We present additional numerical simulations and experiments to support our results discussed in the main text. We also provide a detailed description of the analysis technique to process the beam pictures acquired in the experiments.

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I. NUMERICAL SIMULATIONS FOR SINGLE BEAM EXCITATION

We simulated light propagation for different input powers and different cell length to confirm our theoretical explanation for the formation of Pancharatnam-Berry phase (PBP) solitons. We limit our simulations to the NLC mixture E7 which was employed in our experiments. For the material we then used the two refractive indices $n_\perp = 1.5$ and $n_\parallel = 1.7$, valid for NIR illumination at room temperature. For the single elastic constant we took the value $K = 12 \times 10^{-12}$ N. All of these parameters are kept fixed hereafter. On the edges of the numerical grid we set the hard boundary condition $\theta = 0$. Figure S1 reports the light evolution for growing powers. At $P = 4$ mW self-focusing with no self-collimation is observed, with the beam spreading with a lower divergence than in the linear case. At $P = 10$ mW and $P = 12$ mW the optical reorientation is strong enough to induce self-trapping. In all of these cases, the beam polarization and, consequently, the reorientation angle evolve periodically, with a period given by $\Lambda$ as theoretically predicted. The maximum rotation angle on a given section $z = \text{constant}$ slowly decays along the propagation distance. This can be attributed to the fact that we are launching a CP wave, whereas the self-trapped beam is actually a structured beam with polarization varying across the cross-section (i.e., versus the transverse coordinate $x$), see the polarization evolution plotted in Fig. 1(d) in the main text. Physically, for plane waves the eigensolutions in sinusoidally twisted anisotropic media correspond to circular polarizations only in the limit of vanishing rotation angles. In general, the eigenwaves possess all the three Stokes parameters, with the coefficients of the superposition depending on the oscillation amplitude of the rotation angle $\theta$. To confirm our interpretation, polarization on the beam centre follows a spiraling trajectory on the Poincaré sphere, with the polarization slowly converging towards the coordinate $S = (1, 0, 0)$ (see Fig. S2). The motion strongly depends on the input power, with larger powers evolving on tighter orbits than low powers.

For large powers, the code does not achieve convergence. During the iterations, the polarization periodicity along $z$ slowly changes. When the director is almost relaxed all along the sample (only the zone nearby the output facet is not relaxed), a sudden change occurs: the beam undergoes strong spreading for long $z$ despite an initial confinement. For the numerical algorithm we are using, the evolution with the iterations does not correspond to temporal evolution of the beam [1]. Nonetheless, the absence of convergence suggests that the stationary regime is not static, i.e., the coupled light-matter system enters into a stationary regime consisting of periodic or chaotic oscillations [2].

The distribution of the Stokes parameters along the beam cross-section is shown in Fig. S3. We note that, whenever the reorientation is appreciable, strong variations with respect to the linear case $\Gamma \to 0$ (i.e., vanishing reorientation...
**Supplementary Figure S1.** Numerical simulations in a 1 mm long cell. Light behavior versus the input effective power $P$ (increasing from left to right, as labelled): top and bottom rows correspond to $xz$ distribution of the overall intensity and reorientation angle $\theta$, respectively. Wavelength is 1064 nm and the input waist is $w_{in} = 4 \, \mu m$.

**Supplementary Figure S2.** Evolution of the normalized Stokes parameters on the Poincaré sphere. The polarization evolution in the beam centre $x = 0$ along $z$ for different input powers. The propagation distance is color-coded, going from blue (initial position $z = 0$) to red (final section $z = 1$ mm). Simulations parameters are the same as in Fig. S1.

angle) are visible. The Stokes parameters are no more homogeneously distributed along the beam cross-section, but they are strongly modulated owing to the complicated interplay between nonlinearity and spin-orbit interaction. In other words, in our system the self-trapped waves are inherently structured beams [3, 4], paving the way to the generation of new kind of of waveguides capable of supporting spatial multiplexing with classically entangled beams [5], i.e., both spin and orbital angular momentum can be jointly modulated. The possible existence of shape-preserving (i.e., both intensity and polarization distributions exactly conserved during propagation) nonlinear beams in our structure will be investigated in future works.

To summarize the beam behavior versus the input power, in Fig. S4 we showed the behavior of the beam width versus $z$. Due to the small degree of nonlocality, the beam propagation in 100 $\mu$m long cell is almost identical to the
Supplementary Figure S3. Self-trapped waves as structured beams. Normalized Stokes parameters versus the transverse direction $x$ for three different input powers as indicated. The cut is taken in $z = 0.5$ mm, corresponding to the middle section of the cell. The simulations parameters are as in Fig. S1.

Supplementary Figure S4. Dependence on the cell length and input polarization. First and second column: Beam width $w$ versus the propagation distance $z$ and input power $P$ when the input is CP, for two different lengths of the cell (the length of the short cell is 100 µm and that of the long one is 1 mm). Third column: width evolution versus $z$ in a long cell for a linear input polarization with different direction as labelled (the angles are with respect to $\hat{x}$); the input power is fixed to $P = 8$ mW.

propagation in the initial 100 µm of a long sample. Noticeably, the maximum power $P_{th}$ corresponding to a stable solution is larger in shorter cells. The nonlinear propagation also strongly depends on the input polarization due to the nonlinear nature of the reorientational equation. For example, both for vertical and horizontal linear polarizations no nonlinear effects are observed (the input power is below the Fréedericksz threshold). As theoretically predictable, the maximum self-trapping is achieved for diagonal polarization, with the beam width following a trend very close to the CP case (see Fig. 2 in the main text). We also notice a slightly different behavior for 30° and 60° owing to the different effective self-phase modulation between the ordinary and the extraordinary component [6].

II. OBSERVATION OF SELF-TRAPPING FOR WIDER INPUT AND FOCUS INSIDE THE CELL

In the main text, single-beam confinement has been shown for an input waist of $w_{in} = 2$ µm (see Fig. 4 in the main text): in fact, due to the tighter focusing, the differences between linear spreading and nonlinear self-trapping are more evident for $w_{in} = 2$ µm than for 4 µm. Figure S5 shows how light propagation changes when the input
**Supplementary Figure S5.** Experimental acquisitions. The input beam waist is $4 \mu m$ and the focus is placed in $z = 200 \mu m$. Input polarization is circularly polarized and the applied bias is $9 V$.

**Supplementary Figure S6.** Overview of the light self-trapping. Beam width versus $z$ computed via best-fitting (left panel) with the function $F(x) + I_G(x)$ (see the appendix to the main text for their definition) accounting for the superposed diffusive radiation or (middle panel) with the Gaussian function $I_G(x)$. Right panel: cross-section at $z = 940 \mu m$ for $P = 2 mW$ (linear regime, dashed red line) and $P = 80 mW$ (nonlinear trapping, blue solid line).

power is increased. In this experiment, the beam waist has been shifted inside the sample, at about $z = 200 \mu m$. Light behavior for the focus placed in $z = 0$ are presented in Fig. S9 below. With respect to the latter, self-trapping is achieved for larger power ($\approx 70 mW$ compared with $\approx 40 mW$), in agreement with Ref. [7]. The position of the beam waist is confirmed by the behavior of the beam width versus $z$, see Fig. S6. For $P = 50 mW$, a wider beam than for $20 mW$ is observed owing to the strong focusing at the initial stage, yielding a narrower waist than in the linear regime. Noteworthy, the beam trapping is much more clear when the diffusive background is accounted for: in fact, due to the beam curvature at the cell input, the diffusive component is much stronger, yielding large errors when simple Gaussian functions are used to describe the beam propagation. The spatial extension of the diffusive tails can be appreciated in the last panel of Fig. S6.
Supplementary Figure S7. Experimental results for linearly polarized inputs at different angle. Left panel: beam width $w$ versus the propagation distance $z$ and the direction of the input polarization with respect to the $y$ axis. Right panel: beam width measured via best-fitting at the final section $z = 960$ µm versus the input angle; solid and dashed lines correspond to best-fits with respect to the function $F(x) + I_G(x)$ (see Methods) and to a Gaussian $I_G(x)$, respectively. The input beam waist is 4 µm and the focus is placed in $z = 200$ µm. The input power is 50 mW and the applied bias is 9 V.

III. EXPERIMENTS WITH LINEARLY POLARIZED INPUTS

In Section I, specifically in Fig. S4, we discussed the behavior of the system when excited with a linear polarization of variable direction. Here we show the corresponding experimental results. Figure S7 shows the evolution of the beam width versus the distance from the input interface and the angle of the linear polarization at the input (a null angle corresponds to a polarization parallel to $\hat{y}$). The input power is 50 mW. In good agreement with the simulations, the most confined beam is observed for an input polarization between 30° and 40° (deviation from the theoretical prediction 45° is ascribable to slight depolarizing effects coming from the input interface). When the input is parallel to the axis $y$, the beam diffracts linearly, given that no reorientational effects can occur. Noteworthy, the input corresponds to a purely extraordinary wave, which is subject to a defocusing thermal nonlinearity [8]. The absence of any appreciable defocusing effect proves that thermal effects in our case are negligible. As the input angle is increased, the beam undergoes strong confinement. After achieving a minimum, the beam continues to be self-focused (final width lower than in the linear case). For input parallel to the axis $x$, nonlinear effects still take place owing to the fact that the input power is above the Fréedericksz threshold.

IV. 3D REORIENTATION IN BIASED CELLS

Let us first describe the director rotation induced by the external voltage at 1 KHz. We introduce the angle $\varphi = \angle (\hat{n} \cdot \hat{z})$. In the absence of optical field, director rotates on the plane $yz$ and the problem is effectively monodimensional, i.e., the rotation angle depends only on $y$, $\varphi = \varphi(y)$. In the quasi-static limit, the low-frequency (LF) electric field is $E_{LF} = -\frac{dV(y)}{dy}\hat{y}$. The director distribution is then ruled by [9]

$$\left(\epsilon_{\parallel}^{LF} \sin^2 \varphi + \epsilon_{\perp}^{LF} \cos^2 \varphi\right) \frac{d^2V}{dy^2} + \Delta \epsilon_{LF} \sin(2\varphi) \frac{d\varphi}{dy} \frac{dV}{dy} = 0, \quad (S1)$$

$$\left(K_3 \sin^2 \varphi + K_1 \cos^2 \varphi\right) \frac{d^2\varphi}{dy^2} + \frac{K_3 - K_2}{2} \sin(2\varphi) \left(\frac{d\varphi}{dy}\right)^2 + \frac{\Delta \epsilon_{LF}}{2} \sin(2\varphi) \left(\frac{dV}{dy}\right)^2 = 0 \quad (S2)$$

Equation (S1) is the Poisson’s equation for the given geometry, where the potential $V$ is time-independent and corresponding to the RMS value of the sinusoidal field; Eq. (S2) states the equilibrium between electrical and elastic torque. Here $K_1$, $K_2$ and $K_3$ are the elastic constants for bend, splay and twist deformations, respectively. The dielectric permittivities $\epsilon_{\parallel}^{LF}$, $\epsilon_{\perp}^{LF}$ and $\Delta \epsilon_{LF} = \epsilon_{\parallel}^{LF} - \epsilon_{\perp}^{LF}$ are absolute and calculated at the field frequency. Numerical solution of Eqs. (S1-S2) corresponding to an applied bias of 9 V (peak value) is shown in Fig. S8. The next step is to consider the director rotation when both the quasi-static and the optical electric fields are applied simultaneously. For the sake of simplicity, here we solve the reorientational equations by supposing a uniform quasi-static field inside the sample and a $z$–independent distribution for the optical intensity. The problem is then
Supplementary Figure S8. Electric rotation of the director. Potential distribution (left panel) and the corresponding director angle \( \varphi \) (middle panel) versus the normalized position \( y/L_y \) when the applied voltage is 9 V (the corresponding RMS value is 6.37 V). Right panel: derivative of the rotation angle \( \varphi \) versus \( y/L_y \) for \( L_y = 75 \mu m \). The reorientational equation is solved by setting \( \varphi = 2^\circ \) at the boundaries.

bidimensional, that is, the solutions depend only on \( x \) and \( y \). Under this condition the director rotation is fully three dimensional, and we need two angles to describe the director distribution in space. Thus, we write the director as \( \hat{n} = (\sin \varphi \cos \theta, \sin \varphi \sin \theta, \cos \varphi) \). Under the single elastic constant approximation \( K_1 = K_2 = K_3 = K \), the direction distribution is found by solving [10]

\[
K \nabla^2 \varphi - \frac{K}{2} \sin(2\varphi) \left[ \left( \frac{\partial \varphi}{\partial x} \right)^2 + \left( \frac{\partial \varphi}{\partial y} \right)^2 \right] + \frac{\Delta \epsilon_{LF}}{2} \sin(2\varphi) E_{LF}^2 + \frac{\epsilon_0 \epsilon_a}{4} \sin(2\varphi) \left[ |E_x|^2 + |E_y|^2 - \sin(2\theta) \text{Re} \left( E_x E_y^* \right) \right] = 0,
\]

(S3)

\[
K \nabla^2 \theta + 2K \cos(\varphi) \left( \frac{\partial \varphi}{\partial x} \frac{\partial \varphi}{\partial y} + \frac{\partial \theta}{\partial y} \frac{\partial \varphi}{\partial y} \right) + \frac{\Delta \epsilon_{LF}}{2} \sin^2 \varphi \sin(2\theta) |E_{LF}|^2 + \frac{\epsilon_0 \epsilon_a}{4} \sin^2(\varphi) \left[ \sin(2\theta) \left( |E_x|^2 - |E_y|^2 \right) + 2 \cos(2\theta) \text{Re} \left( E_x E_y^* \right) \right] = 0.
\]

(S4)

In finding Eqs. (S3-S4) we neglected the walk-off angle, the latter being negligibly small when the director is almost parallel to the axis \( \hat{y} \), i.e., for large applied biases.

The director angles can be expanded in a power series with respect to the optical power \( P \), providing \( \theta = \theta^{(0)} + \theta^{(1)} P + \theta^{(2)} P^2 + \ldots \) and \( \varphi = \varphi^{(0)} + \varphi^{(1)} P + \varphi^{(2)} P^2 + \ldots \). In our case, it is \( \theta^{(0)} = 0 \) and \( \varphi^{(0)} = \varphi_{LF}(y) \), where \( \varphi_{LF}(y) \) is solution of Eq. (S2) under the single elastic constant approximation and for uniform \( E_{LF} \). In the perturbative limit we retain only the terms proportional to the input power \( P \). Equations (S3-S4) turn into

\[
K \nabla^2 \varphi_{opt} + \Delta \epsilon_{LF} \cos(2\varphi_{LF}) E_{LF}^2 \varphi_{opt} + \frac{\epsilon_0 \epsilon_a}{4} \sin(2\varphi_{LF}) \left( |E_x|^2 + |E_y|^2 \right) = 0,
\]

(S5)

\[
K \nabla^2 \theta_{opt} + 2K \cos(\varphi_{LF}) \frac{\partial \varphi_{opt}}{\partial y} \frac{\partial \theta_{opt}}{\partial y} + \frac{\Delta \epsilon_{LF}}{2} \sin^2 \varphi_{LF} |E_{LF}|^2 \sin(2\theta_{opt}) + \frac{\epsilon_0 \epsilon_a}{4} \sin^2(\varphi_{LF}) \left[ \sin(2\theta_{opt}) \left( |E_x|^2 - |E_y|^2 \right) + 2 \cos(2\theta_{opt}) \text{Re} \left( E_x E_y^* \right) \right] = 0,
\]

(S6)

where we have set \( \varphi^{(1)} = \varphi_{opt} \) and \( \theta^{(1)} = \theta_{opt} \). Let us start from Eq. (S5). This equation is a Yukawa (aka screened Poisson) equation, but with a screening length function of coordinate \( y \) through \( \varphi_{LF} \), and a forcing term given by the light intensity. An analogous equation for the director reorientation is found in the case of dynamic-phase-based solitons in biased cell [11]. In our case, the forcing term is multiplied by the additional coefficient \( \sin(2\varphi_{LF}) \), the latter vanishing when \( \varphi_{LF} \to \pi/2 \). Thus, the optical field does not significantly change the rotation angle of the director on the plane \( yz \), confirming that in our geometry the observed nonlinear effects must be ascribed to geometric phase variations related with rotations of the optical axis on the plane \( xy \).

Let us now show that Eq. (S6) corresponds for large voltage to the reorientation equation used in the main text. The second term in Eq. (S6) vanishes for the flatness of the electric orientation around the cell mid-plane (see Fig. S8). The term containing \( E_{LF} \) is due to the nonlinear interaction between the quasi-static and the optical electric field.
Supplementary Figure S9. Two-beam repulsion. Top three rows: Acquired light distribution inside the NLC visualized on the plane \(xz\) for growing powers, from \(a\) to \(d\) as labelled in the panels. The first and second rows are single beam propagation, whereas in the third row the two beams are launched simultaneously. Bottom row: cross-sections in \(z = 960\ \mu m\). The input beam waist is 4 \(\mu m\) and the applied bias is 9 V.

Physically, the low-frequency field and the director are no more aligned on the plane \(xy\) due to the rotation induced by the optical field on the plane \(xy\), accounted by the term \(\sin(2\theta_{opt})\). First thing to notice, the rotation is on the plane \(xy\), thus no dynamic phase changes are associated with it. Secondly, the induced torque is with good accuracy spatially homogeneous (see Fig. S8 for the spatial distribution of \(E_{LF}\)). The net effect is a uniform rotation of the NLC principal dielectric axes, with no effects on the wave confinement due to the vanishing spatial gradient. Third, this torque -proportional to \(\sin(2\theta_{opt})\) with \(\theta_{opt} \approx 2^\circ\) - is negligible with respect to the pure optical torque, the latter being proportional to \(\sin(\pi/2)\) when the polarization is diagonal. Summarizing, the term can be safely neglected. Noteworthy, the factor \(\sin(\varphi_{LF})\) multiplying the optical torque tends to unity. Finally, the appropriateness of our perturbative approach is confirmed \textit{a posteriori} by the magnitude of optical rotations (few degrees) predicted by the numerical simulations.

V. TWO-BEAM INTERACTION

The set-up for the two-beam interaction is shown in the main text.

A. Overlapping beam versus the input power

Figure S9 shows the acquired intensity distribution when the two input beams are launched with opposite helicity (i.e., one RCP and one LCP). Self-trapping is achieved for input powers around 40 mW, a smaller value than the necessary power when the focus is positioned inside the sample (see Sec. II). At small powers (Fig. S9a) beams undergo a weak self-focusing effect and no significant interaction takes place. When beams are self-trapped (Fig. S9b), a reciprocal repulsion is taking place. For higher powers, the repulsion is slightly attenuated (shifts of few microns with respect to Fig. S9b). Moreover, an exchange of energy between the two beams is observed for large input powers (Fig. S9d), together with a stronger time variability, i.e., the system tends to a time-periodic regime. Numerical simulations plotted in Fig. 6 in the main text confirm that the repulsive nature of the interaction when the input
Supplementary Figure S10. Two-beam interaction in the presence of dynamic phase. Top three rows: Acquired light distribution inside the NLC visualized on the plane $xz$. The first and second rows are single beam propagation, whereas in the third row the two beams are launched simultaneously. a-b The two beams are launched parallel with an initial distance of (a) $40 \, \mu m$ and (b) $20 \, \mu m$. c Crossed-beam interaction. d Overlapping beams. Bottom row: corresponding cross-sections in $z = 960 \, \mu m$. The input beam waist is $4 \, \mu m$ and the applied bias is $2 \, V$.

helicity is opposite in sign. Additional simulations show that, when the two beams are launched with identical circular polarization, a strong attraction and eventually merging of the two beams is observed.

B. Interaction in the presence of dynamic phase

In Fig. 7 in the main text we showed that PBP-solitons do not interact significantly when they are not overlapping due to the low degree of nonlocality, experimentally corresponding to an applied bias of $9 \, V$. We actually showed that the interaction is repulsive when their helicity is opposite, see also Fig. S9. In the main text (Fig. 8) we also verified the PBP nature of the observed solitons by observing light propagation within the same experimental conditions used in Fig. S9, but with an applied bias of $2 \, V$ [12]. Here we provide further details about the soliton interaction when the dynamic phase is dominant over the geometric phase. The relatively low voltage provides a non-vanishing angle (about $45^\circ$) between the director and $\hat{z}$ (i.e., the wavevector), thus allowing a non-negligible molecular reorientation on the $yz$ plane [11]. Such reorientation in turn provides an index distribution capable to confine the beam via the dynamic phase [12, 13]. Results are plotted in Fig. S10. Unlike standard experiments [12, 14, 15], here we launch circular polarizations, thus the observed solitons are actually all hybrids, whose features are dictated by the interplay between dynamic and geometric phase. For an applied bias of $2 \, V$, the spatial nonlocality (the latter being proportional to the lateral extension of the light-written index well) is much wider than for PBP solitons (see Fig. 1(f) in the main text) [16]. Accordingly, a long range interaction is observed, with the beam colliding at $z \approx 1 \, mm$ and $z \approx 500 \, \mu m$ when the initial distance is $40 \, \mu m$ (Fig. S10a) and $20 \, \mu m$ (Fig. S10b), respectively. In the same conditions except for an applied bias of $9 \, V$ (Fig. 7 in the main text), no significant interaction is observed, in agreement with the weak nonlocality associated with pure PBP solitons. For crossed beams, Fig. S10c shows that the inter-soliton distance at $z = 1 \, mm$ is increased by the interaction, whereas for the PBP solitons a slight decrease is observed (Fig. 7(d) in the main text). Finally, when the two beams are overlapping at the entrance of the cell, the two solitons merge, giving birth to a single soliton at higher power. In fact, the observed trajectory is not perfectly straight due to the temporal instability caused by the interaction between NLC intrinsic disorder and nonperturbative reorientation [1, 13]. This is in stark contrast with the PBP case showed in the main text, where a repulsion between the two beams is observed.
VI. FORMATION OF VECTORIAL SOLITONS IN TILTED CELLS

In this section we will discuss the behavior of light when the cell is tilted by an angle $\beta_{in}$ with respect to the input wavevector. At the interface, the Poynting vectors of the ordinary and extraordinary beam (let us call their angle with respect to the axis $z$ $\alpha_{ord}$ and $\alpha_{ext}$, respectively) are not parallel due to the birefringence of the material. Specifically, calling $\beta$ the angle between the wavevector and the axis $z$ (the latter corresponding to the normal to the input interface), for the extraordinary we get $\beta_{ext} = \arctan \left( \frac{\sqrt{n_{\perp}^2 \sin^2 \beta_{in} - \epsilon_{\perp} \epsilon_{\parallel}}}{-1} \right)$ [17]. Experimentally, the angle between the two Poynting vectors can be directly measured. We find

$$\alpha_{ext} - \alpha_{ord} = \beta_{ext} - \arctan \left( \frac{\sin \beta_{in}}{n_{\perp}} \right) - \arctan \left( \frac{\epsilon_a \sin (\beta_{in})}{\epsilon_a + 2\epsilon_{\perp} + \epsilon_a \cos (2\beta_{in})} \right).$$

From Eq. (S7) we can find $\beta_{in}$, given that the LHS is provided by experimental measurements. In our case we find $\beta_{in} = 4^\circ$.

Figure S11 shows the comparison between the linear and the nonlinear regime when the input beam is linearly polarized. At small powers, diffractive spreading is observed, the amount of widening depending on the polarization due to the different refractive index for ordinary and extraordinary waves. The beam trajectory also depends on the input polarization, as discussed above. When power is increased, no significant changes are observed for vertical polarization (i.e., parallel to axis $y$), given that in this condition the electric field and the director are already aligned. When the input polarization is horizontal (i.e., parallel to axis $x$), a progressive defocusing effects for increasing power is observed. Noteworthy, the observed defocusing cannot be attributed to thermo-optic effects, given that for the ordinary beam the thermal nonlinearity is focusing [8]. Direct comparison with numerical simulations suggest that the defocusing effect takes place at the input interface, where there is a transition on the director distribution due to the hard anchoring parallel to $\hat{z}$ imposed by the input interface. Due to the input tilt, the optical torque now induces a full 3D rotation on the director, in turn leading to a complicated interplay between transverse-dependent spatial

Supplementary Figure S11. Vector solitons in a tilted cell. Intensity acquired on the plane $xz$ in the linear regime (top row) and at $P = 50$ mW (middle row). The bottom row is the intensity cross-section at $z = 960 \mu m$. Applied bias is 9 V and the cell is tilted by $4^\circ$ with respect to the input beam.

(see also Fig. S9). We finally note that the exponential tails due to the diffused light is much more pronounced with respect to the PBP solitons owing to the wider photonic potential supporting the self-trapping [16].
Supplementary Figure S12. Comparison between diagonal and circular polarizations in a tilted cell. Intensity acquired on the plane $xz$ for diagonal polarized input (top row) and for circularly polarized input (middle row). The bottom row is the intensity cross-section at $z = 960 \mu m$. Applied bias is 9 V and the cell is tilted by 4$^\circ$ with respect to the input beam.

walk-off and gradients in both dynamic (index gradient) and geometric (polarization variation) phase. In fact, in the nonlinear case several humps are observed in the beam cross-sections, probably due to a strong transverse modulation of the walk-off, in turn deforming the beam shape. Experiments clearly show that this complicated interaction effectively behaves like a defocusing nonlinearity. The details of the phenomenon will be further investigated in future publications.

In Fig. S12 light behavior for diagonal (D) and circularly polarized (CP) inputs is compared. The self-trapping due to the spatial locking between the ordinary and the extraordinary component occurs at lower power when the input is diagonally polarized. As a matter of fact, CP inputs do not induce any reorientation just at the entrance of the cell, differently from diagonal inputs, where the maximum torque is achieved in the section $z = 0$. Accordingly, for $P = 70 \text{ mW}$ a vector soliton with a very similar cross-section is observed for both the polarizations. Nonetheless, in the two cases the beam is transversely shifted due to the nonlinear interaction occurring at the interface.

VII. BEAM BEHAVIOR WITH TIME AND THE EMERGENCE OF THE UNSTABLE REGIME IN THE EXPERIMENTS

Until now, the shown experimental measurements consisted in the average of 100 beam pictures, acquired consecutively in time. As shown in Refs. [1, 18] and discussed in the methods with respect to the light diffusion, the NLC optical response is time dependent owing to the thermal fluctuations in the molecular field [8]. The top row in Figure S13 show the single realizations when the beam undergoes self-trapping. Minimum variations in the beam profile are observed, with a standard deviation on the beam width equal to 0.15 $\mu m$ in $z = 200 \mu m$, and to $2 \mu m$ in $z = 900 \mu m$ (in the intermediate points the growth is very close to a linear trend). This observation demonstrates the overall temporal stability of the self-trapped waves in this regime. Noteworthy, in NLCs the Berry solitons are more stable than standard solitons based upon the dynamic phase, where the fluctuations are observed even in the beam trajectories [1]. For higher powers the spatial self-localization of the beam (Fig. S13b) strongly fluctuates in time. In fact, a quasi-periodic temporal oscillation in the beam width is observed, the light behavior spanning from free diffraction (see e.g. the second panel from the left) to self-collimation (see e.g. the rightmost panel). Mixed states, with the co-existence of self-trapped beams superposed over a diffractive component, are observed as well (see the first panel). These hybrid states are detected only by the ansatz accounting for the light diffusion. The behavior is quite similar to the instability regime found in the numerical simulations, see Sec. I: the beam self-confinement is
Supplementary Figure S13. **Outlook of the beam behavior for different realizations.** Top and middle rows: Series of beam snapshots on the plane $xz$ when the input power is (top) 50 mW and (middle) 80 mW. Bottom row: The corresponding beam width versus $z$, computed both with the Gaussian fit (solid lines) and with the function $F(x) + I_G(x)$ (dashed lines). Applied bias is 9 V and the input beam waist is 4 $\mu$m.

always present in the first 200-300 $\mu$m, whereas the fluctuations start after this distance. Accordingly, the standard deviation on the beam width $w$ is almost flat and lower than 1 $\mu$m for $z < 220$ $\mu$m; then a linear growth takes place, achieving the value of about 20 $\mu$m in $z = 900$ $\mu$m. The instability regime can originate both from the temporal noise related to multiple scattering in NLC [18] or to the slow variations in the Stokes parameters, as in our numerical simulations. Future studies will address the real cause behind the observed temporal instability.

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