ON $k$-CAPS IN $\text{PG}(n,q)$, WITH $q$ EVEN AND $n \geq 4$

J. A. THAS
GHENT UNIVERSITY

ABSTRACT. Let $m_2(n,q)$, $n \geq 3$, be the maximum size of $k$ for which there exists a complete $k$-cap in $\text{PG}(n,q)$. In this paper the known bounds for $m_2(n,q)$, $n \geq 4$, $q$ even and $q \geq 2048$, will be considerably improved.

Keywords: projective space, finite field, $k$-cap.

1. INTRODUCTION

A $k$-arc of $\text{PG}(2,q)$ is a set of $k$ points, no three of which are collinear; a $k$-cap of $\text{PG}(n,q)$, $n \geq 3$, is a set of $k$ points, no three of which are collinear. A $k$-arc or $k$-cap is complete if it is not contained in a $(k+1)$-arc or $(k+1)$-cap. The largest value of $k$ for which a $k$-arc of $\text{PG}(2,q)$, or a $k$-cap of $\text{PG}(n,q)$ with $n \geq 3$, exists is denoted by $m_2(n,q)$. The size of the second largest complete $k$-arc of $\text{PG}(2,q)$ or $k$-cap of $\text{PG}(n,q)$, $n \geq 3$, is denoted by $m_0^2(n,q)$.

For any $k$-arc $K$ in $\text{PG}(2,q)$ or $k$-cap $K$ in $\text{PG}(n,q)$, $n \geq 3$, a tangent of $K$ is a line which has exactly one point in common with $K$. Let $t$ be the number of tangents of $K$ through a point $P$ of $K$ and let $\sigma_1(Q)$ be the number of tangents of $K$ through a point $Q \not \in K$. Then for a $k$-arc $K$ $t + k = q^{n-1} + q^{n-2} + \cdots + q + 2$.

**Theorem 1.1** ([6]). If $K$ is a complete $k$-arc in $\text{PG}(2,q)$, $q$ even, or a complete $k$-cap in $\text{PG}(n,q)$, $n \geq 3$ and $q$ even, then $\sigma_1(Q) \leq t$ for each point $Q$ not in $K$.

**Theorem 1.2.**

(i) $m_2(2,q) = q + 2$, $q$ even [5];

(ii) $m_2(3,q) = q^2 + 1$, $q$ even, $q > 2$ [4, 1, 8];

(iii) $m_2(n,2) = 2^n$ [1];

(iv) $m_2(4,4) = 41$ [3];

(v) $m_2^2(n,2) = 2^{n-1} + 2^{n-3}$ [2];

(vi) $m_2^2(3,4) = 14$ [6].

**Theorem 1.3** ([9, 11, 5]). Let $K$ be a $k$-arc of $\text{PG}(2,q)$, $q$ even and $q > 2$, with $q - \sqrt{q+1} < k \leq q + 1$. Then $K$ can be uniquely extended to a $(q+2)$-arc of $\text{PG}(2,q)$.

The following result is the Main Theorem of [12].

**Theorem 1.4** ([12]).

(1) $m_2^2(3,q) < q^2 - (\sqrt{q} - 1)q + 5$, $q$ even, $q \geq 8$.
As a corollary new bounds for \( m_2(n, q) \), \( q \) even, \( q \geq 8 \) and \( n \geq 4 \), are obtained.

**Theorem 1.5 ([12]).**

(i) \( m_2(4, 8) \geq 479 \); 
(ii) for \( q \) even, \( q > 8 \), 
\[ m_2(4, q) < q^3 - q^2 + 2\sqrt{5}q - 8; \]
(iii) \( m_2(n, 8) \leq 478.8^{n-4} - 2(8^{n-5} + \cdots + 8 + 1) + 1, n \geq 5; \)
(iv) for \( q \) even, \( q > 8, n \geq 5 \), 
\[ m_2(n, q) < q^{n-1} - q^{n-2} + 2\sqrt{5}q^{n-3} - 9q^{n-4} - 2(q^{n-5} + \cdots + q + 1) + 1. \]

Combining the main theorem of [10] with Theorem 1.4, there is an immediate improvement of the upper bound for \( m'_2(3, q), q \geq 2048 \). This important remark is due to T. Szönyi.

**Theorem 1.6 ([12]).**

(2) \( m'_2(3, q) < q^3 - 2q + 3\sqrt{q} + 2, q \) even, \( q \geq 2048 \).

Relying on Theorem 1.6, in the underlying paper new bounds for \( m_2(n, q), q \) even, \( q \geq 2048, n \geq 4 \), will be obtained.

**Theorem 1.7.** For \( q \) even, \( q \geq 2048 \),

(i) \( m_2(4, q) < q^3 - 2q^2 + 3q\sqrt{q} + 8q - 9\sqrt{q} - 6 \), 
(ii) \( m_2(n, q) < q^{n-1} - 2q^{n-2} + 3q^{n-3}\sqrt{q} + 8q^{n-3} - 9q^{n-4}\sqrt{q} - 7q^{n-4} - 2(q^{n-5} + \cdots + q + 1) + 1, n \geq 5. \)

2. NEW BOUND FOR \( m_2(4, q) \)

**Theorem 2.1.** For \( q \) even, \( q \geq 2048 \),

(3) \( m_2(4, q) < q^3 - 2q^2 + 3q\sqrt{q} + 8q - 9\sqrt{q} - 6. \)

**Proof.** Assume, by way of contradiction, that \( K \) is a complete \( k \)-cap of \( \text{PG}(4, q) \) with \( k \geq q^3 - 2q^2 + 3q\sqrt{q} + 8q - 9\sqrt{q} - 6 \). Then \( t \leq q^3 + q^2 + q + 2 - q^3 + 2q^2 - 3q\sqrt{q} - 8q + 9\sqrt{q} + 6 \), so \( t \leq 3q^2 - 3q\sqrt{q} - 7q + 9\sqrt{q} + 8 \). We obtain a contradiction in several stages.

(I) \( K \) contains no plane \( q \)-arc

Assume, by way of contradiction, that \( \pi \) is a plane with \( |\pi \cap K| = q \); let \( \pi \cap K = Q \).

(a) **Suppose that** \( \delta_1, \delta_2, \ldots, \delta_5 \) **are distinct hyperplanes containing** \( \pi \), **such that**

(7) \( |\delta_i \cap K| \geq q^2 - 2q + 3\sqrt{q} + 2, i = 1, 2, \ldots, 5 \)

By Theorem 1.6 each \( \delta_i \cap K \) can be extended to an ovoid \( O_i \) of \( \delta_i, i = 1, 2, \ldots, 5 \). Hence \( O_i \cap \pi \) is a \((q+1)\)-arc \( Q \cup \{N_i\}, i = 1, 2, \ldots, 5 \). Since \( Q \) is contained in two
(q + 1)-arcs at least three of the points \(N_i\) coincide, say \(N_1 = N_2 = N_3\). The joins of \(N_1\) to the points of \(\delta_i \cap K\), with \(i = 1, 2, 3\), are tangents of \(K\). Hence

\[
\sigma_1(N_i) \geq 3(q^2 - 3q + 3\sqrt{q} + 2) + q.
\]

so

\[
\sigma_1(N_1) \geq 3q^2 - 8q + 9\sqrt{q} + 6.
\]

As \(K\) is complete, \(\sigma_1(N_1) \leq t\). So

\[
3q^2 - 8q + 9\sqrt{q} + 6 \leq 3q^2 - 3q\sqrt{q} - 7q + 9\sqrt{q} + 8,
\]

that is

\[
3q\sqrt{q} - q - 2 \leq 0,
\]

clearly a contradiction.

(b) Assume that there are at most 4 hyperplanes \(\delta\) of \(\text{PG}(4, q)\) containing \(\pi\) with \(|\delta \cap K| \geq q^2 - 2q + 3\sqrt{q} + 2\)

Then counting points of \(K\) in hyperplanes containing \(\pi\) gives

\[
k < (q - 3)(q^2 - 3q + 3\sqrt{q} + 2) + 4(q^2 - q) + q,
\]

that is,

\[
k < q^3 - 2q^2 + 3q\sqrt{q} + 8q - 9\sqrt{q} - 6.
\]

(Remark that any hyperplane containing \(\pi\), has at most \(q^2\) points in common with \(K\).)

But \(k \geq q^3 - 2q^2 + 3q\sqrt{q} + 8q - 9\sqrt{q} - 6\), clearly a contradiction.

(II) There exists no hyperplane \(\delta\) of \(\text{PG}(4, q)\) such that

\[
q^2 + 1 > |\delta \cap K| \geq q^2 - 2q + 3\sqrt{q} + 2
\]

Suppose, by way of contradiction, that such a \(\delta\) exists. Let \(\delta \cap K = K'\). Then \(K'\) can be extended to an ovoid \(O\) of \(\delta\). Let \(N \in O \setminus K'\) and let \(N' \in K'\). Consider the \(q + 1\) planes of \(\delta\) containing the line \(NN'\). Each of these planes meets \(O\) in a \((q+1)\)-arc, so by I each such plane meets \(K'\) in at most a \((q-1)\)-arc.

Assume, by way of contradiction, that none of these intersections is a \((q-1)\)-arc. Counting the points of \(K'\) on these \(q + 1\) planes gives

\[
|K'| \leq (q + 1)(q - 3) + 1,
\]

so

\[
|K'| \leq q^2 - 2q - 2.
\]

As \(|K'| \geq q^2 - 2q + 3\sqrt{q} + 2\), there arises \(3\sqrt{q} + 4 \leq 0\), a contradiction.

So we may assume that \(|\pi \cap K'| = q - 1, \pi \subset \delta, NN' \subset \pi\). Consider all hyperplanes of \(\text{PG}(4, q)\) containing the plane \(\pi\). Let \(\theta\) be the number of such hyperplanes \(\pi'\) for which

\[
|\pi' \cap K| \geq q^2 - 2q + 3\sqrt{q} + 2.
\]

By assumption \(\theta \geq 1\).
First assume \( \theta \geq 4 \), hence there are at least 4 hyperplanes \( \pi'_1, \pi'_2, \pi'_3, \pi'_4 \) containing \( \pi \) such that
\[
|\pi'_i \cap K| \geq q^2 - 2q + 3\sqrt{q} + 2.
\]
Consequently \( \pi'_i \cap K \) can be extended to an ovoid \( O_i \) of \( \pi'_i \), with \( i = 1, 2, 3, 4 \). It follows that \( O_i \cap \pi \) is a \((q + 1)\)-arc \((\pi \cap K) \cup \{N_i', N''_i\}, i = 1, 2, 3, 4 \). The \((q - 1)\)-arc \( \pi \cap K \) is extendable to a unique \((q + 2)\)-arc \( R \) of \( \pi \), and each \((q + 1)\)-arc of \( \pi \) containing \( \pi \cap K \) belongs to \( R \). So \( \pi \cap K \) is contained in exactly \( 3(q+1) \)-arcs of \( \pi \). It follows that there is at least one point \( N \) which belongs to 3 of the 4 pairs \( \{N_i', N''_i\} \). So the number of tangents \( \sigma_1(N) \) of \( K \) containing \( N \) is at least
\[
(19) 3(q^2 - 2q + 3\sqrt{q} + 2 - q + 1) + q - 1 = 3q^2 - 8q + 9\sqrt{q} + 8.
\]
As \( \sigma_1(N) \leq t \), there arises
\[
(20) 3q^2 - 3q\sqrt{q} - 7q + 9\sqrt{q} + 8 \geq t \geq \sigma_1(N) \geq 3q^2 - 8q + 9\sqrt{q} + 8,
\]
so
\[
(21) 3q\sqrt{q} - q \leq 0,
\]
a contradiction.

Finally, assume \( \theta \leq 3 \). Counting the points of \( K \) in the \( q + 1 \) hyperplanes containing \( \pi \), we obtain
\[
(22) k < (q - 2)(q^2 - 2q + 3\sqrt{q} + 2 - q + 1) + 3(q^2 - q + 1) + q - 1,
\]
so
\[
(23) k < q^3 - 2q^2 + 3q\sqrt{q} + 7q - 6\sqrt{q} - 4.
\]
As
\[
(24) k \geq q^3 - 2q^2 + 3q\sqrt{q} + 8q - 9\sqrt{q} - 6,
\]
there arises
\[
(25) q - 3\sqrt{q} - 2 < 0,
\]
a final contradiction.

(III) For a point \( N \) not on \( K \), there do not exist planes \( \pi_1 \) and \( \pi_2 \) such that \( \pi_1 \cap \pi_2 = \{N\} \) and such that \( \pi_1 \cap K \) is a \((q + 1)\)-arc with nucleus \( N \), \( i = 1, 2 \).

Suppose, by way of contradiction, that such planes \( \pi_1, \pi_2 \) exist. Let \( \delta \) be a hyperplane containing \( \pi_1 \). Then \( \delta \cap K \) contains the \( q + 1 \) tangents of \( \pi_1 \cap K \) through \( N \) and one tangent of \( \pi_2 \cap K \) through \( N \). So \( \delta \cap K \) has at least \( q + 2 \) tangents through \( N \). Hence \( |\delta \cap K| < q^2 + 1 \).

Suppose that
\[
(26) |\delta \cap K| < q^2 - 2q + 3\sqrt{q} + 2
\]
for any such hyperplane \( \delta \). Counting points of \( K \) in hyperplanes containing \( \pi_1 \) gives
\[
(27) k < (q + 1)(q^2 - 3q + 3\sqrt{q} + 1) + q + 1,
\]
so
\[
(28) k < q^3 - 2q^2 + 3q\sqrt{q} - q + 3\sqrt{q} + 2.
\]
As \( k \geq q^3 - 2q^2 + 3q\sqrt{q} + 8q - 9\sqrt{q} - 6 \), there arises a contradiction.
Consequently there exists a hyperplane \( \delta \) through \( \pi_1 \) for which
\[
q^2 + 1 > |\delta \cap K| \geq q^2 - 2q + 3\sqrt{q} + 2,
\]
contradicting II.

(IV) The tangents of \( K \) through any point \( N \) not on \( K \) lie in a hyperplane
Let \( \delta \) be a hyperplane not containing \( N \) and let \( \mathcal{V} \) be the set of the intersections of \( \delta \) with all tangents of \( K \) through \( N \). We will show that each point of \( \mathcal{V} \) is on at least two lines contained in \( \mathcal{V} \).

Let \( R \in \mathcal{V} \) and let \( r = RN \). Assume, by way of contradiction, that for at most one plane \( \pi \) containing \( r \) we have \( |\pi \cap K| \geq q - 1 \). So
\[
k \leq (q^2 + q)(q - 3) + (q + 1),
\]
that is,
\[
k \leq q^3 - 2q^2 - 2q + 1.
\]
As \( k \geq q^3 - 2q^2 + 3q\sqrt{q} + 8q - 9\sqrt{q} - 6 \), there arises \( 3q\sqrt{q} + 10q - 9\sqrt{q} - 7 \leq 0 \), a contradiction.

Hence we may assume that for distinct planes \( \pi, \pi' \) containing \( r \) we have
\[
|\pi \cap K|, |\pi' \cap K| \in \{q - 1, q + 1\}.
\]
(By I no plane intersects \( K \) in a \( q \)-arc.)

We distinguish two cases.

(a) At least one of the planes \( \pi, \pi' \) intersects \( K \) in a \( (q - 1) \)-arc
Say \( |\pi \cap K| = q - 1 \). Assume, by way of contradiction, that for no hyperplane \( \delta' \) containing \( \pi \) we have \( |\delta' \cap K| = q^2 + 1 \). Counting points of \( K \) in hyperplanes containing \( \pi \) gives by II
\[
k < (q + 1)(q^2 - 2q + 3\sqrt{q} + 2 - q + 1) + q - 1,
\]
that is,
\[
k < q^3 - 2q^2 + 3q\sqrt{q} + q + 3\sqrt{q} + 2.
\]
As \( k \geq q^3 - 2q^2 + 3q\sqrt{q} + 8q - 9\sqrt{q} - 6 \), there arises \( 7q - 12\sqrt{q} - 8 < 0 \), a contradiction.

So for at least one hyperplane \( \delta' \) containing \( \pi \) we have \( |\delta' \cap K| = q^2 + 1 \). But then \( |\pi \cap K| = q + 1 \), again a contradiction.

(b) \( |\pi \cap K| = |\pi' \cap K| = q + 1 \)
If \( N \) is the nucleus of both \( \pi \cap K \) and \( \pi' \cap K \), then there are two lines of \( \mathcal{V} \) through \( R \), namely \( \pi \cap \delta \) and \( \pi' \cap \delta \).

Therefore suppose that \( N \) is not the nucleus of \( \pi \cap K \). If for at most one hyperplane \( \delta' \) containing \( \pi \) we have \( |\delta' \cap K| = q^2 + 1 \), then counting points of \( K \) in hyperplanes containing \( \pi \) gives
\[
k < q^2 + 1 + q(q^2 - 3q + 3\sqrt{q} + 1),
\]
so
\[
k < q^3 - 2q^2 + 3q\sqrt{q} + q + 1.
\]
As \( k \geq q^3 - 2q^2 + 3q\sqrt{q} + 8q - 9\sqrt{q} - 6 \), there arises \( 7q - 9\sqrt{q} - 7 < 0 \), a contradiction.
Consequently there are at least two hyperplanes \( \delta_1 \) and \( \delta_2 \) containing \( \pi \) for which \( \delta_i \cap K = O_i \) is an ovoid, \( i = 1, 2 \). then there is a plane \( \pi_i \) of \( \delta_i \) containing \( N \) such that \( N \) is the nucleus of the \( (q+1) \)-arc \( \pi_i \cap O_i = K_i, i = 1, 2 \). As \( N \) is not the nucleus of \( \pi \cap K \), we have \( \pi \neq \pi_1 \neq \pi_2 \neq \pi \). The tangents of \( K_i \) (which contain \( N \)) meet \( \delta \) in the points of a line \( l_i \), containing \( R \), with \( i = 1, 2 \), and \( l_1 \neq l_2 \).

Consequently each point of \( V \) is on at least two lines contained in \( V \).

If there existed two skew lines in \( V \), there would be two planes \( \pi'_1 \) and \( \pi'_2 \) on \( N \), with \( \pi'_1 \cap \pi'_2 = \{N\} \) and \( N \) the nucleus of the \( (q+1) \)-arcs \( \pi'_1 \cap K \) and \( \pi'_2 \cap K \). This is in contradiction with III. It follows that the lines of \( V \) all have a common point or all lie in a common plane. As each point of \( V \) is on at least two lines of \( V \), all lines of \( V \) lie in a plane. Hence \( V \) is subset of a plane, and so all tangents of \( K \) containing \( N \) lie in a hyperplane.

(V) The final contradiction

The final contradiction will be obtained by counting all tangents of \( K \).

Consider the function

\[
G(x) = x(q^3 + q^2 + q + 2 - x).
\]

It attains its maximum value for

\[
x = \frac{1}{2}(q^3 + q^2 + q + 2).
\]

We have

\[
q^3 > k \geq q^3 - 2q^2 + 3q\sqrt{q} + 8q - 9\sqrt{q} - 6 > \frac{1}{2}(q^3 + q^2 + q + 2),
\]

so

\[
kt = k(q^3 + q^2 + q + 2 - k) = G(k) > G(q^3) = q^3(q^2 + q + 2).
\]

All tangents containing a point \( N \) not on \( K \) lie in a hyperplane, which contains at most \( q^2 + 1 \) points of \( K \). An ovoid of a hyperplane containing \( N \) has exactly \( q+1 \) tangents containing \( N \). Hence \( N \) is contained in at most \( q^2 \) tangents of \( K \).

Counting the pairs \((N,l)\), with \( N \notin K \), \( l \) a tangent of \( K \) containing \( N \), there arises

\[
(q^4 + q^3 + q^2 + q + 1 - k)q^2 \geq k tq,
\]

so

\[
(q^4 + q^3 + q^2 + q + 1 - q^3 + 2q^2 - 3q\sqrt{q} - 8q + 9\sqrt{q} + 6)q \geq kt,
\]

so

\[
(q^4 + 3q^2 - 3q\sqrt{q} - 7q + 9\sqrt{q} + 7)q \geq kt > q^3(q^2 + q + 2),
\]

that is,

\[
q^3 - q^2 + 3q\sqrt{q} + 7q - 9\sqrt{q} - 7 < 0,
\]

a contradiction.
3. New bound for \( m_2(n, q) \), \( n \geq 5 \)

**Theorem 3.1.** For \( q \) even, \( q \geq 2048 \), \( n \geq 5 \)

\[
m_2(n, q) < q^{n-1} - 2q^{n-2} + 3q^{n-3} - 9q^{n-4} - 2(q^{n-5} + \cdots + q + 1) + 1.
\]

**Proof**  By 6.14(ii) of [7] for \( n \geq 5 \) and \( q > 2 \), we have

\[
m_2(n, q) \leq q^{n-4} m_2(4, q) - q^{n-4} - 2(q^{n-5} + \cdots + q + 1) + 1.
\]

From Theorem 2.1 the result follows.  

**References**

[12] J. A. Thas, On \( k \)-caps in \( \text{PG}(n, q) \), with \( q \) even and \( n \geq 3 \), *Discrete Math.*, to appear.

Ghent University, Department of Mathematics, Krijgslaan 281, S25, B-9000 Ghent, Belgium

E-mail address: joseph.thas@ugent.be