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1 Introduction

In this article all triangulations are simple triangulations of the plane with at least 4 vertices. A triangulation or a graph is said to be *Hamiltonian* if it contains a Hamiltonian cycle. For a triangulation G with a Hamiltonian cycle C of G, a *type-i triangle* with $i \in \{0,1,2\}$ is defined as a facial triangle of G which shares exactly i edges with C. We define $t_i(G,C)$ as the number of type-i triangles. If the triangulation and Hamiltonian cycle are clear from the context, we will also just write t_i.

A triangulation G can be extended by inserting a 4-connected triangulation or polyhedron in a triangle T to obtain a larger graph G'. If there is a Hamiltonian cycle C in G, then we can extend C to a Hamiltonian cycle of G' – unless T is a type-0 triangle. If there is a Hamiltonian cycle C without any type-0 triangles such as in a double wheel or the majority of small 4-connected triangulations (e.g. more than 80% for 4-connected triangulations on 20 vertices), then for the graph G' obtained by inserting a 4-connected triangulation or polyhedron in each triangle in a set of disjoint facial triangles we can extend C to a Hamiltonian cycle of G'. In [3] it is proven that the – still open – question whether all triangulations with at most four 3-cuts are Hamiltonian can be reduced to the question whether for each set of four disjoint triangles in a 4-connected triangulation there is a Hamiltonian cycle so that none of them is a type-0 triangle. More properties of triangulations with a Hamiltonian cycle with few or even without type-0 triangles are described in Section 4. Investigating whether there always exists a Hamiltonian cycle with few type-0 triangles is the main target of this paper.

We denote the number of facial triangles of G by $t(G)$. Euler's formula implies that (with $|G|$ the number of vertices of G), $t(G) = 2|G| - 4$, so it is always an even number. For $i \in \{0,2\}$ we further define

$$t_i(G) = \min\{t_i(G,C) | C \text{ is a Hamiltonian cycle of } G\},$$

and for even $t \geq 4$

$$t_i(t) = \max\{t_i(G) | G \text{ is a Hamiltonian triangulation with exactly } t \text{ facial triangles}\}.$$

In some cases we might want to restrict the class to 4- or 5-connected triangulations. Note that there are no 4-connected triangulations G with $t(G) < 8$ and no 5-connected triangulations G with $t(G) < 20$. So for $j = 4$ and even $t \geq 8$, and for $j = 5$ and even $t \geq 20$ we define

$$t_i^j(t) = \max\{t_i(G) | G \text{ is a } j\text{-connected triangulation with exactly } t \text{ facial triangles}\}.$$

Note that there are no 4-connected triangulations G with $t(G) \leq 6$, and no 5-connected triangulations G with $t(G) \leq 18$. In this paper, we show the following theorem.

Theorem 1.1. *Let t be an integer. Then the following hold.*

(i) *For $t \geq 8$ we have $t_0(t) \leq \frac{t-8}{3}$, and for $4 \leq t < 8$ we have $t_0(t) = 0$.*

(ii) *For $t \geq 10$ we have $t_0^4(t) \leq \frac{t-10}{3}$, and for $t = 8$ we have $t_0^4(t) = 0$.*

(iii) *For $t \geq 20$ we have $t_0^5(t) \leq \frac{t-12}{3}$.*

In Section 3, we discuss lower bounds on $t_0(t), t_0^4(t)$ and $t_0^5(t)$.

As we will see in Section 4.1, also the number of type-i triangles on one side of a Hamiltonian cycle is relevant, so we also define $\bar{t}_i(G,C)$ as the number of type-i triangles on that side of C with fewer type-i triangles. The numbers $\bar{t}_i(G)$, $\bar{t}_i(t)$, and $\bar{t}_i^j(t)$ are defined correspondingly. By definition $\bar{t}_i(G,C) \leq t_i(G,C)/2$, $\bar{t}_i(G) \leq t_i(G)/2$, $\bar{t}_i(t) \leq t_i(t)/2$ and $\bar{t}_i^j(t) \leq t_i^j(t)/2$ for $i \in \{0,2\}$ and $j \in \{4,5\}$.

An *outer plane graph* is a plane graph in which all vertices are incident with the outer face. In particular, an outer plane graph with maximal number of edges is called a *maximal outer plane graph*, which is, in other words, an outer plane graph in which all inner faces are triangles. For a triangulation G with a Hamiltonian cycle C, the inside as well as the outside of C together with C form a maximal outer plane graph. For a 2-connected plane graph G, the boundary of the outer face is called the *boundary cycle of* G. In particular, vertices and edges in the boundary cycle of G are *boundary vertices* resp. *boundary edges* in G. A cycle C in a plane graph such that the inside as well as the outside (not including C) contain a vertex is called a *separating cycle*. Note that in a triangulation, any triangle that is not facial is a separating cycle.

Let G be a triangulation with a Hamiltonian cycle C. If we take the dual of the maximal outer plane graph consisting of the inside of C together with C and delete the vertex corresponding to the outer face, then we obtain a subcubic tree in which the vertices of degree $(3-i)$ correspond to type-i triangles of the triangulation. Using these relations, we get the following proposition.

Proposition 1.2. *Let G be a triangulation with a Hamiltonian cycle C. Then*

$$\bar{t}_2(G,C) = \bar{t}_0(G,C) + 2 \text{ and } t_2(G,C) = t_0(G,C) + 4.$$

Note that the number of facial triangles on the inside is equal to the number of facial triangles on the outside. As $t(G) = t_0(G,C) + t_1(G,C) + t_2(G,C))$, we have

$$t_1(G,C) = t(G) - 2t_0(G,C) - 4.$$

So finding the minimum value for $t_0(G,C)$ is equivalent to finding the minimum value for $t_2(G,C)$, and finding the maximum value for $t_1(G,C)$.

Let G be a triangulation and let C be a Hamiltonian cycle in G. We say that two facial triangles are *adjacent* if they share an edge. An *(i,j)-pair* ($i,j \in \{1,2\}$) is defined as a pair of adjacent facial triangles consisting of a type-i triangle and a type-j triangle such that the common edge is contained in C. Note that each type-1 triangle is contained in at most one (1,2)-pair.

2 Upper bounds for $t_0(t)$, $t_0^4(t)$,and $t_0^5(t)$

To prove Theorem 1.1 in this section, we first show some lemmas. A vertex v in a graph G is said to be *dominating* if v is adjacent to all other vertices in G.

If a type-2 triangle T is contained in two $(2,2)$-pairs, we call the three triangles involved a $(2,2,2)$-*triple* and T the *central triangle* of the triple.

Restricted to minimum degree 4 the first part of the following lemma was proven in [13], Lemma 2.1.

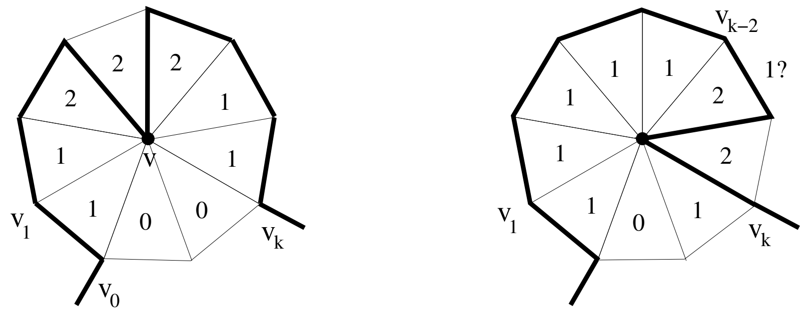

Figure 1: Rerouting a Hamiltonian cycle to remove a $(2, 2, 2)$-triple.

Lemma 2.1. *Let G be a triangulation with a Hamiltonian cycle C, but without a dominating vertex. Then there exists a Hamiltonian cycle C' in G such that C' has no $(2, 2, 2)$-triples.*

If G has minimum degree 4, then C' can be chosen in a way that it also has at least as many $(1, 1)$-pairs as C.

Proof. Assume that there is a $(2, 2, 2)$-triple with central triangle T and let v denote the vertex contained in all three triangles involved. As v is not dominating, there is a first vertex v_0 in counterclockwise orientation from T around v that has a neighbour on C that is not a neighbour of v. Numbering the neighbours of v in clockwise orientation around v as $v_0, v_1, \dots, v_{\deg(v)-1}$, there is also a first vertex v_k with $k > 0$ and a neighbour on C that is not a neighbour of v. We can reroute the part of C containing $v, v_0, \dots, v_k$ along the path $v_0, v_1, \dots, v_{k-1}, v, v_k$. This operation is displayed in Figure 1. Of course the roles of v_0 and v_k are symmetric and we could do the same with their roles interchanged.

If all vertices have degree at least 4, any new type-2 triangle contains v and the number of $(2, 2, 2)$-triples is decreased. Furthermore, no $(1, 1)$-pairs without a triangle containing v can be destroyed and after rerouting at least the edges v_1v_2, $v_2v_3, \dots, v_{k-3}v_{k-2}$ are common edges of a $(1, 1)$-pair. These are $k - 3$ $(1, 1)$-pairs, but note that $k - 3$ can be 0. Depending on whether v_0v_1 is the common edge of a $(1, 1)$-pair in C, the triangles under discussion can belong to $k - 3$ or $k - 4$ $(1, 1)$-pairs before rerouting – so the number of $(1, 1)$-pairs does not decrease.

The vertices v_0 and v_k always have degree at least 4, but if one of $v_1, \dots, v_{k-2}$ has degree 3, it is contained in a type-2 triangle not containing v. For $v_1, \dots, v_{k-3}$ (note that this set of vertices can be empty) this type-2 triangle has type-1 triangles on the other side of the edges in the Hamiltonian cycle and is therefore not contained in a $(2, 2)$-pair. If v_{k-2} has degree 3 we would produce a $(2, 2, 2)$-triple. If v_2 has degree larger than 3, we can apply the operation with the role of v_0 and v_k interchanged, so let us assume that v_2 as well as v_{k-2} have degree 3. As no two vertices of degree 3 can be neighbours in a triangulation different from K_4, this implies that $k > 3$.

Let $i > 0$ be minimal so that there is an edge v_iv_{k-1}. Such an i is sure to exist, as $k-3$ is a candidate. We then reroute the cycle along $v_0, v_1, \dots, v_i, v_{k-1}, v_{k-2}, \dots, v_{i+1}, v, v_k$ to obtain C'. An example of this rerouting is given in Figure 2.

After rerouting, the only edges that can be the common edge of the two triangles in a new $(2, 2)$-pair are $v_{i+1}v_{i+2}$ and v_iv_{k-1}. As v_iv_j is not in C' for any $i < j < k - 1$,

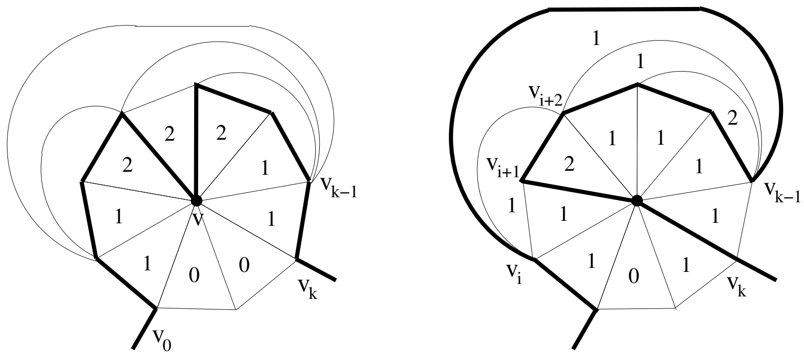

Figure 2: Rerouting a Hamiltonian cycle to remove a $(2,2,2)$-triple if v_{k-2} has degree 3.

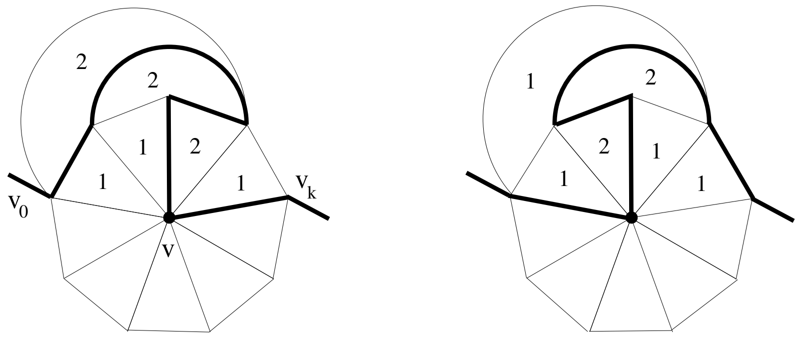

Figure 3: Rerouting a Hamiltonian cycle to remove a $(2,2,2)$-triple if v_{k-2} has degree 3 and the default method produces a $(2,2,2)$-triple.

v_iv_{k-1} can only be in a $(2,2)$-pair if $v_{k-1}v_{k-2}$ is contained in the same triangle, which gives $i = k-3$, so $v_{i+1}v_{i+2} = v_{k-2}v_{k-1}$ is the common edge of a $(2,2)$-pair too and only the case that $v_{i+1}v_{i+2}$ is the common edge of a $(2,2)$-pair remains to be discussed.

Assume that $v_{i+2}v_{i+1}$ is contained in two type-2 triangles — $v_{i+2}v_{i+1}v$ and T'. If the degree of v_{i+2} is 3, then $T' = v_{i+1}v_{i+2}v_{i+3}$ and the second neighbour triangle of T' along C' is a type-1 triangle, so in that case $v_{i+2}v_{i+1}$ is not part of a $(2,2,2)$-triple.

If the degree of v_{i+2} is at least 4, the other edge of T' in the Hamiltonian cycle must be $v_{i+2}v_i$, which can only be contained in a type-2 triangle $v_{i+2}v_{i+1}v_i$ if $i+2 = k-1$, that is $i = k-3$. In order to be contained in a second type-2 triangle, there must be an edge $v_{k-1}v_{k-4}$. Due to The minimality of i implies $k = 4$, so we have the situation depicted in Figure 3 on the left hand side. Rerouting the Hamiltonian cycle along $v_0, v, v_2, v_1, v_3, v_4$ (right hand side of Figure 3) gives a Hamiltonian cycle with one $(2,2,2)$-triple less.

□

Using a result by Whitney [17], we can prove the existence of a Hamiltonian cycle with at least one $(1,1)$-pair in a 4-connected triangulation. Below we first give the lemma by Whitney, but use a simplified version of the formulation from [7].

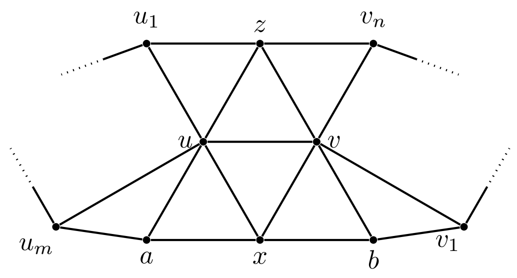

Figure 4: Construction of a Hamiltonian cycle with at least one $(1, 1)$-pair in a 4-connected triangulation.

Lemma 2.2. *Let G be a 4-connected triangulation. Consider a cycle D in G together with the vertices and edges on one side of D (referred to as the outside of D). Let a and b be two vertices of D dividing D into two paths P_1 and P_2 each of which contains both a and b. If*

- *no two vertices of P_1 are joined by an edge which lies outside of D and*
- *there is a vertex z (distinct from a and b) dividing P_2 into two paths P_3 and P_4 each of which contains z such that no pair of vertices in P_3 and no pair of vertices in P_4 are joined by an edge which lies outside of D,*

then there is a path from a to b using only edges on and outside of D which passes through every vertex on and outside of D.

Using this lemma, we can give the following result. Note that for triangulations being k-connected is equivalent to having no separating cycles of length shorter than k.

Lemma 2.3. *Let G be a 4-connected triangulation which is not isomorphic to the octahedron. There exists a Hamiltonian cycle C in G such that C has at least one $(1, 1)$-pair.*

Proof. As a consequence of the Euler formula and the fact that G is not isomorphic to the octahedron, there exists a vertex x of degree at least 5 in G. Let uvx be an arbitrary triangle containing x. The edge uv is contained in a second triangle, say uvz. Let the vertices adjacent to u (in counter clockwise order) be $v, z, u_1, \dots, u_m, a, x$ (note that there are no u_i vertices if u has degree 4), and let the vertices adjacent to v be $u, x, b, v_1, \dots, v_n, z$ (note that there are no v_i vertices if v has degree 4) (see Figure 4).

As G is 4-connected, $D = axbv_1 \cdots v_n z u_1 \cdots u_m a$ is a cycle in G. The vertices a and b partition D into two paths satisfying the conditions of Lemma 2.2 with $P_1 = axb$. Indeed, the path P_2 is divided into P_3 and P_4 by the vertex z. As x has degree at least 5, a and b are not adjacent. All vertices of P_3, resp. P_4, are adjacent to u, resp. v, so any edge which lies outside of D and joins two vertices of P_3 or two vertices of P_4 would be part of a separating triangle.

Let P be the path from a to b described in Lemma 2.2. The Hamiltonian cycle $C = P \cup auvb$ contains the $(1, 1)$-pair (uvx, uvz). $\square$

In the case of 5-connected triangulations, we can prove a slightly stronger result.

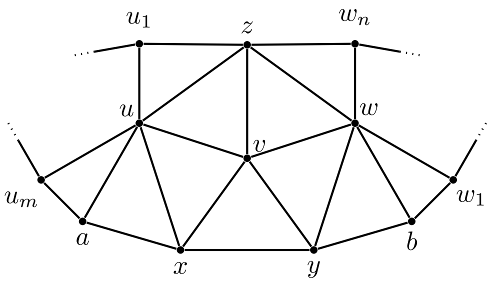

Figure 5: Construction of a Hamiltonian cycle with at least two $(1, 1)$-pairs in a 5-connected triangulation.

Lemma 2.4. *Let G be a 5-connected triangulation. There exists a Hamiltonian cycle C in G such that C has at least two $(1, 1)$-pairs.*

Proof. Let v be a vertex of G which has degree 5, and let u and w be two neighbouring vertices of v which are not adjacent to each other. Let the vertices adjacent to u be $v, z, u_1, \ldots, u_m, a, x$, and let the vertices adjacent to w be $v, y, b, w_1, \ldots, w_n, z$ (see Figure 5).

As G is 5-connected, $D = axybw_1 \cdots w_n z u_1 \cdots u_m a$ is a cycle in G. The vertices a and b partition D into two paths satisfying the conditions of Lemma 2.2 with $P_1 = axyb$. Indeed, the path P_2 is divided into P_3 and P_4 by the vertex z. As all vertices have degree at least 5, any edge outside of D connecting two vertices of P_1 is contained in a separating triangle or a separating quadrangle. All vertices of P_3, resp. P_4, are adjacent to u, resp. w, so any edge which lies outside of D and joins two vertices of P_3 or P_4 would be part of a separating triangle.

Let P be the path from a to b described in Lemma 2.2. The Hamiltonian cycle $C = P \cup auvwb$ contains the $(1, 1)$-pairs (uvx, uvz) and (vwy, vwz). □

Lemma 2.5. *Let G be a triangulation with a dominating vertex v and t triangles. Then $t_0(G) < \frac{t}{4} - 1$ if G is not K_4 and $t_0(K_4) = 0$.*

Proof. We can easily check K_4 by hand, so assume that G is not K_4.

$G - \{v\}$ is an outer plane graph, so it has a vertex w of degree 2. Let w' be a vertex sharing a boundary edge of $G - \{v\}$ with w and let C be the Hamiltonian cycle of G containing $\{v, w\}$, $\{v, w'\}$ and the boundary cycle of $G - \{v\}$ without the edge $\{w, w'\}$. Let $t_{0,\Delta}, t_{1,\Delta}$ and $t_{2,\Delta}$ be the number of facial triangles of type 0, 1 and 2 on the side of C containing the triangle v, w, w'. All triangles on the other side of C contain v and as no type-0 triangle in G contains v, we have $t_0(G) = t_{0,\Delta}$. Since each side of C contains exactly $t(G)/2$ facial triangles, we have $t_{0,\Delta} + t_{1,\Delta} + t_{2,\Delta} = \frac{t(G)}{2}$. Furthermore (as G is not K_4) we have $t_{1,\Delta} \geq 1$ (the unique triangle containing w but not v). So $t_{0,\Delta} + t_{2,\Delta} < t_{0,\Delta} + t_{1,\Delta} + t_{2,\Delta} = \frac{t}{2}$. By Proposition 1.2, we have $t_{2,\Delta} = t_{0,\Delta} + 2$, and hence we get $2t_{0,\Delta} + 2 = 2t_0 + 2 < \frac{t}{2}$ and finally $t_0(G) < \frac{t}{4} - 1$.

□

By combining the results above, we are now ready to prove Theorem 1.1.

Proof of Theorem 1.1:

For $t < 20$ the theorem was checked by testing all triangulations. The triangulations were generated by the program *plantri* [2] and a straightforward exhaustive search for Hamiltonian cycles with the smallest number of type-0 triangles was performed. Thus, we may assume $t \geq 20$. Let G be a Hamiltonian triangulation with $t \geq 20$ facial triangles.

Suppose that G has a dominating vertex v. Since $G - \{v\}$ has a vertex of degree two, G has a 3-cut, and hence G is not 4-connected. Since $\frac{t}{4} - 1 \leq \frac{t-8}{3}$, Lemma 2.5 implies the result.

Assume now that G has no dominating vertex. Suppose that G has a Hamiltonian cycle with p $(1,1)$-pairs. Lemmas 2.3 and 2.4 imply that $p \geq 1$ if G is 4-connected, and $p \geq 2$ if G is 5-connected. Due to Lemma 2.1, G contains a Hamiltonian cycle C' which has at least p $(1,1)$-pairs and in which each type-2 triangle is contained in at least one $(1,2)$-pair. A type-1 triangle is contained in a $(1,1)$-pair or a $(1,2)$-pair. There are at least $2p$ type-1 triangles in $(1,1)$-pairs of C' and therefore at most $(t_1(G,C') - 2p)$ type-1 triangles in $(1,2)$-pairs. Since each type-2 triangle forms a $(1,2)$-pair with at least one of the type-1 triangles in a $(1,2)$-pair, we get

$$t_2(G,C') \leq t_1(G,C') - 2p.$$

By Proposition 1.2, we have $t_2(G,C') = t_0(G,C') + 4$, and hence

$$t_1(G,C') \geq t_0(G,C') + 4 + 2p.$$

Combining these results with $t(G) = t_0(G,C') + t_1(G,C') + t_2(G,C')$, we get

$$t(G) \geq t_0(G,C') + t_0(G,C') + 4 + 2p + t_0(G,C') + 4.$$

This can be rewritten as

$$t_0(G,C') \leq \frac{t(G) - 8 - 2p}{3},$$

and so we also have

$$t_0(t) \leq \frac{t - 8 - 2p}{3}.$$

Using the values for p from Lemma 2.3 and Lemma 2.4, we get the given bounds. ■

3 Lower bounds for $t_0(t)$, $t_0^4(t)$ and $t_0^5(t)$

In order to prove lower bounds for $t_0(t)$, $t_0^4(t)$ and $t_0^5(t)$, we will construct families of graphs in which each Hamiltonian cycle has at least a certain number of type-0 triangles.

Theorem 3.1.

- *Let $t \geq 16$ be even. Then $t_0(t) \geq \lfloor \frac{t}{3} \rfloor - 5$ and $\bar{t}_0(t) \geq \lfloor \frac{t+2}{6} \rfloor - 3$. We have $t_0(14) = 1$ and $\bar{t}_0(14) = 0$. For $t < 14$ we have $t_0(t) = \bar{t}_0(t) = 0$.*
- *Let $t \geq 18$ be even. Then $t_0^4(t) \geq 2(\lfloor \frac{t}{6} \rfloor - 3)$ and $\bar{t}_0^4(t) \geq \lfloor \frac{t}{6} \rfloor - 3$. For $t < 18$ we have $t_0^4(t) = \bar{t}_0^4(t) = 0$.*
- *Let $t \geq 20$ be even. Then $t_0^5(t) \geq 2\lfloor \frac{t}{12} \rfloor - 20$. For $t \leq 66$ we have that $\bar{t}_0^5(t) = 0$.*

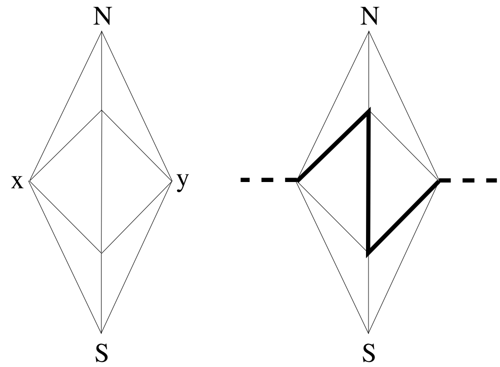

Figure 6: The fragment B used to construct a family of triangulations establishing a lower bound on $t_0^4(t)$ and $\bar{t}_0^4(t)$ and the most common way for a Hamiltonian cycle to pass through this fragment.

Proof. $\mathbf{t_0^4(t)}$ **and** $\mathbf{\bar{t}_0^4(t)}$**:**

The results for $t < 18$ were determined by a computer using the program *plantri* [2] for the generation of all 4-connected triangulations and a straightforward algorithm to compute t_0 and $\bar{t}_0$.

First consider the case where t is a multiple of six, and let $k = \frac{t}{6}$. Consider the fragment B shown in the left part of Figure 6. Take k copies $B_0, \ldots, B_{k-1}$ of B and identify all vertices labelled N and all vertices labelled S, respectively, (we call the resulting vertices the poles) and for $0 \leq i < k$ identify vertex y in B_i with vertex x in $B_{i+1(\text{mod } k)}$. This graph has $6k$ facial triangles, and we denote it by G_k. It is easy to check that G_k is 4-connected.

We show $t_0(G_k) \geq 2(\frac{t}{6} - 3)$ and $\bar{t}_0(G_k, C) \geq \frac{t}{6} - 3$ by induction on k. Computational results give that for $3 \leq k \leq 8$ we have $t_0(G_k) = 2k - 6$ and $\bar{t}_0(G_k) = k - 3$. Since G_k contains $6k$ triangles, we can also write this as $t_0(G_k) = \frac{t}{3} - 6$ and $\bar{t}_0(G_k) = \frac{t}{6} - 3$, and we are done. So we may assume that $k \geq 9$.

Let C be a Hamiltonian cycle in G_k. An edge of C which is incident to a pole is contained in at most two fragments. Since there are two edges incident to each pole, there are at most 8 fragments that contain an edge of C that is incident to a pole. Since $k \geq 9$, we may assume that C visits the fragment B_{k-1} – up to symmetry – as shown in the right part of Figure 6. This part of the Hamiltonian cycle C produces two type-0 triangles in B_{k-1} – one on each side of C. So, by removing two inner vertices of B_{k-1}, identifying the vertex y in the copy B_{k-2} and the vertex x in the copy B_0, we obtain a Hamiltonian cycle, say C', in G_{k-1}. By the induction hypothesis, $t_0(G_{k-1}, C') \geq 2(\frac{t-6}{6} - 3)$ and $\bar{t}_0(G_{k-1}, C') \geq \frac{t-6}{6} - 3$. Since $t_0(G_k, C) = t_0(G_{k-1}, C') + 2$ and $\bar{t}_0(G_k, C) = \bar{t}_0(G_{k-1}, C') + 1$, we obtain the desired inequality.

For the case where t is not a multiple of six, we let $k = \left\lfloor \frac{t}{6} \right\rfloor$. We apply the same construction, but for a pair of neighbouring fragments we connect the x- and y-vertex by an edge instead of identifying them – see the left part of Figure 7 – or with an extra vertex of degree 4 that is also connected to the poles. This gives 2, resp. 4 extra triangles. Confirming the formulas for these modified triangulations with 3 to 8 fragments with a computer, one can apply the same argumentation as above to prove the equations in the lemma.

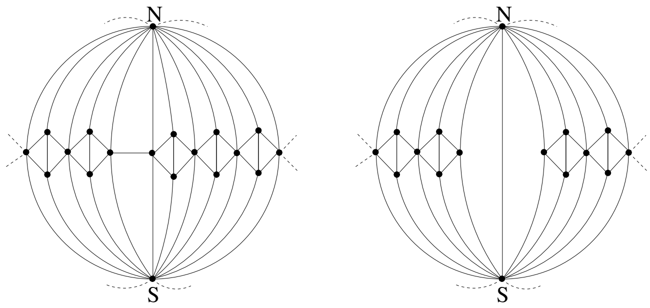

Figure 7: Modifications of the construction for the 4-connected case when t is not a multiple of 6 and for the 3-connected case.

$\mathbf{t_0(t)}$ **and** $\mathbf{\bar{t}_0(t)}$**:**

For $t_0(t)$ and $\bar{t}_0(t)$, where 3-cuts are allowed, we use the same fragment and the same constructions as for $t_0^4(t)$ and $\bar{t}_0^4(t)$, but for two fragments we do not identify x and y but instead connect N and S by an edge between these segments – see the right part of Figure 7. This construction with k fragments gives triangulations with $6k+2$ facial triangles that can be extended to triangulations with $6k+4$ and $6(k+1)$ facial triangles by inserting vertices of degree 3 in one or both triangles containing the edge between the poles.

Computational results for $k \leq 8$ fragments combined with the same reduction argument as before give that $t_0(t) \geq \lfloor \frac{t}{3} \rfloor - 5$ and $\bar{t}_0(t) \geq \lfloor \frac{t+2}{6} \rfloor - 3$.

Remark: For small values of t a double wheel where triangles are subdivided with a vertex of degree 3 alternatingly on both sides of the rim gives a larger result for $t_0(t)$ and $\bar{t}_0(t)$, but the linear factor is only $\frac{1}{4}$, so that the advantage compared to the sequence described is only for small values.

$\mathbf{t_0^5(t)}$ **and** $\mathbf{\bar{t}_0^5(t)}$**:**

For $t \leq 130$ we have that $t_0^5(t) \geq 0 \geq 2\lfloor \frac{t}{12} \rfloor - 20$. So assume that t is even and $t > 130$.

For even $t > 130$ we can construct triangulations in a similar way as for the cases $t_0^4(t)$ and $\bar{t}_0^4(t)$, but use the fragments depicted in Figure 8. We use $r = (t - 12\lfloor \frac{t}{12} \rfloor)/2$ copies $B'_0, \ldots, B'_{r-1}$ of the right fragment with 14 triangles and $l = \lfloor \frac{t}{12} \rfloor - r$ copies $B'_r, \ldots, B'_{r+l-1}$ of the left fragment with 12 triangles.

We identify all vertices labelled N and all vertices labelled S, respectively, and for $0 \leq i < r+l$ identify the vertices y, y' in B'_i with the vertices x, x' in $B'_{i+1(\text{mod } (r+l))}$ respectively. It is easy to check that the resulting graph $G_{r,l}$ is 5-connected.

Checking the different ways how a Hamiltonian cycle can pass the left fragment in Figure 8 without using the poles and saturate the 4 interior vertices (some boundary vertices can also be saturated from outside the segment), gives that each such segment contains at least 2 type-2 triangles. As the fragment on the right hand side of Figure 8 contains the one on the left hand side, the same is true for the fragment on the right hand side too.

So for $t > 130$ and consequently $r+l \geq 11$ any Hamiltonian cycle C in $G_{r,l}$ has at least $r+l-8$ fragments not containing an edge of C incident with a pole and therefore containing at least 2 type-2 triangles. So $t_2(G_{r,l}, C) \geq 2(r+l-8)$ and therefore $t_0(G_{r,l}, C) \geq 2(r+l-8) - 4 = 2(r+l) - 20$. As $r+l = \lfloor \frac{t}{12} \rfloor$ we get $t_0^5(t) \geq 2\lfloor \frac{t}{12} \rfloor - 20$.

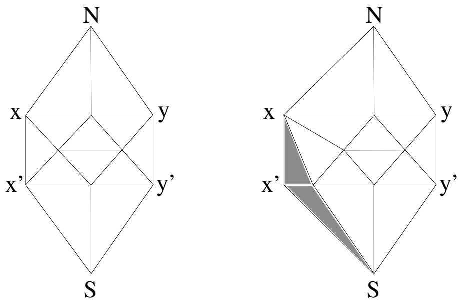

Figure 8: The fragments used for the 5-connected case.

The result for $\bar{t}_0^5(t)$ was proven by a computer search testing graphs constructed by the program plantri [2]. All 5-connected triangulations G with up to 66 triangles were found to have $\bar{t}_0(G) = 0$. It should also be noted that the graphs $G_{r,l}$ constructed for the first part all allow a Hamiltonian cycle C with $\bar{t}_0(G, C) = 0$.

□

Computational results for $r = 0$ and $l \leq 8$ suggest that $t_0^5(t) \geq 2\lfloor \frac{t}{12} \rfloor - 8$, but a proof similar to the one for $t_0(t)$ and $t_0^4(t)$ is out of reach on the computational side for the basic step in the induction and would be very lengthy on the theoretical side.

For $t_0(t)$, $\bar{t}_0(t)$, $t_0^4(t)$, and $\bar{t}_0^4(t)$ the upper and lower bounds differ only by an additive constant, so there is not much room for improvement. For $t_0^5(t)$, and especially $\bar{t}_0^5(t)$ the upper and lower bounds are far apart and have a different growth rate. In these cases there is not only room, but also need for improvement.

4 Applications different from Hamiltonian cycles

Type-0 triangles are of their own interest in the context of Hamiltonicity of triangulations, as they are the problematic case for the extendability of partial Hamiltonian cycles to the inside of separating triangles (see e.g. [9]), but the number $t_0(G)$ has also an impact on invariants that are not that obviously related to Hamiltonian cycles. In this section, we describe two other topics in graph theory for which the value of $t_0(G)$ is relevant.

4.1 The domination number of a triangulation

A vertex subset S of a graph G is said to be *dominating* if every vertex in $G - S$ has a neighbour in S. The cardinality of a minimum dominating set of G is called the *domination number* of G and is denoted by $\gamma(G)$. For a triangulation G, Matheson and Tarjan [11] proved that $\gamma(G) \leq \frac{|G|}{3}$ and they conjectured that $\gamma(G) \leq \frac{|G|}{4}$. This conjecture is still open, even when restricted to 4- or 5-connected triangulations.

Plummer, Ye and Zha [13] proved that $\gamma(G) \leq \min\left\{\lceil \frac{2|G|}{7} \rceil, \lfloor \frac{5|G|}{16} \rfloor\right\}$ for any 4-connected triangulation G. This is the currently best approach towards the Matheson-Tarjan conjecture. The idea of their inductive proof is to find a Hamiltonian cycle with certain properties of type-2 triangles and to use these for reduction of the graph.

If we can find a Hamiltonian cycle with few type-2 triangles, then (as implicitly used in [13]) we can bound the size of a dominating set as follows: Let C be a Hamiltonian cycle. By symmetry we can assume that the number of type-2 triangles on the inside of C is less than or equal to that on the outside of C. Let G' be the maximal outer plane graph consisting of the inside of C together with C. Note that G' contains $\bar{t}_2(G,C)$ type-2 triangles. It is shown in [5, 16] that any maximal outer plane graph H satisfies $\gamma(H) \leq \frac{|H|+k(H)}{4}$, where $k(H)$ denotes the number of vertices of degree 2 in H. Any vertex of degree two in G' is the common end vertex of two edges of C in a type-2 triangle. Thus, we have $k(G') = \bar{t}_2(G,C)$. Since $\bar{t}_2(G,C) = \bar{t}_0(G,C) + 2$, we obtain by Proposition 1.2

$$\gamma(G) \leq \gamma(G') \leq \frac{|G| + k(G')}{4} = \frac{|G| + \bar{t}_2(G,C)}{4} = \frac{|G| + \bar{t}_0(G,C) + 2}{4}$$
$$\leq \frac{2|G| + t_0(G,C) + 4}{8}.$$

So for a given Hamiltonian triangulation, a Hamiltonian cycle C with few type-0 triangles possibly gives a good upper bound on the domination number in that triangulation. In general though, the impact of the values of $t_0(t)$ is a negative one: the lower bounds given in Theorem 3.1 show that at least for 4-connected triangulations a direct application of this method cannot lead to improved bounds for the domination number.

4.2 3-walks with few vertices visited more than once

A *k-tree* of a graph G is a spanning tree of G in which every vertex has degree at most k. A *k-walk* is a spanning closed walk that visits every vertex at most k times. It is well-known that a graph that contains a k-walk also contains a $(k+1)$-tree, see [8] (but the converse does not hold in general). Furthermore, the vertices visited k times in a k-walk correspond to vertices of degree $k+1$ in the $(k+1)$-tree that is constructed.

Every 3-connected planar graph admits a 3-tree [1] and a 2-walk [6]. The result about 3-trees was strengthened in [12] where it is shown that every 3-connected planar graph G admits a 3-tree with at most $\frac{|G|-7}{3}$ vertices of degree 3.

As in the construction of 3-trees from 2-walks in [8], vertices visited twice in a 2-walk correspond to vertices of degree 3 in the 3-tree, it was natural to consider the following problem, which was already mentioned in [12].

Problem 4.1. Is there for every 3-connected planar graph G a 2-walk such that the number of vertices visited twice is at most $\frac{|G|}{3} - c$ for a constant c?

Note that for a 2-walk in a graph G, the number of vertices visited twice is at most t if and only if its length is at most $|G| + t$. With this formulation of the problem in mind, the result that every 3-connected planar graph G contains a spanning closed walk of length at most $\frac{4|G|-4}{3}$ (proven in [10]) can be considered as a first step towards the solution of Problem 4.1. However, a spanning closed walk constructed in [10] may visit a vertex many times, so Problem 4.1 is still open.

In this section we describe a different step towards the solution of Problem 4.1, by limiting the number of times a vertex is visited to 3. The class for which the result is proven is a subclass of all triangulations, but in fact a class containing cases for which Problem 4.1 would hold with equality. Type-0 triangles play an important role in the construction of the walks.

In the language of [9] the triangulations in the class of graphs we will describe now are those triangulations where the so-called decomposition tree is a star. In order not to refer the reader to [9] and to fix notation, we will give an independent description of the class here. To simplify notation, we consider K_4 also as a 4-connected graph in this section. Let $\mathcal{K}$ be the set of all graphs G that can be constructed as follows: Take any 4-connected triangulation H and let F be a subset of facial triangles of H. For each facial triangle $f = xyz \in F$, take a 4-connected plane graph G_f (not necessarily a triangulation) where the outer face is a triangle and let x_f, y_f and z_f be the three boundary vertices of G_f. Then G is obtained from H by adding G_f inside f for $f \in F$, so that x, y, z are identified with x_f, y_f, z_f, respectively. Except for the case when G is a triangulation with exactly one separating triangle the graph H is uniquely defined for each $G \in \mathcal{K}$ and we write $H(G)$ for it. In the case of one separating triangle there are two possible candidates for H and $H(G)$ denotes an arbitrary one of them.

For example, the face subdivision of a 4-connected triangulation H belongs to $\mathcal{K}$. In the definition above, F is the set of all facial triangles of H and for any face f we have $G_f \simeq K_4$. As in [12, Section 2], the face subdivision of a 4-connected triangulation shows that we cannot decrease the coefficient $\frac{1}{3}$ of $|G|$ in Problem 4.1. So, in this sense, some graphs in $\mathcal{K}$ belong to the most difficult ones for Problem 4.1.

The following result shows that a Hamiltonian cycle C in a 4-connected triangulation T with small $t_0(T, C)$ can be used to construct a 3-walk of short length for the graphs $G \in \mathcal{K}$ with $H(G) = T$. Using Theorem 1.1, in Corollary 4.5 we obtain a general upper bound depending only on the number of vertices in G.

Theorem 4.2. *Let $G \in \mathcal{K}$ be given and C a Hamiltonian cycle in $H = H(G)$. We write $t'_0(H, C)$ (or short t'_0) for the number of those type-0 triangles of H that are not faces in G. Then G contains a 3-walk of length at most $\frac{4|G|+t'_0-4}{3}$ which visits each vertex not in H exactly once.*

Proof. Let F, H, and for each facial triangle $f \in F$ also G_f, x_f, y_f, and z_f be as in the definition of $\mathcal{K}$. We denote the length of a walk W by $l(W)$, and let $|R|_- = |R| - 3$ for a plane graph R. With this notation we have $|G| = |H| + \sum_{f \in F} |G_f|_-$.

Claim 4.3. *For a 4-connected plane graph R where the outer face is a triangle (including K_4) with vertices x, y, z in the boundary and $a, b \in \{x, y, z\}$ (with possibly $a = b$), there is a (possibly closed) walk $P_{R,a,b}$ of length $|R|_- + 1$ from a to b in R visiting exactly all vertices in R except those in $\{x, y, z\} \setminus \{a, b\}$ and visiting vertices not in the boundary exactly once.*

Proof. The case $G = K_4$ can be easily checked by hand, so assume that G is not K_4.

If $a = b$ (w.l.o.g. $a = b = x$) then according to [15, (3.4)] there exists a Hamiltonian cycle in $G - \{y, z\}$, which is a closed walk with the given properties starting and ending in a.

If $a \neq b$ (w.l.o.g. $a = x$, $b = y$), due to [14, Corollary 2] there is a Hamiltonian cycle C in G through $\{a, z\}$ and $\{b, z\}$. $C - \{\{a, z\}, \{b, z\}\}$ is the walk $P_{G,a,b}$.

□

For a given cycle C with a fixed vertex c_1 we define a linear order along one of the directions of C starting from c_1 as $c_1 < c_2 < \cdots < c_n$. For each facial triangle f of H we fix the notation of x_f, y_f, z_f so that $x_f < y_f < z_f$.

With this notation we have:

Claim 4.4. *For any two triangles f and f' that belong to the same side of C we have $y_f \neq y_{f'}$.*

Proof. Assume $x_f \leq x_{f'}$. C is divided into three segments by the vertices x_f, y_f and z_f and – as $x_{f'}, y_{f'}$ and $z_{f'}$ are all at least x_f and smaller than c_n, they occur in one of these segments in the order $x_{f'}, y_{f'}, z_{f'}$. This implies that only $x_{f'}$ and $z_{f'}$ can be one of the end vertices of the segment and $y_{f'}$ is in fact different from each of x_f, y_f and z_f.

□

We consider the following spanning subgraph H_C^* of the dual of H: The vertex set of H_C^* is the set of triangles of H, and two faces are adjacent in H_C^* if and only if they share an edge in C. Note that for $i \in \{0, 1, 2\}$, a type-i triangle has degree exactly i in H_C^*. In particular, each component of H_C^* is an isolated vertex, a path or a cycle. We can give an orientation to the edges of such a component P^*, so that each vertex in P^*, except for isolated vertices and one of the end vertices when P^* is a path, has out-degree one. In cases where only one end vertex v of such a path P^* belongs to F, we choose v to have out-degree one.

Recall that F is the set of facial triangles of H into which a graph was inserted. We can partition F into two sets F_0 and F_1. We define for $i \in \{0, 1\}$:

$$F_{-} = \{f \in F : f \text{ has out-degree exactly } i\}.$$

With $t_1'(H, C)$ (or short t_1') for the number of those type-1 triangles of H that are no faces in G our construction gives $|F_0| \leq t_0' + \frac{t_1'}{2}$.

Now we modify C using Claim 4.3 so that for each triangle $f \in F$ it visits each vertex inside G_f exactly once:

- Suppose that $f \in F_0$. Then we add the walk P_{G_f, y_f, y_f} to C. This increases the length of C by $|G_f|_- + 1$.
- Suppose that $f \in F_1$. Let f' be the out-neighbour of f, and let $\{a, b\}$ be the edge in C that is shared by f and f'.

  Then we replace $\{a, b\}$ in C by $P_{G_f, a, b}$. This increases the length of C by only $|G_f|_-$ as one edge in C is also deleted.

The resulting walk C' is a 3-walk because, by Claim 4.4, the number of times a vertex is visited is increased by at most 1 for each side of C.

We will first give some equations we will use to compute the length of C'. For the given Hamiltonian cycle C we denote $t_0(H, C)$, $t_1(H, C)$ and $t_2(H, C)$, by t_0, t_1, t_2, respectively.

As $t_0 + t_1 + t_2 = t(H) = 2|H| - 4$ and $t_2 = t_0 + 4$ (by Proposition 1.2), we get $|H| = \frac{2t_0 + t_1}{2} + 4 \geq \frac{2t_0' + t_1'}{2} + 4$.

As in each face of F at least one vertex is inserted, we get $|G| \geq |H| + t_0' + t_1'$. So together with the previous equation $|G| \geq \frac{4t_0' + 3t_1'}{2} + 4 = \frac{6t_0' + 3t_1'}{2} + 4 - t_0'$ which can be rewritten as $t_0' + \frac{t_1'}{2} \leq \frac{|G| + t_0' - 4}{3}$ we get

$$l(C') = l(C) + \sum_{f\in F_1} |G_f|_- + \sum_{f\in F_0} (|G_f|_- + 1) = l(C) + \sum_{f\in F} |G_f|_- + |F_0|$$
$$= |H| + \sum_{f\in F} |G_f|_- + |F_0| = |G| + |F_0| \le |G| + t_0' + \frac{t_1'}{2}$$
$$\le |G| + \frac{|G| + t_0' - 4}{3} = \frac{4|G| + t_0' - 4}{3}$$

This completes the proof of Theorem 4.2. □

Using Theorem 1.1 (ii) we obtain the following corollary.

Corollary 4.5. *Except for K_4, any graph $G \in \mathcal{K}$ contains a 3-walk of length at most $\frac{22|G|-34}{15}$.*

Proof. Applying the construction of a walk from Theorem 4.2, we get that any graph $G \in \mathcal{K}$ with $K_4 = H(G)$ is Hamiltonian, so we just have to check whether $|G| \le \frac{22|G|-34}{15}$, which is the case for all graphs except K_4 itself (that is if $F = \emptyset$).
Assume now that $t(H(G)) = 8$, so $H(G)$ is the octahedron. If G is Hamiltonian, then there is a walk of length $|G| \le \frac{22|G|-34}{15}$. Otherwise it follows directly from Theorem 16 in [4] that $|F| \ge 4$. As that result is still unpublished, one can alternatively use our construction of a walk from Theorem 4.2 together with Theorem 4.1 from [9] to obtain that $|F| \ge 4$. Furthermore one can easily find a Hamiltonian cycle with $|F_0| \le 2$. With $v_i \ge 4$ the number of vertices added inside a triangle of the octahedron, the construction gives a 3-walk with length at most $6 + v_i + 2$ and $6 + v_i + 2 \le \frac{22(6+v_i)-34}{15}$ for $v_i \ge 4$. From now on assume that $H(G)$ has at least 10 faces.

Let C be a Hamiltonian cycle in $H = H(G)$ with $t_0(H)$ type-0 triangles. Let t_0' denote the number of type-0 triangles of C in H that are not faces in G. As each triangle in F contains at least one vertex, we have that $|G| \ge |H| + t_0'$. By Theorem 1.1 (ii), we get $t_0^4(t(H)) \le \frac{t(H)-10}{3}$.

$$t_0' \le t_0^4(t(H)) = t_0^4(2|H| - 4) \le \frac{2|H| - 14}{3} \le \frac{2(|G| - t_0') - 14}{3}$$

which implies

$$t_0' \le \frac{2|G| - 14}{5}.$$

Substituting this into the equation given in Theorem 4.2, we get Corollary 4.5. □

5 Correctness of the computer programs used

The programs constructing Hamiltonian cycles and computing $t_0()$ and $\bar{t}_0()$ are straightforward branch and bound programs that can be obtained from the authors or be downloaded from `http://caagt.ugent.be/type0/` to check the source code, to check the computational results in this paper, or to be used otherwise. Two independent programs were developed and implemented and the results were compared for each of the around $150.000.000$ triangulations with up to 30 triangles generated by *plantri*. There was full agreement. The computation of $\bar{t}_0()$ for 5-connected triangulations was done independently up to 60 triangles and for larger values only by the faster of the two programs.

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