THE $\mathbb{Z}_2^n$ DIRAC–DUNKL OPERATOR
AND A HIGHER RANK BANNAI–ITO ALGEBRA

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ABSTRACT. The kernel of the $\mathbb{Z}_2^n$ Dirac–Dunkl operator is examined. The symmetry algebra $\mathcal{A}_n$ of the associated Dirac–Dunkl equation on $\mathbb{S}^{n-1}$ is determined and is seen to correspond to a higher rank generalization of the Bannai–Ito algebra. A basis for the polynomial null-solutions of the Dirac–Dunkl operator is constructed. The basis elements are joint eigenfunctions of a maximal commutative subalgebra of $\mathcal{A}_n$ and are given explicitly in terms of Jacobi polynomials. The symmetry algebra is shown to act irreducibly on this basis via raising/lowering operators. A scalar realization of $\mathcal{A}_n$ is proposed.

1. INTRODUCTION

The goal of the present paper is to examine the kernel of the n-dimensional Dirac–Dunkl operator with $\mathbb{Z}_2^n$ reflection group and to study the corresponding Dirac–Dunkl equation on the $(n-1)$-sphere. The invariance algebra $\mathcal{A}_n$ of this equation will be obtained and identified as a higher rank generalization of the Bannai–Ito algebra (see Definition 1). Basis functions for the space of homogeneous polynomial null solutions of the Dirac–Dunkl operator will be constructed using a Cauchy–Kovalevskaia extension theorem and will be given explicitly in terms of Jacobi polynomials. This set of basis functions, characterized as the joint eigenvectors of a maximal commutative subalgebra of $\mathcal{A}_n$, will be shown to transform irreducibly under $\mathcal{A}_n$ through the action of raising and lowering operators.

1.1. The $\mathbb{Z}_2^n$ Laplace–Dunkl operator on $\mathbb{R}^n$.
Dunkl operators are families of differential-difference operators associated with finite reflection groups. Introduced in [6], these operators appear in many areas: they are at the core of the study of multivariate special functions associated with root systems [7, 18], they arise in harmonic analysis and integral transforms [3, 20], they are closely related to representations of double affine Hecke algebras [5], they are at the origin of Dunkl processes [14], and they play a central role in the analysis of Calogero–Sutherland systems [22]. Consider the Abelian reflection group $\mathbb{Z}_2^n = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$; the corresponding Dunkl operators $T_1, \ldots, T_n$ acting on $\mathbb{R}^n$ are defined as follows:

$$T_i = \partial_{x_i} + \frac{\mu_i}{x_i}(1 - r_i), \qquad i = 1, \ldots, n,$$

where $\mu_1, \ldots, \mu_n$ with $\mu_i > 0$ are real parameters, ∂_{x_i} is the partial derivative with respect to x_i and where r_i is the reflection operator in the $x_i = 0$ hyperplane, i.e. $r_i f(x_i) = f(-x_i)$. It is obvious that one has $T_i T_j = T_j T_i$ for all $i, j \in \{1, \ldots, n\}$.

The Laplace–Dunkl operator Δ associated to $\mathbb{Z}_2^n$ is defined by

$$\Delta = \sum_{i=1}^{n} T_i^2.$$

1.2. The $\mathbb{Z}_2^n$ Dirac–Dunkl operator on $\mathbb{R}^n$.

The Clifford algebra $\mathcal{C}\ell_n$ of negative signature is generated by the elements $e_1, \ldots, e_n$ which satisfy the defining relations

$$(1) \qquad \{e_i, e_j\} = -2\delta_{ij}, \qquad i, j \in \{1, \ldots, n\},$$

where $\{x, y\} = xy + yx$ stands for the anticommutator and where δ_{ij} is the Kronecker delta. The Dirac–Dunkl operator $\underline{D}$ on $\mathbb{R}^n$ associated to the reflection group $\mathbb{Z}_2^n$ and the position operator $\underline{x}$ are respectively defined as

$$(2) \qquad \underline{D} = \sum_{i=1}^{n} e_i T_i, \qquad \underline{x} = \sum_{i=1}^{n} e_i x_i.$$

The Dirac–Dunkl operator $\underline{D}$ squares to the Laplace–Dunkl operator; indeed, it is verified that $\underline{D}^2 = -\Delta$. One has also $\underline{x}^2 = -\|x\|^2$, where $\|x\|^2 = \sum_{i=1}^{n} x_i^2$. The "intermediate" Dirac operators and position operators are defined as follows. Let $A \subset [n]$, where $[n] = \{1, \ldots, n\}$, and define

$$(3) \qquad \underline{D}_A = \sum_{i \in A} e_i T_i, \qquad \underline{x}_A = \sum_{i \in A} e_i x_i.$$

In similar fashion, define

$$\Delta_A = \sum_{i \in A} T_i^2, \qquad \|x_A\|^2 = \sum_{i \in A} x_i^2.$$

In this notation, the "full" n-dimensional Dirac–Dunkl operator $\underline{D}$ given in (2) can be written as $\underline{D}_{[n]}$; similarly, one has $\underline{x} = \underline{x}_{[n]}$. For $\ell \leq n$, we shall also use the notation $\underline{D}_{[\ell]}$ and $\underline{x}_{[\ell]}$ for $\underline{D}_{\{1,\ldots,\ell\}}$ and $\underline{x}_{\{1,\ldots,\ell\}}$, respectively.

1.3. The $\mathbb{Z}_2^n$ Dirac–Dunkl operator on $\mathbb{S}^{n-1}$.

The Dirac–Dunkl operator on the $(n-1)$-sphere and its "intermediate" analogs are most naturally defined in the context of the $\mathfrak{osp}(1|2)$ realizations generated by the intermediate Dirac–Dunkl operators and position operators $\underline{D}_A$ and $\underline{x}_A$. These realizations, which occur for any root system, are due to [19] (see also [3]). For the particular case of $\mathbb{Z}_2^n$, one has the following.

Proposition 1 ([3]). *For $A \subset [n]$, the operators $\underline{D}_A$ and $\underline{x}_A$ generate the Lie superalgebra $\mathfrak{osp}(1|2)$ with defining relations*

$$(4) \qquad \begin{aligned} \{\underline{x}_A, \underline{x}_A\} &= -2\|x_A\|^2 & \{\underline{D}_A, \underline{D}_A\} &= -2\Delta_A \\ \{\underline{x}_A, \underline{D}_A\} &= -2(\mathbb{E}_A + \gamma_A) & [\underline{D}_A, \mathbb{E}_A + \gamma_A] &= \underline{D}_A \\ [\underline{D}_A, \|x_A\|^2] &= 2\underline{x}_A & [\mathbb{E}_A + \gamma_A, \underline{x}_A] &= \underline{x}_A \\ [\Delta_A, \underline{x}_A] &= 2\underline{D}_A & [\Delta_A, \mathbb{E}_A + \gamma_A] &= 2\Delta_A \\ [\Delta_A, \|x_A\|^2] &= 4(\mathbb{E}_A + \gamma_A) & [\mathbb{E}_A + \gamma_A, \|x_A\|^2] &= 2\|x_A\|^2 \end{aligned}$$

where $\mathbb{E}_A = \sum_{i \in A} x_i \partial_{x_i}$ is the Euler operator for the set A, $[x, y] = xy - yx$ stands for the commutator and where

$$(5) \qquad \gamma_A = \frac{|A|}{2} + \sum_{i \in A} \mu_i.$$

The sCasimir operator $\underline{S}_A$ for the $\mathfrak{osp}(1|2)$ realization (4) is defined as [17]

$$(6) \qquad \underline{S}_A = \frac{1}{2}\Big([\underline{x}_A, \underline{D}_A] - 1\Big).$$

As follows from (4), this operator satisfies the anticommutation relations

$$(7) \qquad \{\underline{S}_A, \underline{D}_A\} = 0, \qquad \{\underline{S}_A, \underline{x}_A\} = 0.$$

The Dirac–Dunkl operator on $\mathbb{S}^{n-1}$, which will also be referred to as the spherical Dirac–Dunkl operator, will be denoted by $\Gamma_{[n]}$. It is defined as

$$(8) \qquad \Gamma_{[n]} = \underline{S}_{[n]} \prod_{i=1}^{n} r_i.$$

Since one has $\{\prod_{i\in A} r_i, \underline{x}_A\} = \{\prod_{i\in A} r_i, \underline{D}_A\} = 0$, it follows from (7) that $\Gamma_{[n]}$ commutes with the full Dirac–Dunkl operator $\underline{D}$ and position operator $\underline{x}$; that is

$$(9) \qquad [\Gamma_{[n]}, \underline{D}] = 0, \quad [\Gamma_{[n]}, \underline{x}] = 0.$$

Since $\Gamma_{[n]}$ also commutes with the Euler operator $\mathbb{E}_{[n]}$ on $\mathbb{R}^n$, it has a well defined action on the $(n-1)$-dimensional sphere. Similarly, the intermediate spherical Dirac–Dunkl operators Γ_A for $A \subset [n]$ are defined by

$$(10) \qquad \Gamma_A = \underline{S}_A \prod_{i\in A} r_i.$$

1.4. **Dunkl monogenics and the spherical Dirac–Dunkl equation.** Consider $\mathcal{P}(\mathbb{R}^n) = \mathbb{R}[x_1, \dots, x_n]$, the ring of polynomials in the variables $x_1, \dots, x_n$. One has $\mathcal{P}(\mathbb{R}^n) = \bigoplus_{k=0}^{\infty} \mathcal{P}_k(\mathbb{R}^n)$, where $\mathcal{P}_k(\mathbb{R}^n)$ is the space of homogeneous polynomials of degree k in $x_1, \dots, x_n$. Let V be a representation space for $\mathcal{Cl}_n$, fixed once and for all. For example, one could fix V as the spinor space or as the Clifford algebra itself. The space $\mathcal{M}_k(\mathbb{R}^n; V)$ of Dunkl monogenics of degree k associated with the reflection group $\mathbb{Z}_2^n$ is defined as

$$\mathcal{M}_k(\mathbb{R}^n; V) = \mathrm{Ker}\, \underline{D} \cap (\mathcal{P}_k(\mathbb{R}^n) \otimes V).$$

Similarly, the space $\mathcal{H}_k(\mathbb{R}^n)$ of Dunkl harmonics of degree k is defined as $\mathcal{H}_k(\mathbb{R}^n) = \mathrm{Ker}\, \Delta \cap \mathcal{P}_k(\mathbb{R}^n)$. The space of V-valued Dunkl harmonics of degree k decomposes as follows [19]:

$$\mathcal{H}_k(\mathbb{R}^n) \otimes V = \mathcal{M}_k(\mathbb{R}^n; V) \oplus \underline{x}\, \mathcal{M}_{k-1}(\mathbb{R}^n; V).$$

This leads to the Fischer decomposition of V-valued homogeneous polynomials.

Proposition 2 ([19])**.** *The space $\mathcal{P}_k(\mathbb{R}^n) \otimes V$ of V-valued homogeneous polynomials of degree k has the direct sum decomposition*

$$\mathcal{P}_k(\mathbb{R}^n) \otimes V = \bigoplus_{j=0}^{k} \underline{x}^j\, \mathcal{M}_{k-j}(\mathbb{R}^n; V).$$

The next proposition states that the space of Dunkl monogenics $\mathcal{M}_k(\mathbb{R}^n; V)$ is an eigenspace for the spherical Dirac–Dunkl operator $\Gamma_{[n]}$. This means in particular that Dunkl monogenics of degree k satisfy a Dirac–Dunkl equation on $\mathbb{S}^{n-1}$.

Proposition 3. *Let $\Psi_k \in \mathcal{M}_k(\mathbb{R}^n; V)$ be a monogenic polynomial of degree k, then Ψ_k satisfies the spherical Dirac–Dunkl equation*

$$\Gamma_{[n]} \Psi_k = (-1)^k (k + \gamma_{[n]} - 1/2) \Psi_k, \tag{11}$$

where $\gamma_{[n]}$ is as in (5).

Proof. By a direct calculation, one finds

$$
\begin{aligned}
\Gamma_{[n]} \Psi_k &= \frac{(-1)^k}{2} \big(\underline{x}_{[n]} \underline{D}_{[n]} - \underline{D}_{[n]} \underline{x}_{[n]} - 1 \big) \Psi_k = \frac{(-1)^k}{2} \big(- \underline{D}_{[n]} \underline{x}_{[n]} - 1 \big) \Psi_k \\
&= \frac{(-1)^k}{2} \big(- \underline{x}_{[n]} \underline{D}_{[n]} - \underline{D}_{[n]} \underline{x}_{[n]} - 1 \big) \Psi_k = \frac{(-1)^{k+1}}{2} \big(\{ \underline{x}_{[n]}, \underline{D}_{[n]} \} + 1 \big) \Psi_k \\
&= \frac{(-1)^{k+1}}{2} \big(- 2 (\mathbb{E}_{[n]} + \gamma_{[n]}) + 1 \big) \Psi_k = (-1)^k (k + \gamma_{[n]} - 1/2) \Psi_k,
\end{aligned}
$$

where we successively used the definition (8), the relation $(\prod_{j=1}^n r_j) p = (-1)^k p$ valid for $p \in \mathcal{P}_k(\mathbb{R}^n)$, the kernel property $\underline{D} \Psi_k = 0$, the $\mathfrak{osp}(1|2)$ relations (4) and the relation $\mathbb{E}_{[n]} q = k \, q$ valid for $q \in \mathcal{P}_k(\mathbb{R}^n)$. $\square$

One of the main results of this paper is the presentation and characterization of the symmetry algebra of the Dirac–Dunkl equation (11) associated with the kernel of the n-dimensional Dirac–Dunkl operator $\underline{D}$. This novel abstract structure, defined in Section 2, will be identified as a higher rank generalization of the Bannai–Ito algebra. Let us now introduce the basics of this algebra in the rank-one case.

1.5. **The Bannai–Ito algebra.** The Bannai–Ito algebra was originally introduced in [21] as the algebraic structure underlying the bispectral property of the Bannai–Ito polynomials, which sit atop of the hierarchy of "-1" orthogonal polynomials [9, 21]. This algebra has generators A_1, A_2, A_3 and defining relations

$$\{A_1, A_2\} = A_3 + \alpha_3, \qquad \{A_2, A_3\} = A_1 + \alpha_1, \qquad \{A_3, A_1\} = A_2 + \alpha_2, \tag{12}$$

where $\alpha_1, \alpha_2, \alpha_3$ are central elements. The algebra (12) has appeared in [11] as a duality algebra for the Racah problem of $\mathfrak{osp}(1|2)$ (then called $sl_{-1}(2)$) and in [10, 12] as an invariance algebra for superintegrable systems with reflections; when $\alpha_i = 0$, it can be viewed as the $q \to -1$ limit of $U'_q(\mathfrak{so}(3))$ [8]. Recently, it has also been recognized that the Bannai–Ito algebra corresponds to a degeneration of the rank-one double affine Hecke algebra of type $(\check{C}_1, C_1)$ [13].

In the paper [2] by the authors, it was shown that the Bannai–Ito algebra is the symmetry algebra of the Dirac–Dunkl equation on the 2-sphere associated to the $\mathbb{Z}_2^3$ reflection group. In that context, relevant finite-dimensional representations of (12) were constructed, explicit formulas for the basis vectors of these representations were found, and a connection with the Bannai–Ito polynomials was established.

1.6. **Goal and outline.** The main goal of the present paper is to extend the results of [2] to arbitrary dimension. As shall be seen this extension is involved, and yet the results are instructive and lend themselves to an elegant presentation. The outline is as follows. In Section two, the symmetry algebra of the Dirac–Dunkl equation is investigated. In Section three, an explicit basis for the space of Dunkl monogenics is constructed. In Section four, the higher rank Bannai–Ito algebra obtained in Section two is shown to act irreducibly on the basis found in Section three. In Section five, a scalar version of the model is presented. A short conclusion follows.

2. Symmetry algebra of the Dirac–Dunkl operator

In this section, the joint symmetries of the Dirac–Dunkl and spherical Dirac–Dunkl operators are obtained. These symmetries are shown to close under anti-commutation. The resulting invariance algebra $\mathcal{A}_n$ is interpreted as a higher rank extension of the Bannai–Ito algebra. Two Casimir operators are exhibited.

Let us begin by showing that the intermediate spherical Dirac–Dunkl operators Γ_A for $A \subset [n]$ introduced in (10) are in fact symmetries of $\underline{D}$ and $\Gamma_{[n]}$.

Lemma 1. *For $A \subset [n]$, the operator Γ_A defined in (10) satisfies*

i) $[\Gamma_A, \underline{D}] = [\Gamma_A, \underline{x}] = 0$

ii) $[\Gamma_A, \Gamma_{[n]}] = 0$

where $\underline{D}$ and $\underline{x}$ are as in (2).

Proof. For $A \subset [n]$, one has $\underline{D} = \underline{D}_A + \underline{D}_{[n]\setminus A}$ and $\underline{x} = \underline{x}_A + \underline{x}_{[n]\setminus A}$. Consequently, one can write

$$[\Gamma_A, \underline{D}] = [\Gamma_A, \underline{D}_A] + [\Gamma_A, \underline{D}_{[n]\setminus A}] = [\Gamma_A, \underline{D}_A] = 0,$$

where the relations (1) and the property (7) were used; the result for $\underline{x}$ is proven in a similar fashion, yielding *i)*. In view of the definition (10) of Γ_A, *ii)* is seen to follow directly from *i)*. $\qquad\square$

Remark 1. When $A = \emptyset$ or $A = \{k\}$, it follows from the definition (10) that

$$(13) \qquad\qquad \Gamma_\emptyset = -\frac{1}{2}, \qquad \Gamma_{\{k\}} = \mu_k.$$

For two distinct subsets $A, B \subset [n]$, it is seen that the operators Γ_A and Γ_B will generally not commute. The symmetries Γ_A for $A \subset [n]$ hence generate the non-Abelian invariance (or symmetry) algebra of the Dirac–Dunkl and spherical Dirac–Dunkl operators. For concreteness, we give below the explicit expression for the intermediate spherical Dirac–Dunkl operators. For $A \subset [n]$, Γ_A has the form

$$(14) \qquad\qquad \Gamma_A = \left(\sum_{\{i,j\} \subset A} M_{ij} + \frac{|A| - 1}{2} + \sum_{k \in A} \mu_k r_k \right) \prod_{i \in A} r_i,$$

where the first summation is performed on all 2-subsets of A and where M_{ij} reads

$$(15) \qquad\qquad M_{ij} = e_i e_j (x_i T_j - x_j T_i).$$

As is seen from (14), Γ_A is invariant under permutations of the indices.

We now prove a series of lemmas unveiling the commutation relations between the symmetry operators Γ_A.

Lemma 2. *For $A \subset [n]$, $j \in A$ and $k \notin A$, one has*

$$(16) \qquad \{\Gamma_A, \Gamma_{\{j,k\}}\} = \Gamma_{(A\cup\{k\})\setminus\{j\}} + 2\,\Gamma_{\{j\}}\Gamma_{A\cup\{k\}} + 2\,\Gamma_{\{k\}}\Gamma_{A\setminus\{j\}}.$$

Proof. The proof of (16) follows by a direct calculation using (14). For simplicity, one can perform the calculation with $A = [\ell]$ and $\{j, k\} = \{1, \ell+1\}$ and extend the result by symmetry. Proceeding with this calculation, it is useful to observe that

$$\{M_{ij}r_i r_j, M_{jk}r_j r_k\} = (1 + 2\mu_j r_j)M_{ik}r_i r_k, \qquad i \neq k.$$

$$\square$$

Lemma 3. *For $A \subset B$, $B \subset A$ or $A \cap B = \emptyset$, one has*

$$[\Gamma_A, \Gamma_B] = 0.$$

Proof. When $A \subset B$ or $B \subset A$, the result follows directly from lemma 1, with $\underline{D}$, $\underline{x}$ and $\Gamma_{[n]}$ replaced by $\underline{D}_B$, $\underline{x}_B$ and Γ_B or $\underline{D}_A$, $\underline{x}_A$ and Γ_A, respectively. When $A \cap B = \emptyset$, the result is directly obtained using the explicit expression (14) for the intermediate spherical Dirac–Dunkl operators. $\square$

Lemma 4. *For $A, B \subset [n]$ such that $A \cap B = \{k\}$, one has*

(17) $$\{\Gamma_A, \Gamma_B\} = \Gamma_{(A \cup B) \setminus \{k\}} + 2\Gamma_{\{k\}}\Gamma_{A \cup B} + 2\Gamma_{A \setminus \{k\}}\Gamma_{B \setminus \{k\}}.$$

Proof. By induction on $|B|$. When $|B| = 1$, the result follows from (13). When $|B| = 2$, the result holds by Lemma 2. Suppose that the result holds for $|B| = n$ and consider the set $\widetilde{B} = B \cup \{x\}$ with $x \notin A$, $A \cap B = \{k\}$ and $|B| = n$. Let $y \in B$ with $y \notin A$, it follows from Lemma 2 that

(18) $$\Gamma_{\widetilde{B}} = \frac{1}{2\mu_y} \left(\{\Gamma_B, \Gamma_{\{x,y\}}\} - \Gamma_{\widetilde{B} \setminus \{y\}} - 2\mu_x \Gamma_{B \setminus \{y\}} \right).$$

Using the fact that $[\Gamma_A, \Gamma_{\{x,y\}}] = 0$ and the identity $\{A, \{B, C\}\} = \{\{A, B\}, C\}$ which holds when $[A, C] = 0$, one can write

(19) $$\{\Gamma_A, \Gamma_{\widetilde{B}}\} = \frac{1}{2\mu_y} \left(\{\{\Gamma_A, \Gamma_B\}, \Gamma_{\{x,y\}}\} - \{\Gamma_A, \Gamma_{\widetilde{B} \setminus \{y\}}\} - 2\mu_x \{\Gamma_A, \Gamma_{B \setminus \{y\}}\} \right).$$

Each anticommutator appearing in the above expression can be evaluated using the induction hypothesis. Upon expanding the result, one directly finds (17) with B replaced by $\widetilde{B}$. This completes the induction. $\square$

Lemma 5. *For $A, B \subset [n]$ such that $A \cap B \neq \emptyset$, one has*

(20) $$\{\Gamma_A, \Gamma_B\} = \Gamma_{(A \cup B) \setminus (A \cap B)} + 2\Gamma_{A \cap B}\Gamma_{A \cup B} + 2\Gamma_{A \setminus (A \cap B)}\Gamma_{B \setminus (A \cap B)}.$$

Proof. By induction on $|A \cap B|$. When $|A \cap B| = 1$, the result holds by virtue of Lemma 4. Suppose that (20) holds at level n and consider the set $\widetilde{B} = B \cup \{x\}$ with $x \in A$ and $|A \cap B| = n$. Consider $y \in A \cap B$; one can write $\Gamma_{\widetilde{B}}$ as in (18). Since $x, y \in A$, Γ_A commutes with $\Gamma_{\{x,y\}}$ and one can write $\{\Gamma_A, \Gamma_{\widetilde{B}}\}$ as in (19), the only difference with (19) being that $x, y \in A$. Upon applying the induction hypothesis, one finds (20) with B replaced by $\widetilde{B}$ after a straightforward calculation. $\square$

The results of the five preceding Lemmas can now be combined to give the complete set of relations between the symmetries Γ_A.

Proposition 4. *The symmetries Γ_A with $A \subset [n]$ of the Dirac–Dunkl and spherical Dirac–Dunkl operators satisfy the anticommutation relations*

(21) $$\{\Gamma_A, \Gamma_B\} = \Gamma_{(A \cup B) \setminus (A \cap B)} + 2\Gamma_{A \cap B}\Gamma_{A \cup B} + 2\Gamma_{A \setminus (A \cap B)}\Gamma_{B \setminus (A \cap B)}.$$

Proof. The proposition follows from the combination of Lemmas 2–5. $\square$

This result motivates the following definition for the abstract symmetry algebra of the spherical Dirac–Dunkl operator $\Gamma_{[n]}$, which shall be denoted by $\mathcal{A}_n$.

Definition 1. Let $n \geq 3$ and $A \subset [n]$. Call $\mathcal{A}_n$ the abstract associative algebra with generators Γ_A and defining relations (21).

Remark 2. It is visible from Definition 1 that one has $\mathcal{A}_n \subset \mathcal{A}_{n+1}$. That is $\mathcal{A}_n$ is a subalgebra of $\mathcal{A}_{n+1}$.

We now show that operators of the form $\Gamma_{\{i,j\}}$ are sufficient to generate the entire invariance algebra .

Lemma 6. *The set of operators Γ_B for which B is a 2-subset of $[n]$ constitutes a generating set for $\mathcal{A}_n$.*

Proof. First observe that any generator Γ_A with $|A| = m$ where $m > 2$ can be written as a polynomial in generators $\Gamma_{A'}, \Gamma_{A''}$ with $|A'| = m - 1$ and $|A''| = 2$. This can be done (non uniquely) by choosing two sets C and D with $|C| = m - 1$ and $|D| = 2$ such that $C \cup D = A$ and $C \cap D = \{k\}$, and using the relation

$$\Gamma_A = \Gamma_{C \cup D} = \frac{1}{2\mu_k} \left(\{\Gamma_C, \Gamma_D\} - \Gamma_{(C \cup D) \setminus (C \cap D)} - 2\,\Gamma_{C \setminus (C \cap D)} \Gamma_{D \setminus (C \cap D)} \right).$$

Applying this procedure recursively, one can write any generator Γ_A with $|A| > 2$ as a polynomial in generators Γ_B with $|B| = 2$. $\qquad\square$

Remark 3. The algebra $\mathcal{A}_n$ can be considered as a higher rank generalization of the Bannai–Ito algebra (12). To see this, consider the case $n = 3$. The symmetry algebra $\mathcal{A}_3$ of the spherical Dirac–Dunkl operator in three dimensions $\Gamma_{[3]}$ is generated by the operators $K_3 = \Gamma_{\{1,2\}}$, $K_1 = \Gamma_{\{2,3\}}$ and $K_2 = \Gamma_{\{1,3\}}$ with relations

$$\{K_1, K_2\} = K_3 + \omega_3, \qquad \{K_2, K_3\} = K_1 + \omega_1, \qquad \{K_3, K_1\} = K_2 + \omega_2,$$

where ω_1, ω_2, ω_3 are given by

$$\omega_1 = 2\mu_1 \Gamma_{[3]} + 2\mu_2\mu_3, \quad \omega_2 = 2\mu_2 \Gamma_{[3]} + 2\mu_1\mu_3, \quad \omega_3 = 2\mu_3 \Gamma_{[3]} + 2\mu_1\mu_2.$$

It is clear that ω_1, ω_2 and ω_3 are central elements. Moreover, if one considers the realization of the algebra on the space of Dunkl monogenics $\mathcal{M}_k(\mathbb{R}^3; V)$, one can use the eigenvalue equation (11) to obtain $\mathbb{R}$-valued ω_i's. For a detailed analysis of the $n = 3$ case, the reader can consult [2].

The algebra $\mathcal{A}_n$ has an important Abelian subalgebra $\mathcal{Y}_n$. This subalgebra is generated by the $(n - 2)$ pairwise commuting elements

$$(22) \qquad\qquad \mathcal{Y}_n = \langle \Gamma_{[2]}, \Gamma_{[3]}, \ldots, \Gamma_{[n-1]} \rangle.$$

Given that $[n - 1] \subset [n]$, the commutativity of $\mathcal{Y}_n$ follows directly from Lemma 3. Note that $\Gamma_\emptyset$, $\Gamma_{[1]}$ and $\Gamma_{[n]}$ are not included in the generating set of $\mathcal{Y}_n$, as they are central in $\mathcal{A}_n$. It is manifest from the defining relations (21) that $\mathcal{Y}_n$ is the largest possible Abelian subalgebra of $\mathcal{A}_n$. Thus, borrowing from the terminology of Lie algebras, one can say that the algebra $\mathcal{A}_n$ has rank $(n-2)$. Observe that one can easily define another maximal Abelian subalgebra by applying a permutation of S_n on $\mathcal{Y}_n$; this is done by taking $\pi\mathcal{Y}_n = \langle \Gamma_{\pi[2]}, \Gamma_{\pi[3]}, \ldots, \Gamma_{\pi[n-1]} \rangle$ for $\pi \in S_n$. For example, applying the permutation $\pi = (123 \cdots n)$ on $\mathcal{Y}_n$ gives an algebra that we shall denote by $\mathcal{Z}_n$

$$(23) \qquad\qquad \mathcal{Z}_n = \langle \Gamma_{\{2,3\}}, \Gamma_{\{2,3,4\}}, \ldots, \Gamma_{\{2,3,\ldots,n\}} \rangle.$$

Remark 4. While $\mathcal{A}_3$ coincides with a degenerate double affine Hecke algebra of type $(\check{C}_1, C_1)$ [13], it appears that $\mathcal{A}_n$ does not coincide with the higher rank version of this degenerate double affine Hecke algebra of type $(\check{C}_{n-2}, C_{n-2})$, which has been investigated in [15, 16].

We shall now exhibit two non-trivial central elements of the algebra $\mathcal{A}_n$. These two elements, which shall be referred to as Casimir operators, are expressed as quadratic combinations of the generators.

Lemma 7. *Let $Q_{[n]}$ be the element of $\mathcal{A}_n$ defined as*

$$
(24) \qquad Q_{[n]} = \sum_{\{i,j\} \subset [n]} \Gamma^2_{\{i,j\}}.
$$

Then for $A \subset [n]$ one has $[Q_{[n]}, \Gamma_A] = 0$.

Proof. In view of Lemma 6, it suffices to prove that $Q_{[n]}$ commutes with every generator of the from $\Gamma_{\{i,k\}}$. Using the commutator identity $[A^2, C] = [A, \{A, C\}]$ and the relations (21), one finds

$$
[Q_{[n]}, \Gamma_{\{i,k\}}] = \sum_{\substack{j \in [n] \\ j \neq k \neq i}} \left([\Gamma^2_{\{i,j\}}, \Gamma_{\{i,k\}}] + [\Gamma^2_{\{j,k\}}, \Gamma_{\{i,k\}}] \right)
$$

$$
= \sum_{\substack{j \in [n] \\ j \neq k \neq i}} \left([\Gamma_{\{i,j\}}, \Gamma_{\{j,k\}}] + [\Gamma_{\{j,k\}}, \Gamma_{\{i,j\}}] \right) = 0.
$$

$\square$

Since $\mathcal{A}_n \subset \mathcal{A}_{n+1}$, the previous can be equivalently formulated as follows.

Lemma 8. *Let Q_A be the element defined as*

$$
(25) \qquad Q_A = \sum_{\{i,j\} \subset A} \Gamma^2_{\{i,j\}},
$$

one then has $[Q_A, \Gamma_B] = 0$, for all $B \subset A$.

In the realization (14), the Casimir operator Q_A is related to the spherical Dunkl-Dirac operator Γ_A. The result is as follows.

Lemma 9. *For $A \subset [n]$, one has*

$$
(26) \qquad Q_A = \Gamma_A^2 + (|A| - 2) \sum_{i \in A} \mu_i^2 - \frac{1}{8}(|A| - 1)(|A| - 2).
$$

Proof. The result is proved by a direct calculation using the explicit expression (14) for the operators Γ_A. On the one hand, one has

$$
\Gamma^2_{\{i,j\}} = M_{ij}^2 + M_{ij} + (\mu_i r_i + \mu_j r_j + 1/2)^2,
$$

which gives

$$
(27) \quad Q_A = \sum_{\{i,j\} \subset A} \left[M_{ij}^2 + M_{ij} + 2\mu_i\mu_j r_i r_j \right] + (|A| - 1) \sum_{i \in A} (\mu_i r_i + \mu_i^2) + \frac{|A|(|A|-1)}{8}.
$$

On the other hand, one has

$$
\Gamma_A^2 = \sum_{\{i,j\} \subset A} \left[M_{ij}^2 + \sum_{\substack{k \in A \\ k \neq i \neq j}} \left(\frac{1}{2}[\{M_{ij}, M_{ik}\} + \{M_{ij}, M_{jk}\}] + 2\mu_k M_{ij} r_k \right) \right]
$$

$$
+ (|A| - 1) \sum_{\{i,j\} \subset A} M_{ij} + \left[\frac{|A|-1}{2} + \sum_{k \in A} \mu_k r_k \right]^2.
$$

Since $\{M_{ij}, M_{ik}\} = -(1 + 2\mu_i R_i)M_{jk}$, one obtains

$$(28) \qquad \Gamma_A^2 = \sum_{\{i,j\}\subset A}\left(M_{ij}^2 + M_{ij}\right) + \left(\tfrac{|A|-1}{2} + \sum_{k\in A}\mu_k r_k\right)^2.$$

Upon expanding the last term on the right-hand side of (28) and comparing with (27), one readily finds (26). $\qquad\square$

When $n \geq 4$, one more Casimir operator of $\mathcal{A}_n$ can be identified.

Lemma 10. *Let $C_{[n]}$ be defined as*

$$C_{[n]} = \sum_{\{i,j\}\subset[n]}\Gamma_{\{i,j\}}\Gamma_{[n]\setminus\{i,j\}} - (n-2)\sum_{i\in[n]}\mu_i\,\Gamma_{[n]\setminus\{i\}}.$$

For $A \subset [n]$ one has $[C_{[n]}, \Gamma_A] = 0$.

Proof. In view of Lemma 6, it suffices to prove that $C_{[n]}$ commutes with any generator Γ_A with $|A| = 2$. One has

$$[C_{[n]}, \Gamma_{\{i,k\}}] = \sum_{\substack{j\in[n] \\ j\neq i\neq k}}[\Gamma_{\{i,j\}}\Gamma_{[n]\setminus\{i,j\}}, \Gamma_{\{i,k\}}] + [\Gamma_{\{j,k\}}\Gamma_{[n]\setminus\{j,k\}}, \Gamma_{\{i,k\}}]$$

$$- (n-2)\mu_i\,[\Gamma_{[n]\setminus\{i\}}, \Gamma_{\{i,k\}}] - (n-2)\mu_k\,[\Gamma_{[n]\setminus\{k\}}, \Gamma_{\{i,k\}}].$$

Using the identity $[AB, C] = A\{B, C\} - \{A, C\}B$ and (21), one finds

$$[C_{[n]}, \Gamma_{\{i,k\}}] = -(n-2)\mu_i\,[\Gamma_{[n]\setminus\{i\}}, \Gamma_{\{i,k\}}] - (n-2)\mu_k\,[\Gamma_{[n]\setminus\{k\}}, \Gamma_{\{i,k\}}]$$

$$2\mu_i\sum_{\substack{j\in[n] \\ j\neq i\neq k}}\Gamma_{\{i,j\}}\Gamma_{[n]\setminus\{i,j,k\}} + \Gamma_{\{j,k\}}\Gamma_{[n]\setminus\{j\}} - \Gamma_{\{i,j,k\}}\Gamma_{[n]\setminus\{i,j\}} - \mu_j\Gamma_{[n]\setminus\{j,k\}}$$

$$+ 2\mu_k\sum_{\substack{j\in[n] \\ j\neq i\neq k}}\Gamma_{\{j,k\}}\Gamma_{[n]\setminus\{i,j,k\}} + \Gamma_{\{i,j\}}\Gamma_{[n]\setminus\{j\}} - \Gamma_{\{i,j,k\}}\Gamma_{[n]\setminus\{j,k\}} - \mu_j\Gamma_{[n]\setminus\{i,j\}}$$

Recalling that

$$\Gamma_{[n]\setminus\{i\}} = \frac{1}{2\mu_k}\left[\{\Gamma_{[n]\setminus\{i,j\}}, \Gamma_{\{j,k\}}\} - \Gamma_{[n]\setminus\{i,k\}} - 2\mu_j\Gamma_{[n]\setminus\{i,j,k\}}\right],$$

one can write

$$(n-2)\mu_i[\Gamma_{[n]\setminus\{i\}}, \Gamma_{\{i,k\}}] = \frac{\mu_i}{2\mu_k}\sum_{\substack{j\in[n] \\ j\neq i\neq k}}[\{\Gamma_{[n]\setminus\{i,j\}}, \Gamma_{\{j,k\}}\}, \Gamma_{\{i,k\}}].$$

With the help of the identity $[\{A, B\}, C] = [A, \{B, C\}] + [B, \{A, C\}]$ and the relations (21), one finds

$$(n-2)\mu_i[\Gamma_{[n]\setminus\{i\}}, \Gamma_{\{i,k\}}] = \mu_i\sum_{\substack{j\in[n] \\ j\neq i\neq k}}[\Gamma_{[n]\setminus\{i,j\}}, \Gamma_{\{i,j,k\}}] + [\Gamma_{\{j,k\}}, \Gamma_{[n]\setminus\{j\}}].$$

Using this result, one finds

$$[C_{[n]}, \Gamma_{\{i,k\}}] =$$

$$\mu_i \sum_{\substack{j \in [n] \\ j \neq i \neq k}} \{\Gamma_{\{j,k\}}, \Gamma_{[n]\setminus\{j\}}\} - \{\Gamma_{\{i,j,k\}}, \Gamma_{[n]\setminus\{i,j\}}\} + 2\Gamma_{\{i,j\}}\Gamma_{[n]\setminus\{i,j,k\}} - 2\mu_j\Gamma_{[n]\setminus\{j,k\}}$$

$$+ \mu_k \sum_{\substack{j \in [n] \\ j \neq i \neq k}} \{\Gamma_{\{i,j\}}, \Gamma_{[n]\setminus\{j\}}\} - \{\Gamma_{\{i,j,k\}}, \Gamma_{[n]\setminus\{j,k\}}\} + 2\Gamma_{\{j,k\}}\Gamma_{[n]\setminus\{i,j,k\}} - 2\mu_j\Gamma_{[n]\setminus\{i,j\}}$$

Using (21) one last time confirms that $[C_{[n]}, \Gamma_A] = 0$ when $|A| = 2$, and thus that $[C_{[n]}, \Gamma_A]$ for all $A \subset [n]$. $\square$

The previous Lemma is equivalent to the following.

Lemma 11. *Let C_A be defined as*

$$C_A = \sum_{\{i,j\}\subset A} \Gamma_{\{i,j\}}\Gamma_{A\setminus\{i,j\}} - (|A| - 2)\sum_{i \in A} \mu_i\, \Gamma_{A\setminus\{i\}}.$$

For $B \subset A$ one has $[C_A, \Gamma_B] = 0$.

In the realization (14), the operator C_A can also be expressed in terms of Γ_A.

Lemma 12. *For $A \subset [n]$, one has*

$$(29) \qquad\qquad C_A = \left(\frac{|A|(|A| - 3)}{4}\right)\Gamma_A.$$

Proof. The proof proceeds by a direct calculation using the realization (14) of the operators Γ_A. The main observation is that for a given set A, one has

$$\sum_{\{i,j\}\subset A} \sum_{\substack{\{k,\ell\}\subset A \\ \{k,\ell\}\neq\{i,j\}}} M_{ij}M_{k\ell} = 0.$$

This result follows directly from the Clifford relations (1) and the definition of the operators M_{ij} given in (15). Using this result and combinatorial arguments yields the expression

$$\sum_{\{i,j\}\subset A} \Gamma_{\{i,j\}}\Gamma_{A\setminus\{i,j\}} =$$

$$\left[\frac{|A|(|A|-3)}{4}\sum_{\{i,j\}\subset A} M_{ij} + (|A|-2)\sum_{i\in A}\mu_i\sum_{\substack{\{k,\ell\}\subset A \\ k,\ell\neq i}} M_{k\ell} + 2(|A|-2)\sum_{\{i,j\}\subset A}\mu_i\mu_j r_i r_j\right.$$

$$\left. + \frac{(|A|-1)(3|A|-8)}{4}\sum_{j\in A}\mu_j r_j + \frac{|A|(|A|-1)(|A|-3)}{8}\right]\prod_{i\in A} r_i.$$

Upon expanding the second term $\sum_{i\in A}\mu_i\,\Gamma_{A\setminus\{i\}}$ of the operator C_A using (14), a straightforward calculation shows that one has indeed (29).

$\square$

3. A basis for the space of Dunkl monogenics

In this section, a basis for the space $\mathcal{M}_k(\mathbb{R}^n; V)$ of polynomial null-solutions of the Dirac–Dunkl operator $\underline{D}_{[n]}$ is constructed and explicitly presented in terms of Jacobi polynomials. The construction proceeds from a Cauchy-Kovalevskaia (CK) extension theorem and from the Fisher decomposition. It is shown that the basis functions stemming from that construction are joint eigenfunctions for the generators of the Abelian subalgebra $\mathcal{Y}_n$ and the corresponding eigenvalues are computed explicitly. Another basis for $\mathcal{M}_k(\mathbb{R}^n; V)$ corresponding to the Abelian subalgebra $\mathcal{Z}_n$ defined in (23) is introduced. We conjecture that the expansion coefficients between the two bases are given in terms of some multivariate Bannai–Ito polynomials.

3.1. Basis functions and the CK isomorphism.

There is an isomorphism $\mathbf{CK}_{x_j}^{\mu_j} : \mathcal{P}_k(\mathbb{R}^{j-1}) \otimes V \to \mathcal{M}_k(\mathbb{R}^j; V)$ between the space of V-valued k-homogeneous polynomials in $x_1, \ldots, x_{j-1}$ and the space of k-homogeneous Dunkl monogenics in $x_1, \ldots, x_j$. The mapping between the two spaces is as follows.

Proposition 5 ([2]). *The isomorphism between $\mathcal{P}_j(\mathbb{R}^{j-1}) \otimes V$ and $\mathcal{M}_k(\mathbb{R}^j; V)$ denoted by $\mathbf{CK}_{x_j}^{\mu_j}$ is explicitly defined by the operator*

$$(30) \quad \mathbf{CK}_{x_j}^{\mu_j} = \Gamma(\mu_j + 1/2)\Big[\widetilde{I}_{\mu_j - 1/2}(x_j \underline{D}_{[j-1]}) + \frac{1}{2}e_j x_j \underline{D}_{[j-1]} \widetilde{I}_{\mu_j + 1/2}(x_j \underline{D}_{[j-1]})\Big],$$

with $\widetilde{I}_\alpha(x) = (\frac{2}{x})^\alpha I_\alpha(x)$ and $I_\alpha(x)$ the modified Bessel function [1].

Remark 5. When $\mu_j = 0$, $\mathbf{CK}_{x_j}^{\mu_j}$ reduces to $\mathbf{CK}_{x_j} = \exp(e_j x_j \underline{D}_{[j-1]})$ which corresponds to the well-known $\mathbf{CK}$ extension operator for the standard Dirac operator, as determined in [4].

Consider $p \in \mathcal{P}_k(\mathbb{R}^{j-1}) \otimes V$; the $\mathbf{CK}$ map can be written explicitly as

$$(31) \quad \mathbf{CK}_{x_j}^{\mu_j}(p) = \sum_{\ell=0}^{\lfloor \frac{k}{2} \rfloor} \frac{\Gamma(\mu_j + 1/2)}{2^{2\ell}\ell!\,\Gamma(\ell + \mu_j + 1/2)} x_j^{2\ell} \underline{D}_{[j-1]}^{2\ell}\, p$$

$$+ \frac{e_j x_j \underline{D}_{[j]}}{2} \sum_{\ell=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{\Gamma(\mu_j + 1/2)}{2^{2\ell}\ell!\Gamma(\ell + \mu_j + 3/2)} x_j^{2\ell} \underline{D}_{[j-1]}^{2\ell}\, p.$$

The $\mathbf{CK}$ map can be combined with the Fischer decomposition to construct an explicit basis for the space $\mathcal{M}_k(\mathbb{R}^n; V)$. Let $\{v_s\}$ for $s \in I = \{1, \ldots, \dim V\}$ be a set of basis vectors for the representation space V of $\mathcal{C}\ell_n$. One has the following tower of $\mathbf{CK}$ extensions and Fischer decompositions

$$\mathcal{M}_k(\mathbb{R}^n; V) \cong \mathbf{CK}_{x_n}^{\mu_n}\big[\mathcal{P}_k(\mathbb{R}^{n-1}) \otimes V\big] \cong \mathbf{CK}_{x_n}^{\mu_n}\Big[\bigoplus_{j=0}^{k} \underline{x}_{[n-1]}^{k-j}\mathcal{M}_j(\mathbb{R}^{n-1}; V)\Big]$$

$$(32) \qquad \cong \mathbf{CK}_{x_n}^{\mu_n}\Big[\bigoplus_{j=0}^{k} \underline{x}_{[n-1]}^{k-j}\mathbf{CK}_{x_{n-1}}^{\mu_{n-1}}\big[\mathcal{P}_j(\mathbb{R}^{n-2}) \otimes V\big]\Big]$$

$$\cong \mathbf{CK}_{x_n}^{\mu_n}\Big[\bigoplus_{j=0}^{k} \underline{x}_{[n-1]}^{k-j}\mathbf{CK}_{x_{n-1}}^{\mu_{n-1}}\Big[\bigoplus_{\ell=0}^{j} \underline{x}_{[n-2]}^{j-\ell}\mathcal{M}_\ell(\mathbb{R}^{n-2}; V)\Big]\Big] \cong \cdots$$

until one reaches $\mathcal{P}_{j_1}(\mathbb{R}) \otimes V$. Since the latter space is spanned by the basis vectors $x_1^{j_1} v_s$ with $s \in I$ and $j_1 \in \mathbb{N}$, one concludes the following.

Proposition 6. *Let* $\mathbf{j}$ *be defined as* $\mathbf{j} = (j_1, j_2, \ldots, j_{n-2}, j_{n-1} = k - \sum_{i=1}^{n-2} j_i)$ *where* $j_1, \ldots, j_{n-2}$ *are non-negative integers such that* $\sum_{i=1}^{n-2} j_i \leq k$. *Consider the set of functions* $\Psi_{\mathbf{j}}^s(x_1, \ldots, x_n)$ *defined by*

$$(33) \quad \Psi_{\mathbf{j}}^s(x_1, \ldots, x_n) =$$

$$\mathbf{CK}_{x_n}^{\mu_n} \left[\underline{x}_{[n-1]}^{j_{n-1}} \mathbf{CK}_{x_{n-1}}^{\mu_{n-1}} \left[\underline{x}_{[n-2]}^{j_{n-2}} \left[\cdots \mathbf{CK}_{x_3}^{\mu_3} [\underline{x}_{[2]}^{j_2} \mathbf{CK}_{x_2}^{\mu_2} [x_1^{j_1}]] \right] \right] \right] v_s,$$

with $s \in I$. *Then the functions* $\Psi_{\mathbf{j}}^s$ *form a basis for the space* $\mathcal{M}_k(\mathbb{R}^n; V)$ *of* k-*homogeneous Dunkl monogenics.*

Proof. The result follows from the application of the tower (32). $\qquad \square$

Remark 6. As we shall always work at a given value k of the degree, the dependence on k of the multi-index $\mathbf{j}$ is kept implicit.

Let us now present a Lemma which characterizes the action of the operators $\underline{D}$ and $\underline{x}$ on the space of Dunkl monogenics.

Lemma 13. *Let* $M_\ell \in \mathcal{M}_\ell(\mathbb{R}^n; V)$. *Then for non-negative integers* j, k *such that* $j \leq k$ *one has*

$$\underline{D}^{2j} \underline{x}^{2k} M_\ell = \left[\frac{2^{2j} k! \Gamma(k + \ell + \gamma_{[n]})}{(k-j)! \Gamma(k - j + \ell + \gamma_{[n]})} \right] \underline{x}^{2k-2j} M_\ell$$

$$\underline{D}^{2j+1} \underline{x}^{2k} M_\ell = - \left[\frac{2^{2j+1} k! \Gamma(k + \ell + \gamma_{[n]})}{(k-j-1)! \Gamma(k - j + \ell + \gamma_{[n]})} \right] \underline{x}^{2k-2j-1} M_\ell$$

$$\underline{D}^{2j} \underline{x}^{2k+1} M_\ell = \left[\frac{2^{2j} k! \Gamma(k + \ell + \gamma_{[n]} + 1)}{(k-j)! \Gamma(k - j + \ell + \gamma_{[n]} + 1)} \right] \underline{x}^{2k-2j+1} M_\ell$$

$$\underline{D}^{2j+1} \underline{x}^{2k+1} M_\ell = - \left[\frac{2^{2j+1} k! \Gamma(k + \ell + \gamma_{[n]} + 1)}{(k-j)! \Gamma(k - j + \ell + \gamma_{[n]})} \right] \underline{x}^{2k-2j} M_\ell.$$

Proof. The Lemma easily follows from induction and from the relations (4) $\qquad \square$

In view of (31) and Lemma 13, one can expect the wavefunctions (33) to have an explicit expression in terms of hypergeometric functions. It turns out that this is indeed the case. To present the result, we introduce the following notation

$$|\mathbf{j}_\ell| = j_1 + j_2 + \cdots + j_\ell, \qquad \|x_j\|^2 = x_1^2 + \cdots + x_j^2.$$

Proposition 7. *The basis functions* $\Psi_{\mathbf{j}}^s$ *have the expression*

$$(34) \qquad \Psi_{\mathbf{j}}^s(x_1, \ldots, x_n) = \overrightarrow{\left(\prod_{\ell=n}^{3} \right)} Q_{j_{\ell-1}}(x_\ell, \underline{x}_{[\ell-1]}) m_{j_1}(x_2, x_1) v_s,$$

where

$$(35) \quad Q_{j_{\ell-1}}(x_\ell, \underline{x}_{[\ell-1]}) = \frac{\beta! \|x_\ell\|^{2\beta}}{(\mu_\ell + 1/2)_\beta}$$

$$\times \begin{cases} P_\beta^{(|\mathbf{j}_{\ell-2}|+\gamma_{[\ell-1]}-1, \mu_\ell-1/2)} \left(\frac{x_\ell^2 - \|x_{\ell-1}\|^2}{\|x_\ell\|^2} \right) & j_{\ell-1} = 2\beta \\[2mm] \quad - \frac{e_\ell x_\ell \underline{x}_{[\ell-1]}}{\|x_\ell\|^2} \, P_{\beta-1}^{(|\mathbf{j}_{\ell-2}|+\gamma_{[\ell-1]}, \mu_\ell+1/2)} \left(\frac{x_\ell^2 - \|x_{\ell-1}\|^2}{\|x_\ell\|^2} \right) \\[4mm] \underline{x}_{[\ell-1]} \, P_\beta^{(|\mathbf{j}_{\ell-2}|+\gamma_{[\ell-1]}, \mu_\ell-1/2)} \left(\frac{x_\ell^2 - \|x_{\ell-1}\|^2}{\|x_\ell\|^2} \right) & j_{\ell-1} = 2\beta+1 \\[2mm] \quad -e_\ell x_\ell \left(\frac{\beta + |\mathbf{j}_{\ell-2}| + \gamma_{[\ell-1]}}{\beta + \mu_\ell + 1/2} \right) P_\beta^{(|\mathbf{j}_{\ell-2}|+\gamma_{[\ell-1]}-1, \mu_\ell+1/2)} \left(\frac{x_\ell^2 - \|x_{\ell-1}\|^2}{\|x_\ell\|^2} \right) \end{cases}$$

and

$$(36) \quad m_{j_1}(x_2, x_1) = \frac{(-1)^\beta \beta! (x_1^2 + x_2^2)^\beta}{(\mu_2 + 1/2)_\beta}$$

$$\times \begin{cases} P_\beta^{(\mu_1-1/2, \mu_2-1/2)} \left(\frac{x_2^2 - x_1^2}{x_1^2 + x_2^2} \right) & j_1 = 2\beta \\[2mm] \quad - \frac{e_2 e_1 x_2 x_1}{x_1^2 + x_2^2} \, P_{\beta-1}^{(\mu_1+1/2, \mu_2+1/2)} \left(\frac{x_2^2 - x_1^2}{x_1^2 + x_2^2} \right), \\[4mm] x_1 P_\beta^{(\mu_1+1/2, \mu_2-1/2)} \left(\frac{x_2^2 - x_1^2}{x_1^2 + x_2^2} \right) & j_1 = 2\beta+1 \\[2mm] \quad + \left(\frac{\beta + \mu_1 + 1/2}{\beta + \mu_2 + 1/2} \right) e_2 e_1 x_2 \, P_\beta^{(\mu_1-1/2, \mu_2+1/2)} \left(\frac{x_2^2 - x_1^2}{x_1^2 + x_2^2} \right) \end{cases}$$

In (35) and (36), $P_n^{(\alpha, \beta)}(x)$ are the Jacobi polynomials with $P_{-1}^{(\alpha,\beta)}(x) = 0$ and $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ the Pochhammer symbol [1].

Proof. The result follows from a long but straightforward calculation using (31) and Lemma 13. See also [2], where a similar calculation was performed. $\qquad\square$

Let us now show that the basis functions given by (33) are joint eigenfunctions of the generators of the maximal Abelian subalgebra $\mathcal{Y}_n$ defined in (22).

Proposition 8. *The wavefunctions $\Psi_{\mathbf{j}}^s \in \mathcal{M}_k(\mathbb{R}^n; V)$ satisfy*

$$(37) \qquad\qquad \Gamma_{[\ell]} \Psi_{\mathbf{j}}^s(x_1, \dots, x_n) = \lambda_\ell(\mathbf{j}) \Psi_{\mathbf{j}}^s(x_1, \dots, x_n),$$

where the eigenvalues are given by

$$(38) \qquad\qquad \lambda_\ell(\mathbf{j}) = (-1)^{|\mathbf{j}_{\ell-1}|}(|\mathbf{j}_{\ell-1}| + \gamma_{[\ell]} - 1/2).$$

Proof. For $\ell = n$, the result is equivalent to Proposition 3. For $\ell < n$, observe that $[\Gamma_{[\ell]}, \underline{D}_{[j]}] = [\Gamma_{[\ell]}, \underline{x}_{[j]}] = 0$ for all $j \geq \ell$. As a result, $\Gamma_{[\ell]}$ commutes with $\mathbf{CK}_{x_j}^{\mu_j}$ for $j - 1 \geq \ell$ and one can write

$$\Gamma_{[\ell]} \Psi_{\mathbf{j}}^s(x_1, \dots, x_n) = \mathbf{CK}_{x_n}^{\mu_n} \left[\cdots \Gamma_{[\ell]} \mathbf{CK}_{x_\ell}^{\mu_\ell} \left[\cdots \mathbf{CK}_{x_2}^{\mu_2}[x_1^{j_1}] \right] \right] v_s.$$

As $\mathbf{CK}_{x_\ell}^{\mu_\ell} \left[\cdots \mathbf{CK}_{x_2}^{\mu_2}[x_1^{j_1}] \right] v_s \in \mathcal{M}_{|\mathbf{j}_{\ell-1}|}(\mathbb{R}^\ell; V)$, the result once again follows from Proposition 3. $\qquad\square$

Remark 7. It is easily verified from (33) that for fixed k and s, the dimension of the space spanned by the eigenfunctions $\Psi_{\mathbf{j}}^s(x_1, \dots, x_n)$ is $\binom{n+k-2}{k}$, as expected from the isomorphism between $\mathcal{P}_k(\mathbb{R}^{j-1}) \otimes V$ and $\mathcal{M}_k(\mathbb{R}^j; V)$.

Remark 8. If desired, the wavefunctions (34) can be normalized to satisfy an orthogonality relation of the form

$$\int\limits_{\mathbb{S}^{n-1}} \mathrm{d}x_1 \cdots \mathrm{d}x_n \left[\Psi_{\mathbf{j}'}^{s'}(x_1, \ldots, x_n)\right]^* \cdot \Psi_{\mathbf{j}}^{s}(x_1, \ldots, x_n) = \delta_{\mathbf{j}\mathbf{j}'}\delta_{ss'},$$

where $*$ denotes complex conjugation and where $\cdot$ stands for an appropriately defined inner product on the representation space V. One should also take $e_i^* = -e_i$ for the Clifford generators and set $(ab)^* = b^* a^*$ for $a, b \in \mathcal{C}\ell_n$. The normalization coefficients were given in [2] for the $n = 3$ case and are readily generalized for $n > 3$. The orthogonality property can be proven directly by induction on n and follows from the well-known orthogonality relation satisfied by the Jacobi polynomials [1].

From the common eigenfunctions $\Psi_{\mathbf{j}}^{s}(x_1, \ldots, x_n)$ of the Abelian subalgebra $\mathcal{Y}_n$, one can easily construct a set of joint eigenfunctions for the Abelian subalgebra $\pi\mathcal{Y}_n$, where π is a permutation of S_n. Indeed, it is directly verified that $\pi\Psi_{\mathbf{j}}^{s}(x_1, \ldots, x_n)$ are common eigenfunctions for $\pi\mathcal{Y}_n$, where the permutation π acts on $\Psi_{\mathbf{j}}^{s}(x_1, \ldots, x_n)$ by simultaneously permuting the variables $x_1, \ldots, x_n$, the Clifford generators $e_1, \ldots, e_n$ and the parameters $\mu_1, \ldots, \mu_n$. In light of this observation, consider the functions defined as

$$\Phi_{\mathbf{j}'}^{s}(x_1, \ldots, x_n) = \pi\Psi_{\mathbf{j}'}^{s}(x_1, \ldots, x_n) \qquad \pi = (123\cdots n),$$

which are common eigenfunctions for $\mathcal{Z}_n$, as defined in (23). It is manifest that the functions $\Phi_{\mathbf{j}'}^{s}(x_1, \ldots, x_n)$ also form a basis for $\mathcal{M}_k(\mathbb{R}^n; V)$. As both $\Phi_{\mathbf{j}'}^{s}(x_1, \ldots, x_n)$ and $\Psi_{\mathbf{j}}^{s}(x_1, \ldots, x_n)$ can be normalized to form finite-dimensional orthonormal bases for $\mathcal{M}_k(\mathbb{R}^n; V)$, there exists a unitary transformation, or intertwining operator, connecting these two bases. In [2], the matrix elements of this intertwining operator were seen to involve the Bannai–Ito polynomials. In the n-dimensional case, one can expect that the unitary transformation between $\Phi_{\mathbf{j}'}^{s}(x_1, \ldots, x_n)$ and $\Psi_{\mathbf{j}}^{s}(x_1, \ldots, x_n)$ involves multivariate polynomials, which could be viewed as multivariate analogs of the Bannai–Ito polynomials. Since a theory of Bannai–Ito polynomials in many variables is yet to be worked out, we formulate the following as a conjecture.

Conjecture 1. *The matrix elements of the intertwining operator between the basis functions $\Psi_{\mathbf{j}}^{s}(x_1, \ldots, x_n)$ and $\Phi_{\mathbf{j}'}^{s}(x_1, \ldots, x_n)$ are expressed in terms of orthogonal polynomials which could be taken to define the multivariate extension of the Bannai–Ito polynomials.*

4. Action of the symmetry algebra

In this section, the action of the symmetry algebra $\mathcal{A}_n$ on the basis functions $\Psi_{\mathbf{j}}^{s}(x_1, \ldots, x_n)$ is studied. A set of raising and lowering operators is introduced and their actions are computed explicitly. Using these results, it is observed that $\mathcal{A}_n$ acts irreducibly on the space $\mathcal{M}_k(\mathbb{R}^{d-1}; V)$ of k-homogeneous Dunkl monogenics.

For $\ell \in [n-2]$, consider the set of operators $K_\ell^{\pm}$ defined as

$$(39) \quad K_\ell^{\pm} =$$
$$(\Gamma_{\{\ell+1, \ell+2\}} \pm \Gamma_{[\ell+2]\setminus\{\ell+1\}})(\Gamma_{[\ell+1]} \mp 1/2) - (\Gamma_{\{\ell+2\}} \pm \Gamma_{[\ell+2]})(\Gamma_{[\ell]} \pm \Gamma_{\{\ell+1\}}).$$

For simplicity, we write $K_\ell^\pm = A_\ell^\pm + B_\ell^\pm$ with

$$A_\ell^\pm = (\Gamma_{\{\ell+1,\ell+2\}} \pm \Gamma_{[\ell+2]\setminus\{\ell+1\}})(\Gamma_{[\ell+1]} \mp 1/2),$$
$$B_\ell^\pm = -(\Gamma_{\{\ell+2\}} \pm \Gamma_{[\ell+2]})(\Gamma_{[\ell]} \pm \Gamma_{\{\ell+1\}}).$$

The operators $K_\ell^\pm$ have nice covariance properties with respect to the Abelian subalgebra $\mathcal{Y}_n$. These properties are described in the following proposition.

Proposition 9. *Let $j \in [n]$ and $\ell \in [n-2]$. One has*

$$[K_\ell^\pm, \Gamma_{[j]}] = 0, \qquad j \neq \ell + 1.$$

Conversely when $j = \ell + 1$ one has

$$\{K_\ell^\pm, \Gamma_{[\ell+1]}\} = \pm K_\ell^\pm.$$

Proof. The first relation follows immediately from the definition of $K_\ell^\pm$ and from Lemma 3. For the second identity, one has

$$\{K_\ell^\pm, \Gamma_{[\ell+1]}\} = 2B_\ell^\pm \Gamma_{[\ell+1]} + \{\Gamma_{\{\ell+1,\ell+2\}} \pm \Gamma_{[\ell+2]\setminus\{\ell+1\}}, \Gamma_{[\ell+1]}\}(\Gamma_{[\ell+1]} \mp 1/2)$$
$$= 2B_\ell^\pm \Gamma_{[\ell+1]} \pm A_\ell^\pm + 2(\Gamma_{[\ell]} \pm \Gamma_{\{\ell+1\}})(\Gamma_{\{\ell+2\}} \pm \Gamma_{[\ell+2]})(\Gamma_{[\ell+1]} \mp 1/2)$$
$$= 2B_\ell^\pm \Gamma_{[\ell+1]} \pm A_\ell^\pm - 2B_\ell^\pm (\Gamma_{[\ell+1]} \mp 1/2) = \pm K_\ell^\pm,$$

where the relations (21) were used. □

In order to characterize the action of the operators $K_\ell^\pm$ on the basis functions $\Psi_{\mathbf{j}}^s(x_1, \ldots, x_n)$, we shall need the following technical Lemma.

Lemma 14. *One has the identity*

(40) $$\Gamma_{\{\ell+1,\ell+2\}}^2 + \Gamma_{[\ell+2]\setminus\{\ell+1\}}^2 = \Gamma_{[\ell+2]}^2 - \Gamma_{[\ell+1]}^2 + \Gamma_{[\ell]}^2 + \Gamma_{\{\ell+2\}}^2 + \Gamma_{\{\ell+1\}}^2 - 1/4.$$

Proof. The proof follows from the application of Lemma 9. One first writes

$$\Gamma_{\{\ell+1,\ell+2\}}^2 + \Gamma_{[\ell+2]\setminus\{\ell+1\}}^2 =$$
$$(\Gamma_{\{\ell+1,\ell+2\}}^2 + \sum_{i\in[\ell]}\Gamma_{\{i,\ell+1\}}^2 + \Gamma_{[\ell+2]\setminus\{\ell+1\}}^2) - (\sum_{i\in[\ell]}\Gamma_{\{i,\ell+1\}}^2 + \Gamma_{[\ell]}^2) + \Gamma_{[\ell]}^2.$$

To get the result, one first uses (26) on the first two terms. Applying (26) again on the result and simplifying the expression yields (40). □

Quite strikingly, the squares of the operators $K_\ell^\pm$ have an exact factorization.

Proposition 10. *One has*

$$[K_\ell^\pm]^2 = (\Gamma_{[\ell+2]} - \Gamma_{[\ell+1]} \pm \Gamma_{\{\ell+2\}} \pm 1/2)(\Gamma_{[\ell+2]} + \Gamma_{[\ell+1]} \pm \Gamma_{\{\ell+2\}} \mp 1/2)$$
$$\times (\Gamma_{[\ell]} - \Gamma_{[\ell+1]} \pm \Gamma_{\{\ell+1\}} \pm 1/2)(\Gamma_{[\ell]} + \Gamma_{[\ell+1]} \pm \Gamma_{\{\ell+1\}} \mp 1/2).$$

Proof. Since $A_\ell^\pm$ and $B_\ell^\pm$ commute with one another, one has

$$(K_\ell^\pm)^2 = (A_\ell^\pm)^2 + (B_\ell^\pm)^2 + 2A_\ell^\pm B_\ell^\pm.$$

Upon observing that one has

$$(\Gamma_{[\ell+1]} \mp 1/2)(\Gamma_{\{\ell+1,\ell+2\}} \pm \Gamma_{[\ell+2]\setminus\{\ell+1\}}) = -A_\ell^\pm - 2B_\ell^\pm,$$

which follows directly from the commutation relations (21), one can write

(41) $$[K_\ell^\pm]^2 = (B_\ell^\pm)^2 - (\Gamma_{\{\ell+1,\ell+2\}} \pm \Gamma_{[\ell+2]\setminus\{\ell+1\}})^2(\Gamma_{[\ell+1]} \mp 1/2)^2.$$

Using (40), one has

$$(\Gamma_{\{\ell+1,\ell+2\}} \pm \Gamma_{[\ell+2]\setminus\{\ell+1\}})^2 = (\Gamma_{\{\ell+2\}} \pm \Gamma_{[\ell+2]})^2$$
$$+ (\Gamma_{[\ell]} \pm \Gamma_{\{\ell+1\}})^2 - (\Gamma_{[\ell+1]} \mp 1/2)^2.$$

Upon inserting the above expression in (41), one gets

$$[K_\ell^\pm]^2 = \left[(\Gamma_{[\ell+2]} \pm \Gamma_{\{\ell+2\}})^2 - (\Gamma_{[\ell+1]} \mp 1/2)^2\right]$$
$$\times \left[(\Gamma_{[\ell]} \pm \Gamma_{\{\ell+1\}})^2 - (\Gamma_{[\ell+1]} \mp 1/2)^2\right].$$

Factoring each term gives the anticipated result. □

Lemma 15. *The action of $(K_\ell^\pm)^2$ on the $\Psi_{\mathbf{j}}^s(x_1,\ldots,x_n)$ is given by*

$$\left(K_\ell^\pm\right)^2 \Psi_{\mathbf{j}}^s(x_1,\ldots,x_n) = \alpha_\pm^\ell(\mathbf{j})\Psi_{\mathbf{j}}^s(x_1,\ldots,x_n),$$

where

$$(42) \quad \begin{aligned} \alpha_\pm^\ell(\mathbf{j}) &= (\lambda_{\ell+2}(\mathbf{j}) - \lambda_{\ell+1}(\mathbf{j}) \pm \mu_{\ell+2} \pm 1/2)(\lambda_{\ell+2}(\mathbf{j}) + \lambda_{\ell+1}(\mathbf{j}) \pm \mu_{\ell+2} \mp 1/2) \\ &\times (\lambda_\ell(\mathbf{j}) - \lambda_{\ell+1}(\mathbf{j}) \pm \mu_{\ell+1} \pm 1/2)(\lambda_\ell(\mathbf{j}) + \lambda_{\ell+1}(\mathbf{j}) \pm \mu_{\ell+1} \mp 1/2). \end{aligned}$$

The coefficients $\alpha_\pm^\ell(\mathbf{j})$ are non-negative for all admissible $\mathbf{j}$. In particular, they are non-zero except in the following cases

$$(43) \quad \begin{cases} j_{\ell+1} = 0 \text{ and } |\mathbf{j}_\ell| \text{ even} & \Rightarrow \alpha_-^\ell(\mathbf{j}) = 0 \\ j_{\ell+1} = 0 \text{ and } |\mathbf{j}_\ell| \text{ odd} & \Rightarrow \alpha_+^\ell(\mathbf{j}) = 0 \\ j_\ell = 0 \text{ and } |\mathbf{j}_{\ell+1}| \text{ even} & \Rightarrow \alpha_+^\ell(\mathbf{j}) = 0 \\ j_\ell = 0 \text{ and } |\mathbf{j}_{\ell+1}| \text{ odd} & \Rightarrow \alpha_-^\ell(\mathbf{j}) = 0 \end{cases}$$

Proof. The eigenvalue follows from the combination of Propositions 8 and 10. The vanishing of the coefficients $\alpha_\pm^\ell(\mathbf{j})$ is determined directly with the help of the formula (38) for the eigenvalues. The fact that they are positive everywhere else follows from the fact that the indices j_ℓ are non-negative integers and from the assumption that $\mu_i > 0$. □

We now derive the action of the operators $K_\ell^\pm$ on the basis functions. To describe the result, we introduce the following notation. Let $\mathbf{h}_i = (0,\ldots,1,-1,\ldots,0)$ be the vector with $n-1$ entries that has a 1 at position i, a -1 at position $i+1$, and zeros elsewhere. We also define $\bar{\mathbf{h}}_i = (0,\ldots,1,\ldots,0)$, where the 1 is at position i.

Proposition 11. *The action of the operators $K_\ell^\pm$ on the functions $\Psi_{\mathbf{j}}^s(x_1,\ldots,x_n)$ of $\mathcal{M}_k(\mathbb{R}^n;V)$ is as follows. For $\ell \in \{1,2,\ldots,n-3\}$, one has*

$$(44) \qquad K_\ell^\pm \Psi_{\mathbf{j}}^s(x_1,\ldots,x_n) \propto \begin{cases} \Psi_{\mathbf{j}\pm h_\ell}^s(x_1,\ldots,x_n) & |\mathbf{j}_l| \text{ is odd} \\ \Psi_{\mathbf{j}\mp h_\ell}^s(x_1,\ldots,x_n) & |\mathbf{j}_l| \text{ is even} \end{cases}$$

while for $\ell = n-2$, one has

$$(45) \qquad K_{n-2}^\pm \Psi_{\mathbf{j}}^s(x_1,\ldots,x_n) \propto \begin{cases} \Psi_{\mathbf{j}\pm \bar{h}_{n-2}}^s(x_1,\ldots,x_n) & |\mathbf{j}_l| \text{ is odd} \\ \Psi_{\mathbf{j}\mp \bar{h}_{n-2}}^s(x_1,\ldots,x_n) & |\mathbf{j}_l| \text{ is even} \end{cases}.$$

Proof. Consider the case where $\ell \in [n-3]$. Since $K_\ell^\pm$ commutes with $\Gamma_{[i]}$ for $i = 1,\ldots,\ell$, it preserves the quantum numbers $j_1, j_2,\ldots,j_{\ell-1}$. Since $K_\ell^\pm$ also commutes with Γ_k for $k = \ell+2,\ldots,n$, it must preserve the sum $j_\ell + j_{\ell+1}$, the individual quantum numbers $j_{\ell+2},\ldots,j_{n-2}$ and the total degree k which implicitly

appears in the quantum number j_{n-1}. Thus, $K_\ell^\pm$ can only act on $\Psi_{\mathbf{j}}^s(x_1, \ldots, x_n)$ by changing j_ℓ and $j_{\ell+1}$ while preserving the value of their sum. Furthermore, it follows from Propositions 8 and 9 that

$$\Gamma_{[\ell+1]} K_\ell^\pm \Psi_{\mathbf{j}}^s(x_1, \ldots, x_n) = \left[\{\Gamma_{[\ell+1]}, K_\ell^\pm\} - K_\ell^\pm \Gamma_{[\ell+1]} \right] \Psi_{\mathbf{j}}^s(x_1, \ldots, x_n)$$
$$= \left[\pm 1 - \lambda_{\ell+1}(\mathbf{j}) \right] K_\ell^\pm \Psi_{\mathbf{j}}^s(x_1, \ldots, x_n).$$

The above equation means that $K_\ell^\pm \Psi_{\mathbf{j}}^s(x_1, \ldots, x_n)$ is an eigenfunction of the operator $\Gamma_{[\ell+1]}$ with eigenvalue $[\pm 1 - \lambda_{\ell+1}(\mathbf{j})]$. Upon examining the explicit form (38) of the eigenvalues $\lambda_{\ell+1}(\mathbf{j})$ while taking into account that only j_ℓ and $j_{\ell+1}$ are allowed to change, one writes

$$\left[1 - \lambda_{\ell+1}(\mathbf{j}) \right] = \begin{cases} \lambda_{\ell+1}(\mathbf{j} + \bar{h}_\ell) & |\mathbf{j}_\ell| \text{ odd} \\ \lambda_{\ell+1}(\mathbf{j} - \bar{h}_\ell) & |\mathbf{j}_\ell| \text{ even} \end{cases}$$

$$-\left[1 + \lambda_{\ell+1}(\mathbf{j}) \right] = \begin{cases} \lambda_{\ell+1}(\mathbf{j} - \bar{h}_\ell) & |\mathbf{j}_\ell| \text{ odd} \\ \lambda_{\ell+1}(\mathbf{j} + \bar{h}_\ell) & |\mathbf{j}_\ell| \text{ even} \end{cases}$$

which yields (44). The proof of (45) proceeds similarly, with the difference of notation coming from the special role played by the component j_{n-1}. □

Remark 9. The exact proportionality factor in (44) and (45) is not needed for our purposes. This factor can be evaluated explicitly in terms of the coefficients $\alpha_\pm^\ell(\mathbf{j})$ and is non-zero in view of Lemma 15. This could be done, for example, by demanding that the operators $K_\ell^\pm$ be skew-hermitian in the orthonormalized basis arising from the wavefunctions $\Psi_{\mathbf{j}}^s(x_1, \ldots, x_n)$; this is a natural choice if one wishes to interpret the symmetry operators Γ_A as quantum mechanical observables.

Remark 10. As can be seen from (37), (44), and (45), the quantum number s is not associated to any of the symmetry operators Γ_A. Since s is tied to the choice of a basis for the representation space V, the operator associated to this quantum number depends on that choice. For example, one could take the vectors v_s as eigenvectors of $e_1 e_2$ in the representation V. The corresponding symmetry operator would then be $P = e_1 e_2 r_1 r_2$. Note that the presence of $r_1 r_2$ is necessary to ensure that P commutes with every generator of $\mathcal{Y}_n$.

From the actions (44) and (45) of the raising and lowering operators, the expression (39) and the eigenvalue equations of Proposition 8, it is apparent that one can reconstruct the action of any generator Γ_A on the basis functions $\Psi_{\mathbf{j}}^s(x_1, \ldots, x_n)$. However, since the relation between raising/lowering operators is quadratic, there is no systematic way of doing so and the end result is often very involved. We now present the last proposition for this section.

Proposition 12. *For a fixed s, the higher rank Bannai–Ito algebra $\mathcal{A}_n$ acts irreducibly on the space*

$$\mathcal{M}_k^s(\mathbb{R}^n; V) = \mathrm{Span}\{\Psi_{\mathbf{j}}^s(x_1, \ldots, x_n)\},$$

with $\mathbf{j} = (j_1, \ldots, j_n)$, where $j_1, \ldots, j_{n-1}$ are non-negative integers with j_{n-1} defined as $j_{n-1} = k - \sum_{i=1}^{n-2} j_i$ and $\sum_{i=1}^{n-2} j_i \leq k$. The space of k-homogeneous Dunkl

monogenics in $\mathbb{R}^n$ splits in $\dim V$ copies of this irreducible subspace

$$\mathcal{M}_k(\mathbb{R}^n; V) = \bigoplus_{s=1}^{\dim V} \mathcal{M}_k^s(\mathbb{R}^n; V)$$

Proof. The decomposition of $\mathcal{M}_k(\mathbb{R}^n; V)$ as a direct sum of $\mathcal{M}_k^s(\mathbb{R}^n; V)$ is immediate from remark 10. The irreducibility of the representation is manifest from the form of the actions (44) and (45). $\qquad\square$

5. A SCALAR MODEL OF THE HIGHER RANK BANNAI–ITO ALGEBRA

In this section, an alternative realization of the higher rank Bannai–Ito algebra is proposed. The model does not involve Clifford generators, and is hence "scalar".

Let D, X be defined as

$$D = \sum_{i=1}^{n} T_i R_i, \qquad X = \sum_{i=1}^{n} X_i R_i,$$

where $R_i = \prod_{j=i+1}^{n} r_j$. It is readily verified that D also provides a factorization of the Dunkl–Laplace operator; one has indeed

$$D^2 = \Delta, \qquad X^2 = \sum_{i=1}^{n} x_i^2.$$

Similarly, one can define intermediate operators in the following way. For $A \subset [n]$, we consider

$$D_A = \sum_{i \in A} D_i R_i, \qquad X_A = \sum_{i \in A} x_i R_i.$$

One can verify that the operators D_A and X_A together with the Euler operator $\mathbb{E}_A$, the Dunkl-Laplace operator Δ_A and the square-radius operator $\|x_A\|^2$ provide a realization of the $\mathfrak{osp}(1|2)$ Lie superalgebra as in (4). Following the steps of the first section, one can introduce the sCasimir operator

$$S_A = \frac{1}{2}\Big([D_A, X_A] - 1\Big).$$

which satisfies $\{S_A, D_A\} = 0$ and $\{S_A, X_A\} = 0$. Upon defining,

$$\Gamma_A = S_A \prod_{k \in A} r_k,$$

it follows that $[\Gamma_A, D_A] = [\Gamma_A, X_A] = 0$. With these definitions, it can be verified that the operators Γ_A satisfy the commutation relations (21). Moreover, one can confirm that the realization-dependent values taken by the Casimir operators (26) and (29) are also identical. Furthermore, it is possible to adapt the **CK** extension operator to the scalar case.

Remark 11. The $n = 3$ case of this scalar model was already examined in [12].

Remark 12. The Proposition 12 and the existence of the scalar realization indicate that the Clifford algebra representation space V does not play a significant role.

Remark 13. One might ask why the focus was put on the Dirac–Dunkl model rather than the scalar model, which appears simpler. It turns out that the Dirac–Dunkl is advantageous in the construction of the symmetry algebra. Moreover, this model can be viewed as a deformation of the classical Dirac equation by Dunkl operators, and is hence more interesting "physically". However, let us point out that the scalar realization is likely to be more adapted for the study of Conjecture 1.

6. Conclusion

Summing up, we studied the kernel of the n-dimensional Dirac–Dunkl operator associated to the reflection group $\mathbb{Z}_2^n$ as well as the corresponding Dirac–Dunkl equation on the $(n-1)$-sphere. We obtained the symmetries of this equation and shown that they generate a higher rank extension of the Bannai–Ito algebra. Moreover, a basis for the polynomial null-solutions of the Dirac–Dunkl operator was constructed explicitly in terms of Jacobi polynomials. Furthermore, the action of the higher rank Bannai-Ito algebra on this basis was given through raising and lowering operators for this algebra.

The results of this paper call for a further study of the structure and the properties of the higher rank Bannai–Ito algebra $\mathcal{A}_n$. One of the most interesting question in that regard is that of the classification of its unitary irreducible representations. This has not been investigated so far, even in the rank-one case, which corresponds to $\mathcal{A}_3$. In light of the inclusion $\mathcal{A}_n \subset \mathcal{A}_{n-1}$, the representations of $\mathcal{A}_n$ could be studied using induction. Another interesting avenue is the analysis of the symmetries of Dirac–Dunkl operators associated to other reflection groups.

Acknowledgements

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