Constructing minimal blocking sets using field reduction

Geertrui Van de Voorde∗

December 16, 2015

Abstract

We present a construction for minimal blocking sets with respect to \((k - 1)\)-spaces in \(\text{PG}(n - 1, q^t)\), the \((n - 1)\)-dimensional projective space over the finite field \(\mathbb{F}_{q^t}\) of order \(q^t\). The construction relies on the use of blocking cones in the field reduced representation of \(\text{PG}(n - 1, q^t)\), extending the well-known construction of linear blocking sets. This construction is inspired by the construction for minimal blocking sets with respect to the hyperplanes by Mazzocca, Polverino and Storme (the MPS-construction): we show that for a suitable choice of the blocking cone over a planar blocking set, we obtain larger blocking sets than the ones obtained from planar blocking sets in [15].

Furthermore we show that every minimal blocking set with respect to the hyperplanes in \(\text{PG}(n - 1, q^t)\) can be obtained by applying field reduction to a minimal blocking set with respect to \((nt - t - 1)\)-spaces in \(\text{PG}(nt - 1, q)\). We end by relating these constructions to the linearity conjecture for small minimal blocking sets. We show that if a small minimal blocking set is constructed from the MPS-construction, it is of Rédei-type whereas a small minimal blocking set arises from our cone construction if and only if it is linear.

Keywords: field reduction, blocking set, Desarguesian spread, linear set, linearity conjecture

1 Introduction

This paper is inspired by the paper [16], where Mazzocca, Polverino and Storme construct minimal blocking sets with respect to the hyperplanes in \(\text{PG}(n, q^t)\) by using certain cones in the Barlotti-Cofman representation of \(\text{PG}(n, q^t)\), extending the results of [15] to general dimension. Our paper is organised as follows. In Section 2 we give the necessary background on the Barlotti-Cofman and field reduced representation of \(\text{PG}(n, q)\) and recall the correspondence between these representations. In Section 3 we explain how the construction of Mazzocca, Polverino and Storme (also called the MPS-construction) can be presented in an easier way by making use of field reduction: the obtained blocking set corresponds to the points of a minimal blocking set (with respect to subspaces of a

∗This author is supported by the Fund for Scientific Research Flanders (FWO – Vlaanderen).
particular dimension) in a projective space over $\mathbb{F}_q$, considered over $\mathbb{F}_{q^t}$. We also show that the construction of linear blocking sets and the recent construction by Costa [6] fit in this framework. By using cones in the field reduced representation of $\text{PG}(n, q^t)$ we generalise the MPS-construction in Section 4. Starting from a planar blocking set, we construct non-planar blocking sets with respect to $(k - 1)$-spaces in $\text{PG}(n - 1, q^t)$. In Corollary 4.11 we show that if we choose the defining blocking cone carefully, we construct blocking sets whose size exceeds the ones obtained from the MPS-construction using planar blocking sets.

Finally, in Section 5, we show that every minimal blocking set with respect to the hyperplanes in $\text{PG}(n - 1, q^t)$ can be obtained by applying field reduction to a minimal blocking set with respect to $(nt - t - 1)$-spaces in $\text{PG}(nt - 1, q)$. This also provides us with a different view on the linearity conjecture for small minimal blocking sets. Finally, we show that if a small minimal blocking set is obtained by the MPS-construction then it is of Rédei-type, whereas a small minimal blocking set arises from our Construction 4 if and only if it is a linear blocking set.

2 The Barlotti-Cofman representation and field reduction

2.1 Desarguesian spreads and field reduction

Throughout this paper, we let $\text{PG}(m - 1, q)$ denote the $(m - 1)$-dimensional projective space over the finite field $\mathbb{F}_q$ of order $q$. A $(t - 1)$-spread of a projective space $\text{PG}(m - 1, q)$ is a family of mutually disjoint subspaces of dimension $(t - 1)$ partitioning the space $\text{PG}(m - 1, q)$. It is not hard to show that if a $(t - 1)$-spread of $\text{PG}(m - 1, q)$ exists, then $t$ divides $m$. On the other hand, if $t$ divides $m$, there exists a $(t - 1)$-spread of $\text{PG}(m - 1, q)$. This was already shown by Segre [18], and can also be seen as follows.

By field reduction every point of $\text{PG}(n - 1, q^t)$ corresponds to a 1-dimensional vector space over $\mathbb{F}_{q^t}$, which is a $t$-dimensional vector space over $\mathbb{F}_q$, and hence, also corresponds to a projective $(t - 1)$-dimensional space over $\mathbb{F}_q$. The set of all $(t - 1)$-spaces obtained in this way forms a spread of $\text{PG}(nt - 1, q)$, which is called a Desarguesian $(t - 1)$-spread. Throughout this paper, this $(t - 1)$-spread in $\text{PG}(nt - 1, q)$ is fixed and is denoted by $\mathcal{D}$. A $\mathcal{D}$-subspace of $\text{PG}(nt - 1, q)$ is a space spanned by elements of $\mathcal{D}$. It follows from the construction that a $\mathcal{D}$-subspace is partitioned by elements of $\mathcal{D}$ and corresponds to a field reduced subspace $\pi$ of $\text{PG}(n - 1, q^t)$. If the subspace $\pi$ has dimension $r - 1$, then we say that the $\mathcal{D}$-subspace of dimension $rt - 1$ corresponding to $\pi$ is a $\mathcal{D}_{r-1}$-subspace.

The following statement is well-known and can be proven by a straightforward counting argument. It will be of use later in this paper.

**Lemma 2.1.** Every hyperplane of $\text{PG}(nt - 1, q)$ contains exactly one $\mathcal{D}_{n-2}$-subspace, i.e. an $(nt - t - 1)$-space spanned by elements of $\mathcal{D}$.

If $U$ is a subset of $\text{PG}(nt - 1, q)$, then we define $\mathcal{B}(U) := \{R \in \mathcal{D} \mid U \cap R \neq \emptyset\}$. In this paper, we identify the elements of $\mathcal{B}(U)$ with their corresponding points of $\text{PG}(n - 1, q^t)$. Linear sets can be defined in several equivalent ways, but using the terminology of this paper, an $\mathbb{F}_q$-linear set $S$ in $\text{PG}(n - 1, q^t)$ is a set of points such that $S = \mathcal{B}(\mu)$, where $\mu$
is a subspace of $\text{PG}(nt - 1, q)$. For more information on field reduction and linear sets, we refer to [12].

2.2 The Barlotti-Cofman representation

Let $H$ be a hyperplane of $\text{PG}(n - 1, q^t)$; by field reduction $H$ corresponds to a $\mathcal{D}_{n-2}$-subspace $\Sigma$ of $\text{PG}(nt - 1, q)$. Note that $\Sigma$ has dimension $nt - t - 1$. Let $\Sigma'$ be an $(nt - t)$-space through $\Sigma$ in $\text{PG}(nt - 1, q)$.

Consider the following geometry $\Pi_{n-1} = \Pi_{n-1}(\Sigma', \Sigma, S)$, where $S$ is the set of elements of $\mathcal{D}$ contained in $\Sigma$:

- Points: the points of $\Sigma' \setminus \Sigma$ and the elements of $S$.
- Lines: the $t$-subspaces of $\Sigma'$ meeting $\Sigma$ exactly in an element of $S$, together with the $\mathcal{D}_1$-subspaces contained in $\Sigma$.

Incidence: natural.

The incidence structure $\Pi_{n-1}$ is isomorphic to the design obtained by taking points and lines of $\text{PG}(n - 1, q^t)$ and we say that $\Pi_{n-1}$ is the Barlotti-Cofman representation of $\text{PG}(n - 1, q^t)$ [1].

2.3 The correspondence between the Barlotti-Cofman representation and the representation using field reduction

From the definitions, we get that the geometry $\mathcal{G}$ with as points the spread elements of $\mathcal{D}$ and as lines the $\mathcal{D}_1$-spaces of $\text{PG}(nt - 1, q)$ is isomorphic to the design of points and lines of $\text{PG}(n - 1, q^t)$. Let, as in the previous section, $\Sigma'$ be an $(nt - t)$-space through the $\mathcal{D}_{n-2}$-space $\Sigma$ in $\text{PG}(nt - 1, q)$.

![Figure 1: The Barlotti-Cofman representation inside the field reduction representation](image)

It is clear that the Barlotti-Cofman representation $\Pi_{n-1}$ and $\mathcal{G}$ are isomorphic: consider the following mapping $\phi$.
\[ \phi : \mathcal{G} \rightarrow \Pi_{n-1} \]
\[ R \not\subset \Sigma \mapsto R \cap \Sigma' \text{ for } R \in \mathcal{D} \]
\[ R \subset \Sigma \mapsto R \text{ for } R \in \mathcal{D} \]
\[ L \not\subset \Sigma \mapsto L \cap \Sigma', \text{ for } L \text{ a } \mathcal{D}_1\text{-space} \]
\[ L \subset \Sigma \mapsto L, \text{ for } L \text{ a } \mathcal{D}_1\text{-space}. \]

It is easy to see that $\phi$ defines an isomorphism between $\mathcal{G}$ and $\Pi_{n-1}$. This isomorphism will enable us to describe the MPS-construction in an easier way.

## 3 Blocking sets

A blocking set $B$ in $\text{PG}(n, q)$ with respect to $k$-spaces is a set of points such that every $k$-dimensional space in $\text{PG}(n, q)$ (or $k$-space) contains at least one point of $B$. We also say that the set $B$ blocks all $k$-spaces. If we are considering blocking sets with respect to the hyperplanes, we simply say that $B$ is a blocking set. A minimal blocking set $B$ (w.r.t. $k$-spaces) is a blocking set such that no proper subset of $B$ is a blocking set (w.r.t. $k$-spaces). An essential point of a blocking set with respect to $k$-spaces $B$ is a point lying on a tangent $k$-space to $B$ and we see that $B$ is minimal if and only if every point of $B$ is essential. A blocking set w.r.t. $k$-spaces in $\text{PG}(n, q)$ is called trivial if it contains an $(n-k)$-space. A small blocking set in $\text{PG}(n, q)$ with respect to $k$-spaces is a blocking set of size smaller than $3(q^{n-k}+1)/2$. A blocking set $B$ (w.r.t. hyperplanes) in $\text{PG}(n, q)$ is of Rédei-type if there exists a hyperplane meeting $B$ in $|B| - q$ points.

Most constructions for blocking sets concern the planar case or the case of blocking sets with respect to the hyperplanes in $\text{PG}(n, q)$. For blocking sets with respect to $k$-spaces, $k \neq n - 1$, there are results of Beutelspacher [2] and Heim [8] classifying the smallest non-trivial blocking sets as cones with base a blocking set in a plane. Other results characterise the smallest blocking sets that span a space of a certain fixed dimension [4] or aim at classifying small minimal blocking sets as linear sets (see Section 5).

### 3.1 The cone construction for blocking sets

We recall that the cone $K$ with vertex $\Omega$, where $\Omega$ is a subspace of $\text{PG}(n, q)$ and base $\bar{B}$, contained in a subspace $\Gamma$, skew from $\Omega$, is the set $\bigcup_{\bar{P} \in \bar{B}} \langle \bar{P}, \Omega \rangle$.

The following lemma is well-known, but since we did not find an exact reference, we give a proof for completeness.

**Lemma 3.1.** Let $\Omega$ be an $s$-dimensional subspace of $\text{PG}(n, q)$, let $\Gamma$ be an $(n-s-1)$-space disjoint from $\Omega$. The set $\bar{B}$ is a minimal blocking set with respect to $k$-spaces of the space $\Gamma = \text{PG}(n-s-1, q)$ ($k < n - s - 1$), if and only if the cone $K$ with vertex $\Omega$ and base $\bar{B}$ is a minimal blocking set with respect to $k$-spaces of $\text{PG}(n, q)$.

**Proof.** First assume that $\bar{B}$ is a minimal blocking set with respect to $k$-spaces of $\Gamma$. Let $\mu$ be a $k$-space of $\text{PG}(n, q)$. If $\mu$ meets $\Omega$, then $K$ blocks $\Omega$, so assume that $\mu$ is skew from $\Omega$. The projection of $\mu$ from $\Omega$ onto $\Gamma$ (i.e. $\langle \Omega, \mu \rangle \cap \Gamma$) is a $k$-space $\mu'$, which contains
at least one point \( P \) of \( \mathcal{B} \). This implies that \( \mu \) meets \( \langle \Omega, P \rangle \) in at least one point, hence, \( \mu \) contains a point of \( K \). This shows that \( K \) is a blocking set. We will now show that \( K \) is minimal. Since \( \mathcal{B} \) is a minimal blocking set with respect to \( k \)-spaces in \( \Gamma \), every point \( Q \) of \( \mathcal{B} \) lies on a tangent \( k \)-space \( T_Q \). Let \( \nu \) be a \((k - 1)\)-space in \( T_Q \), not through \( Q \), then for every point \( R \) of \( \Omega \), \( \langle R, \nu \rangle \) is a tangent \( k \)-space to \( K \) through the point \( R \), so every point of \( \Omega \) is essential. Now let \( S \) be a point of \( K \), not in \( \Omega \). The projection of \( S \) onto \( \Gamma \) is a point \( S' \) of \( \mathcal{B} \), lying on a tangent \( k \)-space \( T_{S'} \). The space \( \langle \Omega, T_{S'} \rangle \) is \((k + s + 1)\)-dimensional, hence, we can take a \( k \)-dimensional subspace of \( \langle \Omega, T_{S'} \rangle \), meeting \( \langle \Omega, S' \rangle \) in only the point \( S \), which is a tangent \( k \)-space through \( S \) to \( K \).

Conversely, if \( K \) is a minimal blocking set, every \( k \)-space in \( \text{PG}(n, q) \), and hence in \( \Gamma \) is blocked by the points of \( K \), hence, \( \mathcal{B} \) is a blocking set with respect to \( k \)-spaces. Since \( K \) is minimal, there exists a tangent \( k \)-space \( T_P \) to \( K \) through every point \( P \) of \( \mathcal{B} \). It is clear that the projection of \( T_P \) from \( \Omega \) onto \( \Gamma \) is a tangent \( k \)-space through \( P \) to \( \mathcal{B} \), hence, \( \mathcal{B} \) is minimal.

**Remark 1.** The cone \( K \) with vertex \( \Omega \), where \( \Omega \) is an \( s \)-dimensional subspace of \( \text{PG}(n, q) \), and base \( \mathcal{B} \) (contained in an \((n - s - 1)\)-space \( \Gamma \), skew from \( \Omega \)) has size \( q^{s+1}|\mathcal{B}| + \frac{q^{s+1}-1}{q-1} \).

### 3.2 Linear blocking sets and the MPS-construction

The following lemma is essentially a trivial observation, but it is the key idea behind the constructions presented in this paper.

**Lemma 3.2.** Let \( B' \) be a blocking set with respect to \((kt - 1)\)-spaces in \( \text{PG}(nt - 1, q) \), then \( B = \mathcal{B}(B') \) is a blocking set with respect to \((k - 1)\)-spaces in \( \text{PG}(n - 1, q') \).

**Proof.** As \( B' \) blocks all \((kt - 1)\)-spaces in \( \text{PG}(nt - 1, q) \), it also blocks the \((k - 1)\)-spaces spanned by spread elements (i.e. the \( D_{k-1} \)-spaces), which means that all \((k - 1)\)-spaces of \( \text{PG}(n - 1, q') \) contain at least one point of \( \mathcal{B}(B') \).

The previous lemma provides us with a way of creating blocking sets \( B \) in \( \text{PG}(n - 1, q') \), using blocking sets \( B' \) in \( \text{PG}(nt - 1, q) \). An important problem is to determine when the obtained blocking set \( B \) is minimal. In the MPS-construction and the construction of Costa, particular minimal blocking sets \( B' \) are considered to ensure that \( \mathcal{B}(B') \) is minimal. In the next subsections, we recall these constructions and translate them from the Barlotti-Cofman setting to the setting using field reduction which enables an easier description. Since linear blocking sets also fit in this framework and will be used later in this paper, we discuss them here.

#### 3.2.1 Linear blocking sets

Linear blocking sets with respect to \((k - 1)\)-spaces in \( \text{PG}(n - 1, q') \) were introduced by Lunardon [13]: he argues that an \( \mathbb{F}_q \)-linear set of rank \( nt - kt + 1 \) is a blocking set. In view of Lemma 3.2 this is clear: we take \( B' \) to be an \((nt - kt)\)-dimensional subspace of \( \text{PG}(nt - 1, q) \), which is a blocking set with respect to \((k - 1)\)-spaces, to obtain a (linear) blocking set \( \mathcal{B}(B') \).

The importance of this construction lies in the fact that it provided counterexamples to the belief that all small minimal blocking set were of Rédei-type [17]. We will come back to this in Section 5.2.
It is important to note that linear blocking sets are necessarily minimal blocking sets. This was shown in [14] for $\mathbb{F}_q$-linear blocking sets with respect to lines in $\text{PG}(n - 1, q^t)$. But for blocking sets with respect to $(k - 1)$-spaces in $\text{PG}(n - 1, q^t)$, $k \neq 2$, the minimality is up to our knowledge not directly proven in the literature; it does however follow easily from the following lemma of Szönyi and Weiner.

**Lemma 3.3.** [21, Lemma 3.1] Let $B$ be a blocking set of $\text{PG}(n - 1, q)$ with respect to $(k - 1)$-dimensional subspaces, $q = p^h$, $p$ prime, and suppose that $|B| \leq 2q^{n-k}$. Assume that each $(k - 1)$-dimensional subspace of $\text{PG}(n, q)$ intersects $B$ in $1 \mod p$ points. Then $B$ is minimal.

Since every subspace meets an $\mathbb{F}_q$-linear blocking set in $1 \mod q$ points, and an $\mathbb{F}_q$-linear blocking set with respect to $(k - 1)$-spaces in $\text{PG}(n - 1, q^t)$ has size at most $(q^{nt-kt+1} - 1)/(q - 1)$, this implies the following.

**Corollary 3.4.** An $\mathbb{F}_q$-linear blocking set with respect to $(k - 1)$-spaces in $\text{PG}(n - 1, q^t)$ is minimal.

The fact that linear blocking sets are minimal will follow directly from Theorem 4.4.

### 3.2.2 The MPS-construction

The MPS-construction goes as follows. Let $\mathcal{S}$ be a Desarguesian $(t - 1)$-spread of $\Sigma = \text{PG}(nt - t - 1, q)$, embed $\Sigma$ as a hyperplane in $\Sigma' = \text{PG}(nt - t, q)$ and consider the Barlotti-Cofman representation of $\text{PG}(n - 1, q^t)$ as $\Pi_{n-1}(\Sigma', \Sigma, \mathcal{S})$ as described in Subsection 2. Let $Y$ be a fixed element of $\mathcal{S}$ and let $\Omega$ be a hyperplane of $Y$. Let $\Gamma'$ be an $(nt - 2t + 1)$-dimensional subspace of $\Sigma'$, disjoint from $\Omega$ and denote by $\Gamma$ the $(nt - 2t)$-dimensional subspace intersection of $\Gamma'$ and $\Sigma$ and by $T$ the intersection point of $\Gamma$ and $Y$. Let $B$ be a blocking set of $\Gamma'$ such that $\bar{B} \cap \Gamma = \{Q\}$, $Q$ a point, with the property that $\ell \setminus \{T\} \not\subset \bar{B}$, for every line $\ell$ of $\Gamma'$ through $T$. Denote by $K$ the cone with vertex $\Omega$ and base $\bar{B}$. Let $B$ be the subset of $\Pi_{n-1}$ defined by

$$B = (K \setminus \Sigma) \cup \{X \in \mathcal{S} : X \cap K \neq \emptyset\}.$$  

![Figure 2: The MPS-construction](image)

With these definitions, the authors show:
Theorem 3.5. [16] Proposition 1] The set $B$ is a blocking set of the projective space $\text{PG}(n-1, q^t)$.

Let $B$ be the blocking set obtained by the MPS-construction, then, by using the correspondence between the Barlotti-Cofman-representation and the representation using field reduction (see Subsection 2), we see that $B$ corresponds to $B(K)$ by using the above definitions. We also see that the set $K$ is a blocking set with respect to $(nt - t - 1)$-spaces in $\text{PG}(nt - 1, q^t)$: by Lemma 3.1 $K$ is a blocking set w.r.t. $(nt - 2t)$-spaces in $(\Omega, \Gamma')$. Clearly every $(nt - t - 1)$-space of $\text{PG}(nt - 1, q^t)$ meets $(\Omega, \Gamma')$ in a space of dimension at least $nt - 2t$.

The authors distinguish two different cases of their construction: the case $T = Q$ (which they call Construction A) and the case $T \neq Q$ (Construction B).

The authors show for both cases:

Theorem 3.6. [16] Proposition 2] $B$ is a minimal blocking set of $\text{PG}(n-1, q^t)$ if and only if $B$ is a minimal blocking set of $\Gamma'$.


The notations $S, \Sigma, \Sigma'$ and $Y$ are used as before. Let $\Omega$ be an $s$-dimensional subspace of $Y$, with $0 \leq s \leq t - 2$, let $\Gamma'$ be an $(nt - t - s - 1)$-dimensional subspace of $\Sigma$ skew from $\Omega$, let $\Gamma$ be the $(nt - t - s - 2)$-dimensional intersection of $\Gamma'$ and $\Sigma$, let $\theta$ be the $t - s - 2$-space $Y \cap \Gamma$. Define $R$ to be the set $\{S, \Omega) \cap \Gamma': S$ is a hyperplane of $\Pi_{n-1}$, not containing $Y\}$. Let $B$ be a blocking set with respect to elements of the set $R$, containing the space $\theta$. In the same way as before, $B$ is defined to be the set of points of the cone with vertex $\Omega$ and base $B$, contained in $\Sigma' \setminus \Sigma$, together with the point $Y$.

Costa shows that $B$ is a minimal blocking set in $\Pi_{n-1} = \text{PG}(n-1, q^t)$ if and only if $B$ is a minimal blocking set with respect to elements of $R$.

Remark 3. The MPS-construction and the construction of Costa generalise the cone construction of Lemma 4.1 in some sense. Instead of considering the cone with base a blocking set over $F_q$, these constructions use cones over blocking sets over a subfield $F_{q^t}$, i.e., a line through a point of the vertex and the base is no longer a full line, but a subline. The same idea was used for planar blocking sets before in [12] and [20].

4 Constructing minimal blocking sets with respect to $(k - 1)$-spaces in $\text{PG}(n-1, q^t)$

4.1 A general construction method

Construction 1. Let $\Omega$ be an $(nt - kt - 2)$-dimensional subspace of $\text{PG}(nt - 1, q^t)$, let $B$ be a blocking set, contained in a plane $\Gamma$ which is skew from $\Omega$ and let $K$ be the cone in $\Pi = (\Omega, \Gamma)$ with vertex $\Omega$ and base $\tilde{B}$. Let $B = B(K)$.

Lemma 4.1. The set $B$ of Construction 1 is a blocking set with respect to $(k - 1)$-spaces in $\text{PG}(n-1, q^t)$.

Proof. By Lemma 3.1 the cone with base $\tilde{B}$ and vertex $\Omega$ is a blocking set with respect to lines in $(\Omega, \Gamma)$, hence, with respect to $(kt - 1)$-spaces in $\text{PG}(nt - 1, q^t)$. So the statement follows from Lemma 3.2.
Lemma 4.2. Let $\Pi$ be an $(nt-kt+1)$-dimensional subspace of $\text{PG}(nt-1,q)$, $k \geq 2$. Let $\mathcal{D}$ be a Desarguesian $(t-1)$-spread in $\text{PG}(nt-1,q)$ and let $P$ be a point of $\Omega$. Then there exists a $\mathcal{D}_{k-2}$-space meeting $\Omega$ only in points of $\mathcal{B}(P)$. Moreover, if $\dim(\mathcal{B}(P) \cap \Omega) \geq 1$, i.e. if the spread element through $P$ meets $\Omega$ in a subspace of dimension at least 1, then there is a $\mathcal{D}_{k-1}$-space meeting $\Omega$ only in points of $\mathcal{B}(P)$.

Proof. We will first show that there exists a $\mathcal{D}_s$-space containing only points of $\mathcal{B}(P)$ of $\Pi$, where $0 \leq s \leq k-2$. The statement holds trivially for $s = 0$. Suppose that the statement holds for some $0 < i \leq k-3$ and let $D$ be the obtained $\mathcal{D}_r$-space. The number of $\mathcal{D}_{i+1}$-spaces through $D$ is $\frac{(q^i)^{n-i-1}-1}{q-1}$. If $\mathcal{B}(P)$ meets $\Pi$ in a space of dimension $r$, then the number of $\mathcal{D}_{i+1}$-spaces through $D$ that meet $\Pi$ in a point, not belonging to $\mathcal{B}(P)$ is at most $\frac{q^{n-kt+r+1}-1}{q-1}$, since different $\mathcal{D}_{i+1}$-spaces through $D$ that meet $\Pi$ meet in spaces that have only the points of $\mathcal{B}(P)$ in common. Now $\frac{q^{n-kt-r+1}-1}{q-1} < \frac{(q^i)^{n-i-1}-1}{q-1}$ for all $r \geq 0$, and $i \leq k-3$, hence, at least one of the $\mathcal{D}_{i+1}$-spaces through $D$ meets $\Pi$ only in points of $\mathcal{B}(P)$. By induction we find a $\mathcal{D}_{k-2}$-space $T$ containing only points of $\mathcal{B}(P)$ of $\Pi$.

If $\mathcal{B}(P)$ meets $\Pi$ in a space of dimension at least one, then the above count with $r \geq 1$ and $i = k-2$ shows that there exists a $\mathcal{D}_{k-1}$-space through $T$ meeting $\Pi$ only in $\mathcal{B}(P)$. \hfill \Box

Lemma 4.3. Let $\Omega$ be an $(nt-kt-2)$-dimensional subspace of $\text{PG}(nt-1,q)$, let $\bar{B}$ be a minimal blocking set, contained in the plane $\Gamma$ which is skew from $\Omega$. Let $K$ be the cone in $\Pi = \langle \Omega, \Gamma \rangle$ with vertex $\Omega$ and base $\bar{B}$. Suppose that every point of $\bar{B}$ lies on at least $t$ tangent lines to $\bar{B}$ in $\Gamma$. If $P$ is a point of $K$, then $P$ lies in at least $t$ hyperplanes $H_P$ of $\Pi$ that meet $K$ only in some fixed subspace $\mu$ of $\Pi$ of codimension 2.

Proof. Let $P$ be a point of $K$, not in $\Omega$, and let $P'$ be the point $\langle \Omega, P \rangle \cap \Gamma$, which is contained in $\bar{B}$. By assumption, there are at least $t$ tangent lines $\ell_i^{P'}$, $i = 1, \ldots, t$, through $P'$ to $\bar{B}$. The space $\langle \ell_i^{P'}, \Omega \rangle$ is a hyperplane of $\Pi$ which meets $K$ only in the space $\langle \Omega, P \rangle$ which has codimension 2 in $\Pi$.

Let $P$ be a point of $\Omega$. By choosing all hyperplanes $\langle \Omega, \ell_i^Q \rangle$ with $Q$ arbitrary in $\bar{B}$ we satisfy the required condition. \hfill \Box

Theorem 4.4. If $\bar{B}$ is a minimal blocking set in $\Gamma$ such that every point of $\bar{B}$ lies on at least 2 tangent lines to $\bar{B}$, then the blocking set $B = B(K)$ obtained from Construction 7 is minimal.

Proof. Let $P$ be a point of $K$. We need to show that there is a $\mathcal{D}_{k-1}$-space through $P$ containing only points of $\mathcal{B}(P)$ of $K$. If $\mathcal{B}(P)$ meets $\Pi$ in a space of dimension at least one, then by Lemma 4.2, there exists a $\mathcal{D}_{k-1}$-space meeting $\Pi$ and hence $K$ only in $\mathcal{B}(P)$, which implies that there is a tangent $(k-1)$-space through $\mathcal{B}(P)$ to $\bar{B}$ in $\text{PG}(n-1,q^t)$.

So from now on, we suppose that $\mathcal{B}(P)$ meets $\Pi$ only in the point $P$. By Lemma 4.2, we have that there exists a $\mathcal{D}_{k-2}$-space $T$ meeting $\Pi$ only in the point $P$. By Lemma 4.3, there are (at least) two hyperplanes, say $H_1$ and $H_2$, of $\Pi$ through $P$ which meet $K$ only in a subspace $\mu$ of $\Pi$ of codimension 2.

Consider the quotient $\Pi/T$ in the quotient space $\text{PG}(nt-1,q)/T \cong \text{PG}(nt-kt+t-1,q)$. Note that, as $T$ is spanned by spread elements of the Desarguesian spread $\mathcal{D}$, $\mathcal{D}$ induces a Desarguesian $(t-1)$-spread $\mathcal{D}'$ in $\text{PG}(nt-1,q)/T$. The $\mathcal{D}_{k-1}$-spaces through
$T$ are in one-to-one correspondence with the elements of $\mathcal{D}'$. Since a tangent $\mathcal{D}_{k-1}$-space through $P$ to $K$ corresponds to an element of $\mathcal{D}'$ which meets $\Pi/T$ in a subspace skew to $K/T$, we need to show that there exists an element of $\mathcal{D}'$ meeting $\Pi/T$ in a subspace skew to $K/T$. Note that $\Pi/T$ is $(nt-kt)$-dimensional and that $H_1/T$ and $H_2/T$ are hyperplanes of $\Pi/T$.

Let $A$ be the number of elements of $\mathcal{D}'$ meeting $\Pi/T$ in a point, then expressing that $A + (\frac{q^{nt-kt+1}}{q-1} - A)(q + 1)$ is at most $\frac{q^{nt-kt+1} - 1}{q-1}$, the number of points in $\Pi/T$, yields that $A$ is at least $\frac{q+1}{q} \left( \frac{q^{nt-kt+1}}{q-1} - 1 \right)$. This implies that there are at most $\frac{q^{nt-kt+1}}{q-1} - A < 2q^{nt-kt-1}$ points of $\Pi/T$ that are not the exact intersection of an element of $\mathcal{D}'$ with $\Pi/T$. This implies that there is at least one point of $H_1/T$ or $H_2/T$ that induces a tangent $\mathcal{D}_{k-1}$-space through $T$ which in turn implies that we have found a tangent $(k-1)$-space to $B$ in the point $B(P)$ of $\text{PG}(n-1, q')$ and that $P$ is essential. \hfill \Box

**Remark 4.** By [2], an affine blocking set contains at least $2q-1$ points. This implies that every point of a minimal blocking set $B$ in $\text{PG}(2, q)$ with $|B| = q+k$, $k \leq q$, lies on at least $q-k+1$ tangent lines to $B$. So every minimal blocking set of size at most $2q-1$ satisfies the condition of Construction [2]. In particular, if $B$ is a line, we find that, as announced before, a linear blocking set is minimal.

Using that the blocking set constructed by Construction [2] is contained in $\mathcal{B}(\Pi)$, where $\Pi$ is an $(nt-kt+1)$-dimensional subspace of $\text{PG}(nt-1, q)$, the following corollary easily follows.

**Corollary 4.5.** There exists a minimal $(k-1)$-blocking set in $\text{PG}(n-1, q')$ constructed by Construction [2] which spans an $s$-dimensional space for all $n-k \leq s \leq \min\{n-1, nt-kt+1\}$.

**Remark 5.** If we compare Theorem 4.4 to Theorem 3 of Mazzocca, Polverino and Storme, we see that in the latter theorem the vice versa part also holds: if $B$ is minimal, than $B$ is necessarily minimal. In general this does not hold for our construction.

### 4.2 Construction [1] in a scattered subspace

In general, since the position of the space $\Pi = \langle \Omega, \Gamma \rangle$ with respect to the Desarguesian spread $\mathcal{D}$ is arbitrary, we cannot derive the size of the blocking set $B$ from the size of the blocking set $\tilde{B}$. But if we take the space $\Pi$ to be e.g. a scattered subspace with respect to $\mathcal{D}$ (i.e. every element of $\mathcal{D}$ that meets this subspace, meets it in exactly 1 point), we are able to derive the size of $B$ in terms of $|\tilde{B}|$. Note that this is a restriction: it is clear that not all examples arising from Construction [1] can be obtained from a scattered subspace $\Pi$. In some cases, it is even impossible to find a scattered subspace of the right dimension, in view of the following theorem:

**Theorem 4.6.** [3] A scattered $\mathbb{F}_q$-linear set in $\text{PG}(n-1, q')$ has rank $\leq nt/2$.

In [10] Theorem 2.5.5], Lavrauw shows the following:
**Theorem 4.7.** If \( r \) is even, then there exists a scattered subspace (with respect to a Desarguesian \((t - 1)\)-spread) of dimension \( rt/2 - 1 \) in \( \text{PG}(rt - 1, q) \). If \( r \) is odd, there exists a scattered subspace (with respect to a Desarguesian \((t - 1)\)-spread) of dimension \((rt - t)/2 - 1 \).

**Theorem 4.8.** Let \( k \geq (n + 3)/2 \). If \( B \) is a minimal blocking set in \( \text{PG}(2, q) \) such that every point of \( B \) lies on at least 2 tangent lines to \( B \), then there exists a minimal blocking set with respect to \((k - 1)\)-spaces in \( \text{PG}(n - 1, q') \) with size \(|B|q^{nt-kt-1} + \frac{q^{nt-kt-1-1}}{q-1}\).

**Proof.** Consider an \((nt - kt + 1)\)-dimensional space \( \Pi = \langle \Omega, \Gamma \rangle \), where \( \Omega \) is \((nt - kt - 1)\)-dimensional and \( \Gamma \) a plane skew from \( \Omega \) and such that \( \Pi \) is scattered with respect to \( \mathcal{D} \), which is possible in view of Theorem 4.7 since \( k \geq (n + 3)/2 \). Apply Construction \( \Pi \) with the spaces \( \Omega, \Gamma \) and the minimal blocking set \( B \) in \( \Gamma \). Then, by Theorem 4.4, \( B(K) \) is a minimal blocking set with respect to \((k - 1)\)-spaces. Moreover, since every element of \( \mathcal{D} \) that meets \( \langle \Omega, \Gamma \rangle \), meets it in exactly one point, the number of points in \( B \) is equal to the number of points in \( K \), which is equal to \(|B|q^{nt-kt-1} + \frac{q^{nt-kt-1-1}}{q-1}\). □

The following corollary shows which are the possible dimensions spanned by a blocking set obtained by Construction \( \Pi \) in a scattered subspace, again using Theorem 4.7.

**Corollary 4.9.** If \( B \) is a minimal blocking set in \( \text{PG}(2, q) \) such that every point of \( B \) lies on at least 2 tangent lines to \( B \) and if \( nt - kt + 1 \leq \frac{n't}{q'} - 1 \) and \( n' - 1 \leq nt - kt + 1 \), then there is a \((k - 1)\)-blocking set \( B \) in \( \text{PG}(n - 1, q') \) spanning an \((n' - 1)\)-space, where \( B \) has size \(|B|q^{nt-kt-1} + \frac{q^{nt-kt-1-1}}{q-1}\).

### 4.3 A modification of Construction 1

A disadvantage of Construction 1 is clearly the requirement that every point lies on at least two tangent lines to \( B \). Using the computer package ‘FinInG’ for GAP [7], we found that we cannot remove this condition in general: there are \((t + 1)\)-subspaces in \( \text{PG}(3t-1, q) \) such that a cone over a Hermitian curve does not define a minimal blocking set in \( \text{PG}(2, q') \). In this subsection, we choose a particular subspace of \( \text{PG}(nt - 1, q) \) so that we do not need at least two tangent lines to \( B \) in order to have minimality.

**Construction 2.** Let \( t \geq 4 \). Let \( \nu \) be a \( \mathcal{D}_{n-k-1} \)-space in \( \text{PG}(nt - 1, q) \) and let \( \Pi \) be an \((nt - kt + 1)\)-space through \( \nu \) such that \( \dim(\langle B(\Pi) \rangle) = n - k + 1 \). Let \( \Omega \) be an \((nt - k t - 2)\)-dimensional subspace of \( \Pi \), meeting \( \nu \) in an \((nt - kt - 4)\)-dimensional space, let \( B \) be a minimal blocking set, contained in the plane \( \Gamma \) of \( \Pi \), which is skew from \( \Omega \) and such that \( B \) is skew from \( \nu \). Let \( K \) be the cone with vertex \( \Omega \) and base \( B \) and let \( B = B(K) \).

**Theorem 4.10.** The set \( B \) from Construction 2 is a minimal \((k - 1)\)-blocking set in \( \text{PG}(n - 1, q') \).

**Proof.** The fact that \( B \) is a \((k - 1)\)-blocking set follows from Lemma 3.2.

The set \( B(\Pi) \) in \( \text{PG}(n - 1, q') \) is a cone \( C \) with vertex an \((n - k - 1)\)-space \( B(\nu) \) and base an \( \mathbb{F}_q \)-subline \( \ell \). In the quotient space \( \text{PG}(n - 1, q')/B(\nu) \) we easily find a
(k − 2)-space skew from ℓ/B(ν), which forces the existence of a (k − 2)-space S skew from C. Every point P in K ∩ ν corresponds to a point B(P) of B lying on the tangent (k − 1)-space ⟨B(µ), S⟩, which means that the point B(P) is essential if P ∈ ν ∩ K.

Now let P be a point of K, not contained in ν. As in the proof of Theorem 4.4, we may restrict ourselves to the case where B(P) ∩ Π = {P} and we know that there exists a D_{k−2}-space T through B(P) such that T meets Π only in the point P. We have that the quotient space Π/T in PG(nt − 1, qt) ∼= PG(nt − kt + t − 1, q) is (nt − kt)-dimensional and that D induces a Desarguesian spread D’ in PG(nt − kt + t − 1, q). Since Π/T contains a D_{n−k−1}-space ν/T, every element of D’ not meeting ν/T meets Π/T in a single point. By Lemma 4.3, since B is minimal, P lies on at least one hyperplane H of Π meeting K only in a space of codimension 2.

Since Ω meets ν in an (nt − kt − 4)-space, the (nt − kt − 1)-space H/T, is different from the (nt − kt − 1)-space ν/T. So we can consider a point Q of H/T in Π/T which is not contained in K nor ν/T. This point Q corresponds to a spread element D’ which together with T induces a tangent D_{k−1}-space to B(P) in PG(n − 1, q), which shows that P is essential.

Note that by construction, B(K) contains an (n − k)-space.

**Proposition 1.** The (k − 1)-blocking set B in PG(n − 1, q) obtained from Construction 2 spans a space of dimension (n − k + 1).

**Proof.** The set B is contained in the subspace ⟨B(Π)⟩, which spans an (n − k + 1)-dimensional space. Note that it is by construction not possible that ⟨B(Π)⟩ = n − k + 1 and ⟨K⟩ = n − k. □

**Proposition 2.** Let Π be so that ⟨B(Π)⟩ is (n − k + 1)-dimensional. The (k − 1)-blocking set B of PG(n − 1, q) has size

\[ |\tilde{B}|(q^{nt−kt−1} − q^{nt−kt−3}) + q^{nt−kt−2} + q^{nt−kt−3} + \epsilon, \]

where ε is at least 1.
Proof. Every spread element of \( D \) meeting \( \Pi \) is either entirely contained in \( \Pi \) or meets \( \Pi \) in a point. The cone \( K \) consists of a union of \((nt - kt - 1)\)-spaces through \( \Omega \). Each of these \((nt - kt - 1)\)-spaces contains \( q^{nt - kt - 1} \) points, not in \( \Omega \) of which, since \( \bar{B} \) does not meet \( \nu \), \( q^{nt - kt - 1} - q^{nt - kt - 3} \) are not contained in \( \nu \). There are \( q^{nt - kt - 2} + q^{nt - kt - 3} \) points in \( \Omega \setminus \nu \). The only points of \( B \) that we have not counted yet are the points of \( K \) in \( \nu \). It is clear that, since \( \Omega \) meets \( \nu \) non-trivially, this number is at least 1.

Applying the previous proposition for \( k = n - 1 \), and using that a \( D_{n-k-1} \)-space is a spread element, we find the following corollary. The second part follows from the fact that a Hermitian curve in \( PG(2, q) \), \( q \) square, is a minimal blocking set of size \( q\sqrt{q} + 1 \).

**Corollary 4.11.** The blocking set \( B \) in \( PG(n - 1, q') \) obtained from a planar blocking set \( \bar{B} \) by Construction 2 has size

\[
|\bar{B}|(q^{t-1} - q^{t-3}) + q^{t-2} + q^{t-3} + 1.
\]

In particular, if \( q \) is a square, we find minimal blocking sets in \( PG(2, q') \) of size \( q^t\sqrt{q} + q^{t-1} - q^{t-2}\sqrt{q} + q^{t-2} + 1 \).

In [15], Mazzocca and Polverino construct blocking sets in \( PG(2, q') \) arising from cones in \( PG(2t, q) \). We want to point out that, starting from a Hermitian curve \( \bar{B} \) in \( PG(2, q) \), \( q \) square, they construct minimal blocking sets in \( PG(2, q') \) of size \( q^{t-1}(|\bar{B}|-1)+1=q^t\sqrt{q}+1 \), which is smaller than the size of the minimal blocking set obtained from a Hermitian curve in the previous corollary.

## 5 Minimal blocking sets with respect to the hyperplanes

### 5.1 The main observation

We have seen in Lemma 3.2 that, if \( B' \) is a blocking set with respect to \((nt - t - 1)\)-spaces in \( PG(nt - 1, q) \), then \( B = B(B') \) is a blocking set (w.r.t. hyperplanes) in \( PG(n - 1, q') \). In the following theorem, we show that a kind of reverse statement also holds.

**Theorem 5.1.** Let \( B \) be a minimal blocking set with respect to the hyperplanes of \( PG(n - 1, q') \), then \( B \) can be written as \( B(B') \), where \( B' \) is a minimal blocking set with respect to \((nt - t - 1)\)-spaces of \( PG(nt - 1, q) \).

**Proof.** Let \( S(B) \) denote the set of spread elements corresponding to the points of \( B \) and let \( B \) be the point set of the elements of \( S(B) \). Since \( B \) is a blocking set with respect to the hyperplanes of \( PG(n - 1, q') \), every \( D_n \)-space contains an element of \( S(B) \), and hence, certainly a point of \( \bar{B} \) (in fact, at least \( q^{t-1}/q^{t-1} \) of them).

Now consider an \((nt - t - 1)\)-space \( \pi \) of \( PG(nt - 1, q) \) which is not a \( D_n \)-space. Let \( H \) be a hyperplane of \( PG(nt - 1, q) \) through \( \pi \). We know from Lemma 2.1 that \( H \) contains a \( D_n \)-space \( \pi' \). Since \( B \) is a blocking set with respect to the hyperplanes, the space \( \pi' \) contains at least one element of \( S(B) \), say \( S \). Now \( \pi \cap \pi' \) is at least \((nt - 2t)\)-dimensional, hence, since \( S \) is \((t - 1)\)-dimensional and contained in \( \pi' \), \( S \) meets \( \pi \) non-trivially, and
hence, \( \pi \) contains at least one point of the set \( \tilde{B} \). This implies that \( \tilde{B} \) is a blocking set with respect to \((nt - t - 1)\)-spaces.

Now let \( B' \) be a minimal blocking set with respect to \((nt - t - 1)\)-spaces contained in \( \tilde{B} \) (\( B' \) is not necessarily unique). To show that \( \mathcal{B}(B') = B \), we show that in every element of \( \mathcal{S}(B) \), there lies at least one point of \( B' \). Suppose that there is an element of \( \mathcal{S}(B) \), say \( T \), that does not contain a point of \( B' \). Since \( B \) is a minimal blocking set, there is a tangent hyperplane in PG\((n - 1, q^t)\) to \( B \) in the point corresponding to \( T \). This tangent hyperplane corresponds to a \( \mathcal{D}_{n-2} \)-space (which is an \((nt - t - 1)\)-space) meeting \( \mathcal{S}(B) \) in exactly the element \( T \), hence, does not contain a point of the blocking set with respect to \((nt - t - 1)\)-spaces \( B' \), a contradiction. \( \Box \)

**Remark 6.** The blocking set \( B' \) found in the previous theorem is not necessarily unique. Consider for example the blocking set \( B \) in PG\((2, q^3)\) consisting of all points of a line. Then \( B \) corresponds to the set of spread elements in a \( \mathcal{D}_1 \)-space \( \pi \) of PG\((3t - 1, q)\). It is clear that \( B = \mathcal{B}(\mu) \) for any subspace \( \mu \) of dimension \( t \) in \( \pi \). Take a plane \( \nu \) in \( \pi \) and a \((2t - 4)\)-dimensional subspace \( \nu' \) in \( \pi \), skew from \( \nu \), then the cone with vertex \( \nu' \) and base a large minimal blocking set in \( \nu \) (e.g. a Hermitian curve if \( q \) is a square) is a minimal blocking set \( B'' \) with respect to \((t - 1)\)-spaces in \( \pi \), hence, \( B = \mathcal{B}(B'') \). Note that \( \mu \) is a small minimal blocking set with respect to \((2t - 1)\)-spaces in PG\((3t - 1, q)\), whereas \( B'' \) is a large minimal blocking set with respect to \((2t - 1)\)-spaces in PG\((3t - 1, q)\).

**Remark 7.** Let \( B \) be a Hermitian curve in PG\((2, q^2)\). Then \( B \) is a minimal blocking set of size \( q^3 + 1 \). If we apply field reduction to \( B \), then we obtain a set \( \mathcal{S}(B) \) of \( q^3 + 1 \) lines in PG\((5, q)\) of which the point set \( \tilde{B} \) blocks all \( 3 \)-spaces by Theorem 5.1. The \( (q^3 + 1)(q + 1) \) points do not form a minimal blocking set with respect to \( 3 \)-spaces. It is not too hard to check that \( \tilde{B} \) is the point set of an elliptic quadric in PG\((5, q)\) (see e.g. [12]) and that the \( q^3 + q^2 + q + 1 \) points of a parabolic quadric \( \mathcal{Q}(4, q) \) in \( \tilde{B} \) form a minimal blocking set \( B' \) with respect to \( 3 \)-spaces in PG\((5, q)\), hence, \( B = \mathcal{B}(B') \). But in general, it is not easy to construct a minimal blocking set \( B' \) with respect to \((nt - t - 1)\)-spaces contained in the point set of the spread elements corresponding to \( B \).

### 5.2 Small minimal blocking sets and the linearity conjecture

The linearity conjecture for small blocking sets states that all small minimal blocking sets in PG\((n - 1, q^t)\) are linear sets over \( \mathbb{F}_p \), where \( q^t = p^h \), \( p \) prime (see [22]).

We have the following result of T. Szőnyi and Zs. Weiner [21], showing that the linearity conjecture holds for projective spaces over fields of prime order.

**Theorem 5.2.** [21] Corollary 3.15 \( A \) small minimal blocking set with respect to \( k \)-spaces in PG\((n, p)\), \( p \) prime, is an \((n - k)\)-space.

In general, the linearity conjecture for small minimal blocking sets remains open. For an overview of the cases in which the linearity conjecture has proven to be true, we refer to [12].

If \( B \) is a small minimal blocking set with respect to the hyperplanes of PG\((n - 1, p^h)\), \( p \) prime, by Theorem 5.1, \( B \) can be written as \( \mathcal{B}(B') \) where \( B' \) is a minimal blocking set with respect to \((hn - h - 1)\)-spaces of PG\((nh - 1, p)\). If this latter blocking set is small then by Theorem 5.2 it is an \( h \)-space. Hence, the linearity conjecture can be restated as
follows (compare to the statement of Theorem 5.1):

**Linearity conjecture:** Let $B$ be a small minimal blocking set with respect to the hyperplanes of $\text{PG}(n-1, p^h)$, $p$ prime, then $B$ can be written as $\mathcal{B}(B')$, where $B'$ is a small minimal blocking set with respect to $(nh-h-1)$-spaces of $\text{PG}(nh-1, p)$.

We will now investigate the properties of small minimal blocking sets that arise from the MPS-construction and from Construction 1; we will show that the obtained blocking sets are linear blocking sets.

Define the exponent $e$ of a small minimal blocking set $B$ with respect to $k$-spaces as the largest integer such that every $k$-space meets $B$ in $1$ mod $p^e$ points. Lemma 5.3(1) will show that $e$ is well-defined.

We will need the following properties of small minimal blocking sets in $\text{PG}(n, p^t)$, $p$ prime. For item (2), for simplicity, we state a slightly weaker bound than the one proven in [21, Theorem 3.9].

**Lemma 5.3.**
1. [21, Proposition 2.7] Every $k$-space meets a small minimal blocking set with respect to $k$-spaces in $\text{PG}(n, p^t)$, $p$ prime, in $1$ mod $p^e$ points, hence, $e \geq 1$.
2. [21, Theorem 3.9] A small minimal blocking set in $\text{PG}(n, p^t)$, $p \geq 7$, with exponent $e$ has at most $p^t + 2p^{t-e} + 1$ points.
3. [21, Corollary 3.11] Every subspace meets a small minimal blocking set with respect to $k$-spaces with exponent $e$ in $1$ mod $p^e$ or zero points.
4. [22, Lemma 6] Let $B$ be a small minimal blocking set with exponent $e$ in $\text{PG}(n, p^t)$, $p$ prime. If for a certain line $L$, $|L \cap B| = p^e + 1$, then $\mathbb{F}_{p^e}$ is a subfield of $\mathbb{F}_{p^t}$ and $L \cap B$ is $\mathbb{F}_{p^e}$-linear.
5. [22, Lemma 4] A point of a small minimal blocking set $B$ with exponent $e$ in $\text{PG}(n, p^t)$, $p \geq 7$, $p$ prime, lying on a $(p^e+1)$-secant, lies on at least $p^{t-e} - 4p^{t-e-1} + 1$ $(p^e+1)$-secants.

From Lemma 5.3(3) we easily deduce the following corollary which also reproofs Theorem 5.2.

**Corollary 5.4.** A small minimal blocking set with respect to $k$-spaces with exponent $t$ in $\text{PG}(n, p^t)$, $p$ prime, is an $(n-k)$-space.

**Theorem 5.5.** If a small minimal blocking set $B$ in $\text{PG}(n - 1, p^t)$, $p$ prime, arises from the MPS construction in $\text{PG}(nt - 1, p)$ then it is linear and of Rédei-type.

**Proof.** We use the notations of Section 3.2.2. Let $B$ be a small minimal blocking set with respect to the hyperplanes in $\text{PG}(n-1, p^t)$ arising from the MPS-construction, then $B = \mathcal{B}(K)$, where $K$ is a blocking set with respect to the hyperplanes of $\Sigma' = \text{PG}(nt-t, p)$ and $K$ meets $\Sigma$ in a $(t-1)$-space. Since all points of $B$, not in the hyperplane corresponding to the $\mathcal{D}_{n-2}$-space $\Sigma$ correspond to a unique point of $K$, and $|B| \leq p^t + p^{t-1} + 1$ we have that $K$ has at most $p^t + 2p^{t-1} + \frac{p^{t-1}}{p-1}$ points, which implies that $K$ is small. Since $K$ is a minimal blocking set by Theorem 3.6 it is a $t$-space by Theorem 5.2.
Lemma 5.6. If the \((t-1)\)-space \(K \cap \Sigma\) is not contained in the spread element \(Y\), then \(\Sigma\) corresponds to a hyperplane containing \(p^{t-1} + 1 = |B| - p^t\) points of \(B\), hence, \(B\) is of Rédei-type. If \(K \cap \Sigma\) equals \(Y\), then let \(Q\) be a point of \(K\), not in \(\Sigma\). It is clear that, since \(K\) is a \(t\)-space through the spread element \(Y\), \(K\) is contained in the \(D_t\)-space \(\langle Y, B(Y) \rangle\), hence \(B\) is a line.

Remark 8. The vice versa part of the previous theorem does not hold: let \(\pi\) be a \(t\)-space in \(\text{PG}(nt-1, p)\) such that there is an element \(Y\) of \(D\) meeting \(\pi\) in a \((t-3)\)-space and an element \(Z\) meeting \(\pi\) in a line and such that \(\pi\) is not entirely contained in \(\langle Y, Z \rangle\). Then the line corresponding to the \(D_t\)-space \(\langle Y, Z \rangle\) clearly contains \(|B| - p^t\) points of \(B(\pi)\), hence, the small linear blocking set \(B(\pi)\) is of Rédei-type, but there does not exist an element of \(D\) meeting \(\pi\) in a \((t-2)\)-space, so \(B(\pi)\) cannot be constructed by the MPS-construction.

By Lemma \[\text{Lemma 5.6}(4)\], the points of \(B\) on a \((p^t + 1)\)-secant form an \(F_{p^t}\)-linear set of rank \(2\), i.e., a subline, which by field reduction corresponds to a regulus. We need the following information on the intersection of a regulus with a plane.

Lemma 5.6. [1] Lemma 6, Corollary 13] A plane \(\pi\) meeting all elements of a regulus \(B(\ell)\), meets the point set of \(B(\ell)\) either in a line, or in two lines, or in a conic.

Corollary 5.7. Suppose that \(B(\ell)\) where \(\ell\) is a line, is contained in \(B(K)\), where \(K\) is a blocking set with respect to lines in \(\Pi\). Then the intersection of the point set of \(B(\ell)\) with a plane of \(\Pi\) contains a line.

Proof. From Lemma \[\text{Lemma 5.6}\] we get that the intersection of the point set of \(B(\ell)\) with a plane either contains a line or is a conic. But a conic does not block all lines in the plane \(\pi\) and \(K \cap \pi\) is a blocking set with respect to lines in \(\pi\), hence, this possibility does not occur.

Lemma 5.8. A plane \(\pi\) meets the cone \(K\), with vertex a \((t-2)\)-dimensional space \(\Omega\) and base a small minimal blocking set \(\tilde{B}\) in a plane \(\Gamma\), skew from \(\Omega\), either in the plane \(\pi\) itself, in a number of lines through a fixed point or in a minimal blocking set equivalent to \(\tilde{B}\).

Proof. Let \(\pi\) be a plane in \(\Pi = \langle \Omega, \Gamma \rangle\). We have the following possibilities:

- \(\pi\) is contained in \(\Omega\). In this case, \(\pi\) is contained in \(K\).

- \(\pi\) meets \(\Omega\) in a line \(L\). If \(\pi\) contains a point \(P\) of \(K\), not on \(L\), then \(\langle \Omega, P \rangle\) is contained in \(K\), hence, \(\pi\) is contained in \(K\).

- \(\pi\) meets \(\Omega\) in a point \(P\). Since \(K\) is a blocking set with respect to lines in \(\Pi\), \(K\) forms a blocking set with respect to lines in the plane \(\pi\), so there exists a point of \(K\) in \(\pi\), different from \(P\). It is clear that every point of \(K\) in \(\pi\), different from \(P\), gives rise to a line contained in \(K\) through \(P\).

- \(\pi\) is skew from \(\Omega\). Consider the mapping \(\phi\) from \(\Gamma\) to \(\pi\) defined by mapping the point \(P\) of \(\Gamma\) to the intersection of the cone \(\langle P, \Omega \rangle\) with \(\pi\). It is clear that \(\phi\) defines a collineation mapping \(\tilde{B}\) onto the intersection of \(K\) with \(\pi\), hence, \(K\) and \(\pi\) intersect in a minimal blocking set, equivalent to \(\tilde{B}\).
Lemma 5.9. Let $P$ be a point of a cone $K$ with vertex a $(t-2)$-dimensional space $\Omega$ and base a non-trivial minimal blocking set in a plane $\Gamma$ skew from $\Omega$ and suppose that $P$ is not contained in the vertex of $K$, then $P$ lies on $(q^{t-1} - 1)/(q-1)$ lines contained in $K$.

Proof. Since the vertex $\Omega$ of the cone $K$ is $(t-2)$-dimensional, it is clear that a point $P$, not in $\Omega$, lies on $(q^{t-1} - 1)/(q-1)$ lines of $\langle P, \Omega \rangle$, which are contained in $K$. If $P$ lies on another line $L$ that is contained in $K$, then this line is skew from $\Omega$ and $\langle \Omega, L \rangle$ meets $\Gamma$ in a line $L'$, contained in $K$, a contradiction since $K \cap \Gamma$ is a non-trivial minimal blocking set.

We now extend a property of small minimal planar blocking sets ([22, Proposition 4.17]) to blocking sets in $\text{PG}(n, q)$, $n \geq 2$.

Lemma 5.10. If a small minimal blocking set $B$ in $\text{PG}(n, p^0)$ has exponent $e$, $p_0 := p^e \geq 7$, $p$ prime, then there exists a $(p_0 + 1)$-secant to $B$.

Proof. We proceed by induction, where the case $n = 2$ is Proposition 4.17 of [22]. Since $B$ has exponent $e$, there is a hyperplane $H$ with $|H \cap B| = 1$ mod $p^e$ and $|H \cap B| \neq 1$ mod $p^{e+1}$. It is clear that, since every subspace meets $B$ in $1$ mod $p^e$ points and the number of points in $B$ is equal to $1$ mod $p^e$, that we can find a line $L$ in $H$ with $\lambda = 1$ mod $p^e$ points and $\lambda \neq 1$ mod $p^{e+1}$. If a plane contains a point of $B$ outside of $L$, then this plane contains at least $p^{2e}$ points of $B$ outside of $L$ since the line through 2 points of $B$ contains at least $p^e + 1$ points of $B$ by Lemma 5.3(1). By Lemma 5.3(1), this implies that there exists a plane $\pi$ through $L$ without extra points of $B$. Let $P$ be a point of $\pi$, not on $L$, then the projection $B'$ of $B$ from $P$ onto a hyperplane through $L$ and not through $P$ is a small minimal blocking set in $\text{PG}(n-1, p^0)$ (see e.g. [23, Corollary 3.2]), which has, by our claim, exponent $e$. So by induction, we find a $(p_0 + 1)$-secant to $B'$ in $H'$. Since all points of $B$ in $\pi$ are on the line $L$, we have found a $(p_0 + 1)$-secant to $B$.

Corollary 5.11. Let $B$ be a small minimal blocking set with exponent $e$ in $\text{PG}(n, p^0)$, $p_0 := p^e \geq 7$, $p$ prime, then there are at least $(p_0^{h-1} - 4p_0^{h-2} + 1)p_0 + 1$ points of $B$ that each lie on at least $p_0^{h-1} - 4p_0^{h-2} + 1$ $(p_0 + 1)$-secants to $B$.

Proof. From Lemma 5.10 we get that there exists one $(p_0 + 1)$-secant to $B$. Lemma 5.3 shows that through a point of this $(p_0 + 1)$-secant there are at least $p_0^{h-1} - 4p_0^{h-2} + 1$ $(p_0 + 1)$-secants.

As mentioned before, not all linear blocking sets are of Rédei-type, and we will now show that a small minimal blocking set arises from Construction [1] if and only if it is linear.

Theorem 5.12. If $p \geq 7$, a small minimal blocking set with respect to the hyperplanes in $\text{PG}(n-1, q^t)$, $q = p^h$, $p$ prime, with exponent $e$ arises from Construction [1] in $\text{PG}(nht/e - 1, p^e)$ if and only if it is an $\mathbb{F}_{p^e}$-linear blocking set.

Proof. Put $q_0 = p^e$, $q^{t} = q_0^0$ (hence $t_0 = ht/e$). Let $B$ be an $\mathbb{F}_{p^e}$-linear blocking set, then $B = B_{D'}(\pi)$, where $\pi$ is a $t_0$-space in $\text{PG}(nt_0 - 1, q_0)$ and $D'$ is a Desarguesian $(t_0-1)$-spread in $\text{PG}(nt_0 - 1, q_0)$. Let $\Omega$ be a $(t_0 - 2)$-dimensional subspace of $\pi$ and let $\Gamma$ be a plane meeting $\pi$ in a line $B$, disjoint from $\Omega$. It is clear that $B = B_{D'}(K)$, with $K$ the
cone with vertex $\Omega$ and base $\bar{B}$; note that $\bar{B}$ is a minimal blocking set such that every point of $\bar{B}$ lies on at least 2 tangent lines to $\bar{B}$. So this implies that $B$ is obtainable from Construction 1.

So now assume that $B$ is a small minimal blocking set obtained from Construction 1 for some $\Pi = \langle \Omega, \Gamma \rangle$ and $\bar{B}$ where $\bar{B}$ is a minimal blocking set in the plane $\Gamma$ in $\text{PG}(nt_0 - 1, q_0)$, where $\Gamma$ is skew from the $(t_0 - 2)$-space $\Omega$. We will show that $B = \mathcal{B}_D'(\pi)$, where $\pi$ is a $t_0$-space in $\text{PG}(nt - 1, q_0)$, where $\mathcal{D}'$ is a Desarguesian $(t_0 - 1)$-spread in $\text{PG}(nt_0 - 1, q_0)$. Corollary 5.11 states that there exists a point $P$ of $B$ on at least $q_0^{t_0 - s} - 4q_0^{t_0 - 2} + 1 (q_0 + 1)$-secants, and such that $P$ is not an element of $\mathcal{B}(\Omega)$ since $|\mathcal{B}(\Omega)| \leq q^{t_0 - 1}$. Consider the spread element $\mathcal{B}_D'(P)$ and its intersection $S$ with the space $\Pi = \langle \Omega, \Gamma \rangle$. Let $L_1, \ldots, L_r$ be the $(q_0 + 1)$-secants through $P$ to $B$. Every $L_i$ corresponds to a $(2t_0 - 1)$-dimensional space in $\text{PG}(nt_0 - 1, q_0)$, denote by $\pi_1, \ldots, \pi_r$ the subspaces of $\Pi$ that occur as the intersection of $\Pi$ with the $(2t_0 - 1)$-dimensional space corresponding to $L_i$. Note that two different spaces $\pi_i$ and $\pi_j$ meet exactly in the subspace $S$. From this, it follows that if $S$ has dimension $s$, then the number $r$ can be at most the number of spaces of dimension $s + 1$ through the $s$-dimensional subspace $S$ in $\Pi$. This number equals $(q_0^{t_0 - s} - 1)/(q_0 - 1)$, which is smaller than $q_0^{t_0 - 1} - 4q_0^{t_0 - 2} + 1$ if $s$ is at least 1. So this implies that $s = 0$.

So $\mathcal{B}_D'(P) \cap \langle \Omega, \Gamma \rangle$ is a point. By an easy counting, we find that there are more than $(q_0^{t_0 - 1} - 1)/(q_0 - 1)$ of the spaces $\pi_i$ that are 1-or 2-dimensional. By Corollary 5.7 these all give rise to a line through $P$, contained in $K$. So by Lemma 5.9 we find that $\bar{B}$ is a line which proves the statement.

References


