On the Compatibility of a Ternary Relation with a Binary Fuzzy Relation

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Recently, De Baets et al. have characterized the fuzzy tolerance relations that a given strict order relation is compatible with. In general, the compatibility of a strict order relation with a binary fuzzy relation guarantees also the compatibility of its associated betweenness relation with that binary fuzzy relation. In this paper, we study the compatibility of an arbitrary ternary relation with a binary fuzzy relation. We prove that this compatibility can be expressed in terms of inclusions of the binary fuzzy relation in the traces of the given ternary relation.

Keywords: Ternary relation; binary fuzzy relation; traces of a ternary relation; compatibility.

1. Introduction

The notion of compatibility of a binary fuzzy relation with a fuzzy equality relation is due to Bělohlávek.1 It turns out to be equivalent to the older notion of extensionality introduced by Höhle and Blanchard.19 Daňková12 has further generalized this notion to the extensionality of an n-ary fuzzy relation w.r.t. n binary fuzzy relations. The compatibility property has been extensively studied by Kheniche et al.20 for arbitrary binary fuzzy relations. In particular, they have shown that Fodor’s notion15 of traces of a binary fuzzy relation facilitates the characterization of the compatibility of fuzzy relations in terms of fuzzy relational inclusions.

In general, the notion of compatibility is witnessing increasing attention and appears in various studies on binary fuzzy relations, for instance, in the study of fuzzy ordered

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structures, fuzzy functions and \(L\)-algebras. Given the importance of this notion, De Baets et al. have characterized the fuzzy tolerance and fuzzy equivalence relations that a given strict order relation is compatible with. In a recent paper, they have extended this characterization by considering an arbitrary binary relation instead of a strict order relation.

To any strict order relation \(\prec\) on a set \(X\), we can associate a ternary relation \(\prec\) on \(X\), called strict order betweenness relation, as follows:

\[
(a, b, c) \in \prec \quad \text{if and only if} \quad a < b < c \quad \text{or} \quad c < b < a.
\]

As a consequence, the compatibility of the strict order relation \(\prec\) with a binary fuzzy relation \(S\) implies the compatibility of \(\prec\) with \(S\). More specifically, if

\[
\tau(x < y) \ast S(x, a) \ast S(y, b) \leq \tau(a < b),
\]

for any \(x, y, a, b \in X\), then it holds that

\[
\tau(B_<(x, y, z)) \ast S(x, a) \ast S(y, b) \ast S(z, c) \leq \tau(B_<(a, b, c)),
\]

for any \(x, y, z, a, b, c \in X\). Here, \(\ast\) denotes a t-norm and the symbol \(\tau\) is used to refer to the characteristic mapping of a relation. In particular, \(\tau(B_<(x, y, z)) = 1\) if \((x, y, z) \in B_<\), while \(\tau(B_<(x, y, z)) = 0\) if \((x, y, z) \notin B_<\).

Inspired by the above example, in this paper we study the compatibility of an arbitrary ternary relation with a binary fuzzy relation. After recalling some basic concepts in Section 2, we introduce and study the notions of left-, middle- and right-compatibility of an arbitrary ternary relation with a binary fuzzy relation in Section 3. In Section 4, we provide a characterization of the left-, middle- and right-compatibility of a ternary relation with a binary fuzzy relation in terms of inclusions of the binary fuzzy relation in the left, middle and right traces of the ternary relation. Subsequently, several basic properties of these compatibility relations are discussed. Moreover, in Sections 5–7, we study the relationship between the compatibility of a ternary relation and that of its traces and its projections, as well as the relationship between the compatibility of a binary relation and that of its cylindrical extensions. In Section 8, we study the interaction of the compatibility relations with the basic set operations and relational compositions. We discuss the compatibility of members of certain classes of ternary relations with a binary fuzzy relation in Section 9. Finally, we present some conclusions and discuss future research in Section 10.

2. Basic Concepts

This section serves to recall some basic definitions and properties of binary and ternary relations, as well as binary \(L\)-relations.

2.1. Binary relations

A binary relation \(R\) on a set \(X\) is a subset of \(X^2\). Inclusion, intersection and union of binary relations on \(X\) are defined through the corresponding notions for subsets of \(X^2\). The transpose \(R^t\) of \(R\) is defined as \(R^t = \{(y, x) \in X^2 \mid (x, y) \in R\}\). A binary relation \(R\) on a set \(X\) is called:
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(i) reflexive, if, for any \( x \in X \), it holds that \((x, x) \in R\);
(ii) symmetric, if \( R' = R \);
(iii) antisymmetric, if, for any \( x, y \in X \), it holds that \((x, y) \in R \land (y, x) \in R \) implies that \( x = y \);
(iv) asymmetric, if, for any \( x, y \in X \), it holds that \((x, y) \in R \) implies that \((y, x) \notin R \);
(v) transitive, if, for any \( x, y, z \in X \), it holds that \((x, y) \in R \land (y, z) \in R \) implies that \((x, z) \in R \).

For more details on binary relations and their properties, we refer to Ref. 11, 27.

2.2. Ternary relations

A ternary relation \( T \) on a set \( X \) is a subset of \( X^3 \). Inclusion, intersection and union of ternary relations on \( X \) are defined through the corresponding notions for subsets of \( X^3 \). The transpose \( T' \) of \( T \) is defined as \( T' = \{(x, y, z) \in X^3 \mid (z, y, x) \in T\} \). A ternary relation \( T \) on a set \( X \) is called:

(i) reflexive, if, for any \( x \in X \), it holds that \((x, x, x) \in T\);
(ii) symmetric, if \( T' = T \);
(iii) antisymmetric, if, for any \( x, y, z \in X \), it holds that \((x, y, z) \in T \land (z, y, x) \in T \) implies that \( x = y = z \);
(iv) cyclic, if, for any \( x, y, z \in X \), it holds that \((x, y, z) \in T \) implies that \((y, z, x) \in T \).

For more details on ternary relations, we refer to Refs. 9, 24, 25, 29.

2.3. Binary L-relations

The notion of an \( L \)-relation on a set \( X \) generalizes the notion of a relation by expressing degrees of relationship in some bounded lattice \((L, \leq, \sim, \sim, 0, 1)\). A binary \( L \)-relation \( S \) on a set \( X \) is a mapping \( S : X \times X \rightarrow L \). Obviously, if \( L = \{0, 1\} \), then binary relations are retrieved, often referred to as crisp relations. As is commonly done, we will further restrict to the use of a complete residuated lattice \((L, \leq, \sim, \sim, *, \rightarrow, 0, 1)\). The corresponding infimum and supremum operations are denoted as \( \inf \) and \( \sup \), respectively. The operations \( \ast \) and \( \rightarrow \) are known as a t-norm and its residual implication, and form an adjoint pair, i.e. \( x \ast z \leq y \) if and only if \( z \leq x \rightarrow y \), for any \( x, y, z \in L \).

A binary \( L \)-relation \( S_1 \) is said to be included in a binary \( L \)-relation \( S_2 \), denoted by \( S_1 \subseteq S_2 \), if \( S_1(x, y) \leq S_2(x, y) \), for any \( x, y \in X \). The intersection of two binary \( L \)-relations \( S_1 \) and \( S_2 \) on \( X \) is the binary \( L \)-relation \( S_1 \land S_2 \) on \( X \) defined by \( S_1 \land S_2(x, y) = S_1(x, y) \land S_2(x, y) \). Similarly, the union of two binary \( L \)-relations \( S_1 \) and \( S_2 \) on \( X \) is the binary \( L \)-relation \( S_1 \lor S_2 \) on \( X \) defined by \( S_1 \lor S_2(x, y) = S_1(x, y) \lor S_2(x, y) \), for any \( x, y \in X \).

The transpose \( S' \) of a binary \( L \)-relation \( S \) is the binary \( L \)-relation defined by \( S'(x, y) = S(y, x) \). A binary \( L \)-relation \( S \) on a set \( X \) is called:

(i) reflexive, if, for any \( x \in X \), it holds that \( S(x, x) = 1 \);
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(ii) symmetric, if $S' = S$;

(iii) $\ast$-transitive, if, for any $x, y, z \in X$, it holds that $S(x, y) \ast S(y, z) \leq S(x, z)$.

For more details on binary $L$-relations and their properties, we refer to Refs. 1, 5, 8, 30.

An important role in this paper is played by the Fodor’s notions of left and right trace of a binary $L$-relation. Note that we adopt the change of terminology introduced recently. 29

Definition 1. 18

Let $S$ be a binary $L$-relation on a set $X$.

(i) The left trace of $S$ is the $L$-relation $S'_l$ on $X$ defined by

$$S'_l(x, y) = \inf_{z \in X} S(x, z) \rightarrow S(y, z);$$

(ii) The right trace of $S$ is the $L$-relation $S'_r$ on $X$ defined by

$$S'_r(x, y) = \inf_{z \in X} S(z, x) \rightarrow S(z, y).$$

Theorem 1. 18

Let $S$ be a binary $L$-relation on a set $X$. The following statements are equivalent:

(i) $S$ is reflexive;

(ii) $(S'_l)^f \subseteq S$;

(iii) $S'_r \subseteq S$.

Note that the traces of a binary relation $R$ on $X$ can be obtained as a special case of Definition 1.

3. Compatibility of a Ternary Relation with a Binary $L$-Relation

In this section, we generalize the notion of compatibility of binary $L$-relations by replacing one of the binary $L$-relations by a (crisp) ternary relation. To that end, we first recall the original notion of compatibility of binary $L$-relations as introduced and studied by Kheniche et al. For more details, we refer to Refs. 2, 13 and 20.

Definition 2. 20

Let $S_1$ and $S_2$ be two binary $L$-relations on a set $X$.

(i) $S_1$ is called left-compatible with $S_2$ if it holds that

$$S_1(x, y) \ast S_2(x, a) \leq S_1(a, y),$$

for any $x, y, a \in X$;

(ii) $S_1$ is called right-compatible with $S_2$ if it holds that

$$S_1(x, y) \ast S_2(y, b) \leq S_1(x, b),$$

for any $x, y, b \in X$;

(iii) $S_1$ is called compatible with $S_2$ if it holds that

$$S_1(x, y) \ast S_2(x, a) \ast S_2(y, b) \leq S_1(a, b),$$

for any $x, y, a, b \in X$. 
Note that this definition applies in particular when the binary $L$-relation $S_1$ is crisp. We now intend to generalize this special case by replacing this (crisp) binary relation by a (crisp) ternary relation.

**Definition 3.** Let $T$ be a ternary relation and $S$ be a binary $L$-relation on a set $X$.

(i) $T$ is called left-compatible with $S$ if it holds that

$$\tau(T(x, y, z)) * S(x, a) \leq \tau(T(a, y, z)),$$

for any $x, y, z, a \in X$;

(ii) $T$ is called middle-compatible with $S$ if it holds that

$$\tau(T(x, y, z)) * S(y, b) \leq \tau(T(x, b, z)),$$

for any $x, y, z, b \in X$;

(iii) $T$ is called right-compatible with $S$ if it holds that

$$\tau(T(x, y, z)) * S(z, c) \leq \tau(T(x, y, c)),$$

for any $x, y, z, c \in X$;

(iv) $T$ is called compatible with $S$ if it holds that

$$\tau(T(x, y, z)) * S(x, a) * S(y, b) * S(z, c) \leq \tau(T(a, b, c)).$$

The following proposition identifies two important implications that will be helpful in the study of the compatibility property.

**Proposition 1.** Let $T$ be a ternary relation and $S$ be a binary $L$-relation on a set $X$. The following implications hold:

(i) If $T$ is left-, middle- and right-compatible with $S$, then $T$ is compatible with $S$;

(ii) If $T$ is compatible with $S$ and $S$ is reflexive, then $T$ is left-, middle- and right-compatible with $S$.

**Proof.**

(i) Suppose that $T$ is left-, middle- and right-compatible with $S$. Let $x, y, z, a, b, c \in X$.

Since $T$ is left-compatible with $S$, it holds that

$$\tau(T(x, y, z)) * S(x, a) \leq \tau(T(a, y, z)).$$

Furthermore, since $*$ is increasing and $T$ is middle-compatible with $S$, it follows that

$$\tau(T(x, y, z)) * S(x, a) * S(y, b) \leq \tau(T(a, y, z)) * S(y, b) \leq \tau(T(a, b, z)).$$

Similarly, since $*$ is increasing and $T$ is right-compatible with $S$, we obtain

$$\tau(T(x, y, z)) * S(x, a) * S(y, b) * S(z, c) \leq \tau(T(a, b, z)) * S(z, c) \leq \tau(T(a, b, c)).$$

Hence, $T$ is compatible with $S$. 
(ii) Setting $b = y$ and $c = z$ in (4) and taking into account the reflexivity of $S$, we obtain (1). Similarly, we can obtain (2) and (3).

The following proposition is immediate.

**Proposition 2.** Let $T$ be a ternary relation and $S$ be a binary $L$-relation on a set $X$. The following equivalences hold:

(i) $T$ is left-compatible with $S$ if and only if $T^-$ is right-compatible with $S$;
(ii) $T$ is middle-compatible with $S$ if and only if $T^-$ is middle-compatible with $S$;
(iii) $T$ is right-compatible with $S$ if and only if $T^+$ is left-compatible with $S$.

Next, we recall the definition of the cyclic permutations of a ternary relation $T$.

**Definition 4.** Let $T$ be a ternary relation on a set $X$. The cyclic permutations of $T$ are the ternary relations $T^-$ and $T^+$ on $X$ defined as

$$T^- = \{(x, y, z) \in X^3 \mid (z, x, y) \in T\},$$

$$T^+ = \{(x, y, z) \in X^3 \mid (y, z, x) \in T\}.$$

The following proposition relates the middle-compatibility of a ternary relation with the left- and right-compatibility of its cyclic permutations.

**Proposition 3.** Let $T$ be a ternary relation and $S$ be a binary $L$-relation on a set $X$. The following equivalences hold:

(i) $T$ is middle-compatible with $S$ if and only if $T^-$ is left-compatible with $S$;
(ii) $T$ is middle-compatible with $S$ if and only if $T^+$ is right-compatible with $S$.

**Proof.** We prove (i) only, as both cases are analogous. Let $x, y, z, a \in X$. Suppose that $T$ is middle-compatible with $S$, then it holds that

$$\tau(T(z, x, y)) \ast S(x, a) \leq \tau(T(z, a, y)).$$

From the definition of $T^-$, it follows that

$$\tau(T^-(x, y, z)) \ast S(x, a) \leq \tau(T(z, a, y)) = \tau(T^-(a, y, z)).$$

Hence, $T^-$ is left-compatible with $S$. In a similar way, we can prove the converse implication.

**4. Characterization of Compatibility in Terms of Traces**

In this section, we characterize the left-, middle- and right-compatibility of a ternary relation with a binary fuzzy relation in terms of inclusions of the binary fuzzy relation in the left, middle and right traces of the ternary relation. We first recall the notion of traces of a ternary relation and some related properties for further use. It is important to note that, just as for binary relations, the traces of a ternary relation are defined as binary relations.
**Definition 5.** Let $T$ be a ternary relation on a set $X$.

(i) The left trace of $T$ is the binary relation $T^l$ on $X$ defined as

$$T^l = \{(x, y) \in X^2 | (\forall(a, b) \in X^2)((x, a, b) \in T \Rightarrow (y, a, b) \in T) \};$$

(ii) The middle trace of $T$ is the binary relation $T^m$ on $X$ defined as

$$T^m = \{(x, y) \in X^2 | (\forall(a, b) \in X^2)((a, x, b) \in T \Rightarrow (a, y, b) \in T) \};$$

(iii) The right trace of $T$ is the binary relation $T^r$ on $X$ defined as

$$T^r = \{(x, y) \in X^2 | (\forall(a, b) \in X^2)((a, b, x) \in T \Rightarrow (a, b, y) \in T) \}.$$

The following proposition requires a careful reading. It expresses that the above traces (which are binary relations) of a ternary relation coincide with their right traces (in the sense of Definition 1).

**Proposition 4.** Let $T$ be a ternary relation on a set $X$. The following statements hold:

$$T^l = (T^r)^r, \quad T^m = (T^m)^r \quad \text{and} \quad T^r = (T^r)^r.$$

**Proposition 5.** Let $T$ be a ternary relation on a set $X$. The left, middle and right traces of its transpose and cyclic permutations are listed in the following table:

<table>
<thead>
<tr>
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<th>$(\cdot),^T$</th>
<th>$(\cdot)^m$</th>
<th>$(\cdot)^r$</th>
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</thead>
<tbody>
<tr>
<td>$T^l$</td>
<td>$T^r$</td>
<td>$T^m$</td>
<td>$T^l$</td>
</tr>
<tr>
<td>$T^+$</td>
<td>$T^r$</td>
<td>$T^l$</td>
<td>$T^m$</td>
</tr>
<tr>
<td>$T^-$</td>
<td>$T^m$</td>
<td>$T^r$</td>
<td>$T^l$</td>
</tr>
</tbody>
</table>

The following proposition states that a ternary relation is compatible with its traces, just as is the case for a binary relation or a binary $L$-relation.

**Proposition 6.** Let $T$ be a ternary relation on a set $X$. The following statements hold:

(i) $T$ is left-compatible with $T^l$;
(ii) $T$ is middle-compatible with $T^m$;
(iii) $T$ is right-compatible with $T^r$.

**Proof.** As the proofs of the different statements are similar, we only prove the first one. Let $x, y, z, a \in X$, then we need to show that

$$\tau(T(x, y, z)) \ast \tau(T^l(x, a)) \leq \tau(T(a, y, z)). \quad (5)$$

If $(x, y, z) \notin T$ or $(x, a) \notin T^l$, then inequality (5) trivially holds. If $(x, y, z) \in T$ and $(x, a) \in T^l$, then it holds that $(a, y, z) \in T$, and inequality (5) holds again. $\square$

The above proposition allows to prove an important theorem stating that left-, middle- and right-compatibility can be expressed in terms of inclusions involving traces. This result nicely generalizes corresponding results on the compatibility of two $L$-relations recalled further in Theorem 3.
Theorem 2. Let $T$ be a ternary relation and $S$ be an $L$-relation on a set $X$. The following equivalences hold:

(i) $T$ is left-compatible with $S$ if and only if $S \subseteq T^l$;

(ii) $T$ is middle-compatible with $S$ if and only if $S \subseteq T^m$;

(iii) $T$ is right-compatible with $S$ if and only if $S \subseteq T^r$.

Proof.

(i) Suppose that $T$ is left-compatible with $S$. Let $x, a \in X$, then we need to show that

\[ S(x, a) \leq \tau(T^l(x, a)). \]  

If $S(x, a) = 0$, then inequality (6) trivially holds; if $S(x, a) > 0$, then we need to show that $(x, a) \in T^l$. Let $y, z \in X$ such that $(x, y, z) \in T$, then the left-compatibility of $T$ with $S$ implies that

\[ \tau(T(x, y, z)) \ast S(x, a) = S(x, a) \leq \tau(T(a, y, z)). \]

Hence, $\tau(T(a, y, z)) = 1$, i.e., $(a, y, z) \in T$. Hence, $(x, a) \in T^l$.

For the converse implication, it suffices to realise that if $T$ is left-compatible with a binary $L$-relation $S_2$, then it is also left-compatible with any binary $L$-relation $S_1 \subseteq S_2$. Since Proposition 6 (i) states that $T$ is left-compatible with $T^l$ and $S \subseteq T^l$, it follows that $T$ is left-compatible with $S$.

(ii) Follows from (i), Propositions 3, 5, 6.

(iii) Follows from (i), Propositions 2, 5, 6.

Combining Proposition 1 and Theorem 2 leads to the following corollary.

Corollary 1. Let $T$ be a ternary relation and $S$ be a binary $L$-relation on a set $X$. The following implications hold:

(i) If $S \subseteq T^l \cap T^m \cap T^r$, then $T$ is compatible with $S$;

(ii) If $T$ is compatible with $S$ and $S$ is reflexive, then $S \subseteq T^l \cap T^m \cap T^r$.

Next, we recall some trivial properties of the compatibility of binary $L$-relation for further use.

Proposition 7. For any two binary $L$-relations $S_1$ and $S_2$ on a set $X$, the following equivalences hold:

(i) $S_1$ is left-compatible with $S_2$ if and only if $S_1^l$ is right-compatible with $S_2^r$;

(ii) $S_1$ is right-compatible with $S_2$ if and only if $S_1^l$ is left-compatible with $S_2^r$.

Note that the following theorem has been adopted from Kheniche et al. taking into account the change of terminology of traces of a binary $L$-relation introduced in Ref. 29 to facilitate a more elegant formulation of this theorem and the proper introduction of the traces of a ternary relation.
Theorem 3. For any two binary $L$-relations $S_1$ and $S_2$ on a set $X$, the following equivalences hold:

(i) $S_1$ is left-compatible with $S_2$ if and only if $S_2 \subseteq S_1^l$;
(ii) $S_1$ is right-compatible with $S_2$ if and only if $S_2 \subseteq S_1^r$.

The following proposition shows that the compatibility of a ternary relation with a binary $L$-relation $S_1$ that is itself compatible with a binary $L$-relation $S_2$ implies the compatibility of that ternary relation with $S_2$.

Proposition 8. Let $T$ be a ternary relation and $S_1$ and $S_2$ be two binary $L$-relations on a set $X$ such that $S_1$ is reflexive. The following implications hold:

(i) (a) If $T$ is left-compatible with $S_1$, $S_1$ is left-compatible with $S_2$ and $S_2$ is symmetric, then $T$ is left-compatible with $S_2$;

(b) If $T$ is left-compatible with $S_1$ and $S_1$ is right-compatible with $S_2$, then $T$ is left-compatible with $S_2$;

(ii) (a) If $T$ is middle-compatible with $S_1$, $S_1$ is left-compatible with $S_2$ and $S_2$ is symmetric, then $T$ is middle-compatible with $S_2$;

(b) If $T$ is middle-compatible with $S_1$ and $S_1$ is right-compatible with $S_2$, then $T$ is middle-compatible with $S_2$;

(iii) (a) If $T$ is right-compatible with $S_1$, $S_1$ is left-compatible with $S_2$ and $S_2$ is symmetric, then $T$ is right-compatible with $S_2$;

(b) If $T$ is right-compatible with $S_1$ and $S_1$ is right-compatible with $S_2$, then $T$ is right-compatible with $S_2$.

Proof. We only prove (i)-(a), the other implications can be proved analogously. Suppose that $T$ is left-compatible with $S_1$. $S_1$ is left-compatible with $S_2$ and $S_1$ is reflexive and $S_2$ is symmetric. Since $T$ is left-compatible with $S_1$, it follows from Theorem 2 that $S_1 \subseteq T^r$. Since $S_1$ is left-compatible with $S_2$, it follows from Theorem 3 that $S_2 \subseteq S_1^r$. The symmetry of $S_2$ guarantees that $S_2 \subseteq (S_1^r)^r$. Since $S_1$ is reflexive, it follows from Theorem 1 that $(S_1^r)^r \subseteq S_1$. Hence, $S_2 \subseteq S_1 \subseteq T^r$. Thus, $T$ is left-compatible with $S_2$. $\square$

5. Relationship between the Compatibility of a Ternary Relation and that of Its Traces

In this section, we discuss the relationship between the compatibility of a ternary relation and that of its traces. The following theorem expresses that the compatibility of a ternary relation with a given binary $L$-relation is equivalent to the compatibility of its traces with that binary $L$-relation.

Theorem 4. Let $T$ be a ternary relation and $S$ be a binary $L$-relation on a set $X$. The following equivalences hold:

(i) $T$ is left-compatible with $S$ if and only if $(T^r)^r$ is left-compatible with $S$;
(ii) $T$ is middle-compatible with $S$ if and only if $T^m$ is right-compatible with $S$;
(iii) $T$ is middle-compatible with $S$ if and only if $(T^r)^r$ is left-compatible with $S$;
(iv) $T$ is right-compatible with $S$ if and only if $T^r$ is right-compatible with $S$. 
Proof.

(i) Suppose that $T$ is left-compatible with $S$. Due to Theorem 2 and Proposition 4, it holds that $S \subseteq T^\ell = (T^\ell)^\ell$, i.e. $T^\ell$ is right-compatible with $S$. From Proposition 7 (ii), it follows that $(T^\ell)^\ell$ is left-compatible with $S$. The converse goes in the same way.

(ii) Suppose that $T$ is middle-compatible with $S$. Due to Theorem 2 and Proposition 4, it holds that $S \subseteq T^m = (T^m)^r$, i.e. $T^m$ is right-compatible with $S$. The converse goes in the same way.

(iii) Follows from (ii) and Proposition 7 (ii).

(iv) Suppose that $T$ is right-compatible with $S$. Due to Theorem 2 and Proposition 4, it holds that $S \subseteq T^r = (T^r)^r$, i.e. $T^r$ is right-compatible with $S$. The converse goes in the same way.

6. Relationship between the Compatibility of a Ternary Relation and that of Its Projections

In this section, we discuss the relationship between the compatibility of a ternary relation and that of its projections. The following result expresses that the compatibility of a ternary relation with a given binary $L$-relation implies the compatibility of its projections with that binary $L$-relation. To that end, we need to recall the following definition and proposition.

**Definition 6.** Let $T$ be a ternary relation on a set $X$.

(i) The left projection of $T$ is the binary relation $P^\ell(T)$ on $X$ defined as

$$P^\ell(T) = \{ (x, y) \in X^2 \mid (\exists z \in X)((z, x, y) \in T) \};$$

(ii) The middle projection of $T$ is the binary relation $P^m(T)$ on $X$ defined as

$$P^m(T) = \{ (x, y) \in X^2 \mid (\exists z \in X)((x, z, y) \in T) \};$$

(iii) The right projection of $T$ is the binary relation $P^r(T)$ on $X$ defined as

$$P^r(T) = \{ (x, y) \in X^2 \mid (\exists z \in X)((x, y, z) \in T) \}.$$

**Proposition 9.** Let $T$ be a ternary relation on a set $X$, then the following inclusions hold:

(i) $T^\ell \subseteq (P^m(T))^\ell$;

(ii) $T^m \subseteq (P^r(T))^m \cap (P^l(T))^r$;

(iii) $T^r \subseteq (P^m(T))^r$.

Combining Theorem 3 and Proposition 9 leads to the following proposition.

**Proposition 10.** Let $T$ be a ternary relation and $S$ be a binary $L$-relation on a set $X$. The following implications hold:

(i) If $T$ is left-compatible (resp. right-compatible) with $S$, then $P^m(T)$ is left-compatible (resp. right-compatible) with $S$;

(ii) If $T$ is middle-compatible with $S$, then $P^r(T)$ is left-compatible with $S$ and $P^l(T)$ is right-compatible with $S$. 

7. Relationship between the Compatibility of a Binary Relation and that of Its Cylindrical Extensions

In this section, we discuss the relationship between the compatibility of a binary relation and that of its cylindrical extensions. The cylindrical extensions of a binary relation are recalled in the following definition.

**Definition 7.** Let $R$ be a binary relation on a set $X$.

(i) The left cylindrical extension of $R$ is the ternary relation $C_l(R)$ on $X$ defined as

$$C_l(R) = \{(x, y, z) \in X^3 \mid (y, z) \in R\};$$

(ii) The middle cylindrical extension of $R$ is the ternary relation $C_m(R)$ on $X$ defined as

$$C_m(R) = \{(x, y, z) \in X^3 \mid (x, z) \in R\};$$

(iii) The right cylindrical extension of $R$ is the ternary relation $C_r(R)$ on $X$ defined as

$$C_r(R) = \{(x, y, z) \in X^3 \mid (x, y) \in R\}.$$

The following theorem expresses that the compatibility of a binary relation with a given binary $L$-relation is equivalent to the compatibility of its cylindrical extensions with that binary $L$-relation. To that end, we need to recall the following proposition.

**Proposition 11.** Let $R$ be a binary relation on a set $X$, then the following equalities hold:

(i) $R^f = (C_r(R))^f = (C_m(R))^f = (C_l(R))^m$;

(ii) $R^r = (C_l(R))^r = (C_m(R))^r = (C_r(R))^n$.

Combining Theorem 3 and Proposition 11 leads to the following result.

**Theorem 5.** Let $R$ be a binary relation and $S$ be a binary $L$-relation on a set $X$.

(i) The following statements are equivalent:

(a) $R$ is left-compatible with $S$;

(b) $C_l(R)$ is middle-compatible with $S$;

(c) $C_m(R)$ is left-compatible with $S$;

(d) $C_r(R)$ is left-compatible with $S$;

(ii) The following statements are equivalent:

(e) $R$ is right-compatible with $S$;

(f) $C_r(R)$ is right-compatible with $S$;

(g) $C_m(R)$ is right-compatible with $S$;

(h) $C_l(R)$ is middle-compatible with $S$.

**Proof.** We only prove (a) $\Leftrightarrow$ (b), the other equivalences can be proved similarly. Suppose that $R$ is left-compatible with $S$. Due to Theorem 3 and Proposition 11 (i), this is equivalent to $S \subseteq R^f = (C_r(R))^m$, i.e. $C_r(R)$ is middle-compatible with $S$. □
8. Interaction of Compatibility with the Basic Set Operations and Compositions

In this section, we study the interaction of the compatibility relations with inclusion, set-theoretical operations and relational compositions. The characterizations in terms of inclusions expressed by Theorem 2 turn out to play an important role in this study.

8.1. Interaction of the compatibility relations with inclusion and basic set operations

For the sake of completeness, we mention the following trivial identities.

**Proposition 12.** For any family of ternary relations \((T_i)_{i \in I}\) on a set \(X\), the following equalities hold:

\[
(\bigcup_{i \in I} T_i)^l = \bigcup_{i \in I} T_i^l \quad \text{and} \quad (\bigcap_{i \in I} T_i)^l = \bigcap_{i \in I} T_i^l.
\]

Next, we recall the following inclusions.

**Proposition 13.** For any family of ternary relations \((T_i)_{i \in I}\) on a set \(X\), the following inclusions hold:

\[
(i) \quad T_i^c \subseteq (\bigcap_{i \in I} T_i)^c, \quad T_i^m \subseteq (\bigcap_{i \in I} T_i)^m \quad \text{and} \quad T_i^r \subseteq (\bigcap_{i \in I} T_i)^r;
\]

\[
(ii) \quad T_i^c \subseteq (\bigcup_{i \in I} T_i)^c, \quad T_i^m \subseteq (\bigcup_{i \in I} T_i)^m \quad \text{and} \quad T_i^r \subseteq (\bigcup_{i \in I} T_i)^r.
\]

**Proposition 14.** Let \(T\) be a ternary relation and \(S_1\) and \(S_2\) be two binary \(L\)-relations on a set \(X\) such that \(S_1 \subseteq S_2\). The following implications hold:

\[
(i) \quad \text{If } T \text{ is left-compatible with } S_2, \text{ then } T \text{ is left-compatible with } S_1;
\]

\[
(ii) \quad \text{If } T \text{ is middle-compatible with } S_2, \text{ then } T \text{ is middle-compatible with } S_1;
\]

\[
(iii) \quad \text{If } T \text{ is right-compatible with } S_2, \text{ then } T \text{ is right-compatible with } S_1;
\]

\[
(iv) \quad \text{If } T \text{ is compatible with } S_2 \text{ and } S_2 \text{ is reflexive, then } T \text{ is compatible with } S_1.
\]

**Proof.** We only prove (i), the other implications can be proved analogously. Suppose that \(T\) is left-compatible with \(S_2\). From Theorem 2 (i), it follows that \(S_2 \subseteq T^c\). Since \(S_1 \subseteq S_2 \subseteq T^c\), it then follows that \(T\) is left-compatible with \(S_1\).

**Proposition 15.** For any ternary relation \(T\) and family \((T_i)_{i \in I}\) of ternary relations and any binary \(L\)-relations \(S\) and family \((S_i)_{i \in I}\) of binary \(L\)-relations on a set \(X\), the following implications hold:

\[
(i) \quad \text{If } T_i \text{ is left-/middle-/right-compatible with } S, \text{ for any } i \in I, \text{ then } \bigcap_{i \in I} T_i \text{ and } \bigcup_{i \in I} T_i \text{ are left/middle/right-compatible with } S;
\]

\[
(ii) \quad \text{If } T \text{ is left-/middle-/right-compatible with } S_i, \text{ for any } i \in I, \text{ then } T \text{ is left-/middle-/right-compatible with } \bigcap_{i \in I} S_i \text{ and } \bigcup_{i \in I} S_i.
\]
it follows that analogous. Suppose that

\[ T_i \text{ is left-compatible with } S, \quad \text{for any } i \in I. \]

From Theorem 2 (i), it follows that \( S \subseteq T_i^\ell \), for any \( i \in I \), and, hence, \( S \subseteq \bigcap_{i \in I} T_i^\ell \). From Proposition 13 (i), it then follows that \( S \subseteq \left( \bigcap_{i \in I} T_i \right)^\ell \), i.e. \( \bigcap_{i \in I} T_i \) is left-compatible with \( S \).

**Proposition 16.** For any ternary relation \( T \) and family \((T_i)_{i \in I}\) of ternary relations and any binary \( L \)-relations \( S \) and family \((S_i)_{i \in I}\) of binary \( L \)-relations on a set \( X \), the following implications hold:

(i) If \( T_i \) is compatible with \( S \), for any \( i \in I \), and \( S \) is reflexive, then \( \bigcap_{i \in I} T_i \) and \( \bigcup_{i \in I} T_i \) are compatible with \( S \);

(ii) If \( T \) is compatible with \( S_i \) and \( S_i \) is reflexive, for any \( i \in I \), then \( T \) is compatible with \( \bigcap_{i \in I} S_i \) and \( \bigcup_{i \in I} S_i \).

**Proof.** We only prove (i) for intersection, the other implications can be proved analogously. Suppose that \( S \) is reflexive and that \( T_i \) is compatible with \( S \), for any \( i \in I \). From Proposition 1 (ii), it follows that \( T_i \) is left-, middle- and right-compatible with \( S \), for any \( i \in I \). Since \( T_i \) is left-compatible with \( S \), for any \( i \in I \), it follows from Proposition 15 (i) that \( \bigcap_{i \in I} T_i \) is left-compatible with \( S \). In a similar way, we prove that \( \bigcap_{i \in I} T_i \) is middle- and right-compatible with \( S \), for any \( i \in I \). Therefore, \( \bigcap_{i \in I} T_i \) is compatible with \( S \).

\[ \square \]

8.2. Interaction of compatibility with relational compositions

In this subsection, we show that compatibility is preserved under appropriate relational compositions. We first recall several compositions from our work on traces of ternary relations.

**Definition 8.** Let \( T \) be a ternary relation and \( R \) be a binary relation on a set \( X \).

(i) The \( \bowtie \)-composition of \( T \) and \( R \) is the ternary relation \( T \bowtie R \) on \( X \) defined as

\[ T \bowtie R = \{(x, y, z) \in X^3 \mid (\exists t \in X)((x, y, t) \in T \land (t, z) \in R)\}; \]

(ii) The \( \bowtie \)-composition of \( R \) and \( T \) is the ternary relation \( R \bowtie T \) on \( X \) defined as

\[ R \bowtie T = \{(x, y, z) \in X^3 \mid (\exists t \in X)((x, t, z) \in R \land (y, t) \in T)\}; \]

(iii) The \( \ltimes \)-composition of \( T \) and \( R \) is the ternary relation \( T \ltimes R \) on \( X \) defined as

\[ T \ltimes R = \{(x, y, z) \in X^3 \mid (\exists t \in X)((x, y, t) \in T \land (t, z) \in R)\}; \]

(iv) The \( \ltimes \)-composition of \( R \) and \( T \) is the ternary relation \( R \ltimes T \) on \( X \) defined as

\[ R \ltimes T = \{(x, y, z) \in X^3 \mid (\exists t \in X)((y, t) \in R \land (x, t, z) \in T)\}. \]
Proposition 17. Let $T$ be a ternary relation and $R$ be a binary relation on a set $X$. The following inclusions hold:

(i) $T^c \subseteq (T \Join R)^c$, $T^m \subseteq (T \Join R)^m \cap (R \Join T)^{m}$ and $T^f \subseteq (R \Join T)^{f}$;
(ii) $T^c \subseteq (T \Meet R)^c \cap (R \Meet T)^{c}$, $T^m \subseteq (T \Meet R)^m \cap (R \Meet T)^{m}$ and $T^f \subseteq (T \Meet R)^{f}$;
(iii) $R^c \subseteq (R \Join T)^c$ and $R^f \subseteq (T \Join R)^f$;
(iv) $R^c \subseteq (R \Meet T)^c$ and $R^f \subseteq (T \Meet R)^{f}$.

Proposition 18. Let $T$, $R$ and $S$ be a ternary, binary and binary $L$-relation on a set $X$, respectively. The following implications hold:

(i) If $T$ is left-compatible with $S$, then $T \Join R$, $T \Meet R$ and $R \Meet T$ are left-compatible with $S$;
(ii) If $T$ is middle-compatible with $S$, then $T \Join R$, $R \Join T$, $T \Meet R$ and $R \Meet T$ are middle-compatible with $S$;
(iii) If $T$ is right-compatible with $S$, then $R \Join T$, $T \Join R$ and $R \Meet T$ are right-compatible with $S$;
(iv) If $R$ is left-compatible with $S$, then $R \Join T$ is left-compatible with $S$ and $R \Meet T$ is middle-compatible with $S$;
(v) If $R$ is right-compatible with $S$, then $T \Join R$ is right-compatible with $S$ and $T \Meet R$ is middle-compatible with $S$.

Proof. We only prove (i), the other cases being analogous. Suppose that $T$ is left-compatible with $S$, which implies that $S \subseteq T^c$. From Proposition 17 (i), it follows that $S \subseteq (T \Join R)^c$. Hence, $T \Join R$ is left-compatible with $S$.

From Propositions 1 and 18, the following result follows immediately.

Corollary 2. Let $T$, $R$ and $S$ be a ternary, binary and binary $L$-relation on a set $X$, respectively. If $T$ and $R$ are compatible with $S$ and $S$ is reflexive, then $T \Join R$, $R \Join T$, $T \Meet R$ and $R \Meet T$ are compatible with $S$.

9. Compatibility of Particular Types of Ternary Relations with a Binary $L$-Relation

In this section, we study the compatibility of particular types of ternary relations with a binary $L$-relation. To that end, we recall the definition of transitivity of ternary relations. Note that there exist various definitions of transitivity (see, e.g., Refs. 10, 23), but here we restrict to those definitions from our work on traces of ternary relations. A ternary relation $T$ on a set $X$ is called:

(i) $c_1$-transitive, if, for any $x, y, z, s, t \in X$, it holds that $(x, y, t) \in T \land (s, t, z) \in T$ implies that $(x, y, z) \in T$;
(ii) $c_2$-transitive, if, for any $x, y, z, s, t \in X$, it holds that $(x, y, s) \in T \land (s, t, z) \in T$ implies that $(x, t, z) \in T$;
(iii) $c_3$-transitive, if, for any $x, y, z, s, t \in X$, it holds that $(x, y, s) \in T \land (s, t, z) \in T$ implies that $(x, y, z) \in T$;
(iv) $c_4$-transitive, if, for any $x, y, z, s, t \in X$, it holds that $(x, y, s) \in T \land (s, t, z) \in T$ implies that $(x, y, i) \in T$;
(e) $c_5$-transitive, if, for any $x, y, z, s, t \in X$, it holds that $(x, y, s) \in T \land (s, t, z) \in T$ implies that $(y, t, z) \in T$;
(vi) $c_6$-transitive, if, for any $x, y, z, s, t \in X$, it holds that $(x, s, t) \in T \land (s, t, z) \in T$ implies that $(x, y, z) \in T$;
(vii) $\tau$-transitive, if it is $c_i$-transitive, for all $i \in \{1, ..., 6\}$.

The following result facilitates the computation of the traces of particular types of ternary relations.

**Definition 9.** With a ternary relation $T$ on a set $X$ we associate the following binary relations on $X$:

(i) $W_r(T) = \{(x, y) \in X^2 \mid (x, y, y) \in T\}$;
(ii) $W_m(T) = \{(x, y) \in X^2 \mid (x, y, x) \in T\}$;
(iii) $W_l(T) = \{(x, y) \in X^2 \mid (x, x, y) \in T\}$.

**Theorem 6.** Let $T$ be a ternary relation on a set $X$. The following implications hold:

(i) If $T$ is reflexive and $c_2$- or $c_6$-transitive, then $T^r = (W_r(T))^i$;
(ii) If $T$ is reflexive and $c_1$- and $c_6$-transitive, then $T^m = W_m(T)$;
(iii) If $T$ is reflexive and $c_1$- or $c_3$-transitive, then $T^r = W_r(T)$.

Combining Theorems 2 and 6 easily leads to the following result.

**Proposition 19.** Let $T$ be a ternary relation and $S$ be a binary L-relation on a set $X$. The following implications hold:

(i) If $T$ is reflexive and $c_2$- or $c_6$-transitive, then $T$ is left-compatible with $S$ if and only if $S \subseteq (W_r(T))^i$;
(ii) If $T$ is reflexive and $c_1$- and $c_6$-transitive, then $T$ is middle-compatible with $S$ if and only if $S \subseteq W_m(T)$;
(iii) If $T$ is reflexive and $c_1$- or $c_3$-transitive, then $T$ is right-compatible with $S$ if and only if $S \subseteq W_l(T)$.

Next, we study the case of reflexive and antisymmetric ternary relations. The following proposition generalizes a remarkable result of De Baets et al.

**Proposition 20.** Let $T$ be a reflexive and antisymmetric ternary relation and $S$ be a reflexive binary L-relation on a set $X$. It holds that $T$ is compatible with $S$ if and only if $S$ is the crisp equality on $X$.

**Proof.** Suppose that $T$ is compatible with $S$. Let $x, z \in X$, then it holds that

$\tau(T(z, z, z)) \ast S(z, x) \ast S(z, z) \ast S(z, z) \leq \tau(T(x, z, z))$. 

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Since $T$ and $S$ are reflexive, it then follows that $S(z, x) \leq \tau(T(x, z, z))$. Since
\[
\min(\tau(T(x, z, z)), \tau(T(z, z, x))) = 0 \quad \text{for} \quad x \neq z,
\]
it follows that $S(z, x) = 0$. Since $S$ is reflexive, it follows that $S$ is the crisp equality on $X$. Conversely, if $S$ is the crisp equality on $X$, then it is obvious that $T$ is compatible with $S$.

We conclude with some interesting observations concerning symmetry and cyclicity.

**Proposition 21.** Let $T$ be a ternary relation on a set $X$. The following implications hold:

(i) If $T$ is symmetric, then $T^c = T^r$;

(ii) If $T$ is cyclic, then $T^c = T^m = T^r$.

Combining Proposition 21 and Theorem 2 easily leads to the following result. It expresses that for particular types of ternary relations, two or all notions of compatibility coincide.

**Proposition 22.** Let $T$ be a ternary relation $S$ be a binary $L$-relation on a set $X$. The following implications hold:

(i) If $T$ is symmetric, then $T$ is left-compatible with $S$ if and only if $T$ is right-compatible with $S$;

(ii) If $T$ is cyclic, then the following statements are equivalent:

(a) $T$ is left-compatible with $S$;

(b) $T$ is middle-compatible with $S$;

(c) $T$ is right-compatible with $S$.

**10. Conclusions and Future Lines of Research**

In this paper, we have generalized the notion of compatibility of a binary relation with a binary $L$-relation to the notion of compatibility of a ternary relation with a binary $L$-relation. More precisely, we have introduced the notions of left-, middle- and right-compatibility and have shown that they can be expressed in terms of inclusions of the binary $L$-relation in the left, middle and right traces of the ternary relation. Moreover, the relationship between the compatibility of a ternary relation and that of its traces and its projections, as well as the relationship between the compatibility of a binary relation and that of its cylindrical extensions have been investigated. Also, we have studied the interaction of the compatibility notions with basic set operations and relational compositions. Finally, we have paid attention to several classes of ternary relations.

Future work will be directed towards the characterization of the fuzzy tolerance and fuzzy equivalence relations a given ternary relation is compatible with. This requires some preparatory work, in particular the generalization of the notion of clone relation from binary relations to ternary relations.
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References

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