Local powers of optimal one- and multi-sample tests for the concentration of Fisher-von Mises-Langevin distributions

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Abstract

One-sample and multi-sample tests on the concentration parameter of Fisher-von Mises-Langevin (FvML) distributions on (hyper-)spheres have been well studied in the literature. However, only little is known about their behavior under local alternatives, which is due to complications inherent to the curved nature of the parameter space. The aim of the present paper therefore consists in filling that gap by having recourse to the Le Cam methodol-

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ogy, which has been adapted from the linear to the spherical setup in Ley *et al.* (2013). We obtain explicit expressions of the powers for the most efficient one- and multi-sample tests; these tests are those considered in Watamori and Jupp (2005). As a nice by-product, we are also able to write down the powers (against local FvML alternatives) of the celebrated Rayleigh (1919) test of uniformity. A Monte Carlo simulation study confirms our theoretical findings and shows the empirical powers of the above-mentioned procedures.

Keywords: concentration parameter, directional statistics, Fisher-von Mises-Langevin distributions, Le Cam’s third Lemma, uniform local asymptotic normality.

1 Introduction

In this paper, our aim is to calculate explicit power expressions for the most efficient one- and multi-sample tests on the concentration parameter in Fisher-von Mises-Langevin distributions on (hyper-)spheres. In order to achieve this goal, we shall apply the Le Cam methodology which has recently been extended from the classical linear setting to the spherical one in Ley *et al.* (2013). The optimal tests we shall derive coincide with those in Watamori and Jupp (2005), hence we provide their tests with power expressions and moreover with the Le Cam optimality flavor. For the sake of presentation, we will in this Introduction first describe the Fisher-von Mises-Langevin distributions (Section 1.1), then summarize the main features of the Le Cam approach (Section 1.2) and finally briefly explain how it has been transferred from the linear to the spherical setting (Section 1.3).

1.1 Fisher-von Mises-Langevin distributions in directional statistics

Directional data correspond to multivariate data which are unit vectors in the plane or in higher-dimensional spaces. More precisely, the observations are conceived as realizations of random vectors $\mathbf{X}$ taking values on the surface of the unit hypersphere $\mathcal{S}^{k-1} := \{\mathbf{v} \in \mathbb{R}^k \mid \|\mathbf{v}\| = 1\}$, $k \geq 2$. The most common settings correspond to $k = 2$ (circular data) and $k = 3$ (spherical data). This type of data naturally arises in a plethora of earth sciences such as geology, seismology, astrophysics, oceanography or meteorology, as well as in studies of animal behavior (see Mardia and Jupp 2000 for a description of these domains of application), or more recently in the protein structure prediction problem (see Boomsma *et al.* 2006).

Although primitive statistical analysis of directional data can already be traced back to early 19th century works by the likes of C. F. Gauss, the methodical and systematic study of such non-linear data by means of tools tailored for their specificities only began in the 1950s under the impetus of Sir Ronald Fisher’s pioneering work (see Fisher 1953). This is mainly due to the fact that until then one rather tried to circumvent the “curved” nature of the data by working with linear approximations. Fisher, however, explained that such a simplification of the problem would hamper a correct study of several phenomena such as, e.g., the remanent magnetism found in igneous or sedimentary rocks, because of the dispersion of those data over the earth (see Fisher 1953 for possible explanations of that dispersion). He therefore claims that the actual topology of the measurements, namely the unit sphere, needs

to be taken into account. From that point on, an enormous amount of literature has been dedicated to a more correct study of directional data.

For further (recent) information about directional statistics, its history, existing methods, domains of application and references, we refer for example to Mardia and Jupp (2000), Jammalamadaka and SenGupta (2001) or Mardia (2013).

By far the most popular and most used directional distribution is the *Fisher-von Mises-Langevin* (FvML) distribution (named, according to Watson 1983, after von Mises 1918 for $k = 2$, Fisher 1953 for $k = 3$, and Langevin 1905 for general k), whose density is of the form (with respect to the usual surface area measure on spheres)

$$f_{\kappa, \boldsymbol{\theta}}(\mathbf{x}) = c_{\kappa} \exp(\kappa \mathbf{x}' \boldsymbol{\theta}), \quad \mathbf{x} \in \mathcal{S}^{k-1}, \quad (1.1)$$

where $\kappa > 0$ is a concentration parameter, $\boldsymbol{\theta} \in \mathcal{S}^{k-1}$ a directional location parameter and where the normalizing constant c_{κ} is given by

$$c_{\kappa} = \left(\frac{\kappa}{2}\right)^{k/2-1} \frac{1}{\Gamma(k/2) I_{k/2-1}(\kappa)},$$

with $\Gamma(k/2)$ the Gamma function evaluated at $k/2$ and $I_{k/2-1}(\kappa)$ the modified Bessel function of the first kind and of order $k/2 - 1$. The main reason for this prominent role of the FvML distribution, besides the mathematical tractability of its density (1.1), is the following: the FvML is considered as a directional analogue of the (linear) Gaussian distribution for purposes of mathematical statistics (see Schaeben 1992 for a discussion on directional analogues of the Gaussian distribution). This analogy is mainly due to the fact that the FvML distribution can be characterized by the empirical directional mean $\hat{\boldsymbol{\theta}}_{\text{Mean}} := \sum_{i=1}^n \mathbf{X}_i / \|\sum_{i=1}^n \mathbf{X}_i\|$,

$\mathbf{X}_1, \dots, \mathbf{X}_n \in \mathcal{S}^{k-1}$, as the Maximum Likelihood Estimator (MLE) of its (directional) location parameter, similarly as the Gaussian distribution can be characterized by the empirical mean $n^{-1} \sum_{i=1}^n \mathbf{X}_i$, $\mathbf{X}_1, \dots, \mathbf{X}_n \in \mathbb{R}^k$, as the MLE of its classical (linear) location parameter, a famous result due to Gauss (1809). We refer to Duerinckx and Ley (2013) for a formal proof of this fact and for more details on directional MLE characterizations.

Due to its particular role, the FvML distribution has received a lot of attention in the literature, and inferential procedures involving its concentration and location parameters have been extensively studied in the literature (see for instance Sections 10.4-10.6 in Mardia and Jupp 2000). In the present paper, the parameter of interest is the concentration parameter κ which regulates the probability mass in the vicinity of the modal direction $\boldsymbol{\theta}$. Besides the tests described in Mardia and Jupp (2000), hypothesis testing procedures dealing specifically with the concentration parameter can mainly be found in Stephens (1969), Larsen *et al.* (2002) and Watamori and Jupp (2005). These procedures are either likelihood ratio (in its basic and improved versions) or score tests, ensuring nice efficiency properties. However, even if the asymptotic theory of such tests has been well studied in the above-cited papers, little is known about their asymptotic behavior and power under local alternatives. This absence of result can certainly be explained by the curved nature of the parameter space $\mathbb{R}_0^+ \times \mathcal{S}^{k-1}$.

In this paper, our aim is therefore to fill this gap by providing explicit expressions for the powers of the most efficient tests for both the one-sample problem ($\mathcal{H}_0^{\kappa_0}$:

$\kappa = \kappa_0$ for some fixed $\kappa_0 > 0$ versus $\mathcal{H}_1^{\kappa_0} : \kappa \neq \kappa_0$) and the multi-sample problem ($\mathcal{H}_0^{\text{Hom}} : \kappa_1 = \dots = \kappa_m$ for $m \geq 2$ and $\kappa_1, \dots, \kappa_m > 0$ versus $\mathcal{H}_1^{\text{Hom}} : \kappa_i \neq \kappa_j$ for at least one couple $1 \leq i, j \leq m$). In order to achieve this goal, we shall have recourse to the celebrated Le Cam methodology of asymptotic *experiments* (“models” in a more common language), which has been adapted to the directional setup in Ley *et al.* (2013).

1.2 The Le Cam methodology: a brief review

The central idea of the Le Cam theory is the concept of convergence of statistical experiments. Quoting Le Cam (1960), “the family of probability measures under study can be approximated very closely by a family of a simpler nature”. The key ingredient in this approximation is the *Local Asymptotic Normality* (LAN) property of the parametric model under consideration. In order to ease the reading, we will briefly review here the LAN property and its contribution to the theory of hypothesis testing. The following definition of LAN corresponds to Le Cam and Yang (2000).

For all n , let $\mathcal{E}^{(n)} = \left(\mathcal{X}^{(n)}, \mathcal{A}^{(n)}, \mathcal{P}^{(n)} := \{P_{\boldsymbol{\vartheta}}^{(n)} | \boldsymbol{\vartheta} \in \mathcal{V} \subset \mathbb{R}^k\} \right)$ be a sequence of $\boldsymbol{\vartheta}$ -parametric experiments, and let δ_n be a sequence of positive numbers going to zero. The family $\mathcal{P}^{(n)}$ is called LAN at $\boldsymbol{\vartheta} \in \mathcal{V}$ if there exists a sequence of random vectors $\boldsymbol{\Delta}^{(n)}(\boldsymbol{\vartheta})$, called *central sequence*, and a non-singular symmetric matrix $\mathbf{J}(\boldsymbol{\vartheta})$, the associated *Fisher information matrix*, such that, for every bounded sequence of

vectors $\mathbf{h}_n \in \mathbb{R}^k$,

$$\log \frac{dP_{\boldsymbol{\vartheta} + \delta_n \mathbf{h}_n}^{(n)}}{dP_{\boldsymbol{\vartheta}}^{(n)}} - \mathbf{h}_n' \boldsymbol{\Delta}^{(n)}(\boldsymbol{\vartheta}) + \frac{1}{2} \mathbf{h}_n' \mathbf{J}(\boldsymbol{\vartheta}) \mathbf{h}_n = o_P(1) \quad (1.2)$$

and $\boldsymbol{\Delta}^{(n)}(\boldsymbol{\vartheta}) \xrightarrow{\mathcal{L}} \mathcal{N}_k(0, \mathbf{J}(\boldsymbol{\vartheta}))$, both under $P_{\boldsymbol{\vartheta}}^{(n)}$ as $n \rightarrow \infty$.

We easily see that the log-likelihood ratios $\log \frac{dP_{\boldsymbol{\vartheta} + \delta_n \mathbf{h}_n}^{(n)}}{dP_{\boldsymbol{\vartheta}}^{(n)}}$ in (1.2) of a LAN family behave asymptotically like the log-likelihood ratio of the classical Gaussian shift experiment

$$\mathcal{E}_{\mathbf{J}(\boldsymbol{\vartheta})} = (\mathbb{R}^k, \mathcal{B}_k, \mathcal{P}_{\boldsymbol{\vartheta}} := \{P_{\mathbf{h}, \boldsymbol{\vartheta}} = \mathcal{N}_k(\mathbf{J}(\boldsymbol{\vartheta})\mathbf{h}, \mathbf{J}(\boldsymbol{\vartheta})) | \mathbf{h} \in \mathbb{R}^k\})$$

with a single observation which we denote as $\boldsymbol{\Delta}$. This approximation of the statistical experiments $\mathcal{E}^{(n)}$ by the normal experiment $\mathcal{E}_{\mathbf{J}(\boldsymbol{\vartheta})}$ has important consequences on the construction of locally and asymptotically optimal testing procedures as it means that, asymptotically, all power functions that are implementable in the local experiments $\mathcal{E}^{(n)}$ are the power functions that are possible in the Gaussian shift experiment $\mathcal{E}_{\mathbf{J}(\boldsymbol{\vartheta})}$. In view of these considerations, it follows that asymptotically optimal tests in the local models can be derived by analyzing the Gaussian limit model. More precisely, if a test $\phi(\boldsymbol{\Delta})$ enjoys some exact optimality property in the Gaussian experiment $\mathcal{E}_{\mathbf{J}(\boldsymbol{\vartheta})}$, then the corresponding sequence $\phi(\boldsymbol{\Delta}^{(n)})$ inherits, locally and asymptotically, the same optimality properties in the sequence of experiments $\mathcal{E}^{(n)}$.

A slightly reinforced version of LAN is ULAN, the *Uniform Local Asymptotic Normality*. Not much changes with respect to LAN, except that instead at $\boldsymbol{\vartheta}$ one now has to prove the LAN property at $\boldsymbol{\vartheta}^{(n)} = \boldsymbol{\vartheta} + O(\delta_n)$, that is, uniformly in a neighborhood of $\boldsymbol{\vartheta}$. The main attractive feature of ULAN compared to LAN is that

it entails the *asymptotic linearity property*

$$\Delta_{\boldsymbol{\vartheta} + \delta_n \mathbf{h}_n}^{(n)} - \Delta_{\boldsymbol{\vartheta}}^{(n)} = \mathbf{J}(\boldsymbol{\vartheta}) \mathbf{h}_n + o_P(1)$$

under $P_{\boldsymbol{\vartheta}}^{(n)}$. This asymptotic linearity forms the basis for replacing, in the test statistic based on the central sequence, $\boldsymbol{\vartheta}$ or rather the nuisance part of the parameters within $\boldsymbol{\vartheta}$ with *any* root- n consistent estimator under $P_{\boldsymbol{\vartheta}}^{(n)}$ satisfying some mild regularity conditions (which are not required in the present setting!).

Yet another nice consequence of using the Le Cam methodology lies in the fact that, with the LAN property in hand, it becomes possible to determine explicitly the power of given test procedures $\phi^{(n)}$ under sequences $\boldsymbol{\vartheta} + \delta_n \mathbf{h}_n$ of local alternatives to the value $\boldsymbol{\vartheta}$. This power calculation is achieved via *Le Cam's third Lemma* which not only implies, under LAN, that $P_{\boldsymbol{\vartheta}}^{(n)}$ and $P_{\boldsymbol{\vartheta} + \delta_n \mathbf{h}_n}^{(n)}$ are mutually contiguous, but also provides a means to establish the distribution of the test statistic associated with $\phi^{(n)}$ under $P_{\boldsymbol{\vartheta} + \delta_n \mathbf{h}_n}^{(n)}$. This then readily allows to obtain the desired power expressions. We refer to van der Vaart (2002) for a precise statement of Le Cam's third Lemma and for a more detailed description of the Le Cam approach.

1.3 The Le Cam methodology adapted to directional data

Until recently, the power of the Le Cam approach has not been exploited for directional data. While its application in the circular case does not differ much from the well-trodden univariate case (since the circular observations can be seen as angles lying on $[-\pi, \pi)$, see Ley and Verdebout 2013), the situation strongly changes for (hyper-)spherical observations, that is, when dealing with directional models in

dimensions $k \geq 3$. This is of course due to the already mentioned curved nature of the parameter space. This major difficulty has for the first time been overcome in Ley *et al.* (2013) for the entire class of rotationally symmetric distributions (i.e., with densities proportional to $g(\mathbf{x}'\boldsymbol{\theta})$ for some $g : [-1, 1] \rightarrow \mathbb{R}_0^+$). Their solution consists in (i) expressing all quantities in spherical coordinates (which are defined on a linear domain!), (ii) establishing the ULAN property for these new parameters, and (iii) translating the ULAN property into the original parameterization by having recourse to a lemma proved in Hallin *et al.* (2010). With this “directional ULAN property” in hand, Ley *et al.* (2013) have been able to construct optimal R-estimators for the location parameter of rotationally symmetric distributions. A more general version of this property, in the case of m independent populations, has allowed Ley, Swan and Verdebout (2013) to propose efficient ANOVA for directional distributions.

In the present paper, we shall, for the FvML case, generalize the Ley *et al.* (2013) ULAN result from a pure location to a concentration-location FvML model, and then apply the Le Cam theory, in particular Le Cam’s third Lemma, in order to write out explicitly the asymptotic powers of the testing schemes described at the end of Section 1.1. As we shall see, although the ULAN property does not hold for $\kappa = 0$, we are nevertheless able via Le Cam’s third Lemma to write down the asymptotic powers of the *Rayleigh test* for uniformity, which coincides with the score test for uniformity against FvML alternatives. Our calculations for this test thus complement, in some sense, previous works on the efficiency of the Rayleigh test,

see e.g. Rao (1972) for the circular case.

1.4 Outline of the paper

The rest of the paper is organized as follows. In Section 2, we establish the ULAN property of the concentration-location FvML model. In Sections 3 and 4, we write out the locally and asymptotically optimal tests for the one-sample and multi-sample problems, respectively. By construction, these coincide with the score tests proposed in Watamori and Jupp (2005) which themselves are asymptotically equivalent (the difference is $o_P(1)$) to the likelihood ratio tests under the null (and therefore also under contiguous alternatives). In each section, we then study the asymptotic behavior of these most efficient tests under local alternatives and provide the announced expressions of their powers. In Section 3, we also study the asymptotic properties and powers (against FvML alternatives) of the test for uniformity over $\mathcal{S}^{k-1}$ proposed in Rayleigh (1919). The powers of the tests under moderate-to-small sample sizes are investigated in Section 5 by Monte Carlo simulations, and an appendix collects the technical proofs.

2 The ULAN property of the concentration-location FvML model

Let the data points $\mathbf{X}_1, \dots, \mathbf{X}_n$ be i.i.d. with common FvML density (1.1). We denote their joint distribution by $P_{\boldsymbol{\vartheta}}^{(n)}$ with $\boldsymbol{\vartheta} := (\kappa, \boldsymbol{\theta}')' \in \mathbb{R}_0^+ \times \mathcal{S}^{k-1}$. As announced

in the Introduction, the objective of this section is to establish the ULAN property of the sequence of FvML experiments $\left\{P_{\boldsymbol{\vartheta}}^{(n)}, \boldsymbol{\vartheta} \in \mathbb{R}_0^+ \times \mathcal{S}^{k-1}\right\}$. Such a sequence is ULAN (with contiguity rate $n^{-1/2}$) if, for any sequence $\boldsymbol{\vartheta}^{(n)} \in \mathbb{R}_0^+ \times \mathcal{S}^{k-1}$ such that $\boldsymbol{\vartheta}^{(n)} - \boldsymbol{\vartheta} = O(n^{-1/2})$, the likelihood ratio between $P_{\boldsymbol{\vartheta}^{(n)} + n^{-1/2}\boldsymbol{\tau}^{(n)}}^{(n)}$ and $P_{\boldsymbol{\vartheta}^{(n)}}^{(n)}$ allows a (probabilistic) Taylor expansion as a function of the perturbation $\boldsymbol{\tau}^{(n)} \in \mathbb{R} \times \mathbb{R}^k$. In view of the curved parameter set $\mathbb{R}_0^+ \times \mathcal{S}^{k-1}$, it is clear that the local perturbations $\boldsymbol{\tau}^{(n)}$ must be chosen carefully, as they need to satisfy that $\boldsymbol{\vartheta}^{(n)} + n^{-1/2}\boldsymbol{\tau}^{(n)}$ remains in $\mathbb{R}_0^+ \times \mathcal{S}^{k-1}$. Hence, writing $\boldsymbol{\tau}^{(n)}$ as $(c^{(n)}, (\mathbf{t}^{(n)})')'$ with $c^{(n)} \in \mathbb{R}$ and $\mathbf{t}^{(n)} \in \mathbb{R}^k$, we have the conditions

$$\kappa^{(n)} + n^{-1/2}c^{(n)} > 0 \quad (2.3)$$

and

$$\begin{aligned} 0 &= (\boldsymbol{\theta}^{(n)} + n^{-1/2}\mathbf{t}^{(n)})'(\boldsymbol{\theta}^{(n)} + n^{-1/2}\mathbf{t}^{(n)}) - 1 \\ &= 2n^{-1/2}\boldsymbol{\theta}^{(n)'}\mathbf{t}^{(n)} + n^{-1}(\mathbf{t}^{(n)})'\mathbf{t}^{(n)}. \end{aligned} \quad (2.4)$$

The second condition thus means that the perturbation $\mathbf{t}^{(n)}$ must belong, up to a $o(n^{-1/2})$ quantity, to the tangent space to $\mathcal{S}^{k-1}$ at $\boldsymbol{\theta}^{(n)}$.

In order to ease readability, we introduce some notations. It can be shown that the projections $\mathbf{X}_1'\boldsymbol{\theta}, \dots, \mathbf{X}_n'\boldsymbol{\theta}$ are i.i.d. with common density $\tilde{f}_\kappa(t)$ proportional to $\exp(\kappa t)(1 - t^2)^{(k-3)/2}$ for $t \in [-1, 1]$. It then follows that

$$E[\mathbf{X}_i] = E[\mathbf{X}_i'\boldsymbol{\theta}]\boldsymbol{\theta} = \left(\frac{\int_{-1}^1 t e^{\kappa t} (1 - t^2)^{\frac{k-3}{2}} dt}{\int_{-1}^1 e^{\kappa t} (1 - t^2)^{\frac{k-3}{2}} dt} \right) \boldsymbol{\theta} =: A_k(\kappa)\boldsymbol{\theta},$$

showing that the parameter κ is identified via the function $A_k(\cdot)$. Note in passing

that $A_k(\cdot) = I_{k/2}(\cdot)/I_{k/2-1}(\cdot)$. Similar manipulations yield

$$\text{Var}[\mathbf{X}'_i \boldsymbol{\theta}] = A'_k(\kappa) = 1 - \frac{k-1}{\kappa} A_k(\kappa) - (A_k(\kappa))^2;$$

see Watson (1983) for more details on these results. We are now ready to state the ULAN property of the FvML concentration-location model.

Proposition 2.1 *The family $\left\{ P_{\boldsymbol{\vartheta}}^{(n)} \mid \boldsymbol{\vartheta} \in \mathbb{R}_0^+ \times \mathcal{S}^{k-1} \right\}$ is ULAN; more precisely, for any sequence $\boldsymbol{\vartheta}^{(n)} \in \mathbb{R}_0^+ \times \mathcal{S}^{k-1}$ such that $\boldsymbol{\vartheta}^{(n)} - \boldsymbol{\vartheta} = O(n^{-1/2})$ and any bounded sequence $\boldsymbol{\tau}^{(n)} \in \mathbb{R} \times \mathbb{R}^k$ subjected to the conditions (2.3) and (2.4), we have*

$$\log \left(\frac{dP_{\boldsymbol{\vartheta}^{(n)} + n^{-1/2} \boldsymbol{\tau}^{(n)}}^{(n)}}{dP_{\boldsymbol{\vartheta}^{(n)}}^{(n)}} \right) = (\boldsymbol{\tau}^{(n)})' \boldsymbol{\Delta}_{\boldsymbol{\vartheta}^{(n)}}^{(n)} - \frac{1}{2} (\boldsymbol{\tau}^{(n)})' \boldsymbol{\Gamma}_{\boldsymbol{\vartheta}} \boldsymbol{\tau}^{(n)} + o_P(1) \quad (2.5)$$

and $\boldsymbol{\Delta}_{\boldsymbol{\vartheta}^{(n)}}^{(n)} \xrightarrow{\mathcal{D}} \mathcal{N}_{k+1}(\mathbf{0}, \boldsymbol{\Gamma}_{\boldsymbol{\vartheta}})$ under $P_{\boldsymbol{\vartheta}^{(n)}}^{(n)}$ as $n \rightarrow \infty$. The central sequence $\boldsymbol{\Delta}_{\boldsymbol{\vartheta}}^{(n)} := \left(\left(\Delta_{\boldsymbol{\vartheta}}^{(\text{I})(n)} \right)', \left(\Delta_{\boldsymbol{\vartheta}}^{(\text{II})(n)} \right)' \right)'$ is defined by

$$\Delta_{\boldsymbol{\vartheta}}^{(\text{I})(n)} := n^{-1/2} \sum_{i=1}^n (\mathbf{X}'_i \boldsymbol{\theta} - A_k(\kappa))$$

and

$$\Delta_{\boldsymbol{\vartheta}}^{(\text{II})(n)} := \kappa n^{-1/2} \sum_{i=1}^n (1 - (\mathbf{X}'_i \boldsymbol{\theta})^2)^{1/2} \mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_i)$$

with $\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_i) := (\mathbf{X}_i - (\mathbf{X}'_i \boldsymbol{\theta}) \boldsymbol{\theta}) / \|\mathbf{X}_i - (\mathbf{X}'_i \boldsymbol{\theta}) \boldsymbol{\theta}\|$. The associated Fisher information is given by

$$\boldsymbol{\Gamma}_{\boldsymbol{\vartheta}} := \begin{pmatrix} \Gamma_{\boldsymbol{\vartheta}}^{(\text{I})} & \mathbf{0} \\ \mathbf{0} & \Gamma_{\boldsymbol{\vartheta}}^{(\text{II})} \end{pmatrix},$$

where, putting $\mathcal{J}_k(\kappa) := \int_{-1}^1 (1 - u^2) \tilde{f}_{\kappa}(u) du$,

$$\Gamma_{\boldsymbol{\vartheta}}^{(\text{I})} := 1 - \frac{k-1}{\kappa} A_k(\kappa) - (A_k(\kappa))^2 \quad \text{and} \quad \Gamma_{\boldsymbol{\vartheta}}^{(\text{II})} := \frac{\kappa^2 \mathcal{J}_k(\kappa)}{k-1} (\mathbf{I}_k - \boldsymbol{\theta} \boldsymbol{\theta}').$$

The proof is given in the Appendix. This proposition constitutes, for FvML distributions on the hyperspheres $\mathcal{S}^{k-1}$, the desired extension (for FvML distributions) of Proposition 2.2 in Ley *et al.* (2013) where only the location parameter $\boldsymbol{\theta}$ was taken into account. Note the diagonal structure of the information matrix; it is the structural reason why replacing $\boldsymbol{\theta}$ by a root- n consistent estimator has no asymptotic effect on inferential procedures focussing on κ .

3 One-sample tests on the concentration parameter

In this section, our focus lies on the one-sample testing problem $\mathcal{H}_0^{\kappa_0} : \kappa = \kappa_0$ for some fixed $\kappa_0 > 0$ versus $\mathcal{H}_1^{\kappa_0} : \kappa \neq \kappa_0$ (Section 3.1) and on the Rayleigh (1919) tests of uniformity (Section 3.2). In each case, we analyze the most efficient tests (Watanabe and Jupp 2005 and Rayleigh 1919, respectively) in the light of the Le Cam framework. Besides interesting optimality issues, this will allow us to obtain expressions for the powers of the considered tests under local alternatives.

3.1 The one-sample score tests of Watanabe and Jupp (2005)

Based on the ULAN property in Proposition 2.1, the Le Cam asymptotic theory paves the way towards constructing locally and asymptotically optimal tests, as described in the Introduction. The optimality appearing in this section is the *maximin* optimality. A test ϕ^* is called maximin in the class $\mathcal{C}_\alpha$ of level- α tests for $\mathcal{H}_0$ against

$\mathcal{H}_1$ if (i) ϕ^* has level α and (ii) the power of ϕ^* is such that

$$\inf_{P \in \mathcal{H}_1} E_P[\phi^*] \geq \sup_{\phi \in \mathcal{C}_\alpha} \inf_{P \in \mathcal{H}_1} E_P[\phi].$$

Since κ is the parameter of interest, locally and asymptotically optimal tests for $\mathcal{H}_0^{\kappa_0}$ are built upon $\Delta_{\boldsymbol{\vartheta}}^{(I)(n)}$, the κ -part of the central sequence; see Le Cam (1986), Section 11.9, for details. More concretely, a locally and asymptotically maximin test rejects $\mathcal{H}_0^{\kappa_0}$ at asymptotic level α whenever

$$\begin{aligned} Q_{\kappa_0}^{(n)}(\boldsymbol{\theta}) &:= \frac{(\Delta_{\boldsymbol{\vartheta}}^{(I)(n)})^2}{\Gamma_{\boldsymbol{\vartheta}}^{(I)}} \\ &= \frac{\left(\sum_{i=1}^n (\mathbf{X}_i' \boldsymbol{\theta} - A_k^{-1}(\kappa_0))\right)^2}{n(1 - \frac{k-1}{\kappa_0} A_k(\kappa_0) - (A_k(\kappa_0))^2)} \end{aligned}$$

exceeds the α -upper quantile of the chi-square distribution with 1 degree of freedom. Unfortunately, $Q_{\kappa_0}^{(n)}(\boldsymbol{\theta})$ is not (yet) a genuine test statistic since it still depends on the unknown value of $\boldsymbol{\theta}$. This problem can be solved by replacing $\boldsymbol{\theta}$ with a root- n consistent estimator $\hat{\boldsymbol{\theta}}^{(n)}$ in the central sequence $\Delta_{\boldsymbol{\vartheta}}^{(I)(n)}$, whilst, of course, paying attention to the asymptotic effects of such a substitution. It is here that the ULAN property of the concentration-location FvML model comes in handy. Indeed, as described in the Introduction, it directly entails that the FvML model is locally and asymptotically linear in the sense that

$$\Delta_{\boldsymbol{\vartheta} + n^{-1/2} \boldsymbol{\tau}^{(n)}}^{(n)} - \Delta_{\boldsymbol{\vartheta}}^{(n)} = \boldsymbol{\Gamma}_{\boldsymbol{\vartheta}} \boldsymbol{\tau}^{(n)} + o_P(1) \quad (3.6)$$

under $P_{\boldsymbol{\vartheta}}^{(n)}$ as $n \rightarrow \infty$. Of course, the aim consists in using $\boldsymbol{\tau}^{(n)} = (c^{(n)}, (\mathbf{t}^{(n)})')'$ with $\mathbf{t}^{(n)} = n^{1/2}(\hat{\boldsymbol{\theta}}^{(n)} - \boldsymbol{\theta})$ which satisfies condition (2.4); controlling this replacement however is not straightforward and requires a formal proof. The matters are simplified by the block-diagonality of the Fisher information matrix (see Proposition 2.1),

which implies that the κ -part of the central sequence is not influenced by a local perturbation of $\boldsymbol{\theta}$ (similarly, the $\boldsymbol{\theta}$ -part of the central sequence is not influenced by a local perturbation of κ , hence the results in Ley *et al.* 2013 for the FvML case can be extended by estimating κ). Hence, since our focus lies on $\Delta_{\boldsymbol{\theta}}^{(I)(n)}$ and since, under the null hypothesis $\kappa = \kappa_0$, we only need to show by having recourse to the asymptotic linearity property (3.6) for $\Delta_{\boldsymbol{\theta}}^{(I)(n)}$ that a replacement of $\boldsymbol{\theta}$ with a root- n consistent estimator $\hat{\boldsymbol{\theta}}^{(n)}$ (e.g., the sample spherical mean $\bar{\mathbf{X}}/||\bar{\mathbf{X}}||$ with $\bar{\mathbf{X}} = n^{-1} \sum_{i=1}^n \mathbf{X}_i$) has no asymptotic impact on $\Delta_{\boldsymbol{\theta}}^{(I)(n)}$, which is achieved in the following proposition.

Proposition 3.1 *Let $\hat{\boldsymbol{\theta}}^{(n)} \in \mathcal{S}^{k-1}$ be a root- n consistent estimator of $\boldsymbol{\theta}$ under $\mathbf{P}_{\boldsymbol{\theta}}^{(n)}$. Then, letting $T_{\boldsymbol{\theta}}^{(n)} := n^{-1/2} \sum_{i=1}^n \mathbf{X}_i' \boldsymbol{\theta}$, we have that $T_{\hat{\boldsymbol{\theta}}^{(n)}}^{(n)} - T_{\boldsymbol{\theta}}^{(n)}$ is $o_{\mathbf{P}}(1)$ under $\mathbf{P}_{\boldsymbol{\theta}}^{(n)}$ as $n \rightarrow \infty$.*

See the appendix for the proof. The resulting locally and asymptotically maximin test for testing $\mathcal{H}_0^{\kappa_0}$ rejects the null hypothesis (at asymptotic level α) when

$$Q_{\kappa_0}^{(n)} := \frac{\left(\sum_{i=1}^n \mathbf{X}_i' \hat{\boldsymbol{\theta}} - A_k^{-1}(\kappa_0) \right)^2}{n \left(1 - \frac{k-1}{\kappa_0} A_k(\kappa_0) - (A_k(\kappa_0))^2 \right)}$$

exceeds the α -upper quantile of the chi-square distribution with 1 degree of freedom. The test statistic $Q_{\kappa_0}^{(n)}$ coincides with the score test proposed in Watamori and Jupp (2005). This, in passing, shows the local and asymptotic optimality property of the latter. The following result characterizes the asymptotic properties of $Q_{\kappa_0}^{(n)}$.

Proposition 3.2 *We have that*

- (i) $Q_{\kappa_0}^{(n)}$ is asymptotically chi-square with 1 degree of freedom under $\cup_{\boldsymbol{\theta} \in \mathcal{S}^{k-1}} \mathbf{P}_{(\kappa_0, \boldsymbol{\theta})}^{(n)}$;

- (ii) $Q_{\kappa_0}^{(n)}$ is asymptotically non-central chi-square with 1 degree of freedom and non-centrality parameter $(1 - \frac{k-1}{\kappa_0} A_k(\kappa_0) - (A_k(\kappa_0))^2) c^2$ under $\cup_{\boldsymbol{\theta} \in \mathcal{S}^{k-1}} P_{(\kappa_0 + n^{-1/2} c^{(n)}, \boldsymbol{\theta})}^{(n)}$, where $c := \lim_{n \rightarrow \infty} c^{(n)}$ for $c^{(n)}$ satisfying condition (2.3);
- (iii) the test $\phi_{\kappa_0}^{(n)}$ which rejects the null hypothesis as soon as $Q_{\kappa_0}^{(n)}$ exceeds the α -upper quantile of the chi-square distribution with 1 degree of freedom has asymptotic level α under $\cup_{\boldsymbol{\theta} \in \mathcal{S}^{k-1}} P_{(\kappa_0, \boldsymbol{\theta})}^{(n)}$ and is locally and asymptotically maximin against local alternatives of the form $\cup_{\boldsymbol{\theta} \in \mathcal{S}^{k-1}} P_{(\kappa_0 + n^{-1/2} c^{(n)}, \boldsymbol{\theta})}^{(n)}$.

Proposition 3.2 readily follows from Proposition 2.1, Proposition 3.1 and Le Cam's third Lemma, and is hence left to the reader (if unclear, see the next section where we develop this argument for the Rayleigh 1919 test of uniformity). Note that Proposition 3.2 readily yields the announced expression for the power of $Q_{\kappa_0}^{(n)}$ under local alternatives of the form $\cup_{\boldsymbol{\theta} \in \mathcal{S}^{k-1}} P_{(\kappa_0 + n^{-1/2} c^{(n)}, \boldsymbol{\theta})}^{(n)}$ ($c := \lim_{n \rightarrow \infty} c^{(n)}$):

$$1 - F_{\chi_1^2((1 - \frac{k-1}{\kappa_0} A_k(\kappa_0) - (A_k(\kappa_0))^2) c^2)}(\chi_{1;1-\alpha}^2),$$

where $F_{\chi_\nu^2(z)}$ stands for the distribution function of the non-central chi-square distribution with ν degrees of freedom and with non-centrality parameter z and $\chi_{\nu;1-\alpha}^2$ represents the α -upper quantile of the (central) chi-square distribution with ν degrees of freedom.

3.2 The Rayleigh (1919) test of uniformity

Let us now come to the Rayleigh test of uniformity. Within the FvML family, the boundary distribution obtained when $\kappa = 0$ is the uniform distribution. Unfortu-

nately, the ULAN property of Proposition 2.1 does not hold for $\kappa = 0$ (*inter alia* because the location $\boldsymbol{\theta}$ is not identified under the null of uniformity). Nevertheless, we show in this subsection that a study of the asymptotic local powers of the classical Rayleigh (1919) test which rejects the null hypothesis of uniformity $\mathcal{H}_0^{\text{unif}}$ at asymptotic nominal level α when

$$Q_{\text{unif}}^{(n)} := kn \|\bar{\mathbf{X}}\|^2 > \chi_{k;1-\alpha}^2 \quad (3.7)$$

can be performed using Le Cam's third Lemma. It follows from (3.7) that in order to obtain local powers of $Q_{\text{unif}}^{(n)}$ we have to study the asymptotic behavior of $\mathbf{T}^{(n)} := n^{1/2} \bar{\mathbf{X}}$ under local FvML alternatives since $Q_{\text{unif}}^{(n)} = k(\mathbf{T}^{(n)})' \mathbf{T}^{(n)}$. First, let

$$\Lambda^{(n)} := \log \left(\frac{dP_{(n^{-1/2}c^{(n)}, \boldsymbol{\theta})}^{(n)}}{dP_{\text{unif}}^{(n)}} \right)$$

stand for the log-likelihood ratio between a FvML distribution with parameters $(n^{-1/2}c^{(n)}, \boldsymbol{\theta})$ and the uniform distribution on $\mathcal{S}^{k-1}$. Both distributions are clearly contiguous. Very simple computations yield

$$\Lambda^{(n)} = n^{-1/2}c^{(n)} \sum_{i=1}^n \mathbf{X}_i' \boldsymbol{\theta} + C_{k,c^{(n)}}^{(n)}$$

for some constant $C_{k,c^{(n)}}^{(n)}$ which is $O(1)$ as $n \rightarrow \infty$ under $P_{\text{unif}}^{(n)}$. The multivariate central limit theorem directly entails that the limiting distribution of $((\mathbf{T}^{(n)})', \Lambda^{(n)})'$ is a $(k+1)$ -variate Gaussian distribution with mean $(\mathbf{0}', C_{k,c})'$, where $C_{k,c} := \lim_{n \rightarrow \infty} C_{k,c^{(n)}}^{(n)}$, and covariance matrix $(c := \lim_{n \rightarrow \infty} c^{(n)})$

$$\begin{pmatrix} k^{-1} \mathbf{I}_k & ck^{-1} \boldsymbol{\theta} \\ ck^{-1} \boldsymbol{\theta}' & c^2 k^{-1} \end{pmatrix}$$

under $P_{\text{unif}}^{(n)}$ as $n \rightarrow \infty$ (this holds for any fixed $\boldsymbol{\theta} \in \mathcal{S}^{k-1}$). Then Le Cam's third Lemma entails that the limiting distribution of $\mathbf{T}^{(n)}$ is a k -variate Gaussian distribution with mean $ck^{-1}\boldsymbol{\theta}$ and covariance matrix $k^{-1}\mathbf{I}_k$ under $P_{(n^{-1/2}c^{(n)}, \boldsymbol{\theta})}^{(n)}$ as $n \rightarrow \infty$. Thus, we obtain the following result.

Proposition 3.3 *We have that*

- (i) (Rayleigh 1919) $Q_{\text{unif}}^{(n)}$ is asymptotically chi-square with k degrees of freedom under $\mathcal{H}_0^{\text{unif}}$;
- (ii) $Q_{\text{unif}}^{(n)}$ is asymptotically non-central chi-square with k degrees of freedom and non-centrality parameter c^2/k under $\cup_{\boldsymbol{\theta} \in \mathcal{S}^{k-1}} P_{(n^{-1/2}c^{(n)}, \boldsymbol{\theta})}^{(n)}$, with $c := \lim_{n \rightarrow \infty} c^{(n)}$ for $c^{(n)}$ satisfying condition (2.3).

The power of the Rayleigh test under local alternatives of the form $\cup_{\boldsymbol{\theta} \in \mathcal{S}^{k-1}} P_{(n^{-1/2}c^{(n)}, \boldsymbol{\theta})}^{(n)}$ is given by ($c := \lim_{n \rightarrow \infty} c^{(n)}$)

$$1 - F_{\chi_k^2(c^2/k)}(\chi_{k;1-\alpha}^2).$$

Figure 1 right below shows power curves of the Rayleigh test for different values of the dimension k against local FvML alternatives. Note that the power of the Rayleigh test decreases as the dimension k increases.

Figure 1: Power curve (the local alternative is $P_{(n^{-1/2}(c/2), \theta)}^{(n)}$) of $\phi_{\text{unif}}^{(n)}$ for dimensions $k = 2, \dots, 5$.

4 Multi-sample tests on the equality of concentrations

In this section, our focus lies on the multi-sample testing problem $\mathcal{H}_0^{\text{Hom}} : \kappa_1 = \dots = \kappa_m$ for $m \geq 2$ and $\kappa_1, \dots, \kappa_m > 0$ versus $\mathcal{H}_1^{\text{Hom}} : \kappa_i \neq \kappa_j$ for at least one couple $1 \leq i, j \leq m$. In other words, we are dealing with $m(\geq 2)$ samples of i.i.d. data points $\mathbf{X}_{i1}, \dots, \mathbf{X}_{in_i}$ with common FvML distribution with concentration κ_i and location $\boldsymbol{\theta}_i$ for all $i = 1, \dots, m$, and we are interested in determining whether or not these m samples have the same concentration parameters, *without* assuming equality of the mean directions $\boldsymbol{\theta}_i$. As in the previous section, our way of proceeding consists in “re-discovering” the score tests of Watamori and Jupp (2005) thanks to the ULAN property (which we shall adapt to the multi-sample case) and then unveiling new asymptotic results for these tests.

Let us assume that the samples $(\mathbf{X}_{i1}, \dots, \mathbf{X}_{in_i})$, $i = 1, \dots, m$, are independent samples of i.i.d. random vectors; as already mentioned above, the n_i observations $\mathbf{X}_{ij}$, $j = 1, \dots, n_i$, in sample i are i.i.d. with common FvML density with concentration κ_i and location $\boldsymbol{\theta}_i$. We denote this time by $P_{\boldsymbol{\vartheta}^{(m)}}^{(n)}$ the joint distribution of $(\mathbf{X}_{11}, \dots, \mathbf{X}_{mn_m})$, with $\boldsymbol{\vartheta}^{(m)} := (\kappa_1, \dots, \kappa_m, \boldsymbol{\theta}'_1, \dots, \boldsymbol{\theta}'_m)' \in (\mathbb{R}_0^+)^m \times (\mathcal{S}^{k-1})^m$. In order to be able to state our results, we need to impose a certain amount of control on the respective sample sizes n_i , $i = 1, \dots, m$. This is achieved via the following

ASSUMPTION A. Let $n = \sum_{i=1}^m n_i$. For all $i = 1, \dots, m$, the ratio $r_i^{(n)} := n_i/n$

converges to a non-zero constant r_i as $n \rightarrow \infty$.

A direct consequence of Assumption A is that the specific sizes n_i become somehow irrelevant; hence, in what follows, we simply use the superscript $^{(n)}$ for the different quantities at play and do not specify whether they are associated with a given n_i . Now, let $\text{diag}(\mathbf{A}_1, \dots, \mathbf{A}_m)$ stand for the block-diagonal matrix with blocks $\mathbf{A}_1, \dots, \mathbf{A}_m$, and use the notation $\boldsymbol{\nu}^{(n)} := \text{diag}(\boldsymbol{\nu}_1^{(n)}, \boldsymbol{\nu}_2^{(n)})$, where $\boldsymbol{\nu}_1^{(n)} := \text{diag}((r_1^{(n)})^{-1/2}, \dots, (r_m^{(n)})^{-1/2})$ and $\boldsymbol{\nu}_2^{(n)} := \text{diag}((r_1^{(n)})^{-1/2} \mathbf{I}_k, \dots, (r_m^{(n)})^{-1/2} \mathbf{I}_k)$. As in the one-sample case, we consider only perturbations $\boldsymbol{\tau}^{(n)} = (c_1^{(n)}, \dots, c_m^{(n)}, (\mathbf{t}_1^{(n)})', \dots, (\mathbf{t}_m^{(n)})')' \in \mathbb{R}^m \times (\mathbb{R}^k)^m$ such that, for any $\boldsymbol{\vartheta}^{(m)} \in (\mathbb{R}_0^+)^m \times (\mathcal{S}^{k-1})^m$, $\boldsymbol{\vartheta}^{(m)} + n^{-1/2} \boldsymbol{\nu}^{(n)} \boldsymbol{\tau}^{(n)}$ remains in $(\mathbb{R}_0^+)^m \times (\mathcal{S}^{k-1})^m$ (this is simply an adaptation of the conditions (2.3) and (2.4)). This readily leads us to the following multi-sample version of Proposition 2.1, whose straightforward proof is omitted.

Proposition 4.1 *Let Assumption A hold. Then the family $\left\{ P_{\boldsymbol{\vartheta}^{(m)}}^{(n)} \mid \boldsymbol{\vartheta}^{(m)} \in (\mathbb{R}_0^+)^m \times (\mathcal{S}^{k-1})^m \right\}$ is ULAN; more precisely, for any sequence $(\boldsymbol{\vartheta}^{(m)})^{(n)} \in (\mathbb{R}_0^+)^m \times (\mathcal{S}^{k-1})^m$ such that $(\boldsymbol{\vartheta}^{(m)})^{(n)} - \boldsymbol{\vartheta}^{(m)} = O(n^{-1/2})$ and any bounded sequence $\boldsymbol{\tau}^{(n)}$ as described above,*

$$\log \left(\frac{\text{dP}_{(\boldsymbol{\vartheta}^{(m)})^{(n)} + n^{-1/2} \boldsymbol{\nu}^{(n)} \boldsymbol{\tau}^{(n)}}^{(n)}}{\text{dP}_{(\boldsymbol{\vartheta}^{(m)})^{(n)}}^{(n)}} \right) = (\boldsymbol{\tau}^{(n)})' \Delta_{(\boldsymbol{\vartheta}^{(m)})^{(n)}}^{(n)} - \frac{1}{2} (\boldsymbol{\tau}^{(n)})' \mathbf{\Gamma}_{\boldsymbol{\vartheta}^{(m)}} \boldsymbol{\tau}^{(n)} + o_P(1)$$

and $\Delta_{(\boldsymbol{\vartheta}^{(m)})^{(n)}}^{(n)} \xrightarrow{\mathcal{D}} \mathcal{N}_{m(k+1)}(\mathbf{0}, \mathbf{\Gamma}_{\boldsymbol{\vartheta}^{(m)}})$ under $P_{\boldsymbol{\vartheta}^{(m)}}^{(n)}$ as $n \rightarrow \infty$. The central sequence

$$\Delta_{\boldsymbol{\vartheta}^{(m)}}^{(n)} := \left(\left(\Delta_{\boldsymbol{\vartheta}^{(m)}}^{(\text{I},1)(n)} \right)', \left(\Delta_{\boldsymbol{\vartheta}^{(m)}}^{(\text{II},1)(n)} \right)' \right)', \text{ where } \Delta_{\boldsymbol{\vartheta}^{(m)}}^{(\text{I},1)(n)} := \left(\left(\Delta_{\boldsymbol{\vartheta}^{(m)}}^{(\text{I},1)(n)} \right)', \dots, \left(\Delta_{\boldsymbol{\vartheta}^{(m)}}^{(\text{I},m)(n)} \right)' \right)'$$

and $\Delta_{\boldsymbol{\vartheta}^{(m)}}^{(\text{II},1)(n)} := \left(\left(\Delta_{\boldsymbol{\vartheta}^{(m)}}^{(\text{II},1)(n)} \right)', \dots, \left(\Delta_{\boldsymbol{\vartheta}^{(m)}}^{(\text{II},m)(n)} \right)' \right)'$, is defined by

$$\Delta_{\boldsymbol{\vartheta}^{(m)}}^{(\text{I},i)(n)} := n_i^{-1/2} \sum_{j=1}^{n_i} (\mathbf{X}_{ij}' \boldsymbol{\theta}_i - A_k(\kappa_i))$$

and

$$\Delta_{\boldsymbol{\vartheta}^{(m)}}^{(\text{II},i)(n)} := \kappa_i n_i^{-1/2} \sum_{j=1}^{n_i} (1 - (\mathbf{X}_{ij}' \boldsymbol{\theta}_i)^2)^{1/2} \mathbf{S}_{\boldsymbol{\theta}_i}(\mathbf{X}_{ij})$$

for all $i = 1, \dots, m$, with $\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_{ij}) := (\mathbf{X}_{ij} - (\mathbf{X}_{ij}' \boldsymbol{\theta}_i) \boldsymbol{\theta}_i) / \|\mathbf{X}_{ij} - (\mathbf{X}_{ij}' \boldsymbol{\theta}_i) \boldsymbol{\theta}_i\|$. The associated Fisher information is given by $\boldsymbol{\Gamma}_{\boldsymbol{\vartheta}^{(m)}} := \text{diag}(\boldsymbol{\Gamma}_{\boldsymbol{\vartheta}^{(m)}}^{(\text{I})}, \boldsymbol{\Gamma}_{\boldsymbol{\vartheta}^{(m)}}^{(\text{II})})$, where $\boldsymbol{\Gamma}_{\boldsymbol{\vartheta}^{(m)}}^{(\text{I})} := \text{diag}(\Gamma_{\boldsymbol{\vartheta}^{(m)}}^{(\text{I},1)}, \dots, \Gamma_{\boldsymbol{\vartheta}^{(m)}}^{(\text{I},m)})$ with $\Gamma_{\boldsymbol{\vartheta}^{(m)}}^{(\text{I},i)} := 1 - \frac{k-1}{\kappa_i} A_k(\kappa_i) - (A_k(\kappa_i))^2$ for all $i = 1, \dots, m$ and where $\boldsymbol{\Gamma}_{\boldsymbol{\vartheta}^{(m)}}^{(\text{II})} := \text{diag}(\boldsymbol{\Gamma}_{\boldsymbol{\vartheta}^{(m)}}^{(\text{II},1)}, \dots, \boldsymbol{\Gamma}_{\boldsymbol{\vartheta}^{(m)}}^{(\text{II},m)})$ with

$$\boldsymbol{\Gamma}_{\boldsymbol{\vartheta}^{(m)}}^{(\text{II},i)} := \frac{\kappa_i^2 \mathcal{J}_k(\kappa_i)}{k-1} (\mathbf{I}_k - \boldsymbol{\theta}_i \boldsymbol{\theta}_i')$$

for all $i = 1, \dots, m$.

As for the one-sample case in Section 3, we use the ULAN property to construct a locally and asymptotically optimal test for the homogeneity of concentrations. Here, the underpinning optimality concept provides the *most stringent* test for $\mathcal{H}_0^{\text{Hom}}$. A test ϕ^* is called most stringent in the class of level- α tests $\mathcal{C}_\alpha$ for testing $\mathcal{H}_0$ against $\mathcal{H}_1$ if (i) ϕ^* has level α and (ii) is such that

$$\sup_{\mathbf{P} \in \mathcal{H}_1} r_{\phi^*}(\mathbf{P}) \leq \sup_{\mathbf{P} \in \mathcal{H}_1} r_{\phi}(\mathbf{P}) \quad \forall \phi \in \mathcal{C}_\alpha,$$

where $r_{\phi_0}(\mathbf{P})$ stands for the *regret* of the test ϕ_0 under $\mathbf{P} \in \mathcal{H}_1$ defined as $r_{\phi_0}(\mathbf{P}) := [\sup_{\phi \in \mathcal{C}_\alpha} \mathbb{E}_{\mathbf{P}}[\phi]] - \mathbb{E}_{\mathbf{P}}[\phi_0]$, the deficiency in power of ϕ_0 under $\mathbf{P}$ compared to the highest possible (for tests belonging to $\mathcal{C}_\alpha$) power under $\mathbf{P}$.

Letting $\mathbf{1}_m := (1, \dots, 1)' \in \mathbb{R}^m$, the null hypothesis $\mathcal{H}_0^{\text{Hom}}$ can be rewritten as $\mathcal{H}_0^{\text{Hom}} : (\kappa_1, \dots, \kappa_m)' \in \mathcal{M}(\mathbf{1}_m)$, where $\mathcal{M}(\mathbf{A})$ stands for the linear subspace spanned by the columns of $\mathbf{A}$. Following Le Cam (1986), a locally and asymptotically

most stringent test rejects the null hypothesis $\mathcal{H}_0^{\text{Hom}}$ at asymptotic level α when (writing κ for the common value of $\kappa_1, \dots, \kappa_m$ under the null hypothesis, $D_k := 1 - \frac{k-1}{\kappa} A_k(\kappa) - (A_k(\kappa))^2$, $\mathbf{\Upsilon}^{(n)} := (\boldsymbol{\nu}_1^{(n)})^{-1} \mathbf{1}_m$ and $\bar{\mathbf{X}}_i := n_i^{-1} \sum_{j=1}^{n_i} \mathbf{X}_{ij}$)

$$\begin{aligned} Q_{\text{Hom}}^{(n)}(\boldsymbol{\vartheta}^{(m)}) &:= \left(\Delta_{\boldsymbol{\vartheta}^{(m)}}^{(\text{I})(n)} \right)' \left(\left(\Gamma_{\boldsymbol{\vartheta}^{(m)}}^{(\text{I})} \right)^{-1} - \mathbf{\Upsilon}^{(n)} \left((\mathbf{\Upsilon}^{(n)})' \Gamma_{\boldsymbol{\vartheta}^{(m)}}^{(\text{I})} \mathbf{\Upsilon}^{(n)} \right)^{-1} (\mathbf{\Upsilon}^{(n)})' \right) \Delta_{\boldsymbol{\vartheta}^{(m)}}^{(\text{I})(n)} \\ &= D_k^{-1} \left(\sum_{i=1}^m n_i (\boldsymbol{\theta}_i' \bar{\mathbf{X}}_i)^2 - \frac{1}{n} \left(\sum_{i=1}^m n_i \boldsymbol{\theta}_i' \bar{\mathbf{X}}_i \right)^2 \right) \end{aligned}$$

exceeds the α -upper quantile of the chi-square distribution with $m - 1$ degrees of freedom. As for the one-sample case, the statistic $Q_{\text{Hom}}^{(n)}(\boldsymbol{\vartheta}^{(m)})$ is not (yet) a genuine test statistic since it still depends on the unknown location parameters $\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_m$ and moreover on the quantity D_k . The replacement of the location parameters with root- n consistent estimators (e.g., $\hat{\boldsymbol{\theta}}_1^{(n)} = \bar{\mathbf{X}}_1 / \|\bar{\mathbf{X}}_1\|, \dots, \hat{\boldsymbol{\theta}}_m^{(n)} = \bar{\mathbf{X}}_m / \|\bar{\mathbf{X}}_m\|$ with $\bar{\mathbf{X}}_j / \|\bar{\mathbf{X}}_j\|$ the j th within-sample spherical mean) will not have any asymptotic impact on $Q_{\text{Hom}}^{(n)}(\boldsymbol{\vartheta}^{(m)})$, see Proposition 3.1. As concerns the quantity D_k , it can be estimated consistently by $\hat{D}_k := 1 - \frac{k-1}{\hat{\kappa}} A_k(\hat{\kappa}) - (A_k(\hat{\kappa}))^2$, where $\hat{\kappa} := \sum_{i=1}^m r_i^{(n)} \hat{\kappa}_i$, with $\hat{\kappa}_1 = A_k^{-1}(\|\bar{\mathbf{X}}_1\|), \dots, \hat{\kappa}_m = A_k^{-1}(\|\bar{\mathbf{X}}_m\|)$. The resulting locally and asymptotically most stringent test $\phi_{\text{Hom}}^{(n)}$ rejects the null hypothesis $\mathcal{H}_0^{\text{Hom}}$ at asymptotic level α whenever

$$Q_{\text{Hom}}^{(n)} := \hat{D}_k^{-1} \left(\sum_{i=1}^m n_i (\hat{\boldsymbol{\theta}}_i' \bar{\mathbf{X}}_i)^2 - \frac{1}{n} \left(\sum_{i=1}^m n_i \hat{\boldsymbol{\theta}}_i' \bar{\mathbf{X}}_i \right)^2 \right)$$

exceeds the α -upper quantile of the chi-square distribution with $m - 1$ degrees of freedom. Again, the test statistic $Q_{\text{Hom}}^{(n)}$ coincides with the score test proposed in Watamori and Jupp (2005) which is therefore locally and asymptotically most stringent. The following result characterizes the asymptotic properties of $Q_{\text{Hom}}^{(n)}$ under the null and under a sequence of local alternatives.

Proposition 4.2 *Let Assumption A hold. We have that*

(i) $Q_{\text{Hom}}^{(n)}$ is asymptotically chi-square with $m - 1$ degrees of freedom under $\mathcal{H}_0^{\text{Hom}}$;

(ii) letting $\mathbf{c} = (c_1, \dots, c_m) := \lim_{n \rightarrow \infty} (c_1^{(n)}, \dots, c_m^{(n)})'$, $Q_{\text{Hom}}^{(n)}$ is asymptotically non-central chi-square with $m - 1$ degrees of freedom and non-centrality parameter

$$D_k \left[\sum_{i=1}^m c_i^2 - \left(\sum_{i=1}^m \sqrt{r_i} c_i \right)^2 \right] \quad (4.8)$$

under $\cup_{(\boldsymbol{\theta}'_1, \dots, \boldsymbol{\theta}'_m)' \in (S^{k-1})^m} \cup_{\kappa \in \mathbb{R}_0^+} P_{(\kappa, \dots, \kappa, \boldsymbol{\theta}'_1, \dots, \boldsymbol{\theta}'_m)' + n^{-1/2} \boldsymbol{\nu}^{(n)} \boldsymbol{\tau}^{(n)}}^{(n)}$;

(iii) the test $\phi_{\text{Hom}}^{(n)}$ which rejects the null hypothesis as soon as $Q_{\text{Hom}}^{(n)}$ exceeds the α -upper quantile of the chi-square distribution with $m - 1$ degrees of freedom has asymptotic level α under $\mathcal{H}_0^{\text{Hom}}$ and is locally and asymptotically most stringent against local alternatives of the form $\cup_{(\boldsymbol{\theta}'_1, \dots, \boldsymbol{\theta}'_m)' \in (S^{k-1})^m} \cup_{\kappa \in \mathbb{R}_0^+} P_{(\kappa, \dots, \kappa, \boldsymbol{\theta}'_1, \dots, \boldsymbol{\theta}'_m)' + n^{-1/2} \boldsymbol{\nu}^{(n)} \boldsymbol{\tau}^{(n)}}^{(n)}$.

See the appendix for a proof. Note that, when all quantities $c_i^{(n)} (r_i^{(n)})^{-1/2}$ (and hence the limit $c_i r_i^{-1/2}$) are equal, we are still under the null hypothesis; this is well reflected by the fact that then the non-centrality parameter in (4.8) equals zero. Proposition 4.2 also readily yields the announced expression for the power of $Q_{\text{Hom}}^{(n)}$ under local alternatives of the form $\cup_{(\boldsymbol{\theta}'_1, \dots, \boldsymbol{\theta}'_m)' \in (S^{k-1})^m} \cup_{\kappa \in \mathbb{R}_0^+} P_{(\kappa, \dots, \kappa, \boldsymbol{\theta}'_1, \dots, \boldsymbol{\theta}'_m)' + n^{-1/2} \boldsymbol{\nu}^{(n)} \boldsymbol{\tau}^{(n)}}^{(n)}$:

$$1 - F_{\chi_{m-1}^2} \left(D_k \left[\sum_{i=1}^m c_i^2 - \left(\sum_{i=1}^m \sqrt{r_i} c_i \right)^2 \right] \right) (\chi_{m-1; 1-\alpha}^2).$$

We conclude this section by attracting the reader's attention to the fact that this multi-sample problem here complements, for the FvML case, the ANOVA study in Ley, Swan and Verdebout (2013).

In the next section, we study the empirical powers of the tests constructed here via Monte Carlo simulations.

5 Monte Carlo simulations

In this section, we shall concentrate both on the Rayleigh (1919) test of uniformity and on the Watamori and Jupp (2005) multi-sample test. For these two tests, we complement the theoretical powers provided in the previous sections by a simulation study of their empirical powers.

5.1 Power curve of the Rayleigh (1919) test

The aim of this subsection is to corroborate Proposition 3.3 and the ensuing power curves by showing that empirical power curves, even for reasonably small sample sizes n , are close to the theoretical ones. To do so, we generated $N = 5,000$ independent replications of circular FvML (hence, von Mises) random vectors

$$\mathbf{X}_{c;i}, \quad c = 0, \dots, 10, \quad i = 1, \dots, n,$$

with concentration $n^{-1/2}c/2$ and location $\boldsymbol{\theta} = (1, 0)'$. The vectors $\mathbf{X}_{0;i}$ represent the null hypothesis while the vectors $\mathbf{X}_{c;i}$ for $c = 1, \dots, 10$ are (increasingly) under the alternative. The results using sample sizes $n = 10$, $n = 20$ and $n = 100$ are plotted in Figure 2. They clearly confirm the theoretical power curves and hence Proposition 3.3.

Figure 2: Power curve of $\phi_{\text{unif}}^{(n)}$ for $k = 2$ and various sample sizes $n = 10$, $n = 20$ and $n = 100$. The “n=inf” curve is the true (theoretical) power curve (obtained using Proposition 3.3).

5.2 Power curve of the most stringent test for the homogeneity of concentrations

In this subsection, we investigate the empirical behavior of the test $\phi_{\text{Hom}}^{(n)}$ for the homogeneity of the concentrations. We generated $N = 5,000$ independent replications of three pairs ($m = 2$) of mutually independent samples (we considered two designs; first with respective, and relatively small, sizes $n_1 = 100$ and $n_2 = 150$ and then with respective moderate sample sizes $n_1 = 500$ and $n_2 = 500$) of circular random vectors

$$\mathbf{X}_{\ell;1j_1} \quad \text{and} \quad \mathbf{X}_{\ell;c;2j_2}, \quad \ell = 1, 2, 3, \quad c = 0, \dots, 10, \quad j_i = 1, \dots, n_i, \quad i = 1, 2,$$

with FvML densities such that

- (i) $\mathbf{X}_{1;1j_1}$ and $\mathbf{X}_{10;2j_2}$ have a common concentration $\kappa = 1$ and locations $\boldsymbol{\theta}_1 = (1, 0)'$ and $\boldsymbol{\theta}_2 = (-1, 0)'$. Then for $c = 1, \dots, 10$, the $\mathbf{X}_{1c;2j_2}$'s have concentration $1 + c/10$ and still locations $\boldsymbol{\theta}_1 = (1, 0)'$ and $\boldsymbol{\theta}_2 = (-1, 0)'$.
- (ii) $\mathbf{X}_{2;1j_1}$ and $\mathbf{X}_{20;2j_2}$ have a common concentration $\kappa = 5$ and locations $\boldsymbol{\theta}_1 = (1, 0)'$ and $\boldsymbol{\theta}_2 = (-1, 0)'$. Then for $c = 1, \dots, 10$, the $\mathbf{X}_{2c;2j_2}$'s have concentration $5 + c/10$ and still locations $\boldsymbol{\theta}_1 = (1, 0)'$ and $\boldsymbol{\theta}_2 = (-1, 0)'$.
- (iii) $\mathbf{X}_{3;1j_1}$ and $\mathbf{X}_{30;2j_2}$ have a common concentration $\kappa = 10$ and locations $\boldsymbol{\theta}_1 = (1, 0)'$ and $\boldsymbol{\theta}_2 = (-1, 0)'$. Then for $c = 1, \dots, 10$, the $\mathbf{X}_{3c;2j_2}$'s have concentration $10 + c/10$ and still locations $\boldsymbol{\theta}_1 = (1, 0)'$ and $\boldsymbol{\theta}_2 = (-1, 0)'$.

For all $\ell = 1, 2, 3$, the random vectors $\mathbf{X}_{\ell;1j_1}$ and $\mathbf{X}_{\ell 0;2j_2}$ are under the null hypothesis. Then, for $c = 1, \dots, 10$, the random vectors $\mathbf{X}_{\ell c;2j_2}$ are (increasingly) under the alternative. The results are plotted in Figure 3. Inspection of the Figure reveals that the test $\phi_{\text{Hom}}^{(n)}$ reaches the nominal level constraint even with small sample sizes. The power of the test decreases when the concentration increases.

A Appendix: Proofs

Proof of Proposition 2.1 We clearly need to circumvent the curved nature of the parameter space $\mathbb{R}_0^+ \times \mathcal{S}^{k-1}$, more precisely of $\mathcal{S}^{k-1}$. Fortunately, this has been achieved in Ley *et al.* (2013) by proving ULAN rather for the spherical coordinates $\boldsymbol{\theta} = h(\boldsymbol{\eta})$ for some locally full rank chart h and then returning (via a result in Hallin *et al.* 2010) to the initial $\boldsymbol{\theta}$ -parameterization. Thus, thanks to the developments

Figure 3: Empirical power curves of $\phi_{\text{Hom}}^{(n)}$ for $k = 2$, various concentration parameters $\kappa = 1$ (dashed lines), $\kappa = 5$ (dotted lines) and $\kappa = 10$ (dash-dotted lines) and sample sizes $n_1 = 100$ and $n_2 = 150$ (left figure) and $n_1 = 500$ and $n_2 = 500$ (right figure)

in Ley *et al.* (2013), all we need to do here is to prove ULAN with respect to the “linear” parameters κ and $\boldsymbol{\eta}$ with $\boldsymbol{\eta} \in \mathbb{R}^{k-1}$.

This is very simple since we are not working within a semiparametric family of distributions (hence we do not have to deal with an infinite-dimensional parameter); the problem considered involves a parametric family of distributions with densities meeting the most classical regularity conditions. In particular, one readily obtains that (i) $(\kappa, \boldsymbol{\eta}) \mapsto \sqrt{c_\kappa \exp(\kappa \mathbf{x}' h(\boldsymbol{\eta}))}$ is continuously differentiable for every $\mathbf{x} \in \mathcal{S}^{k-1}$ and (ii) the associated Fisher information matrix is well defined and continuous in κ and $\boldsymbol{\eta}$. Thus, by Lemma 7.6 of van der Vaart (1998), $(\kappa, \boldsymbol{\eta}) \mapsto \sqrt{c_\kappa \exp(\kappa \mathbf{x}' h(\boldsymbol{\eta}))}$ is differentiable in quadratic mean, and the ULAN property follows from Theorem 7.2 of van der Vaart (1998). This completes the proof. $\square$

Proof of Proposition 3.1 First note that, combining the fact that $\boldsymbol{\theta}$ and $\hat{\boldsymbol{\theta}}^{(n)}$ have norm 1 with the delta method applied to the mapping $\mathbf{x} \mapsto \mathbf{x}/\|\mathbf{x}\|$, we have that

$$n^{1/2}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) = n^{1/2} \left(\frac{\hat{\boldsymbol{\theta}}}{\|\hat{\boldsymbol{\theta}}\|} - \frac{\boldsymbol{\theta}}{\|\boldsymbol{\theta}\|} \right) = (\mathbf{I}_k - \boldsymbol{\theta}\boldsymbol{\theta}')n^{1/2}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) + o_P(1) \quad (\text{A.9})$$

under $P_{\boldsymbol{\vartheta}}^{(n)}$ as $n \rightarrow \infty$. Now, the law of large numbers, the fact that $E[\mathbf{X}_i] = E(\mathbf{X}_i'\boldsymbol{\theta})\boldsymbol{\theta}$ and (A.9) readily entail that

$$T_{\hat{\boldsymbol{\theta}}}^{(n)} - T_{\boldsymbol{\theta}}^{(n)} = \bar{\mathbf{X}}'n^{1/2}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) = (E[\mathbf{X}_i])'(\mathbf{I}_k - \boldsymbol{\theta}\boldsymbol{\theta}')n^{1/2}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) + o_P(1) = o_P(1)$$

under $P_{\boldsymbol{\vartheta}}^{(n)}$ as $n \rightarrow \infty$, which is the desired result. $\square$

Proof of Proposition 4.2 We first show Point (i). Since $\hat{\boldsymbol{\theta}}_i$ is a root- n_i consistent estimator of $\boldsymbol{\theta}_i$, following the proof of Proposition 3.1, we readily have that $\bar{\mathbf{X}}_i'n_i^{1/2}(\hat{\boldsymbol{\theta}}_i - \boldsymbol{\theta}_i)$ is $o_P(1)$ under $P_{\boldsymbol{\vartheta}^{(m)}}^{(n)}$ as $n \rightarrow \infty$. As a direct consequence, letting $\hat{\boldsymbol{\vartheta}}^{(m)} := (\hat{\boldsymbol{\theta}}_1', \dots, \hat{\boldsymbol{\theta}}_m', \hat{\kappa}, \dots, \hat{\kappa})'$, we obviously have that

$$\Delta_{\hat{\boldsymbol{\vartheta}}^{(m)}}^{(\text{I},i)(n)} - \Delta_{\boldsymbol{\vartheta}^{(m)}}^{(\text{I},i)(n)} = -(r_i^{(n)})^{1/2}n^{1/2}(A_k(\hat{\kappa}) - A_k(\kappa)) + o_P(1)$$

under $P_{\boldsymbol{\vartheta}^{(m)}}^{(n)}$ with $\boldsymbol{\vartheta}^{(m)} \in \mathcal{H}_0^{\text{Hom}}$ as $n \rightarrow \infty$. It follows directly that from the delta method and from Mardia and Jupp (2000, p. 199) that

$$\begin{aligned} \Delta_{\hat{\boldsymbol{\vartheta}}^{(m)}}^{(\text{I})(n)} - \Delta_{\boldsymbol{\vartheta}^{(m)}}^{(\text{I})(n)} &= -\boldsymbol{\Upsilon}^{(n)}n^{1/2}(A_k(\hat{\kappa}) - A_k(\kappa)) + o_P(1) \\ &= -\boldsymbol{\Upsilon}^{(n)}A_k'(\kappa)n^{1/2}(\hat{\kappa} - \kappa) + o_P(1) \\ &= -\boldsymbol{\Upsilon}^{(n)}D_kn^{1/2}(\hat{\kappa} - \kappa) + o_P(1) \\ &= -\boldsymbol{\Gamma}_{\boldsymbol{\vartheta}^{(m)}}^{(\text{I})}\boldsymbol{\Upsilon}^{(n)}n^{1/2}(\hat{\kappa} - \kappa) + o_P(1) \end{aligned} \quad (\text{A.10})$$

still under $P_{\boldsymbol{\vartheta}^{(m)}}^{(n)}$ as $n \rightarrow \infty$ with $\boldsymbol{\vartheta}^{(m)}$ such that all κ_i components are equal (that

is, we are under $\mathcal{H}_0^{\text{Hom}}$). Therefore, defining

$$(\mathbf{\Gamma}_{\boldsymbol{\vartheta}^{(m)}}^{(I)})^\perp := (\mathbf{\Gamma}_{\boldsymbol{\vartheta}^{(m)}}^{(I)})^{-1} - \mathbf{\Upsilon}^{(n)} \left((\mathbf{\Upsilon}^{(n)})' \mathbf{\Gamma}_{\boldsymbol{\vartheta}^{(m)}}^{(I)} \mathbf{\Upsilon}^{(n)} \right)^{-1} (\mathbf{\Upsilon}^{(n)})',$$

the consistency of $\hat{\boldsymbol{\vartheta}}^{(m)}$ together with (A.10) entails that

$$\begin{aligned} Q_{\text{Hom}}^{(n)} &= Q_{\text{Hom}}^{(n)}(\hat{\boldsymbol{\vartheta}}^{(m)}) \\ &= (\Delta_{\boldsymbol{\vartheta}^{(m)}}^{(I)})' (\mathbf{\Gamma}_{\boldsymbol{\vartheta}^{(m)}}^{(I)})^\perp \Delta_{\boldsymbol{\vartheta}^{(m)}}^{(I)} \\ &= (\Delta_{\boldsymbol{\vartheta}^{(m)}}^{(I)} - \mathbf{\Gamma}_{\boldsymbol{\vartheta}^{(m)}}^{(I)} \mathbf{\Upsilon}^{(n)} n^{1/2} (\hat{\kappa} - \kappa))' (\mathbf{\Gamma}_{\boldsymbol{\vartheta}^{(m)}}^{(I)})^\perp (\Delta_{\boldsymbol{\vartheta}^{(m)}}^{(I)} - \mathbf{\Gamma}_{\boldsymbol{\vartheta}^{(m)}}^{(I)} \mathbf{\Upsilon}^{(n)} n^{1/2} (\hat{\kappa} - \kappa)) + o_P(1) \\ &= (\Delta_{\boldsymbol{\vartheta}^{(m)}}^{(I)})' (\mathbf{\Gamma}_{\boldsymbol{\vartheta}^{(m)}}^{(I)})^\perp \Delta_{\boldsymbol{\vartheta}^{(m)}}^{(I)} + o_P(1) \\ &= Q_{\text{Hom}}^{(n)}(\boldsymbol{\vartheta}^{(m)}) + o_P(1) \end{aligned}$$

under $P_{\boldsymbol{\vartheta}^{(m)}}^{(n)}$ as $n \rightarrow \infty$. Then, Point (i) directly follows from the asymptotic normality of $\Delta_{\boldsymbol{\vartheta}^{(m)}}^{(I)}$ in Proposition 4.1 and from the fact that $(\mathbf{\Gamma}_{\boldsymbol{\vartheta}^{(m)}}^{(I)})^\perp \mathbf{\Gamma}_{\boldsymbol{\vartheta}^{(m)}}^{(I)}$ is idempotent with trace $m - 1$. Point (ii) follows by applying Le Cam's third Lemma as in Proposition 3.3. For Point (iii), see Le Cam (1986) or Hallin and Paindaveine (2008) for a more recent reference. $\square$

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