A new construction of the Clifford-Fourier kernel

Denis Constaes^{1 *} Hendrik De Bie^{1†} Pan Lian^{1,2‡}

1: Department of Mathematical Analysis
Faculty of Engineering and Architecture – Ghent University
Galglaan 2, 9000 Gent, Belgium

2: Department of Mathematics - Harbin Institute of Technology
West Da-Zhi Street 92, 150001 Harbin, P.R.China

Abstract

In this paper, we develop a new method based on the Laplace transform to study the Clifford-Fourier transform. First, the kernel of the Clifford-Fourier transform in the Laplace domain is obtained. When the dimension is even, the inverse Laplace transform may be computed and we obtain the explicit expression for the kernel as a finite sum of Bessel functions. We equally obtain the plane wave decomposition and find new integral representations for the kernel in all dimensions. Finally we define and compute the formal generating function for the even dimensional kernels.

Keywords: Clifford-Fourier transform, Laplace transform, Bessel function, Plane wave decomposition

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*E-mail: denis.constaes@ugent.be

†E-mail: hendrik.debie@ugent.be

‡E-mail: pan.lian@ugent.be

1 Introduction

34 In recent years, quite a bit of attention has been paid to the Clifford-Fourier transform

$$F_-(f)(y) = \int_{\mathbb{R}^m} K_m(x, y) f(x) dx$$

35 with kernel given by

$$K_m(x, y) = e^{i \frac{\pi}{2} \Gamma_y} e^{-i(x, y)}$$

36 where

$$\Gamma_y := - \sum_{j < k} e_j e_k (y_j \partial_{y_k} - y_k \partial_{y_j}).$$

37 Here (x, y) is the standard Euclidean inner product on $\mathbb{R}^m$ and the e_j generate a Clifford algebra.

38 This transform was introduced in [2] as a generalization of the classical Fourier transform (FT) for
39 multichannel signals. Because of the underlying Clifford algebra structure and its geometric interpre-
40 tation, a transform is obtained that properly mixes the signals to be investigated. It nevertheless still
41 satisfies many important properties of the classical FT, such as inversion, Plancherel theorem, behaviour
42 of differentiation and (generalized) translation, etc.

43 It turned out to be a hard problem to compute its integral kernel K_m explicitly. This was first done
44 for $m = 2$ in [3]. In [4] a recursive algorithm was presented to compute the kernel in even dimension.
45 A completely explicit expression for all even dimensions was subsequently obtained in [9] by using plane
46 wave decompositions.

47 In later work, the results were extended to fractional versions of the Clifford-Fourier transform [5, 7]
48 and integral kernels satisfying certain generalized Helmholtz PDEs in Clifford analysis [6]. In [8] an
49 approach using Lie superalgebras and group symmetries led to a complete classification of transforms
50 that behave in the same way as both the Clifford-Fourier transform and the classical Fourier transform.
51 Uncertainty principles for these transforms were obtained in [12, 8].

52 The main aim of the present paper is to develop a new and elegant method to compute the integral
53 kernel K_m . This will be done by introducing an auxiliary variable t and subsequently expressing its
54 Laplace transform

$$\mathcal{L}(t^{m/2-1} e^{-it(x, y)})$$

55 in terms of the Cauchy kernel for the Dirac operator. In the Laplace domain, the action of Γ_y is obtained
56 using a monogenic expansion. Laplace inversion then yields our main result in the even dimensional case.

57 Our approach bypasses the need for a rather complicated induction argument in [9]. As an additional
58 bonus, we are now able to compute an explicit generating function for all even dimensional kernels by
59 again using the Laplace domain expression. This is achieved in Theorem 7 and 8.

60 The paper is organized as follows. In Section 2 we recall basic facts concerning the Laplace transform,
61 Clifford analysis and the Clifford-Fourier transform. In Section 3 we first compute the Laplace domain
62 expression for the fractional Clifford-Fourier kernel. We use this result to reobtain both the plane wave
63 decomposition and the explicit expression through Laplace inversion. Finally we derive a new integral
64 identity for the kernel and we construct the generating function.

2 Preliminaries

2.1 The Laplace transform

67 Suppose that f is a real or complex valued function of the variable $t > 0$ and s is a complex parameter.
68 The Laplace transform of f which has exponential order α , i.e. $|f(t)| \leq C e^{\alpha t}$, $t \geq t_0$ is defined as

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t) dt.$$

By Lerch's theorem [14], if we restrict our attention to functions which are continuous on $[0, \infty)$, then the inverse transform

$$\mathcal{L}^{-1}(F(s)) = f(t)$$

is uniquely defined. We need the following Laplace transform formulas in this paper ([10]):

$$\begin{aligned}\mathcal{L}(t^{k-1}) &= \frac{\Gamma(k)}{s^k}, \quad k > 0, \quad \operatorname{Re}(s) > 0; \\ \mathcal{L}(e^{-\alpha t}) &= \frac{1}{s + \alpha}, \quad \operatorname{Re}(s) > -\operatorname{Re}(\alpha); \end{aligned} \quad (1)$$

$$\mathcal{L}(t^{k-1}e^{-\alpha t}) = \frac{\Gamma(k)}{(s + \alpha)^k}, \quad k > 0, \quad \operatorname{Re}(s) > -\operatorname{Re}(\alpha); \quad (2)$$

$$\mathcal{L}(\cos at) = \frac{s}{s^2 + a^2}, \quad \operatorname{Re}(s) > |\operatorname{Im}(a)|; \quad (3)$$

$$\mathcal{L}(\sin at) = \frac{a}{s^2 + a^2}, \quad \operatorname{Re}(s) > |\operatorname{Im}(a)|; \quad (4)$$

We also need some inverse Laplace transforms, with $r = (s^2 + a^2)^{1/2}$, $R = s + r$ and $g(s) = \mathcal{L}(f(t))$. We have

$$\mathcal{L}^{-1}\left(\frac{1}{r}\left(\frac{a}{R}\right)^\nu\right) = J_\nu(at), \quad \operatorname{Re}(\nu) > -1, \operatorname{Re}(s) > |\operatorname{Im}(a)|; \quad (5)$$

$$\mathcal{L}^{-1}(2^\nu \pi^{-1/2} \Gamma(\nu + \frac{1}{2}) a^\nu r^{-2\nu-1}) = t^\nu J_\nu(at), \quad \operatorname{Re}(\nu) > -1/2, \operatorname{Re}(s) > |\operatorname{Im}(a)|; \quad (6)$$

$$\mathcal{L}^{-1}(2^{\nu+1} \pi^{-1/2} \Gamma(\nu + 3/2) a^\nu r^{-2\nu-3} s) = t^{\nu+1} J_\nu(at), \quad \operatorname{Re}(\nu) > -1, \operatorname{Re}(s) > |\operatorname{Im}(a)|; \quad (7)$$

$$\mathcal{L}^{-1}(r^{-1}g(r)) = \int_0^t J_0[a(t^2 - u^2)^{1/2}] f(u) du; \quad (8)$$

$$\mathcal{L}^{-1}(g(r)) = f(t) - a \int_0^t f[(t^2 - u^2)^{1/2}] J_1(au) du; \quad (9)$$

$$\mathcal{L}^{-1}(sr^{-1}g(r)) = f(t) - at \int_0^t (t^2 - u^2)^{-1/2} J_1[a(t^2 - u^2)^{1/2}] f(u) du. \quad (10)$$

The partial fraction decomposition of a rational polynomial

$$F(s) = \frac{\sum_{k=0}^m a_k s^k}{\sum_{j=0}^n b_j s^j} = \frac{A(s)}{B(s)}, \quad (n > m)$$

expresses $F(s)$ as a sum of fractions with simple denominator. We only show the case when $F(s)$ has a single pole of order m . Then $F(s)$ can be expressed as

$$F(s) = \frac{A(s)}{(s-p)^m} = \sum_{k=1}^m \frac{c_k}{(s-p)^k},$$

with complex constants $c_{m-k} = \frac{1}{k!} \frac{d^{m-k}}{ds^{m-k}} [F(s)(s-p)^m]_{s=p}$, $k = 1, \dots, m$.

The Laplace transform of a matrix valued function is simply the matrix of Laplace transforms of the individual elements. For example

$$\mathcal{L} \begin{pmatrix} e^t \\ te^{-t} \end{pmatrix} = \begin{pmatrix} 1/(s-1) \\ 1/(s+1)^2 \end{pmatrix}.$$

Suppose A is an $n \times n$ matrix. The matrix exponential is interpreted in terms of a power series, namely

$$\exp(At) = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}.$$

By analogy with the scalar case, we have

$$\mathcal{L}(e^{At}) = (sI - A)^{-1}.$$

For more about the Laplace transform of matrix-valued functions, see [16].

2.2 Clifford analysis

Let $\mathbb{R}^m$ be the usual m -dimensional Euclidean space with an orthonormal basis $\{e_1, e_2, \dots, e_m\}$. The Clifford algebra $\mathcal{C}\ell_{0,m}$ associated with $\mathbb{R}^m$ is spanned by the reduced products

$$\bigcup_{j=1}^m \{e_\alpha = e_{i_1} e_{i_2} \dots e_{i_j} : \alpha = \{i_1, i_2, \dots, i_j\}, \quad 1 \leq i_1 < i_2 < \dots < i_j \leq m\}$$

with the relations $e_i e_j + e_j e_i = -2\delta_{ij}$. We have $\mathcal{C}\ell_{0,m} = \{\sum_\alpha e_\alpha x_\alpha; x_\alpha \in \mathbb{R}\}$. The Clifford algebra $\mathcal{C}\ell_{0,m}$ is a graded algebra as $\mathcal{C}\ell_{0,m} = \oplus_l \mathcal{C}\ell_{0,m}^l$ where $\mathcal{C}\ell_{0,m}^l$ is spanned by reduced Clifford products of length l . The Clifford algebra $\mathcal{C}\ell_{0,m}$ is a 2^m -dimensional real associative algebra with identity and contains a copy of $\mathbb{R}^m$ by the canonical mapping $x = \sum_{j=1}^m x_j e_j$. For any $x, y \in \mathcal{C}\ell_{0,m}$, the conjugation is defined by $\overline{(e_{j_1} \dots e_{j_l})} = (-1)^l e_{j_l} \dots e_{j_1}$ as a linear mapping. We have $\overline{(xy)} = \overline{y} \overline{x}$, $\overline{\overline{x}} = x$. Now we can define the inner product and the wedge product of two vectors $x, y \in \mathbb{R}^m$ using the Clifford product:

$$(x, y) := \sum_{j=1}^m x_j y_j = -\frac{1}{2}(xy + yx);$$

$$x \wedge y := \sum_{j < k} e_j e_k (x_j y_k - x_k y_j) = \frac{1}{2}(xy - yx).$$

It is easy to get $xy = -(x, y) + x \wedge y$, $(x \wedge y)^2 = -|x|^2 |y|^2 + (x, y)^2$ and $|x|^2 = x \overline{x} = -x^2$ by definition (see also [9]). Because $\frac{(x \wedge y)^2}{|x \wedge y|^2} = -1$, we can consider $\frac{x \wedge y}{|x \wedge y|}$ as an imaginary unit. Now we can Laplace transform $e^{-(x \wedge y)t}$ and get

$$\mathcal{L}(e^{-(x \wedge y)t}) = \frac{1}{s + x \wedge y}.$$

The complexified Clifford algebra $\mathcal{C}\ell_{0,m}^{\mathbb{C}}$ is defined as $\mathbb{C} \otimes \mathcal{C}\ell_{0,m}$.

The Dirac operator is defined as:

$$D = \sum_{j=1}^m e_j \partial_{x_j}.$$

When u is a scalar C^1 function, Du can be identified with the gradient ∇u . A function is called monogenic if $Du = 0$. An important example of a monogenic function is the generalized Cauchy kernel [1]

$$G(x) = \frac{1}{\omega_m} \frac{\overline{x}}{|x|^m}$$

where ω_m is the surface area of the unit ball in $\mathbb{R}^m$. It is the fundamental solution of the Dirac operator.

Denote by $\mathcal{P}$ the space of polynomials taking values in $\mathcal{C}\ell_{0,m}$, i.e. $\mathcal{P} := \mathbb{R}[x_1, \dots, x_m] \otimes \mathcal{C}\ell_{0,m}$. The space of homogeneous polynomials of degree k is then denoted by $\mathcal{P}_k$. The space $\mathcal{M}_k := (\ker D) \cap \mathcal{P}_k$, is called the space of spherical monogenics of degree k .

The local behaviour of a monogenic function near a point can be investigated by the polynomials introduced above. The following theorem is the analogue of the Taylor series in complex analysis.

Theorem 1. [1] Suppose f is monogenic in an open set Ω containing the origin. Then there exists an open neighbourhood Λ of the origin in which f can be developed into a normally convergent series of spherical monogenics $M_k f(x)$, i.e.

$$f(x) = \sum_{k=0}^{\infty} M_k f(x),$$

with $M_k f(x) \in \mathcal{M}_k$.

We further introduce the Gamma operator (see [1])

$$\Gamma_x := - \sum_{j < k} e_j e_k (x_j \partial_{x_k} - x_k \partial_{x_j}) = -x D_x - E_x.$$

Here $E_x = \sum_{i=1}^m x_i \partial_{x_i}$ is the Euler operator. Note that Γ_x commutes with scalar radial functions. The operator Γ_x has two important eigenspaces:

$$\Gamma_x \mathcal{M}_k = -k \mathcal{M}_k, \quad (11)$$

$$\Gamma_x (x \mathcal{M}_{k-1}) = (k + m - 2) x \mathcal{M}_{k-1} \quad (12)$$

which follows from the definition of Γ_x .

2.3 The Clifford-Fourier transform

The classical Fourier transform

$$\mathcal{F}(f)(y) = (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{-i(x,y)} f(x) dx,$$

with (x, y) the inner product on $\mathbb{R}^m$, can be represented by the operator exponential

$$\mathcal{F} = e^{\frac{i\pi m}{4}} e^{\frac{i\pi}{4}(\Delta - |x|^2)},$$

see e.g. [13], [11]. Brackx, De Schepper and Sommen introduced a pair of Fourier transforms using the angular Dirac operator Γ_x in the Clifford algebra setting in [2]. More precisely, it is defined by

$$\mathcal{F}_{\pm} = e^{\frac{i\pi m}{4}} e^{\mp \frac{i\pi}{2} \Gamma_x} e^{\frac{i\pi}{4}(\Delta - |x|^2)}.$$

For $\mathcal{F}_-$, we denote the kernel as

$$K_m(x, y) = e^{i \frac{\pi}{2} \Gamma_y} e^{-i(x,y)}.$$

In general, it is not easy to compute this kernel explicitly. In [9], the authors derived the kernel for even dimensions as a finite sum of Bessel functions. Later in [7], the fractional Clifford-Fourier transform was introduced as a generalization of the fractional Fourier transform

$$\mathcal{F}_{\alpha, \beta} = e^{\frac{i\alpha m}{2}} e^{i\beta \Gamma_x} e^{\frac{i\alpha}{2}(\Delta - |x|^2)}$$

and the kernels of even dimensions were obtained by a similar method. In [5], a new construction of the fractional Clifford-Fourier kernels was given by solving wave-type problems. In the present paper the fractional Clifford-Fourier kernel is computed as

$$K_m^p(x, y) = e^{ip \Gamma_y} e^{-i(x,y)}.$$

The more general case can also be obtained using our method.

3 Laplace transform method

3.1 The fractional Clifford-Fourier kernel in the Laplace domain

In this section we introduce an auxiliary variable t in the exponent of the classical Fourier transform and then use the Laplace transform to get the Clifford-Fourier kernel in the Laplace domain.

131 We use the notation $\sqrt{-} := \sqrt{s^2 - |x|^2|y|^2}$. By direct computation, we have $(s + \sqrt{-})(s - \sqrt{-}) =$
 132 $|x|^2|y|^2$ and

$$\begin{aligned}
 \left(1 + \frac{yx}{s + \sqrt{-}}\right) \overline{\left(1 + \frac{yx}{s + \sqrt{-}}\right)} &= \left(1 + \frac{yx}{s + \sqrt{-}}\right) \left(1 + \frac{xy}{s + \sqrt{-}}\right) \\
 &= 1 + \frac{yx + xy}{s + \sqrt{-}} + \frac{|x|^2|y|^2}{(s + \sqrt{-})^2} \\
 &= 1 - \frac{2(x, y)}{s + \sqrt{-}} + \frac{(s + \sqrt{-})(s - \sqrt{-})}{(s + \sqrt{-})^2} \\
 &= \frac{2(s - (x, y))}{s + \sqrt{-}}.
 \end{aligned} \tag{13}$$

133 Then using (13), we can express $\mathcal{L}(t^{m/2-1}e^{t(x,y)})$ in terms of the generalized Cauchy kernel introduced
 134 in the previous section. Fixing $x, y \in \mathbb{R}^m$, assuming $\text{Re}(s) > |x||y|$, we may compute

$$\begin{aligned}
 \mathcal{L}(t^{m/2-1}e^{t(x,y)}) &= \frac{\Gamma(m/2)}{(s - (x, y))^{m/2}} \\
 &= \frac{\Gamma(m/2)}{\left(\frac{s + \sqrt{-}}{2}\right)^{m/2} \left(\left(1 + \frac{yx}{s + \sqrt{-}}\right) \overline{\left(1 + \frac{yx}{s + \sqrt{-}}\right)}\right)^{m/2}} \\
 &= \frac{\Gamma(m/2)}{\left(\frac{s + \sqrt{-}}{2}\right)^{m/2} \left(\left(1 + \frac{yx}{s + \sqrt{-}}\right) \left(1 + \frac{xy}{s + \sqrt{-}}\right)\right)^{m/2}} \\
 &= \frac{\Gamma(m/2)}{\left(\frac{s + \sqrt{-}}{2}\right)^{m/2} \left(\left(1 + \frac{yx}{s + \sqrt{-}}\right) \left(1 + \frac{xy}{s + \sqrt{-}}\right)\right)^{m/2}} \\
 &= \frac{\Gamma(m/2)}{\left(\frac{s + \sqrt{-}}{2}\right)^{m/2} \left(\left(1 + \frac{yx}{s + \sqrt{-}}\right) \left(1 + \frac{xy}{s + \sqrt{-}}\right)\right)^{m/2}} \frac{1 + \frac{yx}{s + \sqrt{-}} - \frac{y(1 + \frac{yx}{s + \sqrt{-}})x}{s + \sqrt{-}}}{\frac{2\sqrt{-}}{s + \sqrt{-}}} \\
 &= \frac{2^{m/2-1}\Gamma(m/2)}{\sqrt{-}(s + \sqrt{-})^{m/2-1}} \frac{1 + \frac{yx}{s + \sqrt{-}} - \frac{y(1 + \frac{yx}{s + \sqrt{-}})x}{s + \sqrt{-}}}{\left(\left(1 + \frac{yx}{s + \sqrt{-}}\right) \left(1 + \frac{xy}{s + \sqrt{-}}\right)\right)^{m/2}}.
 \end{aligned} \tag{14}$$

135 The first equality is by (2), the second equality by (13), and the third equality follows by

$$\begin{aligned}
 1 + \frac{yx}{s + \sqrt{-}} - \frac{y(1 + \frac{yx}{s + \sqrt{-}})x}{s + \sqrt{-}} &= 1 + \frac{yx}{s + \sqrt{-}} - \frac{yx}{s + \sqrt{-}} - \frac{yyxx}{(s + \sqrt{-})^2} \\
 &= 1 - \frac{(s + \sqrt{-})(s - \sqrt{-})}{(s + \sqrt{-})^2} \\
 &= \frac{2\sqrt{-}}{s + \sqrt{-}}.
 \end{aligned}$$

136 Next we will compute $\mathcal{L}(t^{m/2-1}e^{ip\Gamma_y}e^{t(x,y)})$ by acting with $e^{ip\Gamma_y}$ on both sides of (14). The generalized
 137 Cauchy kernel $G(y) = \frac{1}{\omega_m} \frac{\bar{y}}{|y|^m}$ is a monogenic function except at the origin. By translation, $\frac{y+x}{|y+x|^m}$ is
 138 monogenic in y except at $-x$, and $\frac{y+\frac{1}{x}}{|y+\frac{1}{x}|^m} \frac{x}{|x|^m}$ is monogenic in y except at $-x^{-1}$. The latter function
 139 equals $\frac{yx+1}{((yx+1)(yx+1))^{m/2}}$. Using Theorem 1, for fixed x , we can express $\frac{yx+1}{((yx+1)(yx+1))^{m/2}}$ as a series of
 140 spherical monogenic polynomials, i.e.

$$\frac{yx+1}{((yx+1)(yx+1))^{m/2}} = M_0(y) + M_1(y) + M_2(y) + \cdots$$

141 where $M_k(y)$ is a spherical monogenic of order k . This series converges uniformly and absolutely for any
 142 compact set in $|y| < \frac{1}{|x|}$ (see [1]). Substituting $\frac{y}{s+\sqrt{-}}$ for y , we have

$$\frac{1 + \frac{yx}{s + \sqrt{-}}}{[(1 + \frac{yx}{s + \sqrt{-}})(1 + \frac{xy}{s + \sqrt{-}})]^{m/2}} = \frac{M_0(y)}{(s + \sqrt{-})^0} + \frac{M_1(y)}{(s + \sqrt{-})^1} + \dots,$$

143 which converges uniformly and absolutely for any compact set when $\text{Re}(s) > |y||x|$. Using (11), we obtain
 144 when $\text{Re}(s) > |y||x|$,

$$\begin{aligned} & \Gamma_y \left(\frac{1 + \frac{yx}{s + \sqrt{-}}}{[(1 + \frac{yx}{s + \sqrt{-}})(1 + \frac{xy}{s + \sqrt{-}})]^{m/2}} \right) \\ &= \Gamma_y \left(\sum_{k=0}^{\infty} \frac{M_k(y)}{(s + \sqrt{-})^k} \right) \\ &= \sum_{k=0}^{\infty} \frac{\Gamma_y M_k(y)}{(s + \sqrt{-})^k} \\ &= \sum_{k=0}^{\infty} \frac{(-k) \cdot M_k(y)}{(s + \sqrt{-})^k} \end{aligned}$$

145 because the denominators in the series expansion are radial functions in y and hence commute with Γ_y .
 146 This leads to

$$\begin{aligned} & e^{ip\Gamma_y} \left(\frac{1 + \frac{yx}{s + \sqrt{-}}}{[(1 + \frac{yx}{s + \sqrt{-}})(1 + \frac{xy}{s + \sqrt{-}})]^{m/2}} \right) \\ &= \frac{e^{ip \cdot 0} M_0(y)}{(s + \sqrt{-})^0} + \frac{e^{ip \cdot (-1)} M_1(y)}{(s + \sqrt{-})^1} + \frac{e^{ip \cdot (-2)} M_2(y)}{(s + \sqrt{-})^2} + \dots \\ &= M_0 \left(\frac{e^{-ip} y}{(s + \sqrt{-})} \right) + M_1 \left(\frac{e^{-ip} y}{(s + \sqrt{-})} \right) + M_2 \left(\frac{e^{-ip} y}{(s + \sqrt{-})} \right) + \dots \\ &= \frac{1 + \frac{e^{-ip} yx}{s + \sqrt{-}}}{[(1 + \frac{e^{-ip} yx}{s + \sqrt{-}})(1 + \frac{e^{-ip} xy}{s + \sqrt{-}})]^{m/2}}. \end{aligned}$$

147 Similarly, by (12),

$$e^{ip\Gamma_y}(yM_k(y)) = e^{i(m-2)p+i(k+1)p}(yM_k(y)) = e^{i(m-2)p}(ye^{ip}M_k(e^{ip}y)).$$

148 Now we can get the desired result

$$\begin{aligned} & \mathcal{L}(t^{m/2-1} e^{ip\Gamma_y} e^{t(x,y)}) \\ &= \frac{2^{m/2-1}(m/2-1)!}{\sqrt{-}(s + \sqrt{-})^{m/2-1}} \left(\frac{1 + \frac{e^{-ip} yx}{s + \sqrt{-}}}{[(1 + \frac{e^{-ip} yx}{s + \sqrt{-}})(1 + \frac{e^{-ip} xy}{s + \sqrt{-}})]^{m/2}} - e^{i(m-2)p} \frac{\frac{e^{ip} y(1 + \frac{e^{ip} yx}{s + \sqrt{-}})x}{s + \sqrt{-}}}{[(1 + \frac{e^{ip} yx}{s + \sqrt{-}})(1 + \frac{e^{ip} xy}{s + \sqrt{-}})]^{m/2}} \right). \end{aligned}$$

149 In order to simplify the expression further, we need the following,

$$\begin{aligned}
& [(1 + \frac{e^{-ip}yx}{s + \sqrt{-}})(1 + \frac{e^{-ip}xy}{s + \sqrt{-}})]^{m/2} \\
&= [1 - \frac{2e^{-ip}(x, y)}{s + \sqrt{-}} + \frac{e^{-2ip}yxx(s - \sqrt{-})}{(s + \sqrt{-})^2(s - \sqrt{-})}]^{m/2} \\
&= [\frac{2e^{-ip}}{s + \sqrt{-}}(1/2e^{ip}(s + \sqrt{-}) - (x, y) + 1/2e^{-ip}(s - \sqrt{-}))]^{m/2} \\
&= [\frac{2e^{-ip}}{s + \sqrt{-}}(s \cos p - (x, y) + i\sqrt{-} \sin p)]^{m/2}, \tag{15}
\end{aligned}$$

150 as well as

$$1 + \frac{e^{-ip}yx}{s + \sqrt{-}} = \frac{s + \sqrt{-} + e^{-ip}yx}{s + \sqrt{-}} \tag{16}$$

151 and

$$\frac{y(1 + \frac{e^{ip}yx}{s + \sqrt{-}})x}{s + \sqrt{-}} = \frac{yx + e^{ip}(s - \sqrt{-})}{s + \sqrt{-}} = \frac{e^{ip}(e^{-ip}yx + s - \sqrt{-})}{s + \sqrt{-}}. \tag{17}$$

152 Combining (15), (16) and (17), we get the following theorem.

qq13 **Theorem 2.** For $x, y \in \mathbb{R}^m$, the Laplace transform of the fractional Clifford-Fourier kernel is given by:

$$\begin{aligned}
\mathcal{L}(t^{m/2-1}e^{ip\Gamma_y}e^{t(x,y)}) &= \frac{\Gamma(m/2)}{2\sqrt{-}} \left(\frac{s + \sqrt{-} + e^{-ip}yx}{(e^{-ip}(s \cos p + i\sqrt{-} \sin p - (x, y)))^{m/2}} \right. \\
&\quad \left. - e^{imp} \frac{s - \sqrt{-} + e^{-ip}yx}{(e^{ip}(s \cos p - i\sqrt{-} \sin p - (x, y)))^{m/2}} \right),
\end{aligned}$$

154 when $\text{Re}(s) > |y||x|$ and here $\sqrt{-} = \sqrt{s^2 - |x|^2|y|^2}$.

155 If we redo the computation with x replaced by $-ix$, similar results can be obtained except that the
156 existence condition $\text{Re}(s) > |x||y|$ is relaxed to $\text{Re}(s) > 0$. The result is collected in the following theorem.

qq15 **Theorem 3.** For $x, y \in \mathbb{R}^m$, the Laplace transform of the fractional Clifford-Fourier kernel is given by:

$$\begin{aligned}
\mathcal{L}(t^{m/2-1}e^{ip\Gamma_y}e^{-it(x,y)}) &= \frac{\Gamma(m/2)}{2\sqrt{+}} \left(\frac{s + \sqrt{+} - ie^{-ip}yx}{(e^{-ip}(s \cos p + i\sqrt{+} \sin p + i(x, y)))^{m/2}} \right. \\
&\quad \left. - e^{imp} \frac{s - \sqrt{+} - ie^{-ip}yx}{(e^{ip}(s \cos p - i\sqrt{+} \sin p + i(x, y)))^{m/2}} \right),
\end{aligned}$$

158 when $\text{Re}(s) > 0$ and with $\sqrt{+} = \sqrt{s^2 + |x|^2|y|^2}$.

159 3.2 Plane wave decomposition of the fractional Clifford-Fourier kernel

160 In this subsection, for $x, y \in \mathbb{R}^m$, we use the notation $\hat{x} = \frac{x}{|x|}$, $\hat{y} = \frac{y}{|y|}$ to denote two unit vectors. For $\hat{x}$,
161 $\hat{y}$ we also have the result in Theorem 3. This time, we could get the kernel by putting $t = |x||y|$. Denote
162 $r = \sqrt{s^2 + 1}$, $R = s + \sqrt{s^2 + 1}$, and $(\hat{x}, \hat{y}) = \cos \theta$. Using $s = \frac{R-1/R}{2}$ and $\sqrt{s^2 + 1} = \frac{R+1/R}{2}$, Theorem 3

163 becomes

$$\begin{aligned}
& \mathcal{L}(t^{m/2-1} e^{ip\Gamma_y} e^{-it(\hat{x}, \hat{y})}) \\
&= \frac{\Gamma(m/2)}{r} 2^{m/2-1} R^{-m/2} \left(\frac{R - ie^{-ip} \hat{y} \hat{x}}{(1 + 2 \frac{ie^{-ip}}{R} \cos \theta - (\frac{e^{-ip}}{R})^2)^{m/2}} + e^{imp} \frac{\frac{1}{R} + ie^{-ip} \hat{y} \hat{x}}{(1 + 2 \frac{ie^{ip}}{R} \cos \theta - (\frac{e^{ip}}{R})^2)^{m/2}} \right) \\
&= \frac{\Gamma(m/2)}{r} 2^{m/2-1} R^{-m/2} \\
&\quad \times \left(\frac{1}{(1 + 2 \frac{ie^{-ip}}{R} \cos \theta - (\frac{e^{-ip}}{R})^2)^{m/2}} [ie^{-ip}(-\cos \theta + \frac{-ie^{-ip}}{R}) + (R + 2ie^{-ip} \cos \theta - \frac{e^{-2ip}}{R}) + ie^{-ip} \hat{x} \wedge \hat{y}] \right. \\
&\quad \left. + e^{imp} \frac{\frac{1}{R} + ie^{-ip}(-\cos \theta + \hat{y} \wedge \hat{x})}{(1 + 2 \frac{ie^{ip}}{R} \cos \theta - (\frac{e^{ip}}{R})^2)^{m/2}} \right) \tag{18}
\end{aligned}$$

164 With the help of the generating function of the Gegenbauer polynomial [15]

$$\frac{1}{(1 - 2xt + t^2)^\lambda} = \sum_{k=0}^{\infty} C_k^{(\lambda)}(x) t^k, \tag{19}$$

165 and its derivative with respect to t

$$-\lambda \frac{-2x + 2t}{(1 - 2xt + t^2)^{\lambda+1}} = \sum_{k=0}^{\infty} k C_k^{(\lambda)}(x) t^{k-1}, \tag{20}$$

166 we could express the right hand side of (18) into series. We have by (20)

$$\begin{aligned}
& \frac{ie^{-ip}(-\cos \theta + \frac{-ie^{-ip}}{R})}{(1 + 2 \frac{ie^{-ip}}{R} \cos \theta - (\frac{e^{-ip}}{R})^2)^{m/2}} \\
&= \frac{-ie^{-ip}}{(m-2)} \sum_{k=0}^{\infty} k C_k^{(\frac{m}{2}-1)}(\cos \theta) \left(\frac{-ie^{-ip}}{R} \right)^k,
\end{aligned}$$

167 and by (19)

$$\begin{aligned}
& \frac{R + 2ie^{-ip} \cos \theta - \frac{e^{-2ip}}{R}}{(1 + 2 \frac{ie^{-ip}}{R} \cos \theta - (\frac{e^{-ip}}{R})^2)^{m/2}} \\
&= R \sum_{k=0}^{\infty} C_k^{(\frac{m}{2}-1)}(\cos \theta) \left(\frac{-ie^{-ip}}{R} \right)^k,
\end{aligned}$$

168 as well as by (19)

$$\begin{aligned}
& \frac{ie^{-ip} \hat{x} \wedge \hat{y}}{(1 + 2 \frac{ie^{-ip}}{R} \cos \theta - (\frac{e^{-ip}}{R})^2)^{m/2}} \\
&= ie^{-ip} \hat{x} \wedge \hat{y} \sum_{k=0}^{\infty} C_k^{(\frac{m}{2})}(\cos \theta) \left(\frac{-ie^{-ip}}{R} \right)^k.
\end{aligned}$$

169 Similarly, we can get the series expression of the remaining part of (18),

$$\begin{aligned}
& e^{imp} \frac{\frac{1}{R} + ie^{-ip}(-\cos \theta + \hat{y} \wedge \hat{x})}{(1 + 2 \frac{ie^{ip}}{R} \cos \theta - (\frac{e^{ip}}{R})^2)^{m/2}} \\
&= ie^{i(m-1)p} \left(\hat{y} \wedge \hat{x} \sum_{k=0}^{\infty} C_k^{(\frac{m}{2})}(\cos \theta) \left(\frac{-ie^{ip}}{R} \right)^k - \frac{1}{m-2} \sum_{k=0}^{\infty} k C_k^{(\frac{m}{2}-1)}(\cos \theta) \left(\frac{-ie^{ip}}{R} \right)^k \right).
\end{aligned}$$

Collecting all we got, we have the series expression in the Laplace domain as

$$\mathcal{L}(t^{m/2-1} e^{ip\Gamma_y} e^{-it(\hat{x}, \hat{y})}) = \hat{A}_m^p + \hat{B}_m^p + \hat{x} \wedge \hat{y} \hat{C}_m^p$$

where

$$\begin{aligned}\hat{A}_m^p &= -2^{\frac{m}{2}-2} \frac{\Gamma(m/2)}{r} \sum_{k=0}^{\infty} i^{-k} (e^{ip(k+m-2)} - e^{-ipk}) C_k^{(\frac{m}{2}-1)}(\cos \theta) \frac{1}{R^{k+\frac{m}{2}-1}}; \\ \hat{B}_m^p &= 2^{\frac{m}{2}-2} \frac{\Gamma(m/2-1)}{r} \sum_{k=0}^{\infty} (k + \frac{m}{2} - 1) i^{-k} (e^{ip(k+m-2)} + e^{-ipk}) C_k^{(\frac{m}{2}-1)}(\cos \theta) \frac{1}{R^{k+\frac{m}{2}-1}}; \\ \hat{C}_m^p &= 2^{\frac{m}{2}-1} \Gamma(m/2) \frac{\hat{x} \wedge \hat{y}}{r} \sum_{k=1}^{\infty} i^{-k} (e^{ip(k+m-2)} - e^{-ipk}) C_{k-1}^{(\frac{m}{2})}(\cos \theta) \frac{1}{R^{k+\frac{m}{2}-1}}.\end{aligned}$$

When transforming back by (5), we get the plane wave decomposition of the fractional Clifford-Fourier kernel as follows which can be compared with Theorem 3.2 in [7].

Theorem 4. *The series representation of the fractional Clifford-Fourier kernel is given by*

$$\begin{aligned}K_m^p(x, y) &= e^{ip\Gamma_y} e^{-i(x, y)} \\ &= A_m^p + B_m^p + x \wedge y C_m^p,\end{aligned}$$

where

$$\begin{aligned}A_m^p &= -2^{m/2-2} \Gamma(m/2) \sum_{k=0}^{\infty} i^{-k} (e^{ip(k+m-2)} - e^{-ipk}) (|x||y|)^{-m/2+1} J_{m/2-1+k}(|x||y|) C_k^{(m/2-1)}(\cos \theta), \\ B_m^p &= 2^{m/2-2} \Gamma(m/2-1) \sum_{k=0}^{\infty} i^{-k} (k + m/2 - 1) (e^{ip(k+m-2)} + e^{-ipk}) (|x||y|)^{-m/2+1} J_{m/2-1+k}(|x||y|) C_k^{(m/2-1)}(\cos \theta), \\ C_m^p &= 2^{m/2-1} \Gamma(m/2) \sum_{k=1}^{\infty} i^{-k} (e^{ip(k+m-2)} - e^{-ipk}) (|x||y|)^{-m/2} J_{m/2-1+k}(|x||y|) C_{k-1}^{(m/2)}(\cos \theta).\end{aligned}$$

Alternatively, using the generating function of Gegenbauer polynomials, we have

$$\begin{aligned}& (1 + 2 \cos \theta \frac{ie^{-ip}}{R} + (\frac{-ie^{-ip}}{R})^2)^{-m/2} \\ &= \sum_{k=0}^{\infty} (\frac{-ie^{-ip}}{R})^k C_k^{(m/2)}(\cos \theta) \\ &= \sum_{k=0}^{\infty} \sum_{a=0}^k (\frac{-ie^{-ip}}{R})^k \frac{(m/2)_a (m/2)_{k-a}}{a! (k-a)!} \cos(k-2a)\theta\end{aligned}$$

which means we can express formula (18) equally as a Fourier series.

3.3 Even dimensional Clifford-Fourier kernel

When $p = \pi/2$, the result in Theorem 3 reduces to

$$\begin{aligned}& \mathcal{L}(t^{m/2-1} e^{i\frac{\pi}{2}\Gamma_y} e^{-it(x, y)}) \\ &= \frac{\Gamma(m/2)}{2\sqrt{+}} \left(\frac{s + \sqrt{+} - yx}{(\sqrt{+} + (x, y))^{m/2}} - e^{im\pi/2} \frac{s - \sqrt{+} - yx}{(\sqrt{+} - (x, y))^{m/2}} \right) \\ &= \frac{\Gamma(m/2)}{2\sqrt{+}} \\ & \quad \left(\frac{(s - yx + \sqrt{+})(\sqrt{+} - (x, y))^{m/2} - e^{im\pi/2} (\sqrt{+} + (x, y))^{m/2} (s - yx - \sqrt{+})}{(s^2 + (ix \wedge y)^2)^{m/2}} \right).\end{aligned}\tag{21}$$

180 When $m/2$ is even, (21) becomes

$$\begin{aligned}
& (m/2 - 1)! \left(\frac{(s - yx) \left(\sum_{j=1,3,5,\dots} \binom{m/2}{j} (\sqrt{+})^{m/2-j-1} (-x, y)^j \right)}{(s^2 + (ix \wedge y)^2)^{m/2}} \right. \\
& \quad \left. + \frac{\sum_{j=0,2,4,\dots} \binom{m/2}{j} (\sqrt{+})^{m/2-j} (-x, y)^j}{(s^2 + (ix \wedge y)^2)^{m/2}} \right) \\
& = (m/2 - 1)! \left(\frac{(s - yx) \left(\sum_{j=1,3,5,\dots} \binom{m/2}{j} (s^2 + (ix \wedge y)^2 + (x, y)^2)^{\frac{m/2-j-1}{2}} (-x, y)^j \right)}{(s^2 + |x \wedge y|^2)^{m/2}} \right. \\
& \quad \left. + \frac{\sum_{j=0,2,4,\dots} \binom{m/2}{j} (\sqrt{+})^{m/2-j} (-x, y)^j}{(s^2 + |x \wedge y|^2)^{m/2}} \right) \\
& = (m/2 - 1)! \left(\frac{(s - yx) \left(\sum_{j=1,3,5,\dots, m/2-1} \binom{m/2}{j} [(s^2 + |x \wedge y|^2) + (x, y)^2]^{\frac{m/2-j-1}{2}} (-x, y)^j \right)}{(s^2 + |x \wedge y|^2)^{m/2}} \right. \\
& \quad \left. + \frac{\sum_{u=0,2,4,\dots, m/2} \binom{m/2}{u} [(s^2 + |x \wedge y|^2) + (x, y)^2]^{\frac{m/2-u}{2}} (-x, y)^u}{(s^2 + |x \wedge y|^2)^{m/2}} \right) \tag{22}
\end{aligned}$$

181 where all the sums are finite. By the binomial theorem,

$$[(s^2 + |x \wedge y|^2) + (x, y)^2]^{\frac{m/2-j-1}{2}} = \sum_{k=0}^{\frac{m/2-j-1}{2}} \binom{\frac{m/2-j-1}{2}}{k} (s^2 + |x \wedge y|^2)^k (x, y)^{\frac{m/2-j-1}{2}-k},$$

182 Formula (22) becomes

$$\begin{aligned}
& (m/2 - 1)! \left((s - yx) \sum_{j=1,3,5,\dots, m/2-1} \binom{m/2}{j} \sum_{k=0}^{\frac{m/2-j-1}{2}} \binom{\frac{m/2-j-1}{2}}{k} \frac{(-1)^j (x, y)^{\frac{m+2j-2}{4}-k}}{(s^2 + |x \wedge y|^2)^{m/2-k}} \right. \\
& \quad \left. + \sum_{u=0,2,4,\dots, m/2} \binom{m/2}{u} \sum_{v=0}^{\frac{m/2-u-1}{2}} \binom{\frac{m/2-u-1}{2}}{v} \frac{(-1)^u (x, y)^{\frac{m/2+u-1}{2}-v}}{(s^2 + |x \wedge y|^2)^{m/2-v}} \right) \tag{23}
\end{aligned}$$

183 which is a finite sum of rational function of type $\frac{(x, y)^k}{(s^2 + |x \wedge y|^2)^q}$, $\frac{s(x, y)^k}{(s^2 + |x \wedge y|^2)^q}$ and $y \wedge x \frac{s(x, y)^k}{(s^2 + |x \wedge y|^2)^q}$. Formulas
184 (6) and (7) show that the kernel can be expressed as a finite sum of Bessel functions. Now we can get
185 the kernel expressed in terms of Bessel functions which has been obtained in a completely different way
186 in [9].

187 **Theorem 5.** *The kernel of the Clifford-Fourier transform for even dimension $m = 4n, n \geq 1$ is given by*

$$\begin{aligned}
K_m(x, y) & = e^{i \frac{\pi}{2} \Gamma_y} e^{-i(x, y)} \\
& = (\pi/2)^{1/2} \left(A_m(u, v) + B_m(u, v) + (x \wedge y) C_m(u, v) \right)
\end{aligned}$$

188 where $u = (x, y)$ and $v = |x \wedge y|$ and

$$A_m(u, v) = \sum_{l=0}^{m/4-1} u^{m/2-2-2l} \frac{1}{2^l l!} \frac{(m/2)!}{(m/2 - 2l - 1)!} \frac{J_{(m-2l-3)/2}(v)}{v^{(m-2l-3)/2}},$$

189

$$B_m(u, v) = - \sum_{l=0}^{m/4-1} u^{m/2-1-2l} \frac{1}{2^l l!} \frac{(m/2)!}{(m/2 - 2l)!} \frac{J_{(m-2l-3)/2}(v)}{v^{(m-2l-3)/2}},$$

$$C_m(u, v) = - \sum_{l=0}^{m/4-1} u^{m/2-1-2l} \frac{1}{2^l l!} \frac{(m/2)!}{(m/2-2l)!} \frac{J_{(m-2l-1)/2}(v)}{v^{(m-2l-1)/2}}.$$

191 Similarly, we can get the result when $m/2$ is odd.

192 We can also obtain an alternative expression using exponentials. When m is even, we have found that
193 formula (21) became

$$\mathcal{L}(t^{m/2-1} e^{i\frac{\pi}{2}\Gamma_y} e^{-ti(x,y)}) = \frac{\text{polynomial of } s}{\text{polynomial of } s}.$$

194 Hence we can use partial fractions to transform back, as

$$\mathcal{L}(t^{m/2-1} e^{i\frac{\pi}{2}\Gamma_y} e^{-ti(x,y)}) = \sum_{j=1}^2 \sum_{k=1}^{m/2} \frac{C_{jk}}{(s - \alpha_j)^k} + yx \sum_{p=1}^2 \sum_{q=1}^{m/2} \frac{\tilde{C}_{pq}}{(s - \alpha_p)^q}.$$

195 Each $C_{jk}, \tilde{C}_{pq}$ can be obtained by the usual technique of partial fractions.

196 In particular, the kernel of the 2-dimensional Clifford-Fourier transform can be obtained as follows.
197 Formula (21) becomes

$$\begin{aligned} & \frac{1}{2\sqrt{+}} \frac{2(s - yx)\sqrt{+} - 2\sqrt{+}(x, y)}{s^2 - (x \wedge y)^2} \\ &= \frac{s - yx - (x, y)}{s^2 - (x \wedge y)^2} = \frac{s + (x \wedge y)}{s^2 - (x \wedge y)^2} = \frac{1}{s - (x \wedge y)}. \end{aligned}$$

198 Transforming back, using (1), we get the kernel

$$K_2(x, y) = e^{x \wedge y}.$$

199 This should be compared with section 4.2 in [2] and Proposition 5.1 in [9].

200 3.4 A new integral representation for the Clifford-Fourier kernel

201 When $p = \pi/2$, Theorem 3 becomes for fixed $x, y \in \mathbb{R}^m$,

$$\begin{aligned} & \mathcal{L}(t^{m/2-1} e^{i\frac{\pi}{2}\Gamma_y} e^{-ti(x,y)}) \\ &= \frac{(m/2-1)!}{2\sqrt{+}} \left(\frac{s + \sqrt{+} - yx}{(\sqrt{+} + (x, y))^{m/2}} - e^{im\pi/2} \frac{s - \sqrt{+} - yx}{(\sqrt{+} - (x, y))^{m/2}} \right) \end{aligned} \quad (24)$$

202 with $\text{Re}(s) > 0$. By (8), (9) and (10), we have

$$\begin{aligned} & \mathcal{L}^{-1}\left(\frac{s}{\sqrt{+}} \frac{1}{(\sqrt{+} + (x, y))^{m/2}}\right) = \frac{t^{m/2-1}}{(m/2-1)!} e^{-(x,y)t} \\ & - |x||y|t \int_0^t (t^2 - u^2)^{-1/2} J_1[|x||y|(t^2 - u^2)^{1/2}] \frac{u^{m/2-1}}{(m/2-1)!} e^{-(x,y)u} du, \end{aligned}$$

203 as well as

$$\begin{aligned} & \mathcal{L}^{-1}\left(\frac{1}{\sqrt{+}} \frac{1}{(\sqrt{+} + (x, y))^{m/2-1}}\right) \\ &= \int_0^t J_0[|x||y|(t^2 - u^2)^{1/2}] \frac{u^{m/2-2}}{(m/2-2)!} e^{-(x,y)u} du, \end{aligned}$$

204 and

$$\begin{aligned} & \mathcal{L}^{-1}\left(\frac{1}{\sqrt{+}} \frac{x \wedge y}{(\sqrt{+} + (x, y))^{m/2}}\right) \\ &= \int_0^t (x \wedge y) J_0[|x||y|(t^2 - u^2)^{1/2}] \frac{u^{m/2-1}}{(m/2-1)!} e^{-(x,y)u} du. \end{aligned}$$

205 Using the above three formulas, we can find the result in the time domain. Then setting $t = 1$, we get a
206 new representation of Clifford-Fourier kernel for both even and odd dimension.

207 **Theorem 6.** For $x, y \in \mathbb{R}^m$, $m \geq 3$, the kernel for the m -dimensional Clifford-Fourier transform is given
208 by

$$\begin{aligned} K_m(x, y) = & \frac{e^{-(x,y)}}{2} - \frac{|x||y|}{2} \int_0^1 (1-u^2)^{-1/2} J_1[|x||y|(1-u^2)^{1/2}] u^{m/2-1} e^{-(x,y)u} du \\ & + \frac{m-2}{4} \int_0^1 J_0[|x||y|(1-u^2)^{1/2}] u^{m/2-2} e^{-(x,y)u} du \\ & + \frac{1}{2} \int_0^1 (x \wedge y) J_0[|x||y|(1-u^2)^{1/2}] u^{m/2-1} e^{-(x,y)u} du \\ & - e^{im\pi/2} \left(\frac{e^{(x,y)}}{2} - \frac{|x||y|}{2} \int_0^1 (1-u^2)^{-1/2} J_1[|x||y|(1-u^2)^{1/2}] u^{m/2-1} e^{(x,y)u} du \right. \\ & - \frac{m-2}{4} \int_0^1 J_0[|x||y|(1-u^2)^{1/2}] u^{m/2-2} e^{(x,y)u} du \\ & \left. + \frac{1}{2} \int_0^1 (x \wedge y) J_0[|x||y|(1-u^2)^{1/2}] u^{m/2-1} e^{(x,y)u} du \right). \end{aligned}$$

209 **Remark 1.** The 2-dimensional kernel was given in the previous section. In this integral representation,
210 the integral is divergent when $m = 2$.

211 3.5 Generating function for the kernels of even dimensions

212 In this section we compute the formal generating function of all even dimensional kernels

$$G_p(x, y, a) = \sum_{m=2,4,6,\dots} \frac{K_m^p(x, y) a^{m/2-1}}{\Gamma(m/2)},$$

213 where $K_m^p(x, y)$ is the kernel of dimension m . Here the formal generating function means one can obtain
214 the kernel from the derivatives of the generating function. Note that the kernel $K_m^p(x, y)$ is in fact a
215 function of (x, y) , $|x||y|$ and $|x \wedge y|$. Recall that $\frac{x \wedge y}{|x \wedge y|}$ can be considered as an imaginary unit. So the
216 sum $G_p(x, y, a)$ is not a sum of functions from different spaces but a sum of functions defined on R^3 .

217 When $p = \pi/2$, the Laplace transform of $G_{\frac{\pi}{2}}(x, y, at)$ with respect to t can be computed by

$$\begin{aligned} & \sum_{m=2,4,6,\dots} \frac{1}{(m/2-1)!} \mathcal{L}(t^{m/2-1} e^{i\frac{\pi}{2}\Gamma_y} e^{-it(x,y)} a^{m/2-1}) \\ &= \frac{1}{2\sqrt{+}} \sum_{m=2,4,6,\dots} a^{m/2-1} \left(\frac{s + \sqrt{+} - yx}{(\sqrt{+} + (x, y))^{m/2}} - e^{im\pi/2} \frac{s - \sqrt{+} - yx}{(\sqrt{+} - (x, y))^{m/2}} \right) \\ &= \frac{s + \sqrt{+} - yx}{2\sqrt{+}(\sqrt{+} + (x, y) - a)} - e^{i\pi} \frac{s - \sqrt{+} - yx}{2\sqrt{+}(\sqrt{+} - (x, y) - ae^{i\pi})} \\ &= \frac{s + \sqrt{+} - yx}{2\sqrt{+}(\sqrt{+} + (x, y) - a)} + \frac{s - \sqrt{+} - yx}{2\sqrt{+}(\sqrt{+} - (x, y) + a)}, \end{aligned} \tag{25}$$

the first equality is by (24). Using (8), (9) and (10), we then get

$$\begin{aligned}
& \mathcal{L}^{-1}\left(\frac{s + \sqrt{+} - yx}{\sqrt{+}(\sqrt{+} + (x, y) - a)}\right) \\
&= 2e^{-(x, y) + a}t - |x||y|t \int_0^t (t^2 - u^2)^{-1/2} J_1[|x||y|(t^2 - u^2)^{1/2}] e^{-(x, y) + a}u du \\
&- |x||y| \int_0^t e^{-(x, y) + a}(t^2 - u^2)^{1/2} J_1(|x||y|u) du \\
&- yx \int_0^t J_0[|x||y|(t^2 - u^2)^{1/2}] e^{-(x, y) + a}u du
\end{aligned} \tag{26}$$

as well as

$$\begin{aligned}
& \mathcal{L}^{-1}\left(\frac{s - \sqrt{+} - yx}{\sqrt{+}(\sqrt{+} - (x, y) + a)}\right) \\
&= -|x||y|t \int_0^t (t^2 - u^2)^{-1/2} J_1[|x||y|(t^2 - u^2)^{1/2}] e^{((x, y) - a)u} du \\
&+ |x||y| \int_0^t e^{((x, y) - a)(t^2 - u^2)^{1/2}} J_1(|x||y|u) du \\
&+ yx \int_0^t J_0[|x||y|(t^2 - u^2)^{1/2}] e^{((x, y) - a)u} du.
\end{aligned} \tag{27}$$

Combining (26) and (27), the generating function is

$$\begin{aligned}
G_{\pi/2}(x, y, a) &= e^{-(x, y) + a} \\
&- |x||y| \int_0^1 (1 - u^2)^{-1/2} J_1[|x||y|(1 - u^2)^{1/2}] \cosh(((x, y) - a)u) du \\
&+ |x||y| \int_0^1 \sinh(((x, y) - a)(1 - u^2)^{1/2}) J_1(|x||y|u) du \\
&+ yx \int_0^1 J_0[|x||y|(1 - u^2)^{1/2}] \sinh(((x, y) - a)u) du
\end{aligned}$$

which only gives an integral representation. In the following, we will use different inverse transform techniques to get the closed form. Simplifying (25) further, we have

$$\begin{aligned}
& \sum_{m=2,4,6,\dots} \frac{1}{(m/2 - 1)!} \mathcal{L}(t^{m/2-1} e^{i\frac{\pi}{2}\Gamma_y} e^{-it(x, y)} a^{m/2-1}) \\
&= \frac{s + \sqrt{+} - yx}{2\sqrt{+}(\sqrt{+} + (x, y) - a)} + \frac{s - \sqrt{+} - yx}{2\sqrt{+}(\sqrt{+} - (x, y) + a)} \\
&= \frac{s - yx - (x, y) + a}{s^2 + |x|^2|y|^2 - ((x, y) - a)^2}.
\end{aligned}$$

Transforming back, we get

$$\begin{aligned}
& \mathcal{L}^{-1}\left(\frac{s - yx - (x, y) + a}{s^2 + |x|^2|y|^2 - ((x, y) - a)^2}\right) \\
&= \cos(\sqrt{|x|^2|y|^2 - ((x, y) - a)^2}t) \\
&+ \frac{-yx - (x, y) + a}{\sqrt{|x|^2|y|^2 - ((x, y) - a)^2}} \sin(\sqrt{|x|^2|y|^2 - ((x, y) - a)^2}t).
\end{aligned}$$

The last equality is by (3) and (4). Note that it equals the case $m = 2$ when $a = 0$.

Alternatively, a tedious computation shows that

$$\frac{s - yx - (x, y) + a}{s^2 + |x|^2|y|^2 - ((x, y) - a)^2} = \begin{pmatrix} 0 & 1 \end{pmatrix} \left(sI + A \right)^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

with I the 2×2 identity matrix and A given by

$$A = \left((a - (x, y)) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & -xy \\ yx & 0 \end{pmatrix} \right) = \begin{pmatrix} a - (x, y) & -xy \\ yx & -a + (x, y) \end{pmatrix}.$$

We get the following

Theorem 7. *The generating function for even dimensional Clifford-Fourier kernels for $p = \frac{\pi}{2}$ is given by*

$$\begin{aligned} G_{\pi/2}(x, y, a) &= \begin{pmatrix} 0 & 1 \end{pmatrix} e^{-A} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \cos(\sqrt{|x|^2|y|^2 - ((x, y) - a)^2}) \\ &\quad + (-yx - (x, y) + a) \frac{\sin(\sqrt{|x|^2|y|^2 - ((x, y) - a)^2})}{\sqrt{|x|^2|y|^2 - ((x, y) - a)^2}}. \end{aligned}$$

We can get a similar result for the fractional case, i.e. general p . Denote

$$G_p(x, y, a) = \sum_{m=2,4,6,\dots} \frac{K_m^p(x, y) a^{m/2-1}}{(m/2-1)!}.$$

Now

$$\begin{aligned} &\sum_{m=2,4,\dots} \mathcal{L} \left(\frac{t^{m/2-1} e^{ip\Gamma_y} e^{-it(x,y)} a^{m/2-1}}{(m/2-1)!} \right) \\ &= \frac{e^{ip}}{2\sqrt{+}} \left(\frac{s + \sqrt{+} - ie^{-ip}yx}{s \cos p + i\sqrt{+} \sin p + i(x, y) - ae^{ip}} \right. \\ &\quad \left. - \frac{s - \sqrt{+} - ie^{-ip}yx}{s \cos p - i\sqrt{+} \sin p + i(x, y) - ae^{ip}} \right) \\ &= e^{ip} \frac{(-is - e^{-ip}yx) \sin p + (s \cos p + i(x, y) - ae^{ip})}{(s \cos p + i(x, y) - ae^{ip})^2 + (\sqrt{+})^2 \sin^2 p} \\ &= \frac{s - yx \sin p + i(x, y)e^{ip} - ae^{2ip}}{(s \cos p + i(x, y) - ae^{ip})^2 + (\sqrt{+})^2 \sin^2 p}, \end{aligned}$$

transforming back by (3) and (4), we have

$$\begin{aligned} &\mathcal{L}^{-1} \left(\frac{s - yx \sin p + i(x, y)e^{ip} - ae^{2ip}}{(s \cos p + i(x, y) - ae^{ip})^2 + (\sqrt{+})^2 \sin^2 p} \right) \\ &= e^{-ct} (\cos(dt) + \frac{(x \wedge y - ia e^{ip}) \sin p}{d} \sin(dt)) \end{aligned}$$

with $c = (i(x, y) - ae^{ip}) \cos p$, $d = \sin p \sqrt{|x|^2|y|^2 + (i(x, y) - ae^{ip})^2}$.

Alternatively, we have

$$\begin{aligned} &\frac{s - yx \sin p + i(x, y)e^{ip} - ae^{2ip}}{(s \cos p + i(x, y) - ae^{ip})^2 + (\sqrt{+})^2 \sin^2 p} \\ &= \begin{pmatrix} 0 & 1 \end{pmatrix} \left(sI + B \right)^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \end{aligned}$$

where

$$B = \begin{pmatrix} -\beta_+ & 0 \\ -(-yx \sin p + i(x, y)e^{ip} - ae^{2ip} + \beta_+) & -\beta_- \end{pmatrix},$$

with $\beta_{\pm}$ the roots of $(s \cos p + i(x, y) - ae^{ip})^2 + (\sqrt{-1})^2 \sin^2 p$, i.e.

$$\beta_{\pm} = (-i(x, y) + ae^{ip}) \cos p \pm \sin p (\sqrt{-|x|^2|y|^2 - (i(x, y) - ae^{ip})^2}).$$

Again, we can have

Theorem 8. *The generating function for the even dimensional fractional Clifford-Fourier kernels is given by*

$$\begin{aligned} G_p(x, y, a) &= \begin{pmatrix} 0 & 1 \end{pmatrix} e^{-B} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= e^{-c} (\cos d + \frac{(x \wedge y - iae^{ip}) \sin p}{d} \sin d) \end{aligned}$$

with $c = (i(x, y) - ae^{ip}) \cos p$ and $d = \sin p \sqrt{|x|^2|y|^2 + (i(x, y) - ae^{ip})^2}$.

At the end of this section, we give the kernel for general p when $m = 2$. It is also the case when $a = 0$ in the generating function. The kernel for dimension 2 is hence given by

$$K_2^p(x, y) = e^{-i(x, y) \cos p} e^{x \wedge y \sin p},$$

which coincides with the work in [5] and [7].

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