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The reconstruction of a time-dependent source from a surface measurement for full Maxwell's equations by means of the potential field method

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Abstract

This paper is devoted to the study of an inverse source problem governed by full Maxwell's equations by means of the potential field method (the A - ϕ method). The source term is assumed to be separable in time and space, in which the unknown part is solely time-dependent and is recovered from a surface measurement. We prove that the solution to the inverse problem based on the A - ϕ formulation is existing and unique. We suggest a constructive scheme for approximating the solution and discuss its convergence. Finally, a few examples are presented to verify the theoretical results.

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1. Introduction

In this contribution, an inverse source problem governed by the full Maxwell's equations is studied by means of the potential field method (the A - ϕ method). A bounded, simply-connected and convex polyhedron $\Omega \subset \mathbb{R}^3$ with connected boundary $\partial\Omega$ is considered. The outer normal vector associated with the boundary is denoted by $\mathbf{n}$. The current density $\mathbf{J}$ satisfies Ohm's law such that $\mathbf{J} = \sigma\mathbf{E} + \mathbf{J}_a$, where the vector function $\mathbf{J}_a$ describes possible current sources in Ω . Electromagnetic fields are described by the full linear Maxwell's equations, i.e.

$$\begin{cases} \partial_t \mathbf{D}(\mathbf{x}, t) - (\nabla \times \mathbf{H})(\mathbf{x}, t) = -\mathbf{J}, & (\mathbf{x}, t) \in \Omega \times (0, T] \\ \partial_t \mathbf{B}(\mathbf{x}, t) + (\nabla \times \mathbf{E})(\mathbf{x}, t) = \mathbf{0}, & (\mathbf{x}, t) \in \Omega \times (0, T] \end{cases}$$

or in terms of $\mathbf{E}$ and $\mathbf{H}$ as

$$\begin{cases} \varepsilon(\mathbf{x})\partial_t \mathbf{E}(\mathbf{x}, t) - (\nabla \times \mathbf{H})(\mathbf{x}, t) = -(\sigma(\mathbf{x})\mathbf{E}(\mathbf{x}, t) + \mathbf{J}_a(\mathbf{x}, t)), & (\mathbf{x}, t) \in \Omega \times (0, T] \\ \mu(\mathbf{x})\partial_t \mathbf{H}(\mathbf{x}, t) + (\nabla \times \mathbf{E})(\mathbf{x}, t) = \mathbf{0}, & (\mathbf{x}, t) \in \Omega \times (0, T] \end{cases} \quad (1.1)$$

where $\mathbf{D}$, $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{H}$ are the electric displacement field, electric field, magnetic induction and the magnetic field, σ , ε and μ denote the electric conductivity, the electric permittivity and the magnetic permeability of the medium, respectively.

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The source term $\mathbf{J}_a$ is assumed to be separable, i.e.

$$\mathbf{J}_a(\mathbf{x}, t) = h(t)\mathbf{f}(\mathbf{x}),$$

where $\mathbf{f}(\mathbf{x})$ is given but $h(t)$ is unknown. The measurement used to recover $h(t)$ is specified later. It is assumed that the field $\mathbf{f}$ obeys

$$\mathbf{f} \times \mathbf{n} = \mathbf{0} \quad \text{on } \partial\Omega.$$

The whole domain Ω can be divided into finite subdomains. In each subdomain, the electric conductivity σ , the magnetic permeability μ and the electric permittivity ε of the medium are positive constants. However, for simplicity, it is assumed that Ω consists of only two subdomains, namely a conducting subdomain $\Omega_1 := \Omega_c$ and a nonconducting subdomain $\Omega_e := \Omega_2 = \Omega \setminus \overline{\Omega}_1$. The electric conductivity σ is thus given by

$$\sigma = \begin{cases} \sigma_c & \text{in } \overline{\Omega}_c, \\ 0 & \text{in } \Omega \setminus \overline{\Omega}_c, \end{cases}$$

with $0 < \sigma_c < \infty$. Further, it is assumed that

$$0 < \mu_{\min} \leq \mu \leq \mu_{\max} \quad \text{and} \quad 0 < \varepsilon_{\min} \leq \varepsilon \leq \varepsilon_{\max},$$

- 4 with $\mu_{\min}, \mu_{\max}, \varepsilon_{\min}$ and $\varepsilon_{\max}$ all strict positive constants. The interface between both domains is denoted by $\Gamma = \overline{\Omega}_1 \cap \overline{\Omega}_2$. Following the setting formulated above, the system (1.1) is equipped with the following boundary and initial conditions

$$\begin{cases} \mathbf{n} \times \mathbf{E}(\mathbf{x}, t) = \mathbf{0}, & (\mathbf{x}, t) \in \partial\Omega \times (0, T], \\ \mathbf{E}(\mathbf{x}, 0) = \mathbf{E}_0, & \mathbf{x} \in \Omega, \\ [\mathbf{n}_{12} \cdot (\varepsilon \partial_t \mathbf{E} + \sigma \mathbf{E} + h\mathbf{f})] = [\mathbf{n}_{12} \cdot (\nabla \times \mathbf{H})] = \mathbf{0}, & \text{on } \Gamma \times (0, T], \\ \mu(\mathbf{x})\mathbf{H}(\mathbf{x}, t) \cdot \mathbf{n} = \mathbf{0}, & (\mathbf{x}, t) \in \partial\Omega \times (0, T], \\ \mathbf{H}(\mathbf{x}, 0) = \mathbf{H}_0, & \mathbf{x} \in \Omega, \\ [\mathbf{n}_{12} \times \mathbf{H}] = \mathbf{0}, & \text{on } \Gamma \times (0, T], \end{cases} \quad (1.2)$$

with

$$(\nabla \cdot \mathbf{B}_0)(\mathbf{x}) = (\nabla \cdot (\mu \mathbf{H}_0))(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega,$$

to ensure that the magnetic induction is divergence free. The unit normal vector $\mathbf{n}_{12}$ points from Ω_1 to Ω_2 , thus the jump is defined as follows

$$[\mathbf{n}_{12} \times \mathbf{u}] = \mathbf{n} \times (\mathbf{u}_2 - \mathbf{u}_1), \quad [\mathbf{n}_{12} \cdot \mathbf{u}] = \mathbf{n} \cdot (\mathbf{u}_2 - \mathbf{u}_1).$$

- 5 Usually, the Maxwell system is transformed to the $\mathbf{E}$ or $\mathbf{H}$ equation and the resulting equation is solved by using edge
6 finite element methods. Besides, it can also be changed into potential formulations by means of a decomposition of the
7 field $\mathbf{E}$ or $\mathbf{H}$ (the so-called $\mathbf{A}$ - ϕ or $\mathbf{T}$ - ψ method). Then, nodal finite elements are used to solve the equation numerically,
8 cf. [1, 3, 7, 8, 13, 22, 23]. There are several advantages for the potential field method. For example, it can deal with
9 the possible discontinuity between different mediums very well and has good numerical accuracy. The method avoids
10 spurious solutions by adding a penalty function term in the dominant equation. Moreover, it also has attractive features
11 including natural coupling to moment and boundary element methods, and global energy conservation.

There are many works on inverse problems for electromagnetic fields based on the $\mathbf{E}$ or $\mathbf{H}$ equation, e.g. [5, 6, 14, 15, 18, 21]. But research upon the potential field method has been rarely found. The purpose of this paper is to study the inverse source problem specified above by employing the $\mathbf{A}$ - ϕ method. To transform the formulation (1.1) to the $\mathbf{A}$ - ϕ method, first a potential vector $\mathbf{A}$ is introduced, defined by (see [4, Theorem 2.2])

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} \quad \text{with} \quad \nabla \cdot \mathbf{A} = 0 \text{ in } \Omega \quad \text{and} \quad \mathbf{n} \times \mathbf{A} = \mathbf{0} \text{ on } \partial\Omega,$$

and thus (using the second equation in (1.1))

$$\mathbf{E} = -\partial_t \mathbf{A} - \nabla \phi$$

where the function ϕ is determined up to an additive constant. To ensure the uniqueness of ϕ , it is assumed that $\phi = 0$ on $\partial\Omega$. Now, the first equation of (1.1) can be rewritten as

$$\varepsilon \partial_{tt}(\mathbf{A} + \nabla \phi) + \sigma \partial_t(\mathbf{A} + \nabla \phi) + \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A} \right) = h\mathbf{f}, \quad (1.3)$$

Moreover, the divergence of both sides of (1.3) has been taken to get that

$$\nabla \cdot (\varepsilon \partial_{tt}(\mathbf{A} + \nabla \phi) + \sigma \partial_t(\mathbf{A} + \nabla \phi)) = h \nabla \cdot \mathbf{f}.$$

12 A penalty function term $-\nabla \cdot (\frac{1}{\mu} \nabla \cdot \mathbf{A})$ is added into the dominant $\mathbf{A}$ - ϕ equation to ensure that the vector potential $\mathbf{A}$ is
13 divergence-free (see, e.g. [23]). The $\mathbf{A}$ - ϕ formulation reads as follows:

The dominant equations in the $\mathbf{A}$ - ϕ formulation are given by

$$\begin{cases} \varepsilon \partial_{tt}(\mathbf{A} + \nabla \phi) + \sigma \partial_t(\mathbf{A} + \nabla \phi) + \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A} \right) - \nabla \cdot \left(\frac{1}{\mu} \nabla \cdot \mathbf{A} \right) = h \mathbf{f}, & \text{in } \Omega \times (0, T], \\ \nabla \cdot (\varepsilon \partial_{tt}(\mathbf{A} + \nabla \phi) + \sigma \partial_t(\mathbf{A} + \nabla \phi)) = h \nabla \cdot \mathbf{f}, & \text{in } \Omega \times (0, T], \\ \mathbf{A}(\mathbf{x}, 0) = \mathbf{A}_0, \quad \partial_t \mathbf{A}(\mathbf{x}, 0) = \mathbf{A}'_0, & \mathbf{x} \in \Omega, \\ \phi(\mathbf{x}, 0) = \phi_0, \quad \partial_t \phi(\mathbf{x}, 0) = \phi'_0, & \mathbf{x} \in \Omega, \end{cases} \quad (1.4)$$

the boundary conditions on $\partial\Omega$ are given by

$$\begin{cases} \mathbf{n} \times \mathbf{A}(\mathbf{x}, t) = \mathbf{0}, & (\mathbf{x}, t) \in \partial\Omega \times (0, T], \\ \nabla \cdot \mathbf{A}(\mathbf{x}, t) = 0, & (\mathbf{x}, t) \in \partial\Omega \times (0, T], \\ \phi(\mathbf{x}, t) = 0, & (\mathbf{x}, t) \in \partial\Omega \times (0, T], \end{cases} \quad (1.5)$$

and the interface conditions on Γ are given by

$$[\phi] = 0, \quad \text{on } \Gamma \times (0, T], \quad (1.6)$$

$$[\mathbf{A}] = \mathbf{0}, \quad \text{on } \Gamma \times (0, T], \quad (1.7)$$

$$\left[\mathbf{n}_{12} \cdot \left(\varepsilon \frac{\partial^2}{\partial t^2} (\mathbf{A} + \nabla \phi) + \sigma \frac{\partial}{\partial t} (\mathbf{A} + \nabla \phi) - h \mathbf{f} \right) \right] = \left[\mathbf{n}_{12} \cdot \left(\nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A} \right) - \nabla \cdot \left(\frac{1}{\mu} \nabla \cdot \mathbf{A} \right) \right) \right] = 0, \quad \text{on } \Gamma \times (0, T], \quad (1.8)$$

$$\left[\mathbf{n}_{12} \times \left(\frac{1}{\mu} \nabla \times \mathbf{A} \right) \right] = \mathbf{0}, \quad \text{on } \Gamma \times (0, T], \quad (1.9)$$

$$\left[\frac{1}{\mu} \nabla \cdot \mathbf{A} \right] = 0, \quad \text{on } \Gamma \times (0, T], \quad (1.10)$$

where the function $\mathbf{A}_0$ is derived from $\mathbf{H}_0 \in L^2(\Omega)$ by

$$\nabla \times \mathbf{A}_0 = \mu \mathbf{H}_0 \quad \text{with} \quad \nabla \cdot \mathbf{A}_0 = 0 \text{ in } \Omega,$$

where the function $\mathbf{A}'_0$ is calculated from $\mathbf{E}_0 \in \mathbf{H}(\text{curl}, \Omega) \cap \mathbf{H}(\text{div}, \Omega)$ by

$$\nabla \times \mathbf{A}'_0 = -\nabla \times \mathbf{E}_0 \quad \text{with} \quad \nabla \cdot \mathbf{A}'_0 = 0 \text{ in } \Omega,$$

where ϕ'_0 is found from

$$-\Delta \phi'_0 = \nabla \cdot \mathbf{E}_0 \quad \text{with} \quad \phi'_0 = 0 \text{ on } \partial\Omega$$

and ϕ_0 is defined by

$$\phi_0 = t_1 \phi'_0$$

14 assuming that $\phi(-t_1) = 0$ is prescribed for small $t_1 > 0$ (in computations usually t_1 is taken the time step).

15 The interface conditions (1.6)-(1.7) are natural conditions when using nodal finite elements, the interface condition
16 (1.8) means the continuity of the current density in normal direction, the condition (1.9) reflects the continuity of magnetic
17 field in tangential direction, and the condition (1.10) ensures that the vector potential field $\mathbf{A}$ is divergence-free on the
18 whole domain.

19 The inverse source problem here is to find a triple $(\mathbf{A}(\mathbf{x}, t), \phi(\mathbf{x}, t), h(t))$ from which $\mathbf{E}$ and $\mathbf{H}$ can be derived. Usually,
20 an integral overdetermination measurement is applied to this type of inverse problems for evolutionary problems, cf.
21 [10, 16, 17, 18, 20, 27] and the references therein. Most of the existing literature shows that the additional integral
22 measurement is taken over the whole domain (or over a subdomain). However, the way to obtain such a measurement
23 is not always obvious from practical point of view. Note that if the unknown source depends on the space variable, one
24 usual needs an additional space measurement (e.g. the solution at the final time), cf. [2, 9, 11, 19, 24]. This means that
25 both kinds of inverse source problems need often totally different additional data.

In this contribution, the missing data function $h(t)$ is recovered by means of the following normal component measurement along a 2D-surface $\gamma \subset \Omega_c$:

$$\int_{\gamma} \mathbf{E} \cdot \mathbf{n} \, d\gamma = - \int_{\gamma} (\partial_t \mathbf{A} + \nabla \partial_t \phi) \cdot \mathbf{n} \, d\gamma = m(t). \quad (1.11)$$

It is assumed that the measurement surface γ is a part of a boundary of a subdomain $G \subset \Omega$, i.e. $\gamma \subset \partial G$. Especially, if $\gamma \subset \partial\Omega_c$, then G can be Ω_c . This type of surface measurement is also considered in another framework [18], where the reconstruction of a solely time dependent source is investigated for quasi-static Maxwell's equations (in terms of $\mathbf{E}$). Note that in this latter paper, the domain is only occupied by a conductor. Moreover, the analysis is made in the space $\mathbf{H}(\text{curl}, \Omega) \cap \mathbf{H}(\text{div}, \Omega)$ and second order Lagrange elements are used in the experiments.

This paper is organized as follows. First, in Section 2, some notations and preliminary results are given. Then, a decoupled semi-discrete $\mathbf{A}$ - ϕ scheme based on Rothe's method [12] is presented for solving the inverse source problem (1.4)-(1.11). Afterwards, it is proven that a unique solution on a single timestep exists. In Section 3, some stability estimates for the approximating solution are established in order to discuss the well-posedness of the problem. In Section 4, the well-posedness of the inverse source problem is proven by arguments of functional analysis. Finally, the numerical scheme proposed in this article is verified by some numerical experiments in Section 5 and some conclusions are given in the last section.

2. Notations and time discretization

For convenience of presentation, we first give some notations that are used throughout this paper. Let $L^2(\Omega)$ be the usual Hilbert space of square integrable functions equipped with the inner product and norm:

$$(u, v) := \int_{\Omega} u(\mathbf{x})v(\mathbf{x})d\mathbf{x} \quad \text{and} \quad \|u\|_{L^2(\Omega)} := (u, u)^{1/2}.$$

Define $H^m(\Omega) := \{v \in L^2(\Omega) : D^{\xi}v \in L^2(\Omega), |\xi| \leq m\}$, which is equipped with the following norm and semi-norm

$$\|u\|_{H^m(\Omega)} := \left(\sum_{|\xi| \leq m} \|D^{\xi}u\|_{L^2(\Omega)}^2 \right)^{1/2} \quad \text{and} \quad |u|_{H^m(\Omega)} := \left(\sum_{|\xi|=m} \|D^{\xi}u\|_{L^2(\Omega)}^2 \right)^{1/2},$$

where ξ represents a non-negative triple index. $H_0^1(\Omega)$ is the subspace of $H^1(\Omega)$ whose functions have zero traces on $\partial\Omega$. We use boldface notation to represent vector-valued quantities such as $\mathbf{L}^2(\Omega) := (L^2(\Omega))^3$. Let $\widehat{\mathbf{H}}_0^1(\Omega) := \{\mathbf{v} \in \mathbf{H}^1(\Omega) : \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\}$ with $\|\mathbf{v}\|_{\widehat{\mathbf{H}}_0^1(\Omega)} := \|\mathbf{v}\|_{\mathbf{H}^1(\Omega)}$ for $\mathbf{v} \in \widehat{\mathbf{H}}_0^1(\Omega)$. The space $\mathbf{V} := \widehat{\mathbf{H}}_0^1(\Omega) \times H_0^1(\Omega)$ is equipped with the inner product

$$((\mathbf{P}, \varphi), (\mathbf{Q}, \psi))_{\mathbf{V}} := (\mathbf{P} + \nabla\varphi, \mathbf{Q} + \nabla\psi) + (\nabla \times \mathbf{P}, \nabla \times \mathbf{Q}) + (\nabla \cdot \mathbf{P}, \nabla \cdot \mathbf{Q})$$

and the norm

$$\|(\mathbf{Q}, \psi)\|_{\mathbf{V}} := \left(\|\mathbf{Q} + \nabla\psi\|_{L^2(\Omega)}^2 + \|\nabla \times \mathbf{Q}\|_{L^2(\Omega)}^2 + \|\nabla \cdot \mathbf{Q}\|_{L^2(\Omega)}^2 \right)^{1/2}.$$

Next, we show the following lemmas borrowed from [8]. A direct application of these lemmas yields that the $\mathbf{V}$ -norm is equivalent to the norm $(\|\cdot\|_{\mathbf{H}^1(\Omega)}^2 + \|\nabla \cdot \cdot\|_{L^2(\Omega)}^2)^{1/2}$.

Lemma 2.1 (coercivity). Let Ω be a convex bounded polyhedron. Then there exists a constant $C > 0$ such that

$$C \|(\mathbf{Q}, \psi)\|_{\mathbf{V}}^2 \geq \|\mathbf{Q}\|_{\mathbf{H}^1(\Omega)}^2 + \|\nabla\psi\|_{L^2(\Omega)}^2$$

for all $(\mathbf{Q}, \psi) \in \mathbf{V}$.

Lemma 2.2 (continuity). Let Ω be a convex bounded polyhedron. There exists a constant $C > 0$ such that

$$((\mathbf{P}, \varphi), (\mathbf{Q}, \psi))_{\mathbf{V}}^2 \leq C \left(\|\mathbf{P}\|_{\mathbf{H}^1(\Omega)}^2 + \|\nabla\varphi\|_{L^2(\Omega)}^2 \right) \left(\|\mathbf{Q}\|_{\mathbf{H}^1(\Omega)}^2 + \|\nabla\psi\|_{L^2(\Omega)}^2 \right)$$

for any $(\mathbf{P}, \varphi), (\mathbf{Q}, \psi) \in \mathbf{V}$.

Now, we can define the weak solution of (1.4) in suitable spaces. First, applying the measurement operator to the first equation of (1.4), we can eliminate the unknown $h(t)$ as follows

$$h(t) = \frac{-\varepsilon_c m'(t) - \sigma_c m(t) + \frac{1}{\mu_c} \int_{\gamma} \nabla \times \nabla \times \mathbf{A} \cdot \mathbf{n} \, d\gamma - \frac{1}{\mu_c} \int_{\gamma} \nabla(\nabla \cdot \mathbf{A}) \cdot \mathbf{n} \, d\gamma}{\int_{\gamma} \mathbf{f} \cdot \mathbf{n} \, d\gamma} \quad (2.1)$$

if $\int_{\gamma} \mathbf{f} \cdot \mathbf{n} \, d\gamma \neq 0$.

Next, we take the scalar product of the first equation in (1.4) with $\mathbf{Q} \in \widehat{\mathbf{H}}_0^1(\Omega_c)$ and the scalar product of the second equation in (1.4) with $\psi \in H_0^1(\Omega_c)$. Afterwards, we integrate the result over the domain Ω_c and we apply Green's theorems. Afterwards, we repeat the previous step for Ω_e . Then, the resulting formulations are added together to obtain for all $(\mathbf{Q}, \psi) \in V$ that

$$\begin{aligned} & (\varepsilon \partial_{tt}(\mathbf{A}(t) + \nabla \phi(t)), \mathbf{Q} + \nabla \psi) + \sigma_c (\partial_t(\mathbf{A}(t) + \nabla \phi(t)), \mathbf{Q} + \nabla \psi)_{\Omega_c} + \left(\frac{1}{\mu} \nabla \times \mathbf{A}(t), \nabla \times \mathbf{Q} \right) \\ & + \left(\frac{1}{\mu} \nabla \cdot \mathbf{A}(t), \nabla \cdot \mathbf{Q} \right) - \left([\mathbf{n}_{12} \times (\nabla \times \mathbf{A}(t))], \mathbf{Q} \right)_{\Gamma} + \left([\nabla \cdot \mathbf{A}(t)], \mathbf{Q} \cdot \mathbf{n}_{12} \right)_{\Gamma} \\ & - \left([h\mathbf{f} \cdot \mathbf{n}_{12} - \varepsilon \partial_{tt}(\mathbf{A} + \nabla \phi) \cdot \mathbf{n}_{12} - \sigma \partial_t(\mathbf{A} + \nabla \phi) \cdot \mathbf{n}_{12}], \psi \right)_{\Gamma} = (h(t)\mathbf{f}, \mathbf{Q} + \nabla \psi). \end{aligned} \quad (2.2)$$

43 The interface integrals cancel out by the interface conditions (1.8)-(1.10) such that the following condition can be stated.

Definition 2.1. The potential field pair $(\mathbf{A}, \phi)$, satisfying

$$(\mathbf{A}(t), \phi(t)) \in V, (\partial_t \mathbf{A}(t), \partial_t \phi(t)) \in L^2(\Omega_c) \times H^1(\Omega_c), (\partial_{tt} \mathbf{A}(t), \partial_{tt} \phi(t)) \in L^2(\Omega) \times H^1(\Omega)$$

is the weak solution of (1.4) if the following identity is fulfilled for a.a. $t \in [0, T]$:

$$\begin{cases} (\varepsilon \partial_{tt}(\mathbf{A}(t) + \nabla \phi(t)), \mathbf{Q} + \nabla \psi) + \sigma_c (\partial_t(\mathbf{A}(t) + \nabla \phi(t)), \mathbf{Q} + \nabla \psi)_{\Omega_c} \\ + \left(\frac{1}{\mu} \nabla \times \mathbf{A}(t), \nabla \times \mathbf{Q} \right) + \left(\frac{1}{\mu} \nabla \cdot \mathbf{A}(t), \nabla \cdot \mathbf{Q} \right) = (h(t)\mathbf{f}, \mathbf{Q} + \nabla \psi), \quad \forall (\mathbf{Q}, \psi) \in V, \\ \mathbf{A}(\mathbf{x}, 0) = \mathbf{A}_0, \partial_t \mathbf{A}(\mathbf{x}, 0) = \mathbf{A}'_0, \phi(\mathbf{x}, 0) = \phi_0, \partial_t \phi(\mathbf{x}, 0) = \phi'_0, \quad \mathbf{x} \in \Omega. \end{cases} \quad (2.3)$$

44 **Remark 2.1.** The variational formulation of the first and second equation of (1.4) is a special case of the formulation
45 (2.3). Derivations similar to (2.2) are not repeated in detail further in the paper. Throughout the paper, the sentence
46 ‘integrate over Ω ’ means sometimes these underlying calculations.

Let n be a positive integer and $\{t_i = i\tau : i = 0, \dots, n\}$ be a equidistant partition of $[0, T]$ with $\tau = T/n$. Set for any function u

$$u_i = u_i(\mathbf{x}) \approx u(\mathbf{x}, t_i), \quad \partial_t u(\mathbf{x}, t_i) \approx \delta u_i = \frac{u_i - u_{i-1}}{\tau}, \quad \partial_{tt} u(\mathbf{x}, t_i) \approx \delta^2 u_i = \frac{\delta u_i - \delta u_{i-1}}{\tau}.$$

With these notations, we are allowed to consider the following time-discrete problem: Given $(\mathbf{A}_0, \phi_0)$ and $(\delta \mathbf{A}_0, \delta \phi_0) = (\mathbf{A}'_0, \phi'_0)$, find $(\mathbf{A}_i, \phi_i) \in V$ for $i = 1, \dots, n$ such that

$$\begin{cases} \varepsilon \delta^2(\mathbf{A}_i + \nabla \phi_i) + \sigma \delta(\mathbf{A}_i + \nabla \phi_i) + \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_i \right) - \nabla \left(\frac{1}{\mu} \nabla \cdot \mathbf{A}_i \right) = h_i \mathbf{f} & \text{in } \Omega, \\ \nabla \cdot (\varepsilon \delta^2(\mathbf{A}_i + \nabla \phi_i) + \sigma \delta(\mathbf{A}_i + \nabla \phi_i)) = h_i \nabla \cdot \mathbf{f} & \text{in } \Omega, \\ \mathbf{n} \times \mathbf{A}_i = \mathbf{0}, \nabla \cdot \mathbf{A}_i = 0, \phi_i = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.4)$$

with

$$h_i = \frac{-\varepsilon_c m'_i - \sigma_c m_i + \frac{1}{\mu_c} \int_{\gamma} \nabla \times \nabla \times \mathbf{A}_{i-1} \cdot \mathbf{n} \, d\gamma - \frac{1}{\mu_c} \int_{\gamma} \nabla(\nabla \cdot \mathbf{A}_{i-1}) \cdot \mathbf{n} \, d\gamma}{\int_{\gamma} \mathbf{f} \cdot \mathbf{n} \, d\gamma}. \quad (2.5)$$

47 Here (2.4)-(2.5) is a linear decoupled scheme. For a given i , we first derive h_i and then solve (2.4). Afterwards, we replace
48 i with $i + 1$ and repeat this process.

49 **Theorem 2.3.** Suppose that $\mathbf{f} \in \widehat{\mathbf{H}}_0^1(\Omega)$, $(\mathbf{A}_0, \phi_0) \in V$, $(\mathbf{A}'_0, \phi'_0) \in V$, $\nabla \times \nabla \times \mathbf{A}_0 \in \widehat{\mathbf{H}}_0^1(\Omega)$, $\nabla \cdot \mathbf{A}_0 = \nabla \cdot \mathbf{A}'_0 = 0$ in Ω ,
50 $m \in C^1([0, T])$ and $\int_{\gamma} \mathbf{f} \cdot \mathbf{n} \, d\gamma \neq 0$. Then, for any $i = 1, \dots, n$, there exists a unique weak solution $(\mathbf{A}_i, \phi_i, h_i) \in V \times \mathbb{R}$ for
51 (2.4) and (2.5). Moreover, we have that $h_i \in \mathbb{R}$, $(\mathbf{A}_i, \phi_i) \in V$, $\nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_i \right) \in \widehat{\mathbf{H}}_0^1(\Omega)$, $\left[\nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_i \right) \times \mathbf{n}_{12} \right] = \mathbf{0}$ on Γ
52 and $\nabla \cdot \mathbf{A}_i = 0$ in Ω .

Proof. For a known $(\mathbf{A}_{i-1}, \phi_{i-1}) \in V$ and a known $(\delta \mathbf{A}_{i-1}, \delta \phi_{i-1})$, satisfying $\nabla \times \nabla \times \mathbf{A}_{i-1} \in \widehat{\mathbf{H}}_0^1(\Omega)$ and $\nabla \cdot \mathbf{A}_{i-1} = 0$, we can compute h_i from (2.5). We successively deduce that

$$\begin{aligned}
 |h_i|^2 &= \left| \frac{-\varepsilon_c m'_i - \sigma_c m_i + \frac{1}{\mu_c} \int_{\gamma} \nabla \times \nabla \times \mathbf{A}_{i-1} \cdot \mathbf{n} \, d\gamma}{\int_{\gamma} \mathbf{f} \cdot \mathbf{n} \, d\gamma} \right|^2 \\
 &\leq C \left(1 + \left\| \nabla \times \nabla \times \mathbf{A}_{i-1} \right\|_{L^2(\gamma)}^2 \right) \\
 &\stackrel{\gamma \subset \partial G}{\leq} C \left(1 + \left\| \nabla \times \nabla \times \mathbf{A}_{i-1} \right\|_{L^2(\partial G)}^2 \right) \\
 &\stackrel{\mathbf{H}^1(G) \hookrightarrow L^2(\partial G)}{\leq} C \left(1 + \left\| \nabla \times \nabla \times \mathbf{A}_{i-1} \right\|_{\mathbf{H}^1(G)}^2 \right) \\
 &\stackrel{G \subset \Omega}{\leq} C \left(1 + \left\| \nabla \times \nabla \times \mathbf{A}_{i-1} \right\|_{\mathbf{H}^1(\Omega)}^2 \right) \\
 &\leq C_i.
 \end{aligned} \tag{2.6}$$

The semi-discrete approximation to (2.4) reads: Given $h_i, (\mathbf{A}_0, \phi_0)$ and $(\delta \mathbf{A}_0, \delta \phi_0) = (\mathbf{A}'_0, \phi'_0)$, find $(\mathbf{A}_i, \phi_i) \in V$, $1 \leq i \leq n$, such that

$$\begin{aligned}
 (\varepsilon \delta^2 (\mathbf{A}_i + \nabla \phi_i), \mathbf{Q} + \nabla \psi) + \sigma_c (\delta (\mathbf{A}_i + \nabla \phi_i), \mathbf{Q} + \nabla \psi)_{\Omega_c} + \left(\frac{1}{\mu} \nabla \times \mathbf{A}_i, \nabla \times \mathbf{Q} \right) \\
 + \left(\frac{1}{\mu} \nabla \cdot \mathbf{A}_i, \nabla \cdot \mathbf{Q} \right) = (h_i \mathbf{f}, \mathbf{Q} + \nabla \psi), \quad \forall (\mathbf{Q}, \psi) \in V.
 \end{aligned} \tag{2.7}$$

53 We apply the Lax-Milgram theorem and Lemma 2.1-Lemma 2.2 to get the existence of a unique weak solution $(\mathbf{A}_i, \phi_i) \in$
 54 V .

Taking the divergence to the both sides of the first equation of (2.4) and taking into account the second equation yield that

$$\nabla^2 \left(\frac{1}{\mu} \nabla \cdot \mathbf{A}_i \right) = 0 \quad \text{in } \Omega.$$

Considering the boundary condition $\nabla \cdot \mathbf{A}_i = 0$, we obtain that

$$\nabla \cdot \mathbf{A}_i = 0 \quad \text{a.e. in } \Omega.$$

Therefore, we get that

$$\nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_i \right) = h_i \mathbf{f} - \varepsilon \delta^2 (\mathbf{A}_i + \nabla \phi_i) - \sigma \delta (\mathbf{A}_i + \nabla \phi_i) \in L^2(\Omega),$$

$$\nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_i \right) = h_i \nabla \times \mathbf{f} - \varepsilon \delta^2 \nabla \times \mathbf{A}_i - \sigma \delta \nabla \times \mathbf{A}_i \in L^2(\Omega),$$

and

$$\nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_i \right) \times \mathbf{n} = h_i \mathbf{f} \times \mathbf{n} - \varepsilon \delta^2 (\mathbf{A}_i + \nabla \phi_i) \times \mathbf{n} - \sigma \delta (\mathbf{A}_i + \nabla \phi_i) \times \mathbf{n} = \mathbf{0} \text{ on } \partial \Omega,$$

due to the assumption $\mathbf{f} \times \mathbf{n} = \mathbf{0}$ on the boundary. Moreover, $\mathbf{f} \in \widehat{\mathbf{H}}_0^1(\Omega)$ implies that $\mathbf{f}$ is well-defined on the interface such that

$$\left[\nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_i \right) \times \mathbf{n}_{12} \right] = \mathbf{0} \text{ on } \Gamma.$$

Putting $\psi = 0$ in Lemma 2.1, we get the inequality

$$\left\| \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_i \right) \right\|_{\mathbf{H}^1(\Omega)}^2 \leq C \left(\left\| \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_i \right) \right\|^2 + \left\| \nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_i \right) \right\|^2 \right),$$

which means that

$$\nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_i \right) \in \widehat{\mathbf{H}}_0^1(\Omega), \text{ i.e. } \nabla \times \nabla \times \mathbf{A}_i \in \widehat{\mathbf{H}}_0^1(\Omega),$$

55 which finish the proof. □

Under the assumptions of Theorem 2.3, the weak form of (2.4)-(2.5) reads as: find $(\mathbf{A}_i, \phi_i) \in V$, $1 \leq i \leq n$, such that

$$(\varepsilon \delta^2(\mathbf{A}_i + \nabla \phi_i), \mathbf{Q} + \nabla \psi) + \sigma_c(\delta(\mathbf{A}_i + \nabla \phi_i), \mathbf{Q} + \nabla \psi)_{\Omega_c} + \left(\frac{1}{\mu} \nabla \times \mathbf{A}_i, \nabla \times \mathbf{Q} \right) + \left(\frac{1}{\mu} \nabla \cdot \mathbf{A}_i, \nabla \cdot \mathbf{Q} \right) = (h_i \mathbf{f}, \mathbf{Q} + \nabla \psi), \quad \forall (\mathbf{Q}, \psi) \in V, \quad (2.8)$$

with

$$h_i = \frac{-\varepsilon_c m'_i - \sigma_c m_i + \frac{1}{\mu_c} \int_{\gamma} \nabla \times \nabla \times \mathbf{A}_{i-1} \cdot \mathbf{n} \, d\gamma}{\int_{\gamma} \mathbf{f} \cdot \mathbf{n} \, d\gamma}. \quad (2.9)$$

Remark 2.2. The term $\left(\frac{1}{\mu} \nabla \cdot \mathbf{A}_i, \nabla \cdot \mathbf{Q} \right)$ is needed in (2.8) to assure that the solution $\mathbf{A}_i$ of problem (2.8) for given h_i is divergence free. This implies (starting from $\nabla \cdot \mathbf{A}_0 = 0$) that the corresponding term can be deleted in (2.9).

3. Stability

In the following lemmas, we give some stability estimates for the solution $(\mathbf{A}_i, \phi_i, h_i)$ in order to prove the well-posedness of the problem.

Lemma 3.1. Let the assumptions of Theorem 2.3 be satisfied. Then there exist positive constants C and τ_0 such that for $1 \leq j \leq n$ and for all $0 < \tau < \tau_0$ holds that

$$\|\mathbf{A}_j + \nabla \phi_j\|^2 + \sum_{i=1}^j \|(\mathbf{A}_i + \nabla \phi_i) - (\mathbf{A}_{i-1} + \nabla \phi_{i-1})\|^2 + \sum_{i=1}^j \|\tau \nabla \times \mathbf{A}_i\|^2 \leq C \left(1 + \sum_{i=1}^j \tau h_i^2 \right). \quad (3.1)$$

Proof. First, we multiply equation (2.7) by τ and sum up for $i = 1, \dots, k$. If we define the sequence $\mathbf{b}_k = \sum_{i=1}^k \tau \nabla \times \mathbf{A}_i$ with $\mathbf{b}_0 = \mathbf{0}$, this results in $(\nabla \cdot \mathbf{A}_i = 0, \forall i)$

$$(\varepsilon \delta(\mathbf{A}_k + \nabla \phi_k), \mathbf{Q} + \nabla \psi) + \sigma_c(\mathbf{A}_k + \nabla \phi_k, \mathbf{Q} + \nabla \psi)_{\Omega_c} + \left(\frac{1}{\mu} \mathbf{b}_k, \nabla \times \mathbf{Q} \right) = \left(\sum_{i=1}^k \tau h_i \mathbf{f}, \mathbf{Q} + \nabla \psi \right) + (\varepsilon(\mathbf{A}'_0 + \nabla \phi'_0), \mathbf{Q} + \nabla \psi) + \sigma_c(\mathbf{A}_0 + \nabla \phi_0, \mathbf{Q} + \nabla \psi)_{\Omega_c} \quad (3.2)$$

for any $(\mathbf{Q}, \psi) \in V$. Then we put $(\mathbf{Q}, \psi) = (\mathbf{A}_k, \phi_k)$ and multiply the result by τ and sum up for $k = 1, \dots, j$. We obtain that

$$\begin{aligned} & \sum_{k=1}^j (\varepsilon \delta(\mathbf{A}_k + \nabla \phi_k), \mathbf{A}_k + \nabla \phi_k) \tau + \sigma_c \sum_{k=1}^j \|\mathbf{A}_k + \nabla \phi_k\|_{L^2(\Omega_c)}^2 \tau + \sum_{k=1}^j \left(\frac{1}{\mu} \mathbf{b}_k, \delta \mathbf{b}_k \right) \tau \\ &= \sum_{k=1}^j \left(\sum_{i=1}^k \tau h_i \mathbf{f}, \mathbf{A}_k + \nabla \phi_k \right) \tau + \sum_{k=1}^j (\varepsilon(\mathbf{A}'_0 + \nabla \phi'_0), \mathbf{A}_k + \nabla \phi_k) \tau + \sum_{k=1}^j \sigma_c(\mathbf{A}_0 + \nabla \phi_0, \mathbf{A}_k + \nabla \phi_k)_{\Omega_c} \tau. \end{aligned} \quad (3.3)$$

Next, we multiply this result by τ again and sum up for $k = 1, \dots, j$. The first and last term on the left-hand side (LHS) of equation (3.3) can be rewritten using Abel's summation rule and estimated as follows:

$$\begin{aligned} & 2 \sum_{k=1}^j \tau (\varepsilon \delta(\mathbf{A}_k + \nabla \phi_k), \mathbf{A}_k + \nabla \phi_k) \\ &= \sum_{k=1}^j \left\| \sqrt{\varepsilon}(\mathbf{A}_k + \nabla \phi_k) - \sqrt{\varepsilon}(\mathbf{A}_{k-1} + \nabla \phi_{k-1}) \right\|^2 + \left\| \sqrt{\varepsilon}(\mathbf{A}_j + \nabla \phi_j) \right\|^2 - \left\| \sqrt{\varepsilon}(\mathbf{A}_0 + \nabla \phi_0) \right\|^2 \end{aligned}$$

and

$$\sum_{k=1}^j \tau \left(\frac{1}{\mu} \mathbf{b}_k, \delta \mathbf{b}_k \right) = \frac{1}{2} \sum_{k=1}^j \left\| \frac{1}{\sqrt{\mu}} (\mathbf{b}_k - \mathbf{b}_{k-1}) \right\|^2 + \frac{1}{2} \left\| \frac{1}{\sqrt{\mu}} \mathbf{b}_j \right\|^2 - \frac{1}{2} \left\| \frac{1}{\sqrt{\mu}} \mathbf{b}_0 \right\|^2.$$

The second term on the LHS of equation (3.3) is positive and can be omitted. For the first term on the right-hand side (RHS), we apply Cauchy's and Young's inequalities together with the assumptions on the source function to get that

$$\left| \sum_{k=1}^j \left(\sum_{i=1}^k \tau h_i f, \mathbf{A}_k + \nabla \phi_k \right) \tau \right| \leq C \sum_{i=1}^j h_i^2 \tau + C \sum_{k=1}^j \|\mathbf{A}_k + \nabla \phi_k\|^2 \tau.$$

The second and third term on the RHS is easily bounded using Cauchy's and Young's inequalities as follows

$$\begin{aligned} \left| \sum_{k=1}^j (\varepsilon(\mathbf{A}'_0 + \nabla \phi'_0), \mathbf{A}_k + \nabla \phi_k) \tau \right| &\leq C + C \sum_{k=1}^j \|\mathbf{A}_k + \nabla \phi_k\|^2 \tau, \\ \left| \sum_{k=1}^j \sigma_c(\mathbf{A}_0 + \nabla \phi_0, \mathbf{A}_k + \nabla \phi_k)_{\Omega_c} \tau \right| &\leq C + C \sum_{k=1}^j \|\mathbf{A}_k + \nabla \phi_k\|^2 \tau. \end{aligned}$$

Finally, we arrive at the following inequality (change index $k \leftrightarrow i$):

$$\begin{aligned} \|\mathbf{A}_j + \nabla \phi_j\|^2 + \sum_{i=1}^j \|(\mathbf{A}_i + \nabla \phi_i) - (\mathbf{A}_{i-1} + \nabla \phi_{i-1})\|^2 + \left\| \sum_{i=1}^j \tau \nabla \times \mathbf{A}_i \right\|^2 \\ + \sum_{i=1}^j \|\tau \nabla \times \mathbf{A}_i\|^2 \leq C + C \sum_{i=1}^j h_i^2 \tau + C \sum_{i=1}^j \|\mathbf{A}_i + \nabla \phi_i\|^2 \tau. \end{aligned}$$

61 An application of Grönwall's inequality concludes the proof. $\square$

Lemma 3.2. Let the assumptions of Lemma 3.1 be satisfied. Then there exist positive constants C and τ_0 such that for $1 \leq j \leq n$ and for all $0 < \tau < \tau_0$ holds that

$$\|\delta(\mathbf{A}_j + \nabla \phi_j)\|^2 + \sum_{i=1}^j \|\delta(\mathbf{A}_i + \nabla \phi_i) - \delta(\mathbf{A}_{i-1} + \nabla \phi_{i-1})\|^2 + \|\nabla \times \mathbf{A}_j\|^2 + \sum_{i=1}^j \|\nabla \times (\mathbf{A}_i - \mathbf{A}_{i-1})\|^2 \leq C \left(1 + \sum_{i=1}^j \tau h_i^2 \right). \quad (3.4)$$

Proof. Setting $(\mathbf{Q}, \psi) = \tau(\delta \mathbf{A}_i, \delta \phi_i)$ in equation (2.7) and summing up for $i = 1, \dots, j$ lead to

$$\sum_{i=1}^j \tau (\varepsilon \delta^2(\mathbf{A}_i + \nabla \phi_i), \delta(\mathbf{A}_i + \nabla \phi_i)) + \sigma_c \sum_{i=1}^j \tau \|\delta(\mathbf{A}_i + \nabla \phi_i)\|_{L^2(\Omega_c)}^2 + \sum_{i=1}^j \tau \left(\frac{1}{\mu} \nabla \times \mathbf{A}_i, \nabla \times \delta \mathbf{A}_i \right) = \sum_{i=1}^j \tau (h_i f, \delta(\mathbf{A}_i + \nabla \phi_i)). \quad (3.5)$$

We also apply Abel's summation rule to the first and third term to have

$$\begin{aligned} \sum_{i=1}^j \tau (\varepsilon \delta^2(\mathbf{A}_i + \nabla \phi_i), \delta(\mathbf{A}_i + \nabla \phi_i)) &= \frac{1}{2} \sum_{i=1}^j \left\| \sqrt{\varepsilon} \delta(\mathbf{A}_i + \nabla \phi_i) - \sqrt{\varepsilon} \delta(\mathbf{A}_{i-1} + \nabla \phi_{i-1}) \right\|^2 \\ &\quad + \frac{1}{2} \left\| \sqrt{\varepsilon} \delta(\mathbf{A}_j + \nabla \phi_j) \right\|^2 - \frac{1}{2} \left\| \sqrt{\varepsilon} \delta(\mathbf{A}'_0 + \nabla \phi'_0) \right\|^2 \end{aligned}$$

and

$$\sum_{i=1}^j \tau \left(\frac{1}{\mu} \nabla \times \mathbf{A}_i, \nabla \times \delta \mathbf{A}_i \right) = \frac{1}{2} \sum_{i=1}^j \left\| \frac{1}{\sqrt{\mu}} \nabla \times (\mathbf{A}_i - \mathbf{A}_{i-1}) \right\|^2 + \frac{1}{2} \left\| \frac{1}{\sqrt{\mu}} \nabla \times \mathbf{A}_j \right\|^2 - \frac{1}{2} \left\| \frac{1}{\sqrt{\mu}} \nabla \times \mathbf{A}_0 \right\|^2.$$

For the RHS of equation (3.5), we use Cauchy's and Young's inequalities to obtain

$$\left| \sum_{i=1}^j \tau (h_i f, \delta(\mathbf{A}_i + \nabla \phi_i)) \right| \leq C \sum_{i=1}^j \tau h_i^2 + C \sum_{i=1}^j \tau \|\delta(\mathbf{A}_i + \nabla \phi_i)\|^2.$$

Eventually, we arrive at the following inequality:

$$\begin{aligned} \|\delta(\mathbf{A}_j + \nabla \phi_j)\|^2 + \sum_{i=1}^j \|\delta(\mathbf{A}_i + \nabla \phi_i) - \delta(\mathbf{A}_{i-1} + \nabla \phi_{i-1})\|^2 + \|\nabla \times \mathbf{A}_j\|^2 \\ + \sum_{i=1}^j \|\nabla \times (\mathbf{A}_i - \mathbf{A}_{i-1})\|^2 \leq C + C \sum_{i=1}^j \tau h_i^2 + C \sum_{i=1}^j \tau \|\delta(\mathbf{A}_i + \nabla \phi_i)\|^2. \end{aligned}$$

62 Applying Grönwall's inequality, we conclude the inequality (3.4). $\square$

Lemma 3.3. Let the assumptions of Lemma 3.2 be satisfied. Moreover, assume that $\nabla \times \nabla \times \mathbf{f} \in L^2(\Omega)$ and $\nabla \times \nabla \times \mathbf{A}'_0 \in L^2(\Omega)$. Then there exist positive constants C and τ_0 such that for $1 \leq j \leq n$ and for all $0 < \tau < \tau_0$ holds that

$$\|\nabla \times \nabla \times \mathbf{A}_j\|^2 + \sum_{i=1}^j \|\nabla \times \nabla \times (\mathbf{A}_i - \mathbf{A}_{i-1})\|^2 + \sum_{i=1}^j \|\tau \nabla \times \nabla \times \nabla \times \mathbf{A}_i\|^2 \leq C + C \sum_{i=1}^j h_i^2 \tau. \quad (3.6)$$

Proof. First, applying the curl operator to both sides of the first equation of (2.4) yields

$$\nabla \times (\varepsilon \delta^2 \mathbf{A}_i) + \nabla \times (\sigma \delta \mathbf{A}_i) + \nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_i \right) = h_i \nabla \times \mathbf{f}. \quad (3.7)$$

We multiply this by τ again and sum up for $i = 1, \dots, k$ to obtain

$$\nabla \times (\varepsilon \delta \mathbf{A}_k) + \nabla \times (\sigma \mathbf{A}_k) + \sum_{i=1}^k \tau \nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_i \right) = \sum_{i=1}^k \tau h_i \nabla \times \mathbf{f} + \nabla \times (\varepsilon \mathbf{A}'_0) + \nabla \times (\sigma \mathbf{A}_0). \quad (3.8)$$

Then taking the scalar product with $\nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_k \right)$ and integrating over Ω , we have that

$$\begin{aligned} & \left(\nabla \times (\varepsilon \delta \mathbf{A}_k), \nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_k \right) \right) + \left(\nabla \times (\sigma \mathbf{A}_k), \nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_k \right) \right) \\ & + \left(\sum_{i=1}^k \tau \nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_i \right), \nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_k \right) \right) = \left(\sum_{i=1}^k \tau h_i \nabla \times \mathbf{f}, \nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_k \right) \right) \\ & + \left(\nabla \times (\varepsilon \mathbf{A}'_0), \nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_k \right) \right) + \left(\nabla \times (\sigma \mathbf{A}_0), \nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_k \right) \right). \end{aligned} \quad (3.9)$$

Next, multiplying (3.9) by τ again and summing up for $k = 1, \dots, j$ lead to

$$\begin{aligned} & \sum_{k=1}^j \tau \left(\nabla \times (\varepsilon \delta \mathbf{A}_k), \nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_k \right) \right) + \sum_{k=1}^j \tau \left(\nabla \times (\sigma \mathbf{A}_k), \nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_k \right) \right) \\ & + \sum_{k=1}^j \tau \left(\sum_{i=1}^k \tau \nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_i \right), \nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_k \right) \right) = \sum_{k=1}^j \tau \left(\sum_{i=1}^k \tau h_i \nabla \times \mathbf{f}, \nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_k \right) \right) \\ & + \sum_{k=1}^j \tau \left(\nabla \times (\varepsilon \mathbf{A}'_0), \nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_k \right) \right) + \sum_{k=1}^j \tau \left(\nabla \times (\sigma \mathbf{A}_0), \nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_k \right) \right). \end{aligned} \quad (3.10)$$

We apply Abel's summation rule to the third term on the LHS of (3.10) to get

$$\begin{aligned} & \sum_{k=1}^j \tau \left(\sum_{i=1}^k \tau \nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_i \right), \nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_k \right) \right) \\ & = \sum_{k=1}^j \frac{\tau}{\mu_c^2} \left(\sum_{i=1}^k \tau \nabla \times \nabla \times \nabla \times \mathbf{A}_i, \nabla \times \nabla \times \nabla \times \mathbf{A}_k \right)_{\Omega_c} + \sum_{k=1}^j \frac{\tau}{\mu_e^2} \left(\sum_{i=1}^k \tau \nabla \times \nabla \times \nabla \times \mathbf{A}_i, \nabla \times \nabla \times \nabla \times \mathbf{A}_k \right)_{\Omega_e} \\ & = \frac{1}{2\mu_c^2} \sum_{k=1}^j \|\tau \nabla \times \nabla \times \nabla \times \mathbf{A}_k\|_{L^2(\Omega_c)}^2 + \frac{1}{2\mu_c^2} \left\| \sum_{k=1}^j \tau \nabla \times \nabla \times \nabla \times \mathbf{A}_k \right\|_{L^2(\Omega_c)}^2 \\ & + \frac{1}{2\mu_e^2} \sum_{k=1}^j \|\tau \nabla \times \nabla \times \nabla \times \mathbf{A}_k\|_{L^2(\Omega_e)}^2 + \frac{1}{2\mu_e^2} \left\| \sum_{k=1}^j \tau \nabla \times \nabla \times \nabla \times \mathbf{A}_k \right\|_{L^2(\Omega_e)}^2 \\ & \geq \frac{1}{2\mu_{\max}^2} \left\{ \sum_{k=1}^j \|\tau \nabla \times \nabla \times \nabla \times \mathbf{A}_k\|^2 + \left\| \sum_{k=1}^j \tau \nabla \times \nabla \times \nabla \times \mathbf{A}_k \right\|^2 \right\}. \end{aligned}$$

Since $\nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_k \right) \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$ and $\left[\nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_k \right) \times \mathbf{n} \right] = \mathbf{0}$ on Γ , we have that

$$\begin{aligned}
& \sum_{k=1}^j \tau \left(\nabla \times (\varepsilon \delta \mathbf{A}_k), \nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_k \right) \right)_{\Omega_e \cup \Omega_c} \\
&= \sum_{k=1}^j \tau \left(\nabla \times \nabla \times (\varepsilon \delta \mathbf{A}_k), \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_k \right) \right)_{\Omega_e \cup \Omega_c} \\
&= \sum_{k=1}^j \tau \frac{\varepsilon_c}{\mu_c} \left(\nabla \times \nabla \times (\delta \mathbf{A}_k), \nabla \times \nabla \times \mathbf{A}_k \right)_{\Omega_c} + \sum_{k=1}^j \tau \frac{\varepsilon_e}{\mu_e} \left(\nabla \times \nabla \times (\delta \mathbf{A}_k), \nabla \times \nabla \times \mathbf{A}_k \right)_{\Omega_e} \\
&= \frac{\varepsilon_c}{2\mu_c} \left(\left\| \nabla \times \nabla \times \mathbf{A}_j \right\|_{L^2(\Omega_c)}^2 - \left\| \nabla \times \nabla \times \mathbf{A}_0 \right\|_{L^2(\Omega_c)}^2 + \sum_{k=1}^j \left\| \nabla \times \nabla \times (\mathbf{A}_k - \mathbf{A}_{k-1}) \right\|_{L^2(\Omega_c)}^2 \right) \\
&+ \frac{\varepsilon_e}{2\mu_e} \left(\left\| \nabla \times \nabla \times \mathbf{A}_j \right\|_{L^2(\Omega_e)}^2 - \left\| \nabla \times \nabla \times \mathbf{A}_0 \right\|_{L^2(\Omega_e)}^2 + \sum_{k=1}^j \left\| \nabla \times \nabla \times (\mathbf{A}_k - \mathbf{A}_{k-1}) \right\|_{L^2(\Omega_e)}^2 \right) \\
&\geq \frac{\varepsilon_{\min}}{2\mu_{\max}} \left\| \nabla \times \nabla \times \mathbf{A}_j \right\|^2 - \frac{\varepsilon_{\max}}{2\mu_{\min}} \left\| \nabla \times \nabla \times \mathbf{A}_0 \right\|^2 + \frac{\varepsilon_{\min}}{2\mu_{\max}} \sum_{k=1}^j \left\| \nabla \times \nabla \times (\mathbf{A}_k - \mathbf{A}_{k-1}) \right\|^2.
\end{aligned}$$

Analogously, we obtain that

$$\sum_{k=1}^j \tau \left(\nabla \times (\sigma \mathbf{A}_k), \nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_k \right) \right) = \frac{\sigma_c}{\mu_c} \sum_{k=1}^j \left\| \nabla \times \nabla \times \mathbf{A}_k \right\|_{L^2(\Omega_c)}^2 \tau \geq 0.$$

For the terms on the RHS of (3.10), we use Green's theorem, Cauchy's and Young's inequalities together with the assumptions $\nabla \times \nabla \times \mathbf{f} \in L^2(\Omega)$ and $\nabla \times \nabla \times \mathbf{A}'_0 \in L^2(\Omega)$ to obtain that

$$\begin{aligned}
& \left| \sum_{k=1}^j \tau \left(\sum_{i=1}^k \tau h_i \nabla \times \mathbf{f}, \nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_k \right) \right) \right| \\
&= \left| \sum_{k=1}^j \tau \left(\sum_{i=1}^k \tau h_i \nabla \times \nabla \times \mathbf{f}, \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_k \right) \right) \right| \\
&\leq \frac{1}{2} \sum_{k=1}^j \left\| \sum_{i=1}^k \tau h_i \nabla \times \nabla \times \mathbf{f} \right\|^2 \tau + \frac{1}{2} \sum_{k=1}^j \left\| \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_k \right) \right\|^2 \tau \\
&\leq \frac{1}{2} \sum_{k=1}^j \left(\sum_{i=1}^k \tau |h_i| \left\| \nabla \times \nabla \times \mathbf{f} \right\| \right)^2 \tau + \frac{1}{2\mu_e} \sum_{k=1}^j \left\| \nabla \times \nabla \times \mathbf{A}_k \right\|_{L^2(\Omega_e)}^2 \tau + \frac{1}{2\mu_c} \sum_{k=1}^j \left\| \nabla \times \nabla \times \mathbf{A}_k \right\|_{L^2(\Omega_c)}^2 \tau \\
&\leq C \sum_{k=1}^j h_k^2 \tau + C \sum_{k=1}^j \left\| \nabla \times \nabla \times \mathbf{A}_k \right\|^2 \tau,
\end{aligned}$$

$$\begin{aligned}
& \left| \sum_{k=1}^j \tau \left(\nabla \times (\varepsilon \mathbf{A}'_0), \nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_k \right) \right) \right| = \left| \sum_{k=1}^j \tau \left(\nabla \times \nabla \times (\varepsilon \mathbf{A}'_0), \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_k \right) \right) \right| \\
&\leq C + C \sum_{k=1}^j \left\| \nabla \times \nabla \times \mathbf{A}_k \right\|^2 \tau,
\end{aligned}$$

$$\begin{aligned}
& \left| \sum_{k=1}^j \tau \left(\nabla \times (\sigma \mathbf{A}_0), \nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_k \right) \right) \right| = \left| \sum_{k=1}^j \tau \left(\nabla \times \nabla \times (\sigma \mathbf{A}_0), \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_k \right) \right) \right| \\
&\leq C + C \sum_{k=1}^j \left\| \nabla \times \nabla \times \mathbf{A}_k \right\|^2 \tau.
\end{aligned}$$

Finally, after collecting all estimates above, we receive that

$$\begin{aligned} \|\nabla \times \nabla \times \mathbf{A}_j\|^2 + \sum_{k=1}^j \|\nabla \times \nabla \times (\mathbf{A}_k - \mathbf{A}_{k-1})\|^2 + \sum_{k=1}^j \|\tau \nabla \times \nabla \times \nabla \times \mathbf{A}_k\|^2 \\ + \left\| \sum_{k=1}^j \tau \nabla \times \nabla \times \nabla \times \mathbf{A}_k \right\|^2 \leq C + C \sum_{k=1}^j h_k^2 \tau + C \sum_{k=1}^j \|\nabla \times \nabla \times \mathbf{A}_k\|^2 \tau. \end{aligned} \quad (3.11)$$

63 An application of Grönwall's inequality concludes the proof. $\square$

Lemma 3.4. Let the assumptions of Lemma 3.3 be satisfied. Then there exist positive constants C and τ_0 such that for $1 \leq j \leq n$ and for all $0 < \tau < \tau_0$ holds that

$$\begin{aligned} \|\nabla \times \nabla \times \delta \mathbf{A}_j\|^2 + \sum_{i=1}^j \|\nabla \times \nabla \times (\delta \mathbf{A}_i - \delta \mathbf{A}_{i-1})\|^2 \\ + \|\nabla \times \nabla \times \nabla \times \mathbf{A}_j\|^2 + \sum_{i=1}^j \|\nabla \times \nabla \times \nabla \times (\mathbf{A}_i - \mathbf{A}_{i-1})\|^2 \leq C + C \sum_{i=1}^j h_i^2 \tau. \end{aligned} \quad (3.12)$$

Proof. First, we take the scalar product of equation (3.7) with $\nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \delta \mathbf{A}_i\right)$ and integrate over Ω . Then, we multiply the result by τ and sum it up for $k = 1, \dots, j$, i.e.

$$\begin{aligned} \sum_{i=1}^j \tau \left(\nabla \times (\varepsilon \delta^2 \mathbf{A}_i), \nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \delta \mathbf{A}_i\right) \right) + \sum_{i=1}^j \tau \left(\nabla \times (\sigma \delta \mathbf{A}_i), \nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \delta \mathbf{A}_i\right) \right) \\ + \sum_{i=1}^j \tau \left(\nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_i\right), \nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \delta \mathbf{A}_i\right) \right) = \sum_{i=1}^j \tau \left(h_i \nabla \times \mathbf{f}, \nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \delta \mathbf{A}_i\right) \right). \end{aligned} \quad (3.13)$$

Note that also $\nabla \times \left(\frac{1}{\mu} \nabla \times \delta \mathbf{A}_i\right) \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$ and $\left[\nabla \times \left(\frac{1}{\mu} \nabla \times \delta \mathbf{A}_i\right) \times \mathbf{n}_{12} \right] = \mathbf{0}$ on Γ . For this, the following compatibility condition needs to be defined:

$$\varepsilon(\delta^2 \mathbf{A}_0 + \nabla \delta^2 \phi_0) := h(0) \mathbf{f} - \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_0\right) - \sigma(\mathbf{A}'_0 + \nabla \phi'_0) \in \widehat{\mathbf{H}}_0^1(\Omega).$$

We apply Abel's summation rule to respectively the first and third term on the LHS of (3.13) to get that

$$\begin{aligned} \sum_{i=1}^j \tau \left(\nabla \times (\varepsilon \delta^2 \mathbf{A}_i), \nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \delta \mathbf{A}_i\right) \right)_{\Omega_e \cup \Omega_c} \\ = \sum_{i=1}^j \tau \left(\nabla \times \nabla \times (\varepsilon \delta^2 \mathbf{A}_i), \nabla \times \left(\frac{1}{\mu} \nabla \times \delta \mathbf{A}_i\right) \right)_{\Omega_e \cup \Omega_c} \\ \geq \frac{\varepsilon_{\min}}{2\mu_{\max}} \sum_{i=1}^j \|\nabla \times \nabla \times (\delta \mathbf{A}_i - \delta \mathbf{A}_{i-1})\|^2 + \frac{\varepsilon_{\min}}{2\mu_{\max}} \|\nabla \times \nabla \times \delta \mathbf{A}_j\|^2 - \frac{\varepsilon_{\max}}{2\mu_{\min}} \|\nabla \times \nabla \times \mathbf{A}'_0\|^2 \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^j \tau \left(\nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}_i\right), \nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \delta \mathbf{A}_i\right) \right) \\ \geq \frac{1}{2\mu_{\max}} \sum_{i=1}^j \|\nabla \times \nabla \times \nabla \times (\mathbf{A}_i - \mathbf{A}_{i-1})\|^2 + \frac{1}{2\mu_{\max}} \|\nabla \times \nabla \times \nabla \times \mathbf{A}_j\|^2 - \frac{1}{2\mu_{\min}} \|\nabla \times \nabla \times \nabla \times \mathbf{A}_0\|^2. \end{aligned}$$

The second term on the LHS of (3.13) simplifies to

$$\sum_{i=1}^j \tau \left(\nabla \times (\sigma \delta \mathbf{A}_i), \nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \delta \mathbf{A}_i\right) \right) = \frac{\sigma_c}{\mu_c} \sum_{i=1}^j \|\nabla \times \nabla \times \delta \mathbf{A}_i\|_{L^2(\Omega_c)}^2 \tau \geq 0.$$

For the term on the RHS, we use the Cauchy's and Young's inequalities to obtain

$$\begin{aligned} \left| \sum_{i=1}^j \tau \left(h_i \nabla \times \mathbf{f}, \nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \delta \mathbf{A}_i \right) \right) \right| &= \left| \sum_{i=1}^j \tau \left(h_i \nabla \times \nabla \times \mathbf{f}, \nabla \times \left(\frac{1}{\mu} \nabla \times \delta \mathbf{A}_i \right) \right) \right| \\ &\leq C \sum_{i=1}^j h_i^2 \tau + C \sum_{i=1}^j \|\nabla \times \nabla \times \delta \mathbf{A}_i\|^2 \tau, \end{aligned}$$

64 Combining all the previous estimates and applying Grönwall's inequality concludes the proof. $\square$

Theorem 3.5. Let the assumptions of Lemma 3.4 be satisfied. Then there exist positive constants C and τ_0 such that for all $0 < \tau < \tau_0$ holds that

$$\sum_{i=1}^n h_i^2 \tau \leq C, \quad (3.14)$$

$$\max_{1 \leq j \leq n} \|\mathbf{A}_j + \nabla \phi_j\|^2 + \sum_{i=1}^n \|(\mathbf{A}_i + \nabla \phi_i) - (\mathbf{A}_{i-1} + \nabla \phi_{i-1})\|^2 \leq C, \quad (3.15)$$

$$\max_{1 \leq j \leq n} \|\delta(\mathbf{A}_j + \nabla \phi_j)\|^2 + \sum_{i=1}^n \|\delta(\mathbf{A}_i + \nabla \phi_i) - \delta(\mathbf{A}_{i-1} + \nabla \phi_{i-1})\|^2 + \max_{1 \leq j \leq n} \|\nabla \times \mathbf{A}_j\|^2 + \sum_{i=1}^n \|\nabla \times (\mathbf{A}_i - \mathbf{A}_{i-1})\|^2 \leq C, \quad (3.16)$$

$$\max_{1 \leq j \leq n} \|\nabla \times \nabla \times \mathbf{A}_j\|^2 + \sum_{i=1}^n \|\nabla \times \nabla \times (\mathbf{A}_i - \mathbf{A}_{i-1})\|^2 \leq C \quad (3.17)$$

and

$$\begin{aligned} \max_{1 \leq j \leq n} \|\nabla \times \nabla \times \delta \mathbf{A}_j\|^2 + \sum_{i=1}^n \|\nabla \times \nabla \times (\delta \mathbf{A}_i - \delta \mathbf{A}_{i-1})\|^2 \\ + \max_{1 \leq j \leq n} \|\nabla \times \nabla \times \nabla \times \mathbf{A}_j\|^2 + \sum_{i=1}^n \|\nabla \times \nabla \times \nabla \times (\mathbf{A}_i - \mathbf{A}_{i-1})\|^2 \leq C. \end{aligned} \quad (3.18)$$

Proof. Starting from equation (2.9), we follow the same steps as in Theorem 2.3 to obtain that

$$\begin{aligned} \sum_{i=1}^j h_i^2 \tau &= \sum_{i=1}^j \left(\frac{-\varepsilon_c m'_i - \sigma_c m_i + \frac{1}{\mu_c} \int_{\gamma} \nabla \times \nabla \times \mathbf{A}_{i-1} \cdot \mathbf{n} \, d\gamma}{\int_{\gamma} \mathbf{f} \cdot \mathbf{n} \, d\gamma} \right)^2 \tau \\ &\leq C \sum_{i=1}^j \left(1 + \|\nabla \times \nabla \times \mathbf{A}_{i-1}\|_{L^2(\gamma)}^2 \right) \tau \\ &\leq C + C \sum_{i=1}^j \|\nabla \times \nabla \times \mathbf{A}_{i-1}\|_{H^1(G)}^2 \tau \\ &\leq C + C \sum_{i=1}^{j-1} \left(\|\nabla \times \nabla \times \mathbf{A}_i\|^2 + \|\nabla \times \nabla \times \nabla \times \mathbf{A}_i\|^2 \right) \tau. \end{aligned} \quad (3.19)$$

From Lemma 3.3 and Lemma 3.4 follows that

$$\sum_{i=1}^j h_i^2 \tau \leq C + \sum_{i=1}^{j-1} \left(\sum_{k=1}^i h_k^2 \tau \right) \tau.$$

Again an application of Grönwall's lemma gives that

$$\sum_{i=1}^j h_i^2 \tau \leq C.$$

65 The other results of this lemma follow now easily from Lemma 3.1-3.4. $\square$

Corollary 3.1. Let the assumptions of Theorem 3.5 be fulfilled. Then there exist positive constants C and τ_0 such that for all $0 < \tau < \tau_0$ holds that

$$\max_{1 \leq j \leq n} \left\{ \|A_j\|_{H^1(\Omega)}^2 + \|\phi_j\|_{H^1(\Omega)}^2 + \|\nabla \times \nabla \times A_j\|_{H^1(\Omega)}^2 \right\} + \sum_{i=1}^n \|A_i - A_{i-1}\|_{H^1(\Omega)}^2 + \sum_{i=1}^n \|\phi_i - \phi_{i-1}\|_{H^1(\Omega)}^2 \leq C.$$

Proof. The result is a consequence of Theorem 3.5 and Lemma 2.1. $\square$

Corollary 3.2. Let the assumptions of Theorem 3.5 be fulfilled. Then there exist positive constants C and τ_0 such that for all $0 < \tau < \tau_0$ holds that

$$\max_{1 \leq j \leq n} \|\nabla \times \delta A_j\|_{L^2(\Omega_c)}^2 + \sum_{i=1}^n \|\nabla \times (\delta A_i - \delta A_{i-1})\|_{L^2(\Omega_c)}^2 \leq C.$$

Therefore,

$$\max_{1 \leq j \leq n} \left\{ \|\delta A_j\|_{H^1(\Omega_c)}^2 + \|\delta \phi_j\|_{H^1(\Omega_c)}^2 \right\} + \sum_{i=1}^n \|\delta A_i - \delta A_{i-1}\|_{H^1(\Omega_c)}^2 \leq C.$$

Proof. We take the scalar product of (3.7) on Ω_c with $\nabla \times \delta A_i$ and integrating over Ω_c . Afterwards, we multiply the result by τ and sum up for $i = 1, \dots, j$ to obtain that

$$\begin{aligned} \sum_{i=1}^j \tau \varepsilon_c \left(\nabla \times \delta^2 A_i, \nabla \times \delta A_i \right)_{\Omega_c} + \sum_{i=1}^j \tau \|\nabla \times \delta A_i\|_{\Omega_c}^2 \\ + \sum_{i=1}^j \frac{\tau}{\mu_c} \left(\nabla \times \nabla \times \nabla \times A_i, \nabla \times \delta A_i \right)_{\Omega_c} = \sum_{i=1}^j \tau \left(h_i \nabla \times f, \nabla \times \delta A_i \right)_{\Omega_c}. \end{aligned} \quad (3.20)$$

Abel's summation rule gives that

$$\sum_{i=1}^j \tau \varepsilon_c \left(\nabla \times \delta^2 A_i, \nabla \times \delta A_i \right)_{\Omega_c} = \frac{\varepsilon_c}{2} \left(\|\nabla \times \delta A_j\|_{L^2(\Omega_c)}^2 - \|\nabla \times A_0'\|_{L^2(\Omega_c)}^2 + \sum_{i=1}^j \|\nabla \times (\delta A_i - \delta A_{i-1})\|_{L^2(\Omega_c)}^2 \right).$$

Thanks to Theorem 3.5, we have that

$$\left| \sum_{i=1}^j \frac{\tau}{\mu_c} \left(\nabla \times \nabla \times \nabla \times A_i, \nabla \times \delta A_i \right)_{\Omega_c} \right| \leq C_\varepsilon + \varepsilon \sum_{i=1}^j \|\nabla \times \delta A_i\|_{L^2(\Omega_c)}^2 \tau$$

and

$$\left| \sum_{i=1}^j \tau \left(h_i \nabla \times f, \nabla \times \delta A_i \right)_{\Omega_c} \right| \leq C_\varepsilon + \varepsilon \sum_{i=1}^j \|\nabla \times \delta A_i\|_{L^2(\Omega_c)}^2 \tau.$$

Fixing ε sufficiently small concludes the proof. $\square$

Corollary 3.3. Let the assumptions of Theorem 3.5 be fulfilled. Then there exist positive constants C and τ_0 such that for all $0 < \tau < \tau_0$ holds that

$$\sum_{i=1}^n \|\delta^2 \phi_i\|_{H^1(\Omega)}^2 \tau + \sum_{i=1}^n \|\delta^2 A_i\|_{H^1(\Omega)}^2 \tau \leq C.$$

Proof. From equation (2.4) and Theorem 3.5 it follows that

$$\sum_{i=1}^n \|\delta^2 (A_i + \nabla \phi_i)\|^2 \tau \leq C.$$

From equation (3.7), Theorem 3.5 and Corollary 3.2 it follows that

$$\sum_{i=1}^n \|\nabla \times \delta^2 A_i\|^2 \tau = \sum_{i=1}^n \|\nabla \times \delta^2 A_i\|_{L^2(\Omega_c)}^2 \tau + \sum_{i=1}^n \|\nabla \times \delta^2 A_i\|_{L^2(\Omega_c)}^2 \tau \leq C.$$

The result is a consequence of Lemma 2.1. $\square$

Corollary 3.4. Let the assumptions of Theorem 3.5 be fulfilled. Then there exist positive constants C and τ_0 such that for all $0 < \tau < \tau_0$ holds that

$$\sum_{i=1}^n \|\delta\phi_i\|_{H^1(\Omega)}^2 \tau + \sum_{i=1}^n \|\delta\mathbf{A}_i\|_{H^1(\Omega)}^2 \tau + \sum_{i=1}^n \|\delta\mathbf{A}_i - \delta\mathbf{A}_{i-1}\|_{H^1(\Omega)}^2 + \sum_{i=1}^n \|\delta\phi_i - \delta\phi_{i-1}\|_{H^1(\Omega)}^2 \leq C.$$

Proof. This result follows from

$$\delta\phi_j = \phi'_0 + \sum_{i=1}^j \delta^2\phi_i\tau \quad \text{and} \quad \delta\mathbf{A}_j = \mathbf{A}'_0 + \sum_{i=1}^j \delta^2\mathbf{A}_i\tau,$$

69 and Corollary 3.3. □

70 4. Well-posedness

Let us define some interpolations of the discretized fields in time by

$$\begin{cases} \mathbf{A}_n(t) = \mathbf{A}_{i-1} + (t - t_{i-1})\delta\mathbf{A}_i, & t \in (t_{i-1}, t_i], \\ \mathbf{A}_n(0) = \mathbf{A}_0, \end{cases} \quad \begin{cases} \bar{\mathbf{A}}_n(t) = \mathbf{A}_i, & t \in (t_{i-1}, t_i], \\ \bar{\mathbf{A}}_n(0) = \mathbf{A}_0, \end{cases}$$

$$\begin{cases} \mathbf{V}_n(t) = \delta\mathbf{A}_{i-1} + (t - t_{i-1})\delta^2\mathbf{A}_i, & t \in (t_{i-1}, t_i], \\ \mathbf{V}_n(0) = \mathbf{A}'_0, \end{cases} \quad \begin{cases} \bar{\mathbf{V}}_n(t) = \delta\mathbf{A}_i, & t \in (t_{i-1}, t_i], \\ \bar{\mathbf{V}}_n(0) = \mathbf{A}'_0, \end{cases}$$

$$\begin{cases} \phi_n(t) = \phi_{i-1} + (t - t_{i-1})\delta\phi_i, & t \in (t_{i-1}, t_i], \\ \phi_n(0) = \phi_0, \end{cases} \quad \begin{cases} \bar{\phi}_n(t) = \phi_i, & t \in (t_{i-1}, t_i], \\ \bar{\phi}_n(0) = \phi_0, \end{cases}$$

$$\begin{cases} v_n(t) = \delta\phi_{i-1} + (t - t_{i-1})\delta^2\phi_i, & t \in (t_{i-1}, t_i], \\ v_n(0) = \phi'_0, \end{cases} \quad \begin{cases} \bar{v}_n(t) = \delta\phi_i, & t \in (t_{i-1}, t_i], \\ \bar{v}_n(0) = \phi'_0. \end{cases}$$

Similarly, we define $\bar{h}_n$, $\bar{m}_n$ and $\bar{m}'_n$. For $(\mathbf{A}_0, \phi_0)$ and $(\delta\mathbf{A}_0, \delta\phi_0) = (\mathbf{A}'_0, \phi'_0)$, the discretized problem (2.8)-(2.9) is rewritten as follows for all $(\mathbf{Q}, \psi) \in V$:

$$\begin{aligned} & (\varepsilon\partial_t(\mathbf{V}_n(t) + \nabla v_n(t)), \mathbf{Q} + \nabla\psi) + \sigma_c(\partial_t(\mathbf{A}_n(t) + \nabla\phi_n(t)), \mathbf{Q} + \nabla\psi)_{\Omega_c} \\ & + \left(\frac{1}{\mu} \nabla \times \bar{\mathbf{A}}_n(t), \nabla \times \mathbf{Q}\right) + \left(\frac{1}{\mu} \nabla \cdot \bar{\mathbf{A}}_n(t), \nabla \cdot \mathbf{Q}\right) = (\bar{h}_n(t)f, \mathbf{Q} + \nabla\psi), \end{aligned} \quad (4.1)$$

with

$$\bar{h}_n(t) = \frac{-\varepsilon_c \bar{m}'_n(t) - \sigma_c \bar{m}_n(t) + \frac{1}{\mu_c} \int_{\gamma} \nabla \times \nabla \times \bar{\mathbf{A}}_n(t - \tau) \cdot \mathbf{n} \, d\gamma}{\int_{\gamma} \mathbf{f} \cdot \mathbf{n} \, d\gamma}. \quad (4.2)$$

The a priori estimates derived in the previous section can be written in the new notations as

$$\int_0^T \bar{h}_n(t)^2 dt \leq C \quad (4.3)$$

and

$$\begin{aligned} & \max_{t \in [0, T]} \left\{ \|\bar{\mathbf{A}}_n(t)\|_{H^1(\Omega)}^2 + \|\nabla \times \nabla \times \bar{\mathbf{A}}_n(t)\|_{H^1(\Omega)}^2 + \|\partial_t \mathbf{A}_n(t)\|_{H^1(\Omega_c)}^2 + \|\nabla \times \nabla \times \partial_t \mathbf{A}_n(t)\|^2 + \|\bar{\phi}_n(t)\|_{H^1(\Omega)}^2 + \|\partial_t \phi_n(t)\|_{H^1(\Omega_c)}^2 \right\} \\ & + \int_0^T \|\partial_t \mathbf{A}_n(s)\|_{H^1(\Omega)}^2 ds + \int_0^T \|\partial_t \phi_n(s)\|_{H^1(\Omega)}^2 ds + \int_0^T \|\partial_t \mathbf{V}_n(s)\|_{H^1(\Omega)}^2 ds \\ & + \int_0^T \|\partial_t v_n(s)\|_{H^1(\Omega)}^2 ds + \sum_{i=1}^n \left\| \int_{t_{i-1}}^{t_i} \partial_t \mathbf{A}_n \right\|_{H^1(\Omega)}^2 + \sum_{i=1}^n \left\| \int_{t_{i-1}}^{t_i} \partial_t \phi_n \right\|_{H^1(\Omega)}^2 + \sum_{i=1}^n \left\| \nabla \times \nabla \times \int_{t_{i-1}}^{t_i} \partial_t \mathbf{A}_n \right\|_{H^1(\Omega)}^2 \\ & + \sum_{i=1}^n \left\| \int_{t_{i-1}}^{t_i} \partial_t v_n \right\|_{H^1(\Omega)}^2 + \sum_{i=1}^n \left\| \int_{t_{i-1}}^{t_i} \partial_t \mathbf{V}_n \right\|_{H^1(\Omega)}^2 + \sum_{i=1}^n \left\| \nabla \times \nabla \times \int_{t_{i-1}}^{t_i} \partial_t \mathbf{V}_n \right\|_{H^1(\Omega)}^2 \leq C. \end{aligned} \quad (4.4)$$

Theorem 4.1. Suppose that $\mathbf{f} \in \widehat{\mathbf{H}}_0^1(\Omega)$, $\nabla \times \nabla \times \mathbf{f} \in \mathbf{L}^2(\Omega)$, $(\mathbf{A}_0, \phi_0) \in \mathbf{V}$, $(\mathbf{A}'_0, \phi'_0) \in \mathbf{V}$, $\nabla \times \nabla \times \mathbf{A}_0 \in \widehat{\mathbf{H}}_0^1(\Omega)$, $\nabla \cdot \mathbf{A}_0 = \nabla \cdot \mathbf{A}'_0 = 0$ in Ω , $\nabla \times \nabla \times \mathbf{A}'_0 \in \mathbf{L}^2(\Omega)$, $m \in C^1([0, T])$ and $\int_{\gamma} \mathbf{f} \cdot \mathbf{n} \, d\gamma \neq 0$. Then there exists a weak solution $(\mathbf{A}, \phi, h)$ to the problem (1.4)-(1.11). Moreover, we have that $h \in L^2([0, T])$, $(\mathbf{A}, \phi) \in C([0, T], \mathbf{V})$, $(\partial_t \mathbf{A}, \partial_t \phi) \in C([0, T], \mathbf{V})$, $(\partial_{tt} \mathbf{A}, \partial_{tt} \phi) \in L^2((0, T), \mathbf{V})$, $\nabla \times \nabla \times \mathbf{A} \in C([0, T], \mathbf{L}^2(\Omega)) \cap L^\infty((0, T), \widehat{\mathbf{H}}_0^1(\Omega))$, $\nabla \times \nabla \times \partial_t \mathbf{A} \in L^2((0, T), \mathbf{L}^2(\Omega))$ and $\nabla \cdot \mathbf{A}(t) = 0$ in Ω for any $t \in (0, T]$.

Proof. Reflexivity of the space $L^2(0, T)$ together with $\int_0^T \bar{h}_n^2(t) dt \leq C$ gives for a subsequence (denoted by the same symbol again)

$$\bar{h}_n \rightharpoonup z \text{ in } L^2(0, T).$$

The Rellich-Kondrachov theorem [25, p. 272] implies that

$$\widehat{\mathbf{H}}_0^1(\Omega) \hookrightarrow \mathbf{L}^2(\Omega).$$

It holds that (see equation (4.4))

$$\max_{t \in [0, T]} \|\bar{\mathbf{A}}_n(t)\|_{\widehat{\mathbf{H}}_0^1(\Omega)}^2 + \int_0^T \|\partial_t \mathbf{A}_n(s)\|^2 ds \leq C,$$

such that the conditions of [26, Lemma 1.3.13] are satisfied. Therefore, there exists $\mathbf{A} \in C([0, T], \mathbf{L}^2(\Omega)) \cap L^\infty((0, T), \widehat{\mathbf{H}}_0^1(\Omega))$ and a subsequence $\{\mathbf{A}_{n_k}\}_{k \in \mathbb{N}}$ of $\{\mathbf{A}_n\}_{n \in \mathbb{N}}$ such that

$$\begin{cases} \mathbf{A}_{n_k} \rightarrow \mathbf{A}, & \text{in } C([0, T], \mathbf{L}^2(\Omega)), \\ \mathbf{A}_{n_k}(t) \rightharpoonup \mathbf{A}(t), & \text{in } \widehat{\mathbf{H}}_0^1(\Omega) \text{ for all } t \in [0, T], \\ \bar{\mathbf{A}}_{n_k}(t) \rightharpoonup \mathbf{A}(t), & \text{in } \widehat{\mathbf{H}}_0^1(\Omega) \text{ for all } t \in [0, T], \\ \partial_t \mathbf{A}_{n_k} = \bar{\mathbf{V}}_{n_k} \rightharpoonup \partial_t \mathbf{A}, & \text{in } L^2((0, T), \mathbf{L}^2(\Omega)). \end{cases}$$

In addition, we have that $\nabla \cdot \bar{\mathbf{A}}_{n_k} \rightharpoonup \nabla \cdot \mathbf{A}$ in $L^2((0, T), \mathbf{L}^2(\Omega))$. This follows from the density argument $\overline{C_0^\infty(\Omega)} = L^2(\Omega)$ and from the Green theorem because we obtain for all $\phi \in C_0^\infty(\Omega)$ that

$$\begin{aligned} \int_0^\eta (\nabla \cdot \bar{\mathbf{A}}_{n_k}(t), \phi) dt &= - \int_0^\eta (\bar{\mathbf{A}}_{n_k}(t), \nabla \phi) dt \\ &\quad \downarrow \\ &= - \int_0^\eta (\mathbf{A}(t), \nabla \phi) dt = \int_0^\eta (\nabla \cdot \mathbf{A}(t), \phi) dt. \end{aligned}$$

In an analogue way, there exist a function $\phi \in C([0, T], L^2(\Omega)) \cap L^\infty((0, T), H_0^1(\Omega))$ and a subsequence $\{\phi_{n_k}\}_{k \in \mathbb{N}}$ of $\{\phi_n\}_{n \in \mathbb{N}}$ such that

$$\begin{cases} \phi_{n_k} \rightarrow \phi, & \text{in } C([0, T], L^2(\Omega)), \\ \phi_{n_k}(t) \rightharpoonup \phi(t), & \text{in } H_0^1(\Omega) \text{ for all } t \in [0, T], \\ \bar{\phi}_{n_k}(t) \rightharpoonup \phi(t), & \text{in } H_0^1(\Omega) \text{ for all } t \in [0, T], \\ \partial_t \phi_{n_k} = \bar{v}_{n_k} \rightharpoonup \partial_t \phi, & \text{in } L^2((0, T), L^2(\Omega)). \end{cases}$$

From $\int_0^T \|\partial_t \phi_n(s)\|_{H_0^1(\Omega)}^2 ds \leq C$, it follows that

$$\partial_t \nabla \phi_{n_k} \rightharpoonup \partial_t \nabla \phi \text{ in } L^2((0, T), \mathbf{L}^2(\Omega)).$$

We denote these subsequences again with $\{\mathbf{A}_n\}$ and $\{\phi_n\}$ to skip double indices. From Corollary 3.4, it follows that

$$\lim_{n \rightarrow \infty} \|\mathbf{V}_n - \bar{\mathbf{V}}_n\|_{L^2((0, T), \widehat{\mathbf{H}}_0^1(\Omega))}^2 = 0 = \lim_{n \rightarrow \infty} \|v_n - \bar{v}_n\|_{L^2((0, T), H_0^1(\Omega))}^2.$$

Thus $\mathbf{V}_n \rightharpoonup \partial_t \mathbf{A}$ in $L^2((0, T), \widehat{\mathbf{H}}_0^1(\Omega))$ and $v_n \rightharpoonup \partial_t \phi$ in $L^2((0, T), H_0^1(\Omega))$. Moreover, thanks to

$$\int_0^T \|\partial_t \mathbf{V}_n(s)\|_{\widehat{\mathbf{H}}_0^1(\Omega)}^2 ds + \int_0^T \|\partial_t v_n(s)\|_{H_0^1(\Omega)}^2 ds \leq C,$$

we have that $\partial_t V_n \rightharpoonup \partial_t A$ in $L^2((0, T), \widehat{H}_0^1(\Omega))$ and $\partial_t v_n \rightharpoonup \partial_t \phi$ in $L^2((0, T), H_0^1(\Omega))$. Now, we integrate (4.1) into time (for any $\xi \in (0, T)$) to get

$$\begin{aligned} \int_0^\xi (\varepsilon \partial_t (V_n(t) + \nabla v_n(t)), \mathbf{Q} + \nabla \psi) + \sigma_c \int_0^\xi (\partial_t (A_n(t) + \nabla \phi_n(t)), \mathbf{Q} + \nabla \psi)_{\Omega_c} \\ + \int_0^\xi \left(\frac{1}{\mu} \nabla \times \bar{A}_n(t), \nabla \times \mathbf{Q} \right) + \int_0^\xi \left(\frac{1}{\mu} \nabla \cdot \bar{A}_n(t), \nabla \cdot \mathbf{Q} \right) = \int_0^\xi (\bar{h}_n(t) f, \mathbf{Q} + \nabla \psi). \end{aligned}$$

Due to the convergence results above, we can pass to the limit $n \rightarrow \infty$ to obtain

$$\begin{aligned} \int_0^\xi (\varepsilon \partial_t (A(t) + \nabla \phi(t)), \mathbf{Q} + \nabla \psi) + \sigma_c \int_0^\xi (\partial_t (A(t) + \nabla \phi(t)), \mathbf{Q} + \nabla \psi)_{\Omega_c} \\ + \int_0^\xi \left(\frac{1}{\mu} \nabla \times A(t), \nabla \times \mathbf{Q} \right) + \int_0^\xi \left(\frac{1}{\mu} \nabla \cdot A(t), \nabla \cdot \mathbf{Q} \right) = \int_0^\xi (z(t) f, \mathbf{Q} + \nabla \psi). \end{aligned}$$

Finally, we have to show that $z = h$. Afterwards, we can differentiate the last equality with respect to obtain the existence of a weak solution to (1.4)–(1.11).

To prove that $z = h$, i.e. the convergence of (4.2) to (2.1) as $n \rightarrow \infty$, we need better convergence of the approximates. From equation (4.4), it follows that

$$\max_{t \in [0, T]} \left\{ \left\| \nabla \times \nabla \times \bar{A}_n(t) \right\|_{\widehat{H}_0^1(\Omega)}^2 + \left\| \nabla \times \nabla \times \partial_t A_n(t) \right\|^2 \right\} \leq C.$$

Again, from Kačur's lemma, there exists $\nabla \times \nabla \times A \in C([0, T], L^2(\Omega)) \cap L^\infty((0, T), H^1(\Omega))$ and a subsequence $\{\nabla \times \nabla \times A_{n_k}\}_{k \in \mathbb{N}}$ (denoted with the same symbol again) of $\{\nabla \times \nabla \times A_n\}_{n \in \mathbb{N}}$ such that

$$\begin{cases} \nabla \times \nabla \times A_n \rightarrow \nabla \times \nabla \times A, & \text{in } C([0, T], L^2(\Omega)), \\ \nabla \times \nabla \times A_n(t) \rightharpoonup \nabla \times \nabla \times A(t), & \text{in } \widehat{H}_0^1(\Omega) \text{ for all } t \in [0, T], \\ \nabla \times \nabla \times \bar{A}_n(t) \rightharpoonup \nabla \times \nabla \times A(t), & \text{in } \widehat{H}_0^1(\Omega) \text{ for all } t \in [0, T], \\ \partial_t \nabla \times \nabla \times A_n = \nabla \times \nabla \times \bar{V}_n \rightharpoonup \nabla \times \nabla \times \partial_t A, & \text{in } L^2((0, T), L^2(\Omega)). \end{cases}$$

Recalling the Nečas inequality

$$\|z\|_{L^2(\partial G)}^2 \leq \eta \|\nabla z\|_{L^2(G)}^2 + C_\eta \|z\|_{L^2(G)}^2, \quad \forall z \in H^1(G), \quad 0 < \eta < \eta_0, \quad (4.8)$$

we have that

$$\begin{aligned} & \left| \int_\gamma \nabla \times \nabla \times (A_n(t) - A_n(t - \tau)) \cdot \mathbf{n} \, d\gamma \right|^2 \\ & \leq C \left\| \nabla \times \nabla \times (A_n(t) - A_n(t - \tau)) \right\|_{L^2(\gamma)}^2 \\ & \stackrel{\gamma \subset G}{\leq} C \left\| \nabla \times \nabla \times (A_n(t) - A_n(t - \tau)) \right\|_{L^2(\partial G)}^2 \\ & \stackrel{(4.8)}{\leq} \eta \left\| \nabla \times \nabla \times (A_n(t) - A_n(t - \tau)) \right\|_{H^1(G)}^2 + C_\eta \left\| \nabla \times \nabla \times (A_n(t) - A_n(t - \tau)) \right\|_{L^2(G)}^2 \\ & \stackrel{G \subset \Omega}{\leq} \eta \left\| \nabla \times \nabla \times (A_n(t) - A_n(t - \tau)) \right\|_{H^1(\Omega)}^2 + C_\eta \left\| \nabla \times \nabla \times (A_n(t) - A_n(t - \tau)) \right\|^2 \\ & \leq \eta \left\| \nabla \times \nabla \times (A_n(t) - A_n(t - \tau)) \right\|_{H^1(\Omega)}^2 + C_\eta \int_{t-\tau}^t \left\| \partial_t \nabla \times \nabla \times A_n(s) \right\|^2 ds \\ & \leq \eta + C_\eta \tau. \end{aligned}$$

Passing to the limit for $n \rightarrow \infty$ and then for $\eta \rightarrow 0$, we find that

$$\lim_{n \rightarrow \infty} \left| \int_\gamma \nabla \times \nabla \times (A_n(t) - A_n(t - \tau)) \cdot \mathbf{n} \, d\gamma \right|^2 = 0.$$

In a similar way, we deduce that

$$\begin{aligned}
 \left| \int_{\gamma} \nabla \times \nabla \times (\mathbf{A}_n(t) - \mathbf{A}(t)) \cdot \mathbf{n} \, d\gamma \right|^2 &\leq C \left\| \nabla \times \nabla \times (\mathbf{A}_n(t) - \mathbf{A}(t)) \right\|_{L^2(\gamma)}^2 \\
 &\leq C \left\| \nabla \times \nabla \times (\mathbf{A}_n(t) - \mathbf{A}(t)) \right\|_{L^2(\partial G)}^2 \\
 &\leq \eta \left\| \nabla \times \nabla \times (\mathbf{A}_n(t) - \mathbf{A}(t)) \right\|_{H^1(G)}^2 + C_{\eta} \left\| \nabla \times \nabla \times (\mathbf{A}_n(t) - \mathbf{A}(t)) \right\|_{L^2(G)}^2 \\
 &\leq \eta \left\| \nabla \times \nabla \times (\mathbf{A}_n(t) - \mathbf{A}(t)) \right\|_{H^1(\Omega)}^2 + C_{\eta} \left\| \nabla \times \nabla \times (\mathbf{A}_n(t) - \mathbf{A}(t)) \right\|^2 \\
 &\leq \eta + C_{\eta} \left\| \nabla \times \nabla \times (\mathbf{A}_n(t) - \mathbf{A}(t)) \right\|^2.
 \end{aligned} \tag{4.9}$$

Passing to the limit for $n \rightarrow \infty$ and using

$$\nabla \times \nabla \times \mathbf{A}_n \rightarrow \nabla \times \nabla \times \mathbf{A} \text{ for a.e. } (\mathbf{x}, t) \in \Omega \times [0, T],$$

we get

$$\lim_{n \rightarrow \infty} \left| \int_{\gamma} \nabla \times \nabla \times (\mathbf{A}_n(t - \tau) - \mathbf{A}(t)) \cdot \mathbf{n} \, d\gamma \right|^2 \leq \eta.$$

The value of $\eta > 0$ can be taken arbitrarily small. Thus, we obtain that

$$\lim_{n \rightarrow \infty} \left| \int_{\gamma} \nabla \times \nabla \times (\mathbf{A}_n(t - \tau) - \mathbf{A}(t)) \cdot \mathbf{n} \, d\gamma \right|^2 = 0.$$

Therefore, we conclude that

$$\lim_{n \rightarrow \infty} \left| \int_{\gamma} \nabla \times \nabla \times (\bar{\mathbf{A}}_n(t - \tau) - \mathbf{A}(t)) \cdot \mathbf{n} \, d\gamma \right|^2 = 0.$$

Since the measurement $m \in C^1([0, T])$, we have that $\bar{m}_n(t) \rightarrow m(t)$ and $\bar{m}'_n(t) \rightarrow m'(t)$ for any $t \in [0, T]$. Collecting the considerations above we see that we may pass to the limit for $n \rightarrow \infty$ in (4.2) to arrive at (2.3), i.e. $z = h$. Note that for the equation system (2.4)

$$\begin{cases} \varepsilon \partial_t (\mathbf{V}_n + \nabla v_n) + \sigma \partial_t (\mathbf{A}_n + \nabla \phi_n) + \nabla \times \left(\frac{1}{\mu} \nabla \times \bar{\mathbf{A}}_n \right) - \nabla \left(\frac{1}{\mu} \nabla \cdot \bar{\mathbf{A}}_n \right) = \bar{h}_n \mathbf{f} & \text{in } \Omega \times (0, T], \\ \nabla \cdot (\varepsilon \partial_{tt} (\mathbf{A}_n + \nabla \phi_n) + \sigma \partial_t (\mathbf{A}_n + \nabla \phi_n)) = \bar{h}_n \nabla \cdot \mathbf{f} & \text{in } \Omega \times (0, T], \end{cases}$$

and the equation system (1.4)

$$\begin{cases} \varepsilon \partial_{tt} (\mathbf{A} + \nabla \phi) + \sigma \partial_t (\mathbf{A} + \nabla \phi) + \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A} \right) - \nabla \left(\frac{1}{\mu} \nabla \cdot \mathbf{A} \right) = h \mathbf{f} & \text{in } \Omega \times (0, T], \\ \nabla \cdot (\varepsilon \partial_{tt} (\mathbf{A} + \nabla \phi) + \sigma \partial_t (\mathbf{A} + \nabla \phi)) = h \nabla \cdot \mathbf{f} & \text{in } \Omega \times (0, T], \end{cases}$$

taking divergence to both side of the first equations of the two systems respectively and considering proper boundary conditions, we conclude that

$$\nabla \cdot \bar{\mathbf{A}}_n(t) = 0, \quad \nabla \cdot \mathbf{A}(t) = 0, \quad \forall t \in (0, T].$$

□

Finally, we discuss the uniqueness of a solution to the inverse problem (1.4)-(1.11).

Theorem 4.2. Let the assumptions of Theorem 4.1 be satisfied. Then, there exists a unique weak solution to (1.4)-(1.11).

Proof. Due to the linearity of the problem it is sufficient to show that if $m = 0$ and $(\mathbf{A}_0, \phi_0) = (\mathbf{0}, 0)$ and $(\mathbf{A}'_0, \phi'_0) = (\mathbf{0}, 0)$ then $(\mathbf{A}(t), \phi(t), h(t)) = (\mathbf{0}, 0, 0)$ for any $t \in [0, T]$. The proof is split into several steps.

1st step: We integrate the system (1.4) over the time variable $t \in (0, s) \subset (0, T)$ to get

$$\varepsilon \partial_t (\mathbf{A}(s) + \nabla \phi(s)) + \sigma (\mathbf{A}(s) + \nabla \phi(s)) + \int_0^s \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A} \right) = \int_0^s h \mathbf{f},$$

$$\nabla \cdot (\varepsilon \partial_t (\mathbf{A}(s) + \nabla \phi(s)) + \sigma (\mathbf{A}(s) + \nabla \phi(s))) = \int_0^s h \nabla \cdot \mathbf{f}.$$

We take the scalar product of the above two equations with $\mathbf{A}(s)$ and $\phi(s)$ respectively, integrate over Ω , make use of the Green theorem and sum them together to obtain

$$(\varepsilon \partial_t (\mathbf{A}(s) + \nabla \phi(s)), \mathbf{A}(s) + \nabla \phi(s)) + (\sigma (\mathbf{A}(s) + \nabla \phi(s)), \mathbf{A}(s) + \nabla \phi(s)) + \left(\int_0^s \frac{1}{\mu} \nabla \times \mathbf{A}, \nabla \times \mathbf{A}(s) \right) = \left(\int_0^s h \mathbf{f}, \mathbf{A}(s) + \nabla \phi(s) \right).$$

Integrating in time over the time variable $s \in (0, t) \subset (0, T)$ yields

$$\frac{\varepsilon_{\min}}{2} \|\mathbf{A}(t) + \nabla \phi(t)\|^2 + \sigma_c \int_0^t \|\mathbf{A}(s) + \nabla \phi(s)\|_{L^2(\Omega_c)}^2 ds + \int_0^t \left(\int_0^s \frac{1}{\mu} \nabla \times \mathbf{A}, \nabla \times \mathbf{A}(s) \right) ds = \int_0^t \left(\int_0^s h \mathbf{f}, \mathbf{A}(s) + \nabla \phi(s) \right) ds.$$

For the term on the RHS, we have that

$$\left| \int_0^t \left(\int_0^s h \mathbf{f}, \mathbf{A}(s) + \nabla \phi(s) \right) ds \right| \leq C \int_0^t \|\mathbf{A}(s) + \nabla \phi(s)\|^2 ds + C \int_0^t h^2(s) ds.$$

We obtain for the third term on the LHS that

$$\begin{aligned} \int_0^t \left(\int_0^s \frac{1}{\mu} \nabla \times \mathbf{A}, \nabla \times \mathbf{A}(s) \right) ds &= \int_0^t \left(\int_0^s \frac{1}{\mu_c} \nabla \times \mathbf{A}, \nabla \times \mathbf{A}(s) \right)_{\Omega_c} ds + \int_0^t \left(\int_0^s \frac{1}{\mu_e} \nabla \times \mathbf{A}, \nabla \times \mathbf{A}(s) \right)_{\Omega_e} ds \\ &= \frac{1}{2\mu_c} \left\| \int_0^t \nabla \times \mathbf{A} \right\|_{L^2(\Omega_c)}^2 + \frac{1}{2\mu_e} \left\| \int_0^t \nabla \times \mathbf{A} \right\|_{L^2(\Omega_e)}^2 \\ &\geq \min \left\{ \frac{1}{2\mu_c}, \frac{1}{2\mu_e} \right\} \left\| \int_0^t \nabla \times \mathbf{A} \right\|^2. \end{aligned}$$

The second term in the LHS gives no better information and can be omitted. Applying the Grönwall inequality, we conclude that

$$\|\mathbf{A}(t) + \nabla \phi(t)\|^2 + \left\| \int_0^t \nabla \times \mathbf{A} \right\|^2 \leq C \int_0^t h^2(s) ds, \quad \forall t \in [0, T].$$

2nd step: We put $\mathbf{Q} = \partial_t \mathbf{A}(t)$ and $\psi = \partial_t \phi(t)$ in (2.3). Then, we integrate in time and estimate the resulting RHS. We obtain that

$$\frac{\varepsilon_{\min}}{2} \|\partial_t (\mathbf{A}(t) + \nabla \phi(t))\|^2 + \sigma_c \int_0^t \|\partial_t (\mathbf{A} + \nabla \phi)\|_{L^2(\Omega_c)}^2 ds + \frac{1}{2\mu_{\max}} \|\nabla \times \mathbf{A}(t)\|^2 \leq C \int_0^t \|\partial_t (\mathbf{A} + \nabla \phi)\|^2 ds + C \int_0^t h^2(s) ds.$$

Applying the Grönwall inequality, we conclude that

$$\|\partial_t (\mathbf{A}(t) + \nabla \phi(t))\|^2 + \|\nabla \times \mathbf{A}(t)\|^2 \leq C \int_0^t h^2(s) ds, \quad \forall t \in [0, T].$$

3rd step: From equation (1.4), it follows that

$$\begin{aligned} \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A} \right) \times \mathbf{n} &= h \mathbf{f} \times \mathbf{n} - \varepsilon \partial_{tt} \mathbf{A} \times \mathbf{n} - \sigma \partial_t \mathbf{A} \times \mathbf{n} = \mathbf{0} \quad \text{on } \partial\Omega, \\ \left[\nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A} \right) \times \mathbf{n}_{12} \right] &= \mathbf{0} \quad \text{on } \Gamma, \end{aligned}$$

Applying the curl operator to the first equation of (1.4) gives

$$\nabla \times (\varepsilon \partial_{tt} \mathbf{A}) + \nabla \times (\sigma \partial_t \mathbf{A}) + \nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A} \right) = h \nabla \times \mathbf{f}. \quad (4.10)$$

Integrating in time over $(0, s)$, we obtain that

$$\nabla \times (\varepsilon \partial_t \mathbf{A}(s)) + \nabla \times (\sigma \mathbf{A}(s)) + \int_0^s \nabla \times \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A} \right) = \int_0^s h \nabla \times \mathbf{f}.$$

Now, we take a scalar product with $\nabla \times \nabla \times (\frac{1}{\mu} \nabla \times \mathbf{A}(s))$, integrate over Ω and make use of the Green theorem to get

$$\begin{aligned} & \left(\nabla \times \nabla \times (\varepsilon \partial_t \mathbf{A}(s)), \nabla \times (\frac{1}{\mu} \nabla \times \mathbf{A}(s)) \right) + \left(\nabla \times \nabla \times (\sigma \mathbf{A}(s)), \nabla \times (\frac{1}{\mu} \nabla \times \mathbf{A}(s)) \right) \\ & + \left(\int_0^s \nabla \times \nabla \times (\frac{1}{\mu} \nabla \times \mathbf{A}), \nabla \times \nabla \times (\frac{1}{\mu} \nabla \times \mathbf{A}(s)) \right) = \left(\int_0^s h \nabla \times \nabla \times \mathbf{f}, \nabla \times (\frac{1}{\mu} \nabla \times \mathbf{A}(s)) \right). \end{aligned}$$

Integrating in time again leads to

$$\begin{aligned} & \left\| \nabla \times \nabla \times \mathbf{A}(t) \right\|^2 + \int_0^t \left\| \nabla \times \nabla \times \mathbf{A}(s) \right\|_{L^2(\Omega_c)}^2 ds + \left\| \int_0^t \nabla \times \nabla \times \mathbf{A} \right\|^2 \\ & \leq C \int_0^t \left\| \int_0^s h \nabla \times \nabla \times \mathbf{f} \right\|^2 ds + C \int_0^t \left\| \nabla \times (\frac{1}{\mu} \nabla \times \mathbf{A}(s)) \right\|^2 ds \\ & \leq C \int_0^t h^2(s) ds + C \int_0^t \left\| \nabla \times \nabla \times \mathbf{A}(s) \right\|^2 ds. \end{aligned}$$

Applying the Grönwall inequality yields

$$\left\| \nabla \times \nabla \times \mathbf{A}(t) \right\|^2 + \left\| \int_0^t \nabla \times \nabla \times \mathbf{A} \right\|^2 \leq C \int_0^t h^2(s) ds, \quad \forall t \in [0, T].$$

4th step: We take the scalar product of equation (4.10) with $\nabla \times \nabla \times (\frac{1}{\mu} \nabla \times \partial_t \mathbf{A}(t))$, integrate over Ω , make use of the Green theorem and integrate in time to get

$$\left\| \nabla \times \nabla \times \partial_t \mathbf{A}(t) \right\|^2 + \int_0^t \left\| \nabla \times \nabla \times \partial_t \mathbf{A}(s) \right\|_{L^2(\Omega_c)}^2 ds + \left\| \nabla \times \nabla \times \mathbf{A}(t) \right\|^2 \leq C \int_0^t h^2(s) ds + C \int_0^t \left\| \nabla \times \nabla \times \partial_t \mathbf{A}(s) \right\|^2 ds.$$

Again, an application of Grönwall's inequality gives

$$\left\| \nabla \times \nabla \times \partial_t \mathbf{A}(t) \right\|^2 + \left\| \nabla \times \nabla \times \mathbf{A}(t) \right\|^2 \leq C \int_0^t h^2(s) ds, \quad \forall t \in [0, T].$$

5th step: We start from (2.3) with $m = 0$ to deduce that

$$\begin{aligned} \int_0^t h^2(s) ds &= \int_0^t \left| \frac{\frac{1}{\mu_c} \int_\gamma \nabla \times \nabla \times \mathbf{A}(s) \cdot \mathbf{n} d\gamma}{\int_\gamma \mathbf{f} \cdot \mathbf{n} d\gamma} \right|^2 ds \\ &\leq C \int_0^t \left(\int_\gamma \left| \nabla \times \nabla \times \mathbf{A}(s) \cdot \mathbf{n} \right|^2 d\gamma \right) ds \\ &\leq C \int_0^t \left\| \nabla \times \nabla \times \mathbf{A}(s) \right\|_{L^2(\gamma)}^2 ds \\ &\stackrel{\gamma \subset \partial G}{\leq} C \int_0^t \left\| \nabla \times \nabla \times \mathbf{A}(s) \right\|_{L^2(\partial G)}^2 ds \\ &\stackrel{H^1(G) \hookrightarrow L^2(\partial G)}{\leq} C \int_0^t \left\| \nabla \times \nabla \times \mathbf{A}(s) \right\|_{H^1(G)}^2 ds \\ &\stackrel{G \subset \Omega}{\leq} C \int_0^t \left\| \nabla \times \nabla \times \nabla \times \mathbf{A}(s) \right\|^2 ds + C \int_0^t \left\| \nabla \times \nabla \times \mathbf{A}(s) \right\|^2 ds. \end{aligned} \tag{4.11}$$

From step 3 and 4 follows for all $t \in [0, T]$ that

$$\int_0^t h^2(s) ds \leq \int_0^t \int_0^s h^2(\xi) d\xi ds.$$

Therefore, by Grönwall's argument, it holds that

$$\int_0^T h^2(s) ds = 0 \quad \text{or} \quad h = 0 \text{ in } L^2(0, T).$$

Moreover, step 1 and 2 imply that

$$\max_{t \in [0, T]} \left\| \mathbf{A}(t) + \nabla \phi(t) \right\|^2 + \max_{t \in [0, T]} \left\| \int_0^t \nabla \times \mathbf{A} \right\|^2 = 0$$

and

$$\max_{t \in [0, T]} \left\| \partial_t (\mathbf{A}(t) + \nabla \phi(t)) \right\|^2 + \max_{t \in [0, T]} \left\| \nabla \times \mathbf{A}(t) \right\|^2 = 0.$$

Then, from Lemma 2.1, it follows that

$$\max_{t \in [0, T]} \left\| \mathbf{A}(t) \right\|_{H^1(\Omega)}^2 + \max_{t \in [0, T]} \|\phi(t)\|_{H^1(\Omega)}^2 = 0,$$

which concludes the proof of uniqueness. $\square$

Corollary 4.1. Let the assumptions of Theorem 4.1 be satisfied. Then, there exists a unique solution $(\mathbf{E}, \mathbf{H}, h)$ to (1.1), (1.2) and (1.11) satisfying $\mathbf{E}, \mathbf{H} \in C([0, T], L^2(\Omega))$ and $h \in L^2(0, T)$.

Remark 4.1. Theorem 3.1 shows the convergence of the Rothe functions for a subsequence. Theorem 3.2 ensures the uniqueness of a solution. Combination of both results implies convergence for the whole sequences of approximation $\mathbf{A}_n$, ϕ_n and h_n .

Remark 4.2. The analysis presented in this paper stays valid when instead one subdomain Ω_c multiple subdomains Ω_{c_i} are considered (where σ , ε and μ are piecewise constant). Moreover, the analysis stays valid when the functions are not piecewise constant. Then, the following vector derivative identities need to be used

$$\begin{aligned} \nabla \times (\phi \mathbf{A}) &= \phi \nabla \times \mathbf{A} + \nabla \phi \times \mathbf{A}, \\ \nabla \times (\mathbf{A} \times \mathbf{B}) &= (\mathbf{B} \cdot \nabla) \mathbf{A} - (\mathbf{A} \cdot \nabla) \mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}). \end{aligned}$$

For instance, the terms $\nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A} \right)$, $\nabla \times \nabla \times (\varepsilon \partial_t \mathbf{A})$ and $\nabla \times \nabla \times (\sigma \partial_t \mathbf{A})$ need to be handled. In that case, the functions σ , ε and μ need to be piecewise twice differentiable.

5. Numerical experiments

In the following, multiple numerical experiments are considered, each in a different setting but with the same exact solution on the whole domain $\Omega = (0, 1)^3$ given by

$$\mathbf{A}_{\text{ex}}(\mathbf{x}, t) = \exp(t) (\sin y, \sin z, \sin x), \quad \phi_{\text{ex}}(\mathbf{x}, t) = -\exp(t) (\cos x + \cos y + \cos z) \quad \text{and} \quad h_{\text{ex}}(t) = \exp(t).$$

Then, the RHS of (1.4) equals

$$h\mathbf{f} = h \begin{pmatrix} (\varepsilon + \sigma) \sin x + \left(\varepsilon + \sigma + \frac{1}{\mu} \right) \sin y \\ (\varepsilon + \sigma) \sin y + \left(\varepsilon + \sigma + \frac{1}{\mu} \right) \sin z \\ (\varepsilon + \sigma) \sin z + \left(\varepsilon + \sigma + \frac{1}{\mu} \right) \sin x \end{pmatrix}.$$

In Experiment 1, it is assumed that

$$\Omega_c = \Omega \text{ with } \gamma = \{(x, y, z) \in \partial\Omega_c = \partial\Omega : x = 0, y \in (0.2, 0.6), z \in (0.4, 0.6)\}, \mu = \sigma = \varepsilon = 1,$$

whilst in Experiment 2 it holds that

$$\Omega_c = \Omega \text{ with } \gamma = \{(x, y, z) \in \Omega_c : x = 0.4, y \in (0.2, 0.4), z \in (0.4, 0.6)\}, \mu = \sigma = \varepsilon = 1,$$

and in the challenging Experiment 3, it is assumed that

$$\Omega_c = (0.4, 0.6)^3 \subset \Omega \text{ with } \gamma = \{(x, y, z) \in \partial\Omega_c : x \in (0.4, 0.6), y = 0.4, z \in (0.4, 0.6)\}, \sigma_c = 1, \mu = \varepsilon = \begin{cases} 1 & \text{in } \overline{\Omega}_c, \\ 0.5 & \text{in } \Omega \setminus \overline{\Omega}_c. \end{cases}$$

Thus in Experiment 1, the measurement is on the surface of the conductor, whilst in Experiment 2, the measurement is on a surface inside the conductor. In Experiment 3, the measurement is again on the surface of the conductor but now the conductor is part of a larger domain and the coefficients are discontinuous. In this latter experiment, the interface conditions need to be handled carefully because the magnetic permeability is discontinuous (but piecewise constant).

First, the variational formulation on Ω_c and Ω_e is multiplied with the corresponding magnetic permeability. Then, the resulting formulations are added together to obtain for all $(\mathbf{Q}, \psi) \in \mathbf{V}$ that

$$\begin{aligned} & (\mu \varepsilon \partial_{tt}(\mathbf{A}(t) + \nabla \phi(t)), \mathbf{Q} + \nabla \psi) + (\mu \sigma \partial_t(\mathbf{A}(t) + \nabla \phi(t)), \mathbf{Q} + \nabla \psi) + (\nabla \times \mathbf{A}(t), \nabla \times \mathbf{Q}) + (\nabla \cdot \mathbf{A}(t), \nabla \cdot \mathbf{Q}) \\ & - ([\mathbf{n}_{12} \times (\nabla \times \mathbf{A}(t))], \mathbf{Q})_{\partial \Omega_c} + ([\nabla \cdot \mathbf{A}(t)], \mathbf{Q} \cdot \mathbf{n}_{12})_{\partial \Omega_c} - ([\nabla \times \nabla \times \mathbf{A}(t) \cdot \mathbf{n}_{12} - \nabla(\nabla \cdot \mathbf{A}(t)) \cdot \mathbf{n}_{12}], \psi)_{\partial \Omega_c} \\ & = (\mu h(t) \mathbf{f}, \mathbf{Q} + \nabla \psi). \end{aligned} \quad (5.1)$$

Finally, thanks to the choice of the exact solution, the interface integrals cancel out.

The exact measurement (1.11) in the different experiments is given by

$$m(t) = 0.03094619258 \exp(t), -0.02737785046 \exp(t) \text{ and } 0.03472180952 \exp(t),$$

respectively. A randomly generated uncorrelated noise with magnitude $\tilde{\varepsilon}$ is added to the exact measurement in order to simulate the inherent errors present in real measurements. The algorithm proposed in this contribution uses first order derivatives of the measurement, which enlarges the noise when using for instance finite difference to approximate the derivatives of noisy data. For this reason, the nonlinear least square method is used to get a sufficiently smooth function $m_{\text{app}, \tilde{\varepsilon}}(t) = \alpha \exp(\beta t)$ approximating the noisy measurement. Thus, the main goal of the experiments is to recover $h(t)$ by means of the measurement $m_{\text{app}, \tilde{\varepsilon}}(t)$ in the different settings.

The solution to the inverse source problems is found by applying the algorithm proposed in Section 2. The problem (1.4) is discretized in time according to the backward Euler method. In each experiment, it is assumed that $T = 1$. The time step for the equidistant time partitioning equals 0.005 in Experiment 1 and 0.0025 in Experiment 2 and 3. At each time step, first h_i is derived by numerical integration (see (2.5)), then the resulting elliptic problem (2.4) is solved numerically by the finite element method using second order (P2–FEM) Lagrange finite elements for the space discretization (for both $\mathbf{A}_i$ and ϕ_i). An uniform mesh for the space domain is created consisting out of 6000 ($= 6 \times 10 \times 10 \times 10$) tetrahedra. For these calculations, the finite element library DOLFIN [28, 29] from the FEniCS project [30, 31] is used. The MUMPS solver is used to solve the linear systems at each timestep in order to speed up the computational process.

The numerical results for different noise levels $\tilde{\varepsilon} = 1\%, 5\%$ and 10% are depicted in Figure 1. In Figures 1(a,c,e), the numerical solution is compared with the exact solution. Moreover, in Figures 1(b,d,f), the absolute error on $\mathbf{E}$ at final time $T = 1$ is presented for each experiment using $\tilde{\varepsilon} = 5\%$. The CPU time (in minutes) and computational errors for the experiments using $\tilde{\varepsilon} = 0\%$ can be found in Table 1. The CPU time is comparable for the similar experiments with noise included. The maximum error on h for all experiments is given in Table 2. It can be seen that, as expected, the error increases when the amount of noise increases. Moreover, it is clear from the figures that an accurate approximation of the source is obtained for each experiment (and each noise level). These experiments illustrate the robustness of the numerical procedure.

Remark 5.1. The use of nodal elements comes from regularizing Equation 1.4 by adding the penalty term. However, the use of nodal finite elements is not state of the art in case of domains with reentrant corners at the boundary (this is excluded by the assumption that the domain is convex) or in case of corners at material interfaces [32]. Also [33] pointed out that application of the linear tetrahedral element to vector problems is problematic. Because of numerical error, the gauge condition is not satisfied exactly. If there exists media having vastly different μ , a small value of $\nabla \cdot \mathbf{A}$ in a medium having a low μ could lead to a very large value of $\nabla \cdot \mathbf{A}$ in a medium having a high μ . As a consequence, it is impossible for $\mathbf{A}$ to satisfy the original problem reasonably, particularly in a medium having a high μ . In all experiments quadratic tetrahedral elements are used (these elements are also necessary for the calculation of h at each timestep) and moreover the variational formulation (5.1) is used to reduce this problem. This latter remark is tested in Experiment 4 wherein Experiment 3 is repeated with higher contrast in the parameters, i.e $\sigma_c = \mu_c = \varepsilon_c = 1000$. The results are depicted in Figure 2. The absolute error on $\mathbf{A}$ and $\mathbf{B}$ at final time is good but the absolute error on $\mathbf{E}$ is less good in comparison with the previous experiments, which is clearly visible at the corners of the conductor. Nevertheless, the results obtained for the recovery of h are fine, which is the main aim of this paper.

6. Conclusion

In this paper, an inverse source problem associated with full Maxwell's equations was considered. More precisely, the theoretical and numerical determination of a time-dependent source in a hyperbolic partial differential equation in terms of the electric field and magnetic field was studied from the knowledge of a surface measurement. An algorithm based on the $\mathbf{A}$ - ϕ method and on Rothe's method was proposed for the recovery of the unknown source. The numerical experiments carried out herein were implemented using the FEM and validated the numerical procedure. The analysis

Figure 1. The exact source $h(t) = \exp(t)$ and its numerical approximations in Experiment 1, 2 and 3 using various levels of noise (a,c,e) and the error $E - E_{\text{ex}}$ (and its absolute value) at final time $T = 1$ for each experiment when $\tilde{\epsilon} = 5\%$ (b,d,f). Only 50 time points are used to plot the numerical approximations.

Table 1. The CPU time (in minutes), the error $e(\mathbf{A}) = \sum_{i=1}^n \|\mathbf{A}_i - \mathbf{A}_{\text{ex}}(t_i)\|^2 \tau$, the error $e(\phi) = \sum_{i=1}^n \|\phi_i - \phi_{\text{ex}}(t_i)\|^2 \tau$, the error $e(\mathbf{E}) = \sum_{i=1}^n \|\mathbf{E}_i - \mathbf{E}_{\text{ex}}(t_i)\|^2 \tau$, and the error $e(h) = \max_{1 \leq i \leq n} |h_i - h_{\text{ex}}(t_i)|$ obtained for the different numerical experiments (without noise).

	CPU time	$e(\mathbf{A})$	$e(\phi)$	$e(\mathbf{E})$	$e(h)$
Experiment 1	91	1.162665e-06	3.720334e-08	1.038982e-04	1.211658e-02
Experiment 2	182	5.734231e-07	1.812047e-08	1.271689e-05	1.649774e-02
Experiment 3	192	3.000904e-08	9.573042e-09	2.687462e-05	2.184774e-02

Table 2. The error $e(h) = \max_{1 \leq i \leq n} |h_i - h_{\text{ex}}(t_i)|$ obtained for the different numerical experiments for various noise levels.

$e(h)$	$\tilde{\epsilon} = 1\%$	$\tilde{\epsilon} = 5\%$	$\tilde{\epsilon} = 10\%$
Experiment 1	2.263878e-05	2.669173e-02	3.786704e-02
Experiment 2	2.173975e-02	2.540045e-02	6.289425e-02
Experiment 3	2.779261e-02	3.292474e-02	7.828984e-02

Figure 2. The exact source $h(t) = \exp(t)$ and its numerical approximations in Experiment 4 using various levels of noise (a) and the error $\mathbf{A} - \mathbf{A}_{\text{ex}}$, $\mathbf{E} - \mathbf{E}_{\text{ex}}$ and $\mathbf{B} - \mathbf{B}_{\text{ex}}$ (and its absolute value) at final time $T = 1$ for the experiment when $\tilde{\epsilon} = 5\%$ (b,c,d). Only 50 time points are used to plot the numerical approximations.

134 stays valid in two dimensions. Future work can be concerned with the derivation of the error estimates for the time and
135 space discretization and with the improvement of the nodal FEM used in this paper.

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