On the complex symmetry of the Dirichlet-to-Neumann operator

L. Knockaert* and D. De Zutter

EM Group, Dept. Information Technology, Ghent University

Introduction

It is well-known that the Stratton-Chu formalism [1,2] allows a complete reconstruction of the interior and exterior electromagnetic fields $(\mathbf{E}, \mathbf{H})$ inside and outside a simply connected isotropic domain Ω with smooth boundary $\partial\Omega$ merely by knowledge of the tangential field components $(\mathbf{E}^t, \mathbf{H}^t)$ [3] or the equivalent magnetic and electric currents [4] on the boundary $\partial\Omega$ in the absence of sources. A still stronger statement, resulting from the inherent duality of both Maxwell's equations and the Stratton-Chu formalism, is that only one tangential field component $\mathbf{E}^t$ or $\mathbf{H}^t$ (in other words only one equivalent magnetic or electric current), is needed in order to describe the complete field configuration in the absence of source terms, except notably under resonance conditions. It follows that there exists a linear operator relationship between the equivalent magnetic and electric currents, called the Dirichlet-to-Neumann or Poincaré-Steklov operator [3]. In [5] it is indicated how the Poincaré-Steklov operator can be extracted from the Calderón projectors [2] by means of a Schur complement method (see also [6] for the 2D case).

In this note we show, by invoking Lorentz reciprocity [7], that the Dirichlet-to-Neumann operator can be interpreted as a complex symmetric (C-symmetric) [8] operator mapping the tangential electric field (instead of the equivalent magnetic current) onto the equivalent electric current. The Dirichlet-to-Neumann operator in the Lorentz reciprocity context is therefore a genuine admittance operator. It is shown that the Calderón projectors can be reformulated as operators with a block Hamiltonian structure, well-known from Hamiltonian dynamical systems [9] and Lagrangian subspace techniques [10]. This results in a novel explicitly C-symmetric Schur complement expression for both the interior and exterior admittance operators.

Lorentz reciprocity and Hamiltonian Schur symmetry

For fields $(\mathbf{E}_1, \mathbf{H}_1)$ and $(\mathbf{E}_2, \mathbf{H}_2)$ both satisfying the source-free time-harmonic ($e^{i\omega t}$ time dependence) Maxwell equations $\nabla \times \mathbf{E} = -i\omega\mu\mathbf{H}$, $\nabla \times \mathbf{H} = i\omega\epsilon\mathbf{E}$ in the simply connected domain $\Omega \subset R^3$ with boundary $\partial\Omega$, the Lorentz reciprocity law [7] can be stated in terms of the tangential components $(\mathbf{E}^t, \mathbf{H}^t)$ as

$$\int_{\partial\Omega} [\mathbf{E}_1^t \cdot (\mathbf{n} \times \mathbf{H}_2^t) - \mathbf{E}_2^t \cdot (\mathbf{n} \times \mathbf{H}_1^t)] dS = 0 \quad (1)$$

Since formula (1) holds for all possible tangential fields, this implies that there is a C-symmetric operator (the Dirichlet-to-Neumann or admittance operator) $\mathcal{Y} = \mathcal{Y}'$ mapping $\mathbf{E}^t = -\mathbf{n} \times (\mathbf{n} \times \mathbf{E})$ onto $\mathbf{n} \times \mathbf{H}^t = \mathbf{n} \times \mathbf{H}$. It is important to note that the transpose operator $\mathcal{Y}'$ is defined with respect to the bilinear [8] or 'pseudo' inner product [11] p. 113 on the boundary, i.e., $\langle \mathbf{X} | \mathbf{Y} \rangle = \int_{\partial\Omega} \mathbf{X} \cdot \mathbf{Y} dS$. Stated otherwise,

the Lorentz reciprocity relationship (1) implies that

$$\mathbf{n} \times \mathbf{H} = \mathcal{Y}[\mathbf{E}^t] \quad (2)$$

where $\mathcal{Y}$ is a C-symmetric operator. Letting $\mathcal{N}$ denote the local operator $\mathbf{n} \times$, the non-C-symmetric operator mapping the equivalent magnetic current $\mathbf{M} = \mathbf{n} \times \mathbf{E}$ onto the equivalent electric current $\mathbf{J} = \mathbf{n} \times \mathbf{H}$ [1,4] is $\mathcal{S} = -\mathcal{Y}\mathcal{N}$. Although it is not C-symmetric, the operator $\mathcal{S}$ has, somewhat misleadingly, been coined admittance operator in [3,5]. Also, interestingly, formula (2) can simply be written as $\mathbf{J} = \mathcal{Y}\mathbf{E}^t$, which has a distinctively nice reciprocal electrical network flavor : an infinity of ports on the boundary $\partial\Omega$ with 'voltages' $\mathbf{E}^t$ (in V/m) and currents $\mathbf{J}$ (in A/m) which are related by means of the C-symmetric admittance operator $\mathcal{Y}$.

Next, define the wavenumber, characteristic impedance and admittance as

$$k = \omega\sqrt{\epsilon\mu} \quad z_0 = \frac{\omega\mu}{k} \quad y_0 = \frac{1}{z_0} = \frac{\omega\epsilon}{k} \quad (3)$$

the scalar Green's function as $g_0(r) = e^{-ikr}/4\pi r$, the operator $\mathcal{T}$ [2] operating on a tangential field $\mathbf{X}$ as

$$\mathcal{T}\mathbf{X} = \mathbf{n} \times \left\{ -ik \int_{\partial\Omega} g_0(R) \mathbf{X}(\mathbf{y}) dS_y + \frac{i}{k} \oint_{\partial\Omega} \nabla_y g_0(R) \nabla_y^t \cdot \mathbf{X}(\mathbf{y}) dS_y \right\} \quad (4)$$

and similarly the operator $\mathcal{K}$ [2] as

$$\mathcal{K}\mathbf{X} = \mathbf{n} \times \int_{\partial\Omega} \nabla_y g_0(R) \times \mathbf{X}(\mathbf{y}) dS_y \quad (5)$$

In the defining equations (4) and (5), R is the Euclidean distance $\|\mathbf{x} - \mathbf{y}\|$ and $\oint$ stands for the Cauchy principal value integral. Note that $\mathcal{K}$ is a compact operator [12], while $\mathcal{T}$ contains a hyper-singular component necessitating the Cauchy principal value operation. Some salient features of the operator $\mathcal{T}$ are described in Lemma 2 of [5].

The Stratton-Chu equations for the equivalent currents $\mathbf{M} = \mathbf{n} \times \mathbf{E}$ and $\mathbf{J} = \mathbf{n} \times \mathbf{H}$ are [2] :

$$\mathcal{P} \begin{pmatrix} \mathbf{M} \\ z_0 \mathbf{J} \end{pmatrix} = \begin{pmatrix} \mathbf{M} \\ z_0 \mathbf{J} \end{pmatrix} \quad (6)$$

where $\mathcal{P}$ is the block Calderón operator ($\mathcal{I}$ is the identity operator) :

$$\mathcal{P} = \begin{pmatrix} \frac{1}{2} \mathcal{I} + \mathcal{K} & -\mathcal{T} \\ \mathcal{T} & \frac{1}{2} \mathcal{I} + \mathcal{K} \end{pmatrix} \quad (7)$$

The operator $\mathcal{P}$ is an oblique projector, i.e., $\mathcal{P}^2 = \mathcal{P}$. This implies the Calderón identities

$$\mathcal{T}\mathcal{K} + \mathcal{K}\mathcal{T} = 0, \quad \mathcal{K}^2 - \mathcal{T}^2 = \frac{1}{4} \mathcal{I} \quad (8)$$

The complementary oblique projector $\mathcal{I} - \mathcal{P}$, of importance for the exterior problem, is obtained by replacing $\mathcal{K}$ with $-\mathcal{K}$ and $\mathcal{T}$ with $-\mathcal{T}$ in (7). Equation (6) can be written as

$$\begin{pmatrix} \frac{1}{2} \mathcal{I} + \mathcal{K} & -\mathcal{T} \\ \mathcal{T} & \frac{1}{2} \mathcal{I} + \mathcal{K} \end{pmatrix} \begin{pmatrix} \mathcal{N}\mathbf{E}^t \\ z_0 \mathbf{J} \end{pmatrix} = \begin{pmatrix} \mathcal{N}\mathbf{E}^t \\ z_0 \mathbf{J} \end{pmatrix} \quad (9)$$

It is easily observed that $\mathcal{T} = \mathcal{N}\mathcal{U}$, where $\mathcal{U} = \mathcal{U}'$ is the C-symmetric operator defined as

$$\mathcal{U}\mathbf{X} = -ik \int_{\partial\Omega} g_0(R) \mathbf{X}(\mathbf{y}) dS_y + \frac{i}{k} \oint_{\partial\Omega} \nabla_y^t g_0(R) \nabla_y^t \cdot \mathbf{X}(\mathbf{y}) dS_y \quad (10)$$

Since $\mathcal{N}' = -\mathcal{N}$ and $\mathcal{N}^2 = -\mathcal{I}$, equation (9) can be reformulated as

$$\mathcal{H} \begin{pmatrix} \mathbf{E}^t \\ z_0 \mathbf{J} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \mathbf{E}^t \\ z_0 \mathbf{J} \end{pmatrix} \quad (11)$$

where the block operator $\mathcal{H}$ is

$$\mathcal{H} = \begin{pmatrix} -\mathcal{N} & 0 \\ 0 & \mathcal{I} \end{pmatrix} \begin{pmatrix} \mathcal{K} & -\mathcal{T} \\ \mathcal{T} & \mathcal{K} \end{pmatrix} \begin{pmatrix} \mathcal{N} & 0 \\ 0 & \mathcal{I} \end{pmatrix} = \begin{pmatrix} -\mathcal{N}\mathcal{K}\mathcal{N} & -\mathcal{U} \\ \mathcal{N}\mathcal{U}\mathcal{N} & \mathcal{K} \end{pmatrix} \quad (12)$$

Defining the dyadic (or matrix) kernel of $\mathcal{K}$ as $\mathbf{K}(\mathbf{x}, \mathbf{y})$, it is clear that the kernel of $\mathcal{K}'$ is $\mathbf{K}^T(\mathbf{y}, \mathbf{x})$, where ^{T} is the matrix transpose. Now, since

$$\mathbf{K}(\mathbf{x}, \mathbf{y}) = \mathbf{n}_x \times \nabla_y g_0(R) \times \quad (13)$$

we have

$$\mathbf{K}^T(\mathbf{y}, \mathbf{x}) = \nabla_x g_0(R) \times \mathbf{n}_y \times \quad (14)$$

$$= -\nabla_y g_0(R) \times \mathbf{n}_y \times \quad (15)$$

$$= (\mathbf{n}_x \times)^2 \nabla_y g_0(R) \times \mathbf{n}_y \times \quad (16)$$

$$= \mathbf{n}_x \times (\mathbf{n}_x \times \nabla_y g_0(R) \times) \mathbf{n}_y \times \quad (17)$$

implying that $\mathcal{N}\mathcal{K}\mathcal{N} = \mathcal{K}'$. The block operator $\mathcal{H}$ can therefore be written as

$$\mathcal{H} = \begin{pmatrix} -\mathcal{K}' & -\mathcal{U} \\ \mathcal{N}\mathcal{U}\mathcal{N} & \mathcal{K} \end{pmatrix} \quad (18)$$

A block operator of the form

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & -\mathcal{A}' \end{pmatrix} \quad (19)$$

is called Hamiltonian [9, 10] provided $\mathcal{B}$ and $\mathcal{C}$ are C-symmetric, i.e., $\mathcal{B}' = \mathcal{B}$ and $\mathcal{C}' = \mathcal{C}$. Since $\mathcal{U}' = \mathcal{U}$ and $(\mathcal{N}\mathcal{U}\mathcal{N})' = \mathcal{N}\mathcal{U}\mathcal{N}$, it is seen from formula (18) that $\mathcal{H}$ has a Hamiltonian structure. Inserting $\mathbf{J} = \mathcal{Y}\mathbf{E}^t$ in formula (11), it is clear that the admittance operator $\mathcal{Y}$ satisfies the two operator equations

$$-\mathcal{K}' - z_0 \mathcal{U}\mathcal{Y} = \frac{1}{2} \mathcal{I} \quad (20)$$

$$\mathcal{N}\mathcal{U}\mathcal{N} + z_0 \mathcal{K}\mathcal{Y} = \frac{1}{2} z_0 \mathcal{Y} \quad (21)$$

Solving (20) for $z_0 \mathcal{Y}$ yields

$$z_0 \mathcal{Y} = -\mathcal{U}^{-1} \left(\frac{1}{2} \mathcal{I} + \mathcal{K}' \right) \quad (22)$$

provided the operator $\mathcal{U}$ is invertible, which we assume (see also [5]). Rewriting the operator equation (21) as

$$z_0 \mathcal{Y} = \mathcal{N} \mathcal{U} \mathcal{N} + z_0 \left(\frac{1}{2} \mathcal{I} + \mathcal{K} \right) \mathcal{Y} \quad (23)$$

and inserting the solution (22) in the r.h.s. of (23) results in the explicitly C-symmetric Schur complement form of the admittance operator

$$\mathcal{Y} = y_0 \left[\mathcal{N} \mathcal{U} \mathcal{N} - \left(\frac{1}{2} \mathcal{I} + \mathcal{K} \right) \mathcal{U}^{-1} \left(\frac{1}{2} \mathcal{I} + \mathcal{K}' \right) \right] \quad (24)$$

Similarly, the exterior problem with Calderón projector $\mathcal{I} - \mathcal{P}$ and Hamiltonian $-\mathcal{H}$ results in the explicitly C-symmetric Schur complement form of the exterior admittance operator

$$\tilde{\mathcal{Y}} = -y_0 \left[\mathcal{N} \mathcal{U} \mathcal{N} - \left(\frac{1}{2} \mathcal{I} - \mathcal{K} \right) \mathcal{U}^{-1} \left(\frac{1}{2} \mathcal{I} - \mathcal{K}' \right) \right] \quad (25)$$

References

- [1] J. A. Stratton and L. J. Chu, "Diffraction theory of electromagnetic waves," *Phys. Rev.*, vol. 56, no. 1, pp. 99-107, July 1939.
- [2] G. C. Hsiao and R. E. Kleinman, "Mathematical foundations for error estimation in numerical solutions of integral equations in electromagnetics" *IEEE Trans. Antennas Propagat.*, vol. 45, no. 3, pp. 316-328, 1997.
- [3] S. Borel, D. P. Levadoux, F. Alouges, "A new well-conditioned integral formulation for Maxwell equations in three dimensions," *IEEE Trans. Antennas Propagat.*, vol. 53, no. 9, pp. 2995-3004, Sept. 2005.
- [4] X. Q. Sheng, J.-M. Jin, J. Song, W. C. Chew, C.-C. Lu, "Solution of combined-field integral equation using multilevel fast multipole algorithm for scattering by homogeneous bodies," *IEEE Trans. Antennas Propagat.*, vol. 46, no. 11, pp. 1718-1726, Nov. 1998.
- [5] A. de La Bourdonnaye, "Some formulations coupling finite element and integral equation methods for Helmholtz equation and electromagnetism," *Numer. Math.*, vol. 69, no. 3, pp. 257-268, 1995.
- [6] L. Knockaert, D. De Zutter, G. Lippens, and H. Rogier, "On the Schur complement form of the Dirichlet-to-Neumann operator," *Wave Motion*, vol. 45, pp. 309-324, 2008.
- [7] J. G. Van Bladel, *Electromagnetic Fields, Second Edition*. Piscataway : IEEE Press, 2007.
- [8] S. R. Garcia and M. Putinar, "Complex symmetric operators and applications," *Trans. Amer. Math. Soc.*, vol. 358, no. 3, pp. 1285-1315, 2005
- [9] K. R. Meyer and G. R. Hall, *Introduction to Hamiltonian dynamical systems and the N-body problem. Applied Mathematical Sciences, vol. 90*. New York : Springer, 1992.
- [10] W. W. Lin and C. S. Wang, "On computing stable Lagrangian subspaces of Hamiltonian matrices and symplectic pencils," *SIAM J. Matrix Anal. Appl.*, vol. 18, pp. 590-614, 1997.
- [11] G. W. Hanson and A. B. Yakovlev, *Operator Theory for Electromagnetics. An Introduction*. New York : Springer, 2002.
- [12] D. Colton and R. Kress, *Integral Equation Methods in Scattering Theory*. New York : Wiley, 1984.

2008 IEEE AP-S International Symposium and USNC/URSI National Radio Science Meeting

Copyright and Reprint Permission: Abstracting is permitted with credit to the source. Libraries are permitted to photocopy beyond the limit of U.S. copyright law for private use of patrons those articles in this volume that carry a code at the bottom of the first page, provided the per-copy fee indicated in the code is paid through the Copyright Clearance Center, 222 Rosewood Drive, Danvers, MA 01923. For other copying, reprint, or replication permission, write to IEEE Copyrights Manager, IEEE Operations Center, 445 Hoes Lane, P.O. Box 1331, Piscataway, NJ, 08855-1331. All rights reserved. Copyright © 2008 by the Institute of Electrical and Electronics Engineers, Inc. Personal use of this material is permitted. However, permission to reprint/republish this material for advertising or promotional purposes or for creating new collective works for sale or distribution to servers or lists, or to reuse any copyrighted component of this work in other works must be obtained from the IEEE.

Preliminary Version

Support: If you have problems or questions related to the installation of this disc, please contact the 3WAIsmen at FAX: (818) 952-0183 or e-mail: wais3men@yahoo.com

WAIs3men

July 5-12, 2008
San Diego, CA