Q- and ω_0 -Sensitivities in Positive Gain Second-Order RC Active Filters.

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Abstract

An active filter configuration with a general three-terminal second-order RC feedback network and a positive voltage gain is considered. The sensitivities of the Q factor and resonant frequency with respect to the voltage gain are calculated, and a relationship between lower bounds on them is formulated.

Some time ago, Bown deduced a lower bound on the sensitivity S_A^Q of the Q factor with respect to the amplifier gain of a resonant active filter having a biquadratic transfer function [1]. He used the general configuration shown in fig.1. The amplifier is assumed to have infinite input impedance, zero output impedance, and a positive voltage gain A . The network N is a second-order transformerless RC network. Assuming certain conditions on the network N (e.g., Y_{32} has a zero in the origin only), he deduced the inequality

$$S_A^Q > 2Q - 1.$$

The conventional definitions for a second-order active filter realizing complex poles $-\alpha \pm j\beta$, where $\alpha \ll \beta$, are being used, namely,

$$Q = \frac{\sqrt{\alpha^2 + \beta^2}}{2\alpha}$$

and

$$S_A^Q = \frac{dQ}{dA} \cdot \frac{A}{Q} \quad S_A^{\omega_0} = \frac{d\omega_0}{dA} \cdot \frac{A}{\omega_0}$$

In this letter, a new relation between bounds on S_A^Q and $S_A^{\omega_0}$ is presented. The derivation of this relation requires fewer restrictive assumptions on N .

N being a linear second-order network, we can write

$$V_3(s) = H_1(s)V_1(s) + H_2(s)V_2(s)$$

where $H_1(s)$ and $H_2(s)$ are voltage transfer functions that can be realized by three-terminal

Fig. 1.

transformerless second-order RC networks. We therefore have

$$H_2(s) = \frac{as^2 + bs + c}{(s + \sigma_1)(s + \sigma_2)}$$

where σ_1, σ_2, a, b and c are real and positive constants satisfying the conditions [2]

$$0 \leq a \leq 1$$

$$0 \leq b \leq \sigma_1 + \sigma_2$$

$$0 \leq c \leq \sigma_1 \sigma_2.$$

The poles $-\alpha \pm j\beta$ of $(V_2(s))/(V_1(s))$ are the roots of

$$(1 - aA)s^2 + (\sigma_1 + \sigma_2 - bA)s + (\sigma_1 \sigma_2 - cA) = 0$$

Therefore,

$$\begin{aligned} \omega_0 &= (1 - aA)^{-1/2}(\sigma_1 \sigma_2 - cA)^{1/2} \\ &= Y(A)/X(A) \end{aligned} \tag{1}$$

$$\begin{aligned} Q &= (1 - aA)^{1/2}(\sigma_1 \sigma_2 - cA)^{1/2} \\ &\quad \cdot (\sigma_1 + \sigma_2 - bA)^{-1} \\ &= X(A).Y(A).Z(A). \end{aligned} \tag{2}$$

The sensitivities S_A^Q and $S_A^{\omega_0}$ can be calculated as follows:

$$\begin{aligned} S_A^Q &= S_A^X + S_A^Y + S_A^Z \\ &= -\frac{1}{2} \frac{aA}{1 - aA} - \frac{1}{2} \frac{cA}{\sigma_1 \sigma_2 - cA} \\ &\quad + \frac{bA}{\sigma_1 + \sigma_2 - bA} \end{aligned}$$

and

$$\begin{aligned} S_A^{\omega_0} &= -S_A^X + S_A^Y \\ &= \frac{1}{2} \frac{aA}{1-aA} - \frac{1}{2} \frac{cA}{\sigma_1\sigma_2 - cA}. \end{aligned}$$

We can now introduce the Q factor and resonant frequency, given by (1) and (2):

$$S_A^Q = \frac{\sigma_1 + \sigma_2}{\omega_0(1-aA)} \left(Q - \frac{1}{2} \frac{\frac{\sigma_1\sigma_2}{\omega_0^2} + 1}{\frac{\sigma_1 + \sigma_2}{\omega_0}} \right) \quad (3)$$

$$S_A^{\omega_0} = \frac{1}{2} \frac{1 - \frac{\sigma_1\sigma_2}{\omega_0^2}}{1-aA} \quad (4)$$

A necessary condition for the absolute stability of the configuration is that

$$\frac{cA}{\sigma_1\sigma_2} < 1.$$

Combining this inequality with (1) we obtain

$$0 < 1 - aA \leq 1 \quad (5)$$

which will be used later on.

Suppose that we want a reasonably small S_A^Q sensitivity, say

$$|S_A^Q| \leq k. \quad (6)$$

According to (3), this requires the following inequality to hold as a necessary (but not sufficient) condition:

$$2 Q \frac{\sigma_1 + \sigma_2}{\omega_0} \leq \frac{\sigma_1\sigma_2}{\omega_0^2} + m \quad (7)$$

with

$$m = 2k(1-aA) + 1. \quad (8)$$

Because $H_2(s)$ is the voltage transfer function of a passive RC structure, we have

$$\sigma_1 + \sigma_2 \geq 2\sqrt{\sigma_1\sigma_2} \quad (9)$$

so that condition (7) leads to

$$\frac{\sqrt{\sigma_1\sigma_2}}{\omega_0} + m \frac{\omega_0}{\sqrt{\sigma_1\sigma_2}} \geq 4 Q \quad (10)$$

If we make the assumption that $m \ll 4Q^2$, a condition that is usually satisfied for high-Q low S_A^Q -sensitivity networks, inequality (10) requires

$$\sqrt{\sigma_1\sigma_2} \leq \omega_0 x_1 \quad \text{or} \quad \sqrt{\sigma_1\sigma_2} \geq \omega_0 x_2$$

where x_1 and x_2 , are given by

$$\begin{aligned} x_1 &= 2Q \left(1 - \sqrt{1 - \frac{m}{4Q^2}}\right) \\ x_2 &= 2Q \left(1 + \sqrt{1 - \frac{m}{4Q^2}}\right) \end{aligned}$$

Because, now,

$$\sqrt{1 - \frac{m}{4Q^2}} > 1 - \frac{m}{4Q^2}$$

we have

$$x_1 < \frac{m}{2Q} \quad x_2 > 4Q - \frac{m}{2Q}.$$

If we, therefore, want S_A^Q to satisfy condition (6), it is necessary that either

$$\sigma_1\sigma_2 < \frac{m^2}{4Q^2}\omega_0^2 \tag{11}$$

or

$$\sigma_1\sigma_2 > (4Q - \frac{m}{2Q})^2\omega_0^2. \tag{12}$$

We now investigate the effect of the foregoing conditions on the sensitivity $S_A^{\omega_0}$. If condition (11) is satisfied, we obtain, using (1) and the fact that c is a positive constant,

$$(1 - aA) < \frac{m^2}{4Q^2} \tag{13}$$

Because, on the other hand,

$$1 - \frac{\sigma_1\sigma_2}{\omega_0^2} > 1 - \frac{m^2}{4Q^2} \geq 0$$

(4) yields

$$S_A^{\omega_0} > 2\frac{Q^2}{m^2} - \frac{1}{2}. \tag{14}$$

If, on the contrary, condition 12 is satisfied, we have

$$1 - \frac{\sigma_1\sigma_2}{\omega_0^2} < -16Q^2 - \frac{m^2}{4Q^2} + 1 + 4m < -16Q^2 + 1 + 4m < 0.$$

Using (4) and inequality (5) we obtain

$$|S_A^{\omega_0}| > 8Q^2 - \frac{1}{2} - 2m. \quad (15)$$

The conclusion is that if the Q -sensitivity S_A^Q is smaller than k , the ω_0 -sensitivity $S_A^{\omega_0}$ is bounded by either

$$\begin{aligned} |S_A^{\omega_0}| &> 2\frac{Q^2}{m^2} - \frac{1}{2} \quad \text{or} \\ |S_A^{\omega_0}| &> 8Q^2 - \frac{1}{2} - 2m. \end{aligned} \quad (16)$$

A short discussion of these bounds is useful. Let us suppose that we want to realize a circuit with a high Q factor. By letting $a = c = 0$ (Sallen-Key filters [3]), we obtain $S_A^{\omega_0} = 0$, but, as shown by Bown, it becomes impossible to obtain a value of S_A^Q (or of k) lower than $2Q - 1$.

If we assume a or c (or both) to be non-zero, we have shown that lower S_A^Q values and thus lower k -values can be obtained. In that case, however, the inequality (16) states that $S_A^{\omega_0}$ rapidly becomes very high. Often, both S_A^Q and $S_A^{\omega_0}$ are high (as is the case in Inigo's circuits [4]. While not desirable, such a situation is not always to be rejected. In a recent paper, Moschytz [5] argued that even a high S_A^Q can lead to a reasonable Q -variation with temperature, if the passive elements that determine A track very closely and if the loop gain remains large enough. The same argument holds for $S_A^{\omega_0}$. Under those conditions high sensitivities $S_A^{\omega_0}$ and S_A^Q can be tolerated.

CONCLUSION

If we try to realize a high Q -factor with a positive gain second-order RC active filter, it is impossible to obtain simultaneously low sensitivities of the Q -factor and the resonant frequency with respect to the amplifier gain.

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