On a Mathematical Method for Discovering Relations Between Physical Quantities: a Photonics Case Study

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Abstract: Quantity calculus defines the rules that apply to SI physical quantities used in physics and engineering. This research aims at the construction of a rigorous mathematical framework explaining the selection rule resulting in realizable constitutive equations. Here, we show that each SI physical quantity, that is represented by a lattice point in a seven dimensional integer lattice, has a unique 7D-hypersphere. The lattice points incident on the 7D-hypersphere are forming rectangles containing the origin \( o \), the lattice point \( z \) representing the selected physical quantity and the lattice point representation of a pair of distinguishable physical quantities \( [x] \), \([y] \) where \( z = x + y \). The resulting rectangles are the geometric representations of the realizable constitutive equations for the selected physical quantity \( [z] \). We apply the ”n-D-hypersphere” method on the physical quantities \( E, H, D, B \) and find the integral forms of Maxwell’s equations. We find an integer sequence of non-degenerated unique rectangles formed by 4 lattice points \( o, x, y, z \) in \( Z^7 \) as function of the infinity norm \( \|z\|_\infty = s \).

Keywords: constitutive equations, SI physical quantities, integer lattice, 7D-hypersphere, infinity norm.

1. Introduction

The SI [1] is used worldwide defining the semantics and syntax in the domains of science and technology. An algebraic structure for quantity calculus was proposed by R. Fleischmann [2], who also introduced the concept of “Verknüpfungsgleichung” that we translate as constitutive equation. This research addresses the question What are realizable constitutive equations?

2. Axioms of the SI physical quantities

We posit from the 8th edition of the SI [1] a set of axioms derived from promoting some of the SI conventions to mathematical axioms.

Axiom 1. The base quantities are length, mass, time, electric current, thermodynamic temperature, amount of substance and luminous intensity.

Axiom 2. The base quantities are independent.

Axiom 3. The physical quantities are organized according to a system of dimensions.

Axiom 4. For each base quantity of the SI, there exists one and only one dimension.

Axiom 5. The product of two quantities is the product of their numerical values and units.

Axiom 6. The quotient of two quantities is the quotient of their numerical values and units.

The uniqueness of the SI symbols forms an alphabet that is the base of any physical expression.

Definition 1. The dimension of a physical quantity \( q \) is expressed as a dimensional product \( [q] \):

\[
\dim q = L^\alpha M^\beta T^\gamma I^\delta \Theta^\epsilon N^\zeta J^\eta \;
\]

where the exponents \( \alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta \in Z \) are called dimensional exponents.

The dimensional exponents of the common SI physical quantities take small integer values. When all the dimensional exponents are zero, we call the physical quantity dimensionless or a physical quantity of dimension one. These dimensionless quantities occur in the celebrated Buckingham theorem [3] also known as the II-theorem.

3. Isomorphism between classes of physical quantities and the \( 7 \)-dimensional integer lattice

Let the set of all physical quantities be denoted by \( Q \). Physical quantities are described by tensors and we can without loss of generality consider a component of a tensor and denote it as \( q \). We know that concepts in physics are labeled in many ways. The physics concept energy has the labels: potential energy, kinetic energy, work, Lagrange function, Hamilton function,…in the formulations of physics. To cope with this multitude of labels, we define an equivalence relation between the physical quantities \( a, b \in Q \) with notation \( a \sim b \) meaning “\( a \) is dimensionally equivalent to \( b \)”. The set of all equivalence classes in \( Q \), given the equivalence relation \( \sim \), is the quotient set \( Q/\sim \). The equivalence class for the concept energy has notation \( [\text{energy}]_{\sim} \). We define the surjective function \( \dim(q) \) from \( Q \) to \( Q/\sim \) given by \( \dim(q) = [q]_{\sim} = L^\alpha M^\beta T^\gamma I^\delta \Theta^\epsilon N^\zeta J^\eta \). In the sequel of this article we omit the symbol for the equivalence relation \( \sim \) and denote the...
equivalece class as \([q]\). The class of dimensionless physical quantities is denoted \([I]\). We consider a multiplicative binary operator \(\{\cdot\}\) between the equivalence classes of \(Q/\sim\). The algebraic properties of the composition of the equivalence classes result in a multiplicative commutative group \(Q/\sim,\{\cdot\}\). We now consider the set of integer septuples \(Z^7 \equiv \{(f_1, \ldots, f_7): f_i \in Z\}\). We know that \(Z^7, \{+\}\) is an additive commutative group. We define a mapping \(\text{dex}(\cdot):\)

**Definition 2** (Mapping \(\text{dex}(\cdot)\)). The mapping \(\text{dex}(\cdot)\) is defined from \(Q/\sim\) into \(Z^7\) and formally as \(\text{dex}(\cdot): Q/\sim \rightarrow Z^7: \text{dex}(q) = f = (f_1, \ldots, f_7)\) where \(f_i \in Z\).

We relabel \(f_i\) such that \(f_1 = \alpha, f_2 = \beta, f_3 = \gamma, \ldots f_7 = \eta\) being the dimensional exponents taken in the correct order of a physical quantity \(q\). Observe that we map the unit element \([I]\) of \(Q/\sim,\{\cdot\}\) on the unit element \(o = (0, \ldots, 0)\) of \(Z^7, \{+\}\) and thus we have \(\text{dex}(\{I\}) = o = (0, \ldots, 0)\). Each element of \(Z^7\) is the image of one and only one class \([q]\) of dimensionally equivalent physical quantities. We define the inverse mapping \(\text{dex}^{-1}(\cdot):\)

**Definition 3** (Mapping \(\text{dex}^{-1}(\cdot)\)). The inverse of the mapping \(\text{dex}(\cdot)\) is a mapping of \(Z^7\) into \(Q/\sim,\{\cdot\}\), and defined as \(\text{dex}^{-1}(\cdot): \forall a \in Z^7, \exists[a] \in Q/\sim: \text{dex}^{-1}(a) = [a]\).

A homomorphism \(f: Q/\sim \rightarrow Z^7\) is an isomorphism if there exists a homomorphism \(g: Z^7 \rightarrow Q/\sim\) such that \(f \circ g = g \circ f\). We identify \(f = \text{dex}(\cdot)\) and \(g = \text{dex}^{-1}(\cdot)\) and infer that a group isomorphism exists between \(Q/\sim\) and \(Z^7\) that we denote \(Z^7 \approx Q/\sim\). The set \(Z^n\) is known as the \(n\)-dimensional integer lattice \([5]\) that is a discrete subgroup of the real vector space \(\mathbb{R}^n\). The properties of the integer lattice \(Z^n\) are found in several publications \([5]\). In the sequel of this article we choose \(n = 7\). We select seven basis lattice points of \(Z^7\) and choose an orthonormal basis and write using the Conway notation \([5]\):

\[ e_1 \equiv \text{dex}[\text{length}] = (1, 0^6), \]
\[ e_2 \equiv \text{dex}[\text{mass}] = (0, 1, 0^5), \]
\[ e_3 \equiv \text{dex}[\text{time}] = (0^2, 1, 0^4), \]
\[ e_4 \equiv \text{dex}[\text{electric current}] = (0^3, 1, 0^3), \]
\[ e_5 \equiv \text{dex}[\text{thermodynamic temperature}] = (0^4, 1, 0^2), \]
\[ e_6 \equiv \text{dex}[\text{amount of substance}] = (0^5, 1, 0), \]
\[ e_7 \equiv \text{dex}[\text{luminous intensity}] = (0^6, 1) \]

with \(e_i \in Z^7\). A set of lattice points is called a lattice constellation \([6]\). An arbitrary set of physical quantities is represented by a constellation of points in \(Z^7\) and not by a set of vectors. We are interested in the properties of these constellations of points and focus on the simplest non-trivial constellation consisting of 4 integer lattice points. Observe that the parallelogram law \(x + y = z\) where \(x, y, z \in Z^7\) is valid. We can prove \([7]\) that ternary equations \([z] = f([I]) [x][y]\) are geometrically represented by parallelograms in \(Z^7\). We can define \([7]\) an inner product \(\langle \cdot, \cdot \rangle\) and \(p\)-norm \(||| \cdot |||_p\) in \(Z^7\) and write \(f = \sum_{i=1}^7 (f \cdot e_i) e_i\).

4. Decomposition of a lattice point in pairwise orthogonal lattice points

We define distinguishable physical quantities as orthogonal lattice points \(\text{dex}(\{x\})\) and \(\text{dex}(\{y\})\). The decomposition of a lattice point \(z\) in two pairwise orthogonal lattice points \(x\) and \(y\) assumes the existence of a system of Diophantine equations:

\[
\begin{align*}
\text{parallelogram law:} & \quad x + y - z = 0, \quad (1a) \\
\text{inner product:} & \quad x \cdot y = 0, \quad (1b)
\end{align*}
\]

where \(x, y, z \in Z^7\). We eliminate \(y\) from the equation \((1b)\) and find:

\[ x \cdot x - x \cdot z = 0 \quad (2) \]

We apply the method of completing the square and write equation \((2)\) as:

\[ (x - \frac{z}{2})^2 = \left(\frac{z}{2}\right)^2, \quad (3) \]

that represents a 7-D-hypersphere in \(\mathbb{R}^7\) with center at \(\frac{z}{2}\) and radius \(\frac{z}{2}\). The center of the 7-D-hypersphere is a lattice point if its coordinates are even. Observe that there exists a unique 7-D-hypersphere \([8]\) for each physical quantity \([z]\). This unique 7-D-hypersphere determines the finite set of pairwise distinguishable physical quantities \([x]\) and \([y]\) that satisfy the realizable constitutive equation \([z] = f([I]) [x][y]\). We call the above method the “\(n\)-D-hypersphere method” as it can be generalized to a \(n\)-dimensional integer lattice.

5. Applications in photonics

We apply the “\(n\)-D-hypersphere method” to the physical quantities \([H], [B], [E], [D]\) occurring in the celebrated Maxwell’s equations and infer relations between the physical quantities. The SI coordinates of the physical quantities \([H], [B], [E], [D]\) are:

- Magnetic field strength: \(\text{dex}([H]) = (-1, 0, 0, 1, 0, 0, 0)\)
- Magnetic induction: \(\text{dex}([B]) = (0, 1, -2, -1, 0, 0, 0)\)
- Electric field: \(\text{dex}([E]) = (1, 1, -3, -1, 0, 0, 0)\)
- Electrical displacement: \(\text{dex}([D]) = (-2, 0, 1, 1, 0, 0, 0)\)

The results are summarized in an isoperimetric distribution (Table \([1]\) giving the frequency of occurrence of rectangles having a perimeter with value \(p\) formed by the lattice points \(\{o, x_i, y, z\}\) where \(\{x_i, y\}\) are the \(i\)-th pair
of lattice points representing uncorrelated physical quantities. We denote electric current $I$, electric charge $q$, electric charge density $\rho_f$, volume $V$, area $S$, time $t$, length $l$, electric current density $J$. We select the smallest non-degenerated rectangle of $[D]$ having $p = 6.472$. We find the lattice points $x = (-2, 0, 0, 1, 0, 0, 0)$ and $y = (0, 0, 1, 0, 0, 0)$. We suggest the realizable constitutive equation:

$$D = f_1(\Pi) \left( \frac{1}{S} \right) (t)$$

$$DS = f_1(\Pi)It$$

$$D \oint_{S(V)} dS = f_1(\Pi)I \int dt = f_1(\Pi)q$$

$$\oint_{S(V)} D \cdot dS = f_1(\Pi) \int V \rho_f dV$$

$$\oint_V \nabla \cdot D dV = f_1(\Pi) \int V \rho_f dV$$

that is the integral form of $\nabla \cdot D = \rho_f$ where $f_1(\Pi) = 1$. We select the smallest non-degenerated rectangle of $[H]$ having $p = 4$. Observe that only $[H]$ has an unique non-degenerated rectangle. We find the lattice points $x = (-1, 0, 0, 0, 0, 0, 0)$ and $y = (0, 0, 0, 1, 0, 0, 0)$. We suggest the realizable constitutive equation:

$$H = f_2(\Pi) \left( \frac{1}{I} \right) (I)$$

$$Hl = f_2(\Pi)It$$

$$\oint_{L(S)} H \cdot dl = f_2(\Pi) \oint_S J \cdot dS$$

$$\oint_{L(S)} \nabla \times H \cdot dS = f_2(\Pi) \oint_S J \cdot dS$$

that is the integral form of $\nabla \times H = J$ where $f_2(\Pi) = 1$. We select a non-degenerated rectangle of $[E]$ having $p = 9.152$. We find the lattice points $x = (-1, 0, -1, 0, 0, 0, 0)$ and $y = (2, 1, -2, -1, 0, 0, 0)$. We suggest the realizable constitutive equation:

$$E = f_3(\Pi) \left( \frac{1}{It} \right) (BS)$$

$$El = f_3(\Pi) \frac{1}{T} BS$$

$$\oint_{L(S)} E \cdot dl = f_3(\Pi) \frac{d}{dt} \oint_S B \cdot dS$$

$$\oint_{L(S)} E \cdot dl = f_3(\Pi) \frac{d}{dt} \oint_S B \cdot dS$$

$$\oint_S (\nabla \times E) \cdot dS = f_3(\Pi) \frac{d}{dt} \oint_S B \cdot dS$$

that is the integral form of $\nabla \times E = -\frac{\partial B}{\partial t}$ where $f_3(\Pi) = -1$. We have not found a realizable constitutive equation as basis for $\nabla \cdot B = 0$. Observe that only $[E]$ and $[H]$ form a pair of orthogonal lattice points because the inner product $\text{dex} ([E]) \cdot \text{dex} ([B]) = 0$. We find that the physical quantities $[D]$ and $[B]$ have the same isoperimetric distributions and thus we find a matrix $M$ such that $\text{dex} ([D])^T = M \text{dex} ([B])^T$ where:

$$M = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

$$\text{dex} ([D])^T = M \text{dex} ([B])^T$$

represents a signed permutation matrix. The automorphism group of the 7-dimensional cubic lattice $\text{Aut}(\mathbb{Z}^7)$

Table 1: Isoperimetric distributions for $[H], [B], [E], [D]$.

<table>
<thead>
<tr>
<th>$[q]$</th>
<th>Perimeter $p$</th>
<th>Frequency $f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[H]$</td>
<td>2.828</td>
<td>1</td>
</tr>
<tr>
<td>$[H]$</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>$[B]$</td>
<td>4.489</td>
<td>1</td>
</tr>
<tr>
<td>$[B]$</td>
<td>6.472</td>
<td>2</td>
</tr>
<tr>
<td>$[B]$</td>
<td>6.828</td>
<td>9</td>
</tr>
<tr>
<td>$[B]$</td>
<td>6.928</td>
<td>8</td>
</tr>
<tr>
<td>$[E]$</td>
<td>6.928</td>
<td>1</td>
</tr>
<tr>
<td>$[E]$</td>
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<td>3</td>
</tr>
<tr>
<td>$[E]$</td>
<td>9.152</td>
<td>19</td>
</tr>
<tr>
<td>$[E]$</td>
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</tr>
<tr>
<td>$[E]$</td>
<td>9.763</td>
<td>42</td>
</tr>
<tr>
<td>$[E]$</td>
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<td>18</td>
</tr>
<tr>
<td>$[D]$</td>
<td>4.489</td>
<td>1</td>
</tr>
<tr>
<td>$[D]$</td>
<td>6.472</td>
<td>2</td>
</tr>
<tr>
<td>$[D]$</td>
<td>6.828</td>
<td>9</td>
</tr>
<tr>
<td>$[D]$</td>
<td>6.928</td>
<td>8</td>
</tr>
</tbody>
</table>
contains all permutations and sign changes of the 7 coordinates and has order $2^7 7! = 645120$. Each signed permutation matrix is an orthogonal matrix \[ A \]

It is known from linear vector quantization \[ B \] that the $\ell_2$-norm and the phase of a lattice point are used to partition a lattice. However, this norm and phase are not the correct classifiers for the physical quantities. If we use as classifier the $\ell_\infty$-norm we obtain equivalence classes for which the elements of the class have the same isoperimetric distribution. In the framework of information theory we state that the lattice points $\text{dex}(\{D\})$ and $\text{dex}(\{B\})$ are elements of the absolute leader class $[21^20^4]$ that has cardinality 840.

6. Distribution of unique rectangles in the 7D integer lattice

We determine the distribution of non-degenerated unique rectangles formed by 4 lattice points $a, x, y, z$ in $\mathbb{Z}^7$ as function of the infinity norm $\|z\|_\infty = s$. We define a sample space $\Omega$ consisting of 7D-hyperspheres with infinity norm $\|z\|_\infty = s$, with $s \in \mathbb{N}$ and search for the event of an unique perimeter $p$ in each hypercube with $\|z\|_\infty = s$. Table 2 gives the result of the search for unique rectangles. We find in the 7D-hypercube where $\|z\|_\infty \leq 7$ a total of 1321 unique rectangles that represent unique realizable constitutive equations of the ternary type $z = f(\Pi)|x|/|y|$ for the selected physical quantity $z$. This sequence of integers is not listed in the OEIS \[ C \] and we suggest further research on it.

<table>
<thead>
<tr>
<th>Infinity norm $|z|_\infty = s$</th>
<th>Frequency $f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
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<tr>
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<tr>
<td>6</td>
<td>333</td>
</tr>
<tr>
<td>7</td>
<td>693</td>
</tr>
</tbody>
</table>

7. Conclusion

We show that each SI physical quantity, that is represented by a lattice point in a seven dimensional integer lattice $\mathbb{Z}^7$, has a unique 7D-hypersphere. The lattice points incident on the 7D-hypersphere are rectangles formed by 4 lattice points $a, x, y, z$ in $\mathbb{Z}^7$ where $z = x + y$. The resulting rectangles are the geometric representation of the realizable constitutive equations of the ternary type $z = f(\Pi)|x|/|y|$ for the selected physical quantity $z$. We apply the “nD-hypersphere” method on the physical quantities $E, H, D, B$ and find the integral forms of Maxwell’s equations. We find in the 7D-hypercube, where $\|z\|_\infty \leq 7$, a total of 1321 unique rectangles that represent unique realizable constitutive equations.

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References


