HÖLDER ESTIMATES FOR PSEUDO-DIFFERENTIAL OPERATORS ON $\mathbb{T}^1$

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ABSTRACT. In this work we provide Hölder estimates for one dimensional pseudo-differential operators defined on the Torus. A priori estimates for toroidal pseudo-differential problems also are considered.

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1. INTRODUCTION

Given a pseudo-differential operator on the torus with symbol in the toroidal Hörmander class $S_{\rho,\delta}^m(\mathbb{T} \times \mathbb{Z})$, in this paper we give sufficient conditions on the degree of the symbol in order to ensure that the corresponding pseudo-differential operator is a bounded operator from $\Lambda_0^s(\mathbb{T})$ into $\Lambda^s(\mathbb{T})$. The toroidal Hölder spaces are the Banach spaces defined for each $0 < s \leq 1$ by

$$\Lambda^s(\mathbb{T}) = \{f : \mathbb{T} \rightarrow \mathbb{C} : |f|_{\Lambda^s} = \sup_{x,h \in \mathbb{T}} |f(x+h) - f(x)| |h|^{-s} < \infty\}$$

and

$$\Lambda_0^s(\mathbb{T}) = \{f \in \Lambda^s(\mathbb{T}) : f(0) = 0\},$$

together with the norm $\|f\|_{\Lambda^s} = |f|_{\Lambda^s} + \sup_{x \in \mathbb{T}} |f(x)|$. It is easy to see that for every $s \in \mathbb{R}^+$, $\Lambda_0^s(\mathbb{T})$ is an algebraic maximal ideal of $\Lambda^s(\mathbb{T})$. We write, $|f|_{\Lambda_0^s} = |f|_{\Lambda^s}$ and $\|f\|_{\Lambda_0^s} = \|f\|_{\Lambda^s}$ if $f \in \Lambda_0^s(\mathbb{T})$. In general, on $\Lambda^s(\mathbb{T})$, $|\cdot|_{\Lambda^s}$ is a seminorm, but $\Lambda_0^s(\mathbb{T})$ together with $|\cdot|_{\Lambda_0^s}$ is a Banach space. Functions on the torus $\mathbb{T}$ may be thought as those functions on $\mathbb{R}$ that are 1-periodic. Here, we consider periodic pseudo-differential operators in the framework of the pseudo-differential calculus developed in [8]. In the theory of pseudodifferential operators, one of the most interesting topics is to investigate the behavior of pseudo-differential operators in different functions spaces. $L^p(\mathbb{T}^n)$ and Sobolev estimates have been obtained by M. Ruzhansky and V. Turunen in [8], M. Wong in [10] and J. Delgado in [4]. Bounds on L^p -spaces for pseudo-differential operator with symbols in the Hörmander class $S_{\rho,\delta}^m(\mathbb{T}^n \times \mathbb{Z}^n)$ for $m < 0$ and $1 \leq \delta < \rho \leq 1$ are provided by J. Delgado in [4]. Estimates of toroidal pseudo-differential operators on toroidal Hölder spaces have not been considered in the literature. The purpose of this paper is to establish Hölder continuity for one-dimensional pseudo-differential operators using standard techniques in pseudo-differential operators theory, and in this context, provide some

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a priori estimate of elliptic toroidal pseudo-differential problems. We note that the results presented here may not be the best possible ones. We will prove that in the periodic case, pseudo-differential operators are bounded on Hölder spaces, more specifically we establish the following theorems:

Theorem I. Let $0 < \varepsilon < 1$ and p be a symbol such that $|\Delta_\xi^\alpha \xi p(x, \xi)| \leq C_\alpha \langle \xi \rangle^{-\frac{\varepsilon}{2} - (1-\varepsilon)|\alpha|}$, $|\partial_x^\beta \xi p(x, \xi)| \leq C_\beta \langle \xi \rangle^{-\frac{\varepsilon}{2}}$, for $0 \leq |\alpha|, |\beta| \leq 1$. If $\frac{1}{2} \leq s < 1$ then $p(x, D) : \Lambda_0^s(\mathbb{T}) \rightarrow \Lambda^s(\mathbb{T})$ is a bounded linear operator.

Theorem II. Let $0 \leq \delta < \rho \leq 1$ and $m > 1$. If $p \in S_{\rho, \delta}^{-m}(\mathbb{T} \times \mathbb{Z})$ then $p(x, D) : \Lambda_0^{\frac{1}{2}}(\mathbb{T}) \rightarrow \Lambda^{\frac{1}{2}}(\mathbb{T})$ is a bounded operator.

Theorem III. Let $0 < s < 1$, $0 \leq \delta < \rho \leq 1$ and $m > 1 + \delta$. If $p \in S_{\rho, \delta}^{-m}(\mathbb{T} \times \mathbb{Z})$, then $p(x, D) : \Lambda_0^s(\mathbb{T}) \rightarrow \Lambda^s(\mathbb{T})$ is a bounded linear operator.

In this paper the boundedness theorems are proved using the compactness of the torus. So, the techniques used here can be not extended to the real line. Boundedness results for classical pseudo-differential operators ($S(m, g)$ -classes) can be found in Beals [2].

2. PRELIMINARIES

In this section we recall some elements of the basic theory of pseudo-differential operators on the torus $\mathbb{T}^n = [0, 1)^n$. We refer the reader to [8] and [10] for a more comprehensive account on this theory.

A version of Hörmander classes $S_{\rho, \delta}^m$ can be defined on the torus, this quantization is recently studied by W. McLean, G. Vainikko, M. Ruzhansky, and V. Turunen (see [8, 9]). For $0 \leq \rho, \delta \leq 1$, the Hörmander toroidal symbols $S_{\rho, \delta}^m(\mathbb{T}^n \times \mathbb{Z}^n)$ is defined by the set of functions $\sigma(x, \xi)$ such that

$$(2.1) \quad |\sigma_\beta^\alpha(x, \xi)| \leq C_{\alpha, \beta} \langle \xi \rangle^{m - \rho|\alpha| + \delta|\beta|}$$

for all $x \in \mathbb{T}^n, \xi \in \mathbb{Z}^n$ where $\sigma_\beta^\alpha = \Delta_\xi^\alpha \partial_x^\beta \sigma$ and $\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$.

For all $\sigma \in S_{\rho, \delta}^m(\mathbb{T}^n \times \mathbb{Z}^n)$, the pseudo-differential operator associated to σ is given by

$$(2.2) \quad \sigma(x, D)f(x) = \sum_{\xi \in \mathbb{Z}^n} e^{i2\pi x \xi} \sigma(x, \xi) \hat{f}(\xi).$$

In a recent paper, J. Delgado establish L^p -boundedness for toroidal pseudo-differential operators with symbols in $S_{\rho, \delta}^m(\mathbb{T}^n \times \mathbb{Z}^n)$, $m < 0$ (see [4]). A proof on L^2 -boundedness of operators with symbols in $S_{\rho, \delta}^0$ is given by D. Cardona in [3].

Theorem 1. (*Bernstein*). If $f \in \Lambda^s(\mathbb{T})$, $s > \frac{1}{2}$ then $|\hat{f}|_{L^1(\mathbb{Z})} \leq C_s \|f\|_{\Lambda^s}$.

The proof of the Bernstein theorem can be found in [5]. The Bernstein Theorem is sharp; there exist functions in $\Lambda^{\frac{1}{2}}(\mathbb{T})$ whose Fourier series does not converges absolutely. A classical example is the Hardy-Littlewood series $\sum_{n=1}^{\infty} \frac{\exp in \log n}{n} \exp int$, (see [11], Vol. 1. p. 197). The following results provide some properties about composition and invertibility of toroidal pseudo-differential operators. Proofs of these assertions can be found in [8, 9].

Theorem 2. (Composition formula). *Let $0 \leq \delta < \rho \leq 1$. The composition $\tau(x, D) \circ \sigma(x, D)$ of two pseudo-differential operators with symbols $\tau \in S_{\rho, \delta}^l(\mathbb{T}^n \times \mathbb{Z}^n)$ and $\sigma \in S_{\rho, \delta}^m(\mathbb{T}^n \times \mathbb{Z}^n)$ is a pseudo-differential operator, and its toroidal symbol $\psi(x, \xi)$ has the following asymptotic expansion,*

$$(2.3) \quad \psi(x, \xi) \approx \sum_{\gamma \geq 0} \frac{1}{\gamma!} \Delta_\xi^\gamma \tau(x, \xi) \cdot D_x^{(\gamma)} \sigma(x, \xi).$$

A pseudo-differential operator $\sigma(x, \xi) \in S_{\rho, \delta}^m$ is called elliptic, if for every $M > 0$, there exists $R > 0$ such that $|\sigma(x, \xi)| \geq R \langle \xi \rangle^m$ if $|\xi| \geq M$.

Theorem 3. (Parametrix existence). *Let $0 \leq \delta < \rho \leq 1$. For every elliptic pseudo-differential operators with symbol $\sigma \in S_{\rho, \delta}^m(\mathbb{T}^n \times \mathbb{Z}^n)$ there exists $\tau \in S_{\rho, \delta}^{-m}(\mathbb{T}^n \times \mathbb{Z}^n)$ such that*

$$(2.4) \quad \sigma(x, D) \circ \tau(x, D) = I + R, \quad \tau(x, D) \circ \sigma(x, D) = I + S,$$

where, S, R are pseudo-differential operators with symbols in $S^{-\infty} = \cap_m S_{\rho, \delta}^m$.

3. MAIN RESULTS

In this section we establish our main results, we study sufficient conditions for Hölder continuity. We write for $h \in \mathbb{T}$,

$$(3.1) \quad \sigma(x, D)f(x+h) = \sum_{\xi \in \mathbb{Z}} \sigma(x+h, \xi) e^{i2\pi x \xi} \int_0^1 e^{-i2\pi y \xi} f(y+h) dy,$$

this assumption may be proved with suitable variable change. We first establish a mild proposition on boundedness of Fourier multipliers.

Proposition 1. *Let $p(x, D) = p(D)$ be a pseudo-differential operator with symbol $p(\xi)$ depending only on the discrete Fourier variable ξ . If $p(\xi) \in L^1(\mathbb{Z})$, then $|p(D)f|_{\Lambda^s} \leq |p|_{L^1(\mathbb{Z})} |f|_{\Lambda^s}$ for $0 < s \leq 1$.*

Proof. Clearly,

$$p(x, D)f(x+h) - p(x, D)f(x) = \sum_{\xi \in \mathbb{Z}} e^{i2\pi x \xi} \left(p(\xi) \int_0^1 e^{-i2\pi y \xi} f(y+h) - f(y) dy \right).$$

So, we get $\frac{|p(x, D)f(x+h) - p(x, D)f(x)|}{|h|^s} \leq \sum_{\xi \in \mathbb{Z}} \left(|p(\xi)| \int_0^1 \frac{|f(y+h) - f(y)|}{|h|^s} dy \right)$. Finally, $|p(D)f|_{\Lambda^s} \leq \left(\sum_{\xi \in \mathbb{Z}} |p(\xi)| \right) |f|_{\Lambda^s}$. $\square$

The following theorem is a result on Hölder boundedness.

Theorem 4. ($|\cdot|_{\Lambda^s}$ -Boundedness). *Let $\frac{1}{2} < s < 1$ and $p \in S_{\rho, \delta}^{-m}(\mathbb{T} \times \mathbb{Z})$, $m \geq 1$. There exists $M > 0$ such that $|p(x, D)f|_{\Lambda^s} \leq M \|f\|_{\Lambda^s}$.*

Proof. By the mean value Theorem, there exists $x_h \in \mathbb{T}$ such that

$$e^{i2\pi(x+h)\xi} p(x+h, \xi) - e^{i2\pi x \xi} p(x, \xi) = h e^{i2\pi x_h \xi} (\partial_x p(x_h, \xi) + i2\pi x \xi p(x, \xi)).$$

By the Bernstein's Theorem (Theorem 1), we have:

$$\begin{aligned}
\frac{|p(x, D)f(x+h) - p(x, D)f(x)|}{|h|^s} &\leq \sum_{\xi \in \mathbb{Z}} |h|^{1-s} |\partial_x p(x_h, \xi) + i2\pi x \xi p(x, \xi)| |\widehat{f}(\xi)| \\
&\leq \sum_{\xi \in \mathbb{Z}} (C\langle \xi \rangle)^{-m+\delta} + C|\xi| \langle \xi \rangle^{-m+1} |\widehat{f}(\xi)| \\
&\leq \sum_{\xi \in \mathbb{Z}} 2C |\widehat{f}(\xi)| \\
&\leq 2C_s C \|f\|_{\Lambda^s}.
\end{aligned}$$

So, $|p(x, D)f|_{\Lambda^s} \leq 2C_s C \|f\|_{\Lambda^s}$.

□

The next theorem gives a single sufficient condition on the symbol p for the corresponding pseudo-differential operator $p(x, D) : \Lambda_0^s(\mathbb{T}) \rightarrow \Lambda^s(\mathbb{T})$ to be bounded for $0 < s < 1$.

Theorem 5. *Let $0 < s < 1$, $0 \leq \delta < \rho \leq 1$ and $m > 1 + \delta$. If $p \in S_{\rho, \delta}^{-m}$, then $p(x, D) : \Lambda_0^s(\mathbb{T}) \rightarrow \Lambda^s(\mathbb{T})$ is a bounded operator.*

Proof. Let $f \in \Lambda_0^s(\mathbb{T})$, we have

$$\begin{aligned}
&p(x, D)f(x+h) - p(x, D)f(x) \\
&= \sum_{\xi \in \mathbb{Z}} e^{i2\pi x \xi} \left(p(x+h, \xi) \int_0^1 e^{-i2\pi y \xi} f(y+h) dy - p(x, \xi) \int_0^1 e^{-i2\pi y \xi} f(y) dy \right) \\
&= \sum_{\xi \in \mathbb{Z}} e^{i2\pi x \xi} \left(p(x+h, \xi) \int_0^1 e^{-i2\pi y \xi} (f(y+h) - f(y)) dy \right) + \\
&\quad \sum_{\xi \in \mathbb{Z}} e^{i2\pi x \xi} \left((p(x+h, \xi) - p(x, \xi)) \int_0^1 e^{-i2\pi y \xi} f(y) dy \right).
\end{aligned}$$

Therefore, using the value mean Theorem, we get

$$\begin{aligned}
&\frac{|p(x, D)u(x+h) - p(x, D)u(x)|}{|h|^s} \\
&\leq \sum_{\xi \in \mathbb{Z}} \left(|p(x+h, \xi)| \int_0^1 \frac{|f(y+h) - f(y)|}{|h|^s} dy + \frac{|p(x+h, \xi) - p(x, \xi)|}{|h|^s} \int_0^1 |f(y)| dy \right) \\
&\leq \sum_{\xi \in \mathbb{Z}} \left(|p(x+h, \xi)| \int_0^1 \frac{|f(y+h) - f(y)|}{|h|^s} dy + \frac{|h| |\partial_x p(x_h, \xi)|}{|h|^s} \int_0^1 |f(y)| dy \right) \\
&\leq \sum_{\xi \in \mathbb{Z}} \left(C\langle \xi \rangle^{-m} |f|_{\Lambda_0^s} + |h|^{1-s} C\langle \xi \rangle^{-m+\delta} \int_0^1 |f(y)| dy \right) \\
&= \sum_{\xi \in \mathbb{Z}} \left(C\langle \xi \rangle^{-m} |f|_{\Lambda_0^s} + |h|^{1-s} C\langle \xi \rangle^{-m+\delta} \int_0^1 \frac{|f(y) - f(0)|}{|y|^s} |y|^s dy \right) \\
&\leq |f|_{\Lambda_0^s} \left(\sum_{\xi \in \mathbb{Z}} C\langle \xi \rangle^{-m+\delta} \right) \left(1 + \frac{1}{s+1} \right).
\end{aligned}$$

Then we have,

$$|p(x, D)f|_{\Lambda^s} \leq |f|_{\Lambda_0^s} \left(\sum_{\xi \in \mathbb{Z}} C\langle \xi \rangle^{-m+\delta} \right) \left(1 + \frac{1}{s+1} \right).$$

Notice that $m > 1$ and we can write:

$$\sup_{x \in \mathbb{T}} |p(x, D)f(x)| \leq \sup_{x \in \mathbb{T}} |p(x, \cdot)|_{L^1(\mathbb{Z})} \sup_{x \in \mathbb{T}} |f(x)|.$$

Thus, for some $M > 0$, $\|p(x, D)f\|_{\Lambda^s} \leq M\|f\|_{\Lambda_0^s}$. $\square$

Remark 1. It follows from the proof of the Theorem 5 that

$$|p(x, D)f|_{\Lambda^s} \leq |f|_{\Lambda_0^s} \sum_{\xi \in \mathbb{Z}} \left(C\langle \xi \rangle^{-m} + |p(\cdot, \xi)|_{\Lambda_0^s} \frac{1}{s+1} \right).$$

So, If $|p(\cdot, \xi)|_{\Lambda_0^s} \in L^1(\mathbb{Z})$, then $|p(x, D)f|_{\Lambda^s} \leq M|f|_{\Lambda_0^s}$, with $M = C \sum (\langle \xi \rangle^{-m} + |p(\cdot, \xi)|_{\Lambda_0^s} \frac{1}{s+1}) < \infty$. In conclusion, the operator $p(x, D)$ will be bounded from $\Lambda_0^s(\mathbb{T})$ into $\Lambda^s(\mathbb{T})$. So, we obtain the next result:

Proposition 2. Let $s \neq 1$, $m > 1$ and $|p(\cdot, \xi)|_{\Lambda_0^s} \in L^1(\mathbb{Z})$. If $p \in S_{\rho, \delta}^{-m}$, then $p(x, D) : \Lambda_0^s(\mathbb{T}) \rightarrow \Lambda^s(\mathbb{T})$ is a bounded operator.

The following result by J. Delgado (see [4]) will be applied to analyze local Hölder estimates.

Theorem 6. Let $0 < \varepsilon < 1$ and $k \in \mathbb{N}$ with $k > \frac{n}{2}$, let a be a symbol such that $|\Delta_\xi^\alpha a(x, \xi)| \leq C_\alpha \langle \xi \rangle^{-\frac{n}{2}\varepsilon - (1-\varepsilon)|\alpha|}$, $|\partial_x^\beta a(x, \xi)| \leq C_\beta \langle \xi \rangle^{-\frac{n}{2}}$, for $|\alpha|, |\beta| \leq k$. Then, $a(x, D)$ is a bounded operator from $L^p(\mathbb{T}^n)$ into $L^p(\mathbb{T}^n)$ for $2 \leq p < \infty$.

In the following theorems, we obtain Hölder boundedness using the Morrey inequality (see [1]): if $1 < p < \infty$ and $s_p = 1 - \frac{1}{p}$, then for $x, y \in \mathbb{R}$ we have

$$\frac{|u(x+h) - u(x)|}{|h|^{s_p}} \leq |\partial_x u|_{L^p(\mathbb{R})}.$$

Functions on the torus may be thought as those functions on $\mathbb{R}$ that are 1-periodic, under these assumptions, we can use a toroidal version of Morrey inequality on $L^p(\mathbb{T})$.

Theorem 7. Let $0 \leq \delta < \rho \leq 1$ and $m > 1$. If $p \in S_{\rho, \delta}^{-m}$ then $p(x, D) : \Lambda_0^{\frac{1}{2}}(\mathbb{T}) \rightarrow \Lambda^{\frac{1}{2}}(\mathbb{T})$ is a bounded operator.

Proof. The composition of the pseudo-differential operators ∂_x and $p(x, D)$ is the pseudo-differential operator $\partial_x p(x, D)$ of degree $-m+1 < 0$, so, $T = \partial_x p(x, D)$ is a bounded operator from $L^2(\mathbb{T})$ into $L^2(\mathbb{T})$, (see [3]). If $u \in \Lambda_0^{\frac{1}{2}}(\mathbb{T})$, then

$$\begin{aligned} \frac{|p(x, D)u(x+h) - p(x, D)u(x)|}{|h|^{s_p}} &\leq C|\partial_x p(x, D)u|_{L^2} \\ &\leq C\|\partial_x p(x, D)\|_{(L^2, L^2)}|u|_{L^2} \\ &= C\|\partial_x p(x, D)\|_{(L^2, L^2)} \left(\int_0^1 \frac{|u(x) - u(0)|^2}{|x|^{2(\frac{1}{2})}} |x|^{2(\frac{1}{2})} \right)^{\frac{1}{2}} \\ &\leq C\|\partial_x p(x, D)\|_{(L^2, L^2)} \left(\frac{1}{2} \right)^{\frac{1}{2}} |u|_{\Lambda_0^{\frac{1}{2}}}. \end{aligned}$$

Hence,

$$|p(x, D)u|_{\Lambda_0^{\frac{1}{2}}} \leq C \|\partial_x p(x, D)\|_{(L^2, L^2)} \left(\frac{1}{2}\right)^{\frac{1}{2}} |u|_{\Lambda_0^{\frac{1}{2}}}.$$

Finally

$$\|p(x, D)f\|_{\Lambda_0^{\frac{1}{2}}(\mathbb{T})} \leq \max\left\{\sup_{x \in \mathbb{T}} |\sigma(x, \cdot)|_{L^1(\mathbb{Z})}, C \|\partial_x p(x, D)\|_{(L^2, L^2)} \left(\frac{1}{2}\right)^{\frac{1}{2}}\right\} \|f\|_{\Lambda_0^{\frac{1}{2}}(\mathbb{T})}.$$

□

As a consequence of the Theorem 6, we obtain:

Theorem 8. *Let $0 < \varepsilon < 1$ and p be a symbol such that $|\Delta_\xi^\alpha \xi p(x, \xi)| \leq C_\alpha \langle \xi \rangle^{-\frac{\varepsilon}{2} - (1-\varepsilon)|\alpha|}$, $|\partial_x^\beta \xi p(x, \xi)| \leq C_\beta \langle \xi \rangle^{-\frac{\varepsilon}{2}}$, for $0 \leq |\alpha|, |\beta| \leq 1$. If $\frac{1}{2} \leq s < 1$, then $p(x, D) : \Lambda_0^s(\mathbb{T}) \rightarrow \Lambda^s(\mathbb{T})$ is a bounded linear operator.*

Proof. If $\frac{1}{2} \leq s < 1$, there exists $2 \leq p < \infty$ such that $s = 1 - \frac{1}{p}$. Applying Theorem 6 to the symbol $i2\pi \xi p(x, \xi)$ we obtain $L^p(\mathbb{T}^n)$ -boundedness for the operator $\partial_x p(x, D)$. If $u \in \Lambda_0^s(\mathbb{T})$, then

$$\begin{aligned} \frac{|p(x, D)u(x+h) - p(x, D)u(x)|}{|h|^s} &\leq C |\partial_x p(x, D)u|_{L^p} \\ &\leq C \|\partial_x p(x, D)\|_{(L^p, L^p)} |u|_{L^p} \\ &= C \|\partial_x p(x, D)\|_{(L^p, L^p)} \left(\int_0^1 \frac{|u(x) - u(0)|^p}{|x|^{ps}} |x|^{ps} \right)^{\frac{1}{p}} \\ &\leq C \|\partial_x p(x, D)\|_{(L^p, L^p)} \left(\frac{1}{p}\right)^{\frac{1}{p}} |u|_{\Lambda_0^s}. \end{aligned}$$

Hence,

$$|p(x, D)u|_{\Lambda^s} \leq C \|\partial_x p(x, D)\|_{(L^p, L^p)} \left(\frac{1}{p}\right)^{\frac{1}{p}} |u|_{\Lambda_0^s}.$$

Now, since $|\xi p(x, \xi)| \leq C \langle \xi \rangle^{-\frac{\varepsilon}{2}}$, we get

$$|p(x, \xi)| \leq C \langle \xi \rangle^{-\frac{\varepsilon}{2}} |\xi|^{-1}, \quad \xi \neq 0.$$

Hence, we obtain

$$M = \sup_{x \in \mathbb{T}} |p(x, \cdot)|_{L^1(\mathbb{Z})} < \infty.$$

Therefore

$$\|p(x, D)f\|_{\Lambda^s(\mathbb{T})} \leq \max\{M, C \|\partial_x p(x, D)\|_{(L^p, L^p)} \left(\frac{1}{p}\right)^{\frac{1}{p}}\} \|f\|_{\Lambda_0^s(\mathbb{T})}.$$

□

4. *A priori* ESTIMATES FOR PSEUDO-DIFFERENTIAL PROBLEMS

Let $0 < s < 1$ and $f \in \Lambda_0^s(\mathbb{T})$. Consider the toroidal pseudo-differential problem

$$(4.1) \quad \begin{cases} p(x, D)u = f, \\ u(0) = f(0) = 0, \end{cases}$$

where $p(x, D)$ is an elliptic operator with symbol $p \in S_{\rho, \delta}^m$, $m > 1$, (In particular, $p(x, D)$ can be an elliptic differential operator of the form $\sum_{0 \leq i \leq m} a_i(x) \partial_x^i$, $m \geq 2$). By the Theorem 3, there exists $q \in S_{\rho, \delta}^{-m}$ and $r \in S^{-\infty}$ such that

$$(4.2) \quad q(x, D) \circ p(x, D) = I + r(x, D).$$

Therefore, $q(x, D) \circ p(x, D)u = u + r(x, D)u = q(x, D)f$. Using the Theorem 5, we have $|q(x, D)f|_{\Lambda_0^s} \leq C|f|_{\Lambda_0^s}$, and considering that the operator $r(x, D)$ is a smoothing operator and can be written as an integral operator with C^∞ kernel ($r(x, D)f(x) = \int_0^1 k(x, y)f(y)dy$, for some kernel $k(x, y) \in C^\infty(\mathbb{T} \times \mathbb{T})$, see [8, 9]), we get $u \in \Lambda_0^s$. In conclusion, under the pseudo-differential problem considered, if $f \in \Lambda_0^s(\mathbb{T})$ then $u \in \Lambda_0^s(\mathbb{T})$.

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