Adaptive logics: a parametric approach

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Abstract

Adaptive logics (ALs) in standard format are defined in terms of a monotonic core logic **L**, a distinct set of 'abnormal' formulas Ω and a strategy, which can be either reliability or minimal abnormality. In this article we we ask under which conditions the consequence relation of two ALs that use the same strategy are identical, and when one is a proper subrelation of the other. This results in a number of sufficient (and sometimes necessary) conditions on **L** and Ω which apply to all ALs in standard format. In addition, we translate our results to the closely related family of default assumption consequence relations.

Keywords: Adaptive logics, standard format, default assumptions, metatheory, nonmonotonic logic.

1 Introduction

Getting to know one's tools: Over the past few decades, the field of non-monotonic logic has grown incessantly, resulting in a wide range of formal systems: default logic, circumscription logic, auto-epistemic logic, inheritance networks, adaptive logics, etc. The study and comparison of these systems at various levels of generality and abstraction has been an integral part of the field, at least since the publication of [19].

In [20, p. 14], Makinson makes the following remark concluding the great variety of non-monotonic systems in the literature (even within one framework):

> Leaving technical details aside, the essential message is as follows. Don't expect to find *the* nonmonotonic consequence relation that will always, in all contexts, be the right one to use. Rather, expect to find several *families* of such relations, interesting *syntactic conditions* that they sometimes satisfy but sometimes fail, and principal *ways of generating* them mathematically from underlying structures.

Still, even if we grant ourselves this multitude of non-monotonic logics and consider it as fruitful rather than problematic, this does not take away the need to bring order in the apparant chaos, and to develop theoretic means for doing so. For instance, it is crucial that one tries to find out which of these logics in the end coincide (at least with respect to the resulting consequence relation), how they translate into one another, which ones are stronger than others, etc. The ultimate goal of such research is perhaps not to end up with a single model of nonmonotonic reasoning, but to obtain better insight into the various tools we have at our disposal, when modelling such reasoning. This article contributes to this general aim.

Narrowing down the scope: In this article, we will mostly be concerned with one type of non-monotonic logics, viz. adaptive logics in *standard format* [10].[1] Such adaptive logics are defined in

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[1] We will actually consider a slightly more general format than the one from [10], allowing arbitrary sets of abnormalities. We return to this point in Section 2.

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terms of (i) a monotonic core logic **L**, (ii) a set Ω of formulas in the object language of **L**, which are taken to be characteristic of abnormality, and (iii) a *strategy*, which can be thought of as a kind of a specific policy for avoiding abnormalities in the face of the available information. The standard format covers two such strategies, viz. *reliability* and *minimal abnormality*. Exact definitions of the standard format will be given in Section 2; here we will briefly explain some of its history and underlying ideas to motivate the technical work that follows.

Adaptive logics can be traced back to [4] and some earlier papers by Batens, where a 'dynamic dialectic proof theory' is presented for reasoning about inconsistent premise sets. The main idea is that we should use a paraconsistent logic **L** to reason sensibly about inconsistent theories, but we can nevertheless assume that inconsistencies are false *unless the premises indicate otherwise*. With hindsight, one may say that in the logics from [4], the set of abnormalities consists of all inconsistencies, i.e. all formulas of the form $A \wedge \sim A$.[2]

In later work, the underlying idea behind the logic from [4] has been used to characterize other types reasoning in which certain logical principles are defeasible. For instance, in the context of a conflict-tolerant, non-aggregative deontic logic, one may assume that any two obligations A and B can be aggregated unless they are incompatible [17, 22]. In the context of first order predicate logic, one can assume that any (possibly complex) property Π holds of all objects whenever there is no counterinstance to Π [8, 11]. In the context of the doxastic logic **K**, one may assume that any proposition A is true whenever it is believed [32].

As explained in the cited works, these are but basic ideas which need further refinement in order to obtain a workable and sensible logic for their respective intended applications. Such refinements moreover give rise to several variants and combinations thereof. The standard format unifies the resulting systems in terms of one basic underlying structure, thereby allowing us to study their generic properties. In addition, the characterization of ALs in standard format by means of a triple (**L**, set of abnormalities, strategy) provides modularity, a simple recipe to develop new logics and variants, and to fine-tune logics whenever the need arises.

The basic metatheory of ALs in standard format is summarized in [10]. Some further results were established in [14], where it is argued that ALs have a number of advantages over alternative approaches to paraconsistent and defeasible reasoning. The interested reader may find a detailed overview of the theory and applications of ALs in Part I of [28]. As the latter work shows in particular, the standard format is by now a well-established framework for defeasible reasoning, which has a place of its own in the field of non-monotonic logic.

Our focus on the standard format of ALs is further motivated by another, independent reason. As shown in [30], the class of Makinson's *default assumption consequence relations* [20] corresponds to the class of ALs that use minimal abnormality. Hence all metatheoretic properties of the standard format nicely carry over to DACRs. We will return to this observation in Section 5, where we apply it to the results of the present study.

Aim of this article: Notwithstanding its relative success, there has been little investigation so far of the standard format *as a parametric framework*, asking how different ALs relate to each other in view of the constants they are defined from. That is, on the assumption that we work within the standard format, how and when do variations on a particular **L** and Ω affect the adaptive consequence relation?[3] When is this consequence relation preserved or conservatively extended, when do

[2]Throughout this paper, we use $\wedge$ for classical conjunction and $\sim$ for a paraconsistent negation, the meaning of which is disambiguated whenever necessary.

[3]The relation between the two adaptive strategies is studied in [29].

we obtain a stronger logic, and when are we guaranteed to end up with (generally) incomparable logics?

Such investigations are interesting not only from a theoretical perspective: they also point at means to change a given AL or enrich its language, while either keeping its consequence relation unchanged or strengthening it. They allow us to simplify our formal models, e.g. when it turns out that we may equivalently express a given AL by using a much simpler or smaller set of abnormalities. Finally, in view of the connections between ALs and the other frameworks mentioned above, the generic metatheory of ALs nicely carries over to those frameworks as well. Each of these points will be illustrated by means of concrete examples below.

Outline: In Section 2, we introduce the standard format in detail, using inconsistency-ALs as our running example. In the next two sections, we focus on the first and second parameter used in the AL framework: the monotonic core **L** (Section 3) and the set of abnormalities (Section 4). Our main technical results are summarized by Corollaries 3.4–3.7, 3.9 and 3.12 in Section 3, and Theorem 4.23 in Section 4. These state a number of sufficient (and, in the case of **L**, necessary) conditions, which are then applied to concrete cases in order to show their usefulness. In Section 5, we show how our results carry over to the class of DACRs from [20], and how they give rise to an interesting variant of those systems which corresponds to the reliability strategy.

2 The adaptive logic framework

In this section, we define the consequence relation of ALs in standard format. In addition, we mention some basic metatheorems concerning the standard format which will be called upon in subsequent sections.

Let us insert some remarks about presentation. First, in the current study, we only define the semantics of ALs and prove all metatheorems on the basis of it. A proof theory in terms of conditional, defeasible derivations and a corresponding syntactic consequence relation can be found e.g. in [10]. Since the standard format warrants soundness and completeness, all results from this paper automatically apply to the syntactic consequence relation of ALs in standard format as well.

Second, we will consider a slightly generalized version of the standard format of ALs, in the sense that we allow the set of abnormalities Ω to be arbitrary.[4] This allows us to present our results in their most generic form, to use simple examples in order to illustrate certain negative results, and to translate our results about ALs in a straightforward way to the DACR-format (we return to this point in Section 5).

Third and last, we will often refer to the same specific example throughout this paper, viz. inconsistency-adaptive logics based on the paraconsistent logic **CLuN** (see below). Apart from the fact that these logics played a prominent role in the development of ALs—see also the introduction of this article—this is mainly motivated pragmatically: they are fairly easy to define, which allows us to focus on the new results and their motivation. However, it should be stressed that none of our technical results hinge on this choice of example: they apply to all ALs in the format defined below.

[4] In most papers on ALs, Ω is supposed to be closed under uniform substitution of non-logical symbols for other non-logical symbols of the same type (e.g. propositional variables for other propositional variables, predicates for other predicates of the same arity, etc). Note that this does not imply that Ω is closed under uniform substitution in general, which does not hold for most ALs in the literature (the **CLuN**-based ALs which we use in this article are rather exceptional in this respect).

2.1 Preliminaries

Where X, Y are sets, we write $X \subseteq_f Y$ ($X \subset_f Y$) to denote that X is a finite (proper) subset of Y. Let $\wp(X)$ be the power set of X, and $\wp_f(X) = \{Y \mid Y \subseteq_f X\}$. Where $\prec$ is a binary relation on the set X, let $\mathsf{min}_\prec(X) = \{x \in X \mid \text{ for no } y \in X, y \prec x\}$.

Where $\mathcal{L}, \mathcal{L}', \ldots$ are formal languages, we use $\Phi, \Phi', \ldots$ to denote the sets of all well-formed formulas in these languages. In this notation, we always assume that each of $\Phi, \Phi', \ldots$ is closed under the unary connective $\neg$ and the binary connectives $\vee, \wedge, \supset, \equiv$. Where Φ is given, we use $A, B, \ldots$ as metavariables for its members and $\Gamma, \Delta, \ldots$ as metavariables for its subsets.

We use $\mathbf{L}, \mathbf{L}', \ldots$ as metavariables for logics which are defined on the basis of the respective languages $\mathcal{L}, \mathcal{L}', \ldots$. Every such $\mathbf{L}$ is defined from a set of models $\mathcal{M}_\mathbf{L}$ and a validity relation $\models_\mathbf{L} \subseteq \mathcal{M}_\mathbf{L} \times \Phi$. It is moreover presupposed that $\models_\mathbf{L}$ satisfies the following truth conditions:

(C$\neg$) for all $M \in \mathcal{M}_\mathbf{L}$, $M \models_\mathbf{L} \neg A$ iff $M \not\models_\mathbf{L} A$;
(C$\vee$) for all $M \in \mathcal{M}_\mathbf{L}$, $M \models_\mathbf{L} A \vee B$ iff $M \models_\mathbf{L} A$ or $M \models_\mathbf{L} B$;
(C$\wedge$) for all $M \in \mathcal{M}_\mathbf{L}$, $M \models_\mathbf{L} A \wedge B$ iff $M \models_\mathbf{L} A$ and $M \models_\mathbf{L} B$;
(C$\supset$) for all $M \in \mathcal{M}_\mathbf{L}$, $M \models_\mathbf{L} A \supset B$ iff $M \not\models_\mathbf{L} A$ or $M \models_\mathbf{L} B$;
(C$\equiv$) for all $M \in \mathcal{M}_\mathbf{L}$, $M \models_\mathbf{L} A \equiv B$ iff ($M \models_\mathbf{L} A$ iff $M \models_\mathbf{L} B$);

and the following semantic version of compactness:

(C$_\mathsf{f}$) for all $\Gamma \subseteq \Phi$: if every $\Gamma' \subseteq_f \Gamma$ has models, then Γ has models.

Where $\Gamma \subseteq \Phi$, let $\mathcal{M}_\mathbf{L}(\Gamma)$ be the set of all $M \in \mathcal{M}_\mathbf{L}$ such that $M \models_\mathbf{L} A$ for all $A \in \Gamma$. We define $\Vdash_\mathbf{L}$ as usual, putting $\Gamma \Vdash_\mathbf{L} A$ iff $M \models_\mathbf{L} A$ for all $M \in \mathcal{M}_\mathbf{L}(\Gamma)$. Let $\mathrm{Cn}_\mathbf{L}(\Gamma) =_\mathsf{df} \{A \mid \Gamma \Vdash_\mathbf{L} A\}$.

In view of the construction of $\Vdash_\mathbf{L}$, it is a Tarskian consequence relation. In other words, $\mathrm{Cn}_\mathbf{L}$ has the following three basic properties: monotonicity ($\mathrm{Cn}_\mathbf{L}(\Gamma) \subseteq \mathrm{Cn}_\mathbf{L}(\Gamma \cup \Gamma')$), transitivity (where $\Gamma' \subseteq \mathrm{Cn}_\mathbf{L}(\Gamma)$, $\mathrm{Cn}_\mathbf{L}(\Gamma') \subseteq \mathrm{Cn}_\mathbf{L}(\Gamma)$), and reflexivity ($\Gamma \subseteq \mathrm{Cn}_\mathbf{L}(\Gamma)$). By (C$\neg$)-(C$\equiv$) respectively, the connectives $\neg, \vee, \wedge, \supset, \equiv$ behave classically in $\mathbf{L}$. Finally, by (C$\neg$) and (C$_\mathsf{f}$) we can derive that $\mathrm{Cn}_\mathbf{L}$ is compact: $A \in \mathrm{Cn}_\mathbf{L}(\Gamma)$ iff there is a $\Gamma' \subseteq_f \Gamma$ such that $A \in \mathrm{Cn}_\mathbf{L}(\Gamma')$.

Remark 2.1
Where a given consequence operation $\mathrm{Cn}_\mathbf{L} : \wp(\Phi) \to \wp(\Phi)$ satisfies each of the conditions from the previous paragraph, we can easily construct a semantics for it in the above sense. This is done by letting $\mathcal{M}_\mathbf{L}$ be the set of all sets $\Theta \subseteq \Phi$ that are maximally consistent w.r.t. $\mathrm{Cn}_\mathbf{L}$, and putting $\Theta \models_\mathbf{L} A$ iff $A \in \Theta$. Conversely, where we have an $\mathbf{L}$-semantics in the above sense, it can easily be verified that each $M \in \mathcal{M}_\mathbf{L}$ corresponds to a maximal $\mathbf{L}$-consistent set Θ (which is just the set of all formulas valid in M). We briefly return to this point in Section 4.3.

Let ${}^\neg\Delta = \{\neg A \mid A \in \Delta\}$. Where Δ is finite and non-empty, let $\bigwedge\Delta$ ($\bigvee\Delta$) denote the classical conjunction (disjunction) of all the members of Δ. Where $\Delta = \{A\}$, let $\bigwedge\Delta = \bigvee\Delta = A$.

Where $\Omega \subseteq \Phi$ and $M \in \mathcal{M}_\mathbf{L}$, let $\mathsf{Ab}_\Omega(M) = \{A \in \Omega \mid M \models_\mathbf{L} A\}$. We will use this notation i.a. to represent what is usually called the *abnormal part* of a model, given a fixed set of abnormalities $\Omega \subseteq \Phi$. Note that, by (C$\neg$), the following holds:

Fact 2.2
($M \in \mathcal{M}_\mathbf{L}(\Gamma)$ and $\mathsf{Ab}_\Omega(M) \subseteq \Omega'$) iff $M \in \mathcal{M}_\mathbf{L}(\Gamma \cup {}^\neg(\Omega - \Omega'))$.

We use **CL** to denote propositional classical logic with the set of propositional variables $\Sigma = \{p, q, r, \ldots\}$ and the connectives $\neg, \vee, \wedge, \supset, \equiv$.

2.2 Setting the stage

Recall that every adaptive logic is defined from a triple: a monotonic core **L** with a compact, supraclassical Tarskian consequence relation $\Vdash_{\mathbf{L}} \subseteq \wp(\Phi) \times \Phi$; a set of abnormalities $\Omega \subseteq \Phi$, and a strategy. In the remainder of Section 2, we assume a fixed **L** and Ω and define the semantic consequence relations $\Vdash_{\mathbf{L},\Omega,\mathsf{m}}$ and $\Vdash_{\mathbf{L},\Omega,\mathsf{r}}$. These correspond to the minimal abnormality strategy, resp. the reliability strategy.

To illustrate certain definitions and properties in the remainder, we will use the well-known inconsistency-adaptive logics $\mathbf{CLuN^r}$ and $\mathbf{CLuN^m}$, which are described e.g. in [7]. Before we define each strategy, let us explain the basic motivation behind both logics (and inconsistency-ALs more generally) in a nutshell.

In $\mathbf{CLuN^r}$ and $\mathbf{CLuN^m}$, **L** is the paraconsistent logic **CLuN**—the name stands for '**CL** with gl**u**ts for the **N**egation'. For the sake of space, we restrict ourselves to the propositional fragment of these three systems. **CLuN** works on the basis of a language $\Phi_\sim$, which is built up from the propositional variables $p, q, \ldots$, a paraconsistent negation $\sim$ and the classical connectives $\neg, \vee, \wedge, \supset, \equiv$.

Semantically, **CLuN** can be characterized as follows. As usual, every model M is associated with a valuation function $v : \Phi_\sim \to \{0,1\}$. However, unlike the case for **CL**, v is not only used to determine the validity of propositional letters in M, but also of formulas of the form $\sim A$. This is done by means of the following clause (where v is the specific valuation function associated with M):

(C$\sim$) $M \models_{\mathbf{CLuN}} \sim A$ iff ($M \not\models_{\mathbf{CLuN}} A$ or $v(\sim A) = 1$)

Here, the first disjunct on the right ensures that excluded middle is valid in **CLuN**, whereas the second disjunct ensures that $\sim$-contradictions can be valid in a model M.

CLuN is a fairly weak logic, in that it invalidates a number of intuitive rules such as disjunctive syllogism ($A, \sim A \vee B$ / B), contraposition ($B \supset A$ / $\sim A \supset \sim B$), double negation introduction and elimination, and De Morgan's rules for $\sim$.[5] The idea behind $\mathbf{CLuN^r}$ and $\mathbf{CLuN^m}$ is to strengthen **CLuN**, by assuming $\sim$-inconsistencies to be false 'as much as possible'. This is done by taking as the set of abnormalities $\Omega_c = \{A \wedge \sim A \mid A \in \Phi_\sim\}$. By assuming these abnormalities to be false, unless they follow (by **CLuN**) from the premise set, we allow for the local validity of classical inferences.

For instance, where $\Gamma_1 = \{p, \sim q, \sim p \vee r, q, s \supset q\}$, we may say that q behaves inconsistently in view of Γ_1, yet there is no reason to also accept an inconsistency w.r.t. p. Hence, although we may apply disjunctive syllogism to p and $\sim p \vee r$ in order to derive r, we cannot apply modus tollens to $\sim q$ and $s \supset q$ in order to derive $\sim s$.

However, things are not always as cut and dry as the example Γ_1 suggests. Sometimes a premise set **CLuN**-entails a disjunction of abnormalities, but none of its disjuncts follow. Consider e.g. $\Gamma_2 = \{\sim p, \sim\sim p \vee t, \sim t\}$. Note that, since $\vee$ and $\wedge$ behave classically in **CLuN**, $\Gamma_2 \Vdash_{\mathbf{CLuN}} (\sim\sim p \wedge \sim p) \vee (t \wedge \sim t)$, and that neither of the disjuncts of this disjunction are **CLuN**-derivable from Γ_2.

[5] As a result, inconsistencies are not 'spread' in **CLuN**: $A \wedge \sim A \nVdash_{\mathbf{CLuN}} B \wedge \sim B$ for $B \neq A$, except when $A \Vdash_{\mathbf{CLuN}} B \wedge \sim B$ or $\sim A \Vdash_{\mathbf{CLuN}} B \wedge \sim B$. This makes **CLuN** particularly suitable to serve as the underlying logic of an inconsistency-AL, as argued in [5].

In cases like Γ_2, the phrase 'to interpret the premises as consistently as possible' can be interpreted in various ways. The two adaptive strategies can be seen as two prototypical specifications of this phrase. We will now define the semantics of both, after which we illustrate it in terms of the **CLuN**-based adaptive logics.

2.3 Minimal abnormality

The minimal abnormality strategy selects from $\mathcal{M}_{\mathbf{L}}(\Gamma)$ those models that verify a $\subset$-minimal set of abnormalities. A formula is a consequence iff it holds in all the selected models. Formally:

Definition 2.3
$\mathcal{M}_{\mathbf{L},\Omega,\mathsf{m}}(\Gamma)=\{M\in\mathcal{M}_{\mathbf{L}}(\Gamma)\mid \text{ for no } M'\in\mathcal{M}_{\mathbf{L}}(\Gamma), \mathsf{Ab}_\Omega(M')\subset\mathsf{Ab}_\Omega(M)\}$.

Definition 2.4
$\Gamma\Vdash_{\mathbf{L},\Omega,\mathsf{m}}A$ ($A\in\mathrm{Cn}_{\mathbf{L},\Omega,\mathsf{m}}(\Gamma)$) iff $M\models A$ for every $M\in\mathcal{M}_{\mathbf{L},\Omega,\mathsf{m}}(\Gamma)$.

Example 2.5
Let $\Gamma_3=\{{\sim}p, {\sim}{\sim}p\vee t, {\sim}t, q, {\sim}q\vee r, {\sim}{\sim}p\vee s, t\vee s\}$. Note that for all $M\in\mathcal{M}_{\mathbf{CLuN}}(\Gamma_3)$, either ${\sim}p\wedge{\sim}{\sim}p\in\mathsf{Ab}_{\Omega_c}(M)$ or $t\wedge{\sim}t\in\mathsf{Ab}_{\Omega_c}(M)$. It can be easily verified that all $M\in\mathcal{M}_{\mathbf{CLuN},\Omega_c,\mathsf{m}}(\Gamma_3)$ are such that either $\mathsf{Ab}_{\Omega_c}(M)=\{{\sim}p\wedge{\sim}{\sim}p\}$ or $\mathsf{Ab}_{\Omega_c}(M)=\{t\wedge{\sim}t\}$. All these models falsify $q\wedge{\sim}q$, which means that they verify r. As a consequence, $\Gamma_3\Vdash_{\mathbf{CLuN},\Omega_c,\mathsf{m}}r$. Moreover, whenever M is a minimally abnormal model of Γ_3, it falsifies either $t\wedge{\sim}t$ or ${\sim}p\wedge{\sim}{\sim}p$. Hence, all $M\in\mathcal{M}_{\mathbf{CLuN},\Omega_c,\mathsf{m}}(\Gamma_3)$ verify s. Equivalently, $\Gamma_3\Vdash_{\mathbf{CLuN},\Omega_c,\mathsf{m}}s$.

From Definitions 2.3 and 2.4, we can derive the following:

Theorem 2.6
$A\in\mathrm{Cn}_{\mathbf{L},\Omega,\mathsf{m}}(\Gamma)$ iff for all $M\in\mathcal{M}_{\mathbf{L},\Omega,\mathsf{m}}(\Gamma)$, $A\in\mathrm{Cn}_{\mathbf{L}}(\Gamma\cup{}^{\neg}(\Omega-\mathsf{Ab}_\Omega(M)))$.

Proof. ($\Rightarrow$) Let $M\in\mathcal{M}_{\mathbf{L},\Omega,\mathsf{m}}(\Gamma)$ be such that $A\notin\mathrm{Cn}_{\mathbf{L}}(\Gamma\cup{}^{\neg}(\Omega-\mathsf{Ab}_\Omega(M)))$. Hence there is an $M'\in\mathcal{M}_{\mathbf{L}}(\Gamma\cup{}^{\neg}(\Omega-\mathsf{Ab}_\Omega(M)))$ such that $M'\not\models_{\mathbf{L}}A$. By the minimality of M, $\mathsf{Ab}_\Omega(M')=\mathsf{Ab}_\Omega(M)$. Hence also $M'\in\mathcal{M}_{\mathbf{L},\Omega,\mathsf{m}}(\Gamma)$, so that $A\notin\mathrm{Cn}_{\mathbf{L},\Omega,\mathsf{m}}(\Gamma)$.

($\Leftarrow$) Suppose $A\notin\mathrm{Cn}_{\mathbf{L},\Omega,\mathsf{m}}(\Gamma)$. Let $M\in\mathcal{M}_{\mathbf{L},\Omega,\mathsf{m}}(\Gamma)$ be such that $M\not\models_{\mathbf{L}}A$. Obviously, $M\in\mathcal{M}_{\mathbf{L}}(\Gamma\cup{}^{\neg}(\Omega-\mathsf{Ab}_\Omega(M)))$ and hence $A\notin\mathrm{Cn}_{\mathbf{L}}(\Gamma\cup{}^{\neg}(\Omega-\mathsf{Ab}_\Omega(M)))$. ■

The semantics of minimal abnormality can be equivalently rephrased as a preferential semantics in the vein of [26]. That is, where $M,M'\in\mathcal{M}_{\mathbf{L}}$, let $M\prec_\Omega M'$ iff $\mathsf{Ab}_\Omega(M)\subset\mathsf{Ab}_\Omega(M')$. It can easily be checked that $\mathsf{min}_{\prec_\Omega}(\mathcal{M}_{\mathbf{L}}(\Gamma))=\mathcal{M}_{\mathbf{L},\Omega,\mathsf{m}}(\Gamma)$.

The following was proven in [6] for a number of inconsistency-adaptive logics (including $\mathbf{CLuN^m}$), and generalized to arbitrary logics $\mathbf{L}$ and sets Ω in [3]:

Theorem 2.7 ([3], Theorem 4.3)
If $M\in\mathcal{M}_{\mathbf{L}}(\Gamma)$, then there is an $M'\in\mathcal{M}_{\mathbf{L},\Omega,\mathsf{m}}(\Gamma)$ with $\mathsf{Ab}_\Omega(M')\subseteq\mathsf{Ab}_\Omega(M)$.

Equivalently, $\prec_\Omega$ is smooth w.r.t. every set $\mathcal{M}_{\mathbf{L}}(\Gamma)$.[6] Hence, $\Vdash_{\mathbf{L},\Omega,\mathsf{m}}$ falls within the well-known class **P** of smooth preferential systems, as defined and studied in the classical paper [18] (note though that unlike [18] we allow for infinite premise sets). As a result, $\Vdash_{\mathbf{L},\Omega,\mathsf{m}}$ satisfies a number of basic meta-theoretic properties such as cumulativity, left and right absorption, etc. We refer to [19] for definitions and an elaborate discussion of these properties.

[6] $\prec\,\subseteq X\times X$ is *smooth* w.r.t. X iff for all $x\in X$, either x is $\prec$-minimal in X, or there is a $\prec$-minimal y in X such that $y\prec x$.

In Section 3, we will sometimes rely on the fact that $\Vdash_{\mathbf{L},\Omega,\mathsf{m}}$ preserves consistency w.r.t. $\Vdash_{\mathbf{L}}$. This follows immediately from Theorem 2.7:

COROLLARY 2.8
For all $\Gamma\cup\{A\}\subseteq\Phi$: if $\Gamma\nVdash_{\mathbf{L}} A\wedge\neg A$, then $\Gamma\nVdash_{\mathbf{L},\Omega,\mathsf{m}} A\wedge\neg A$.

2.4 Reliability

The original idea behind the *reliability* strategy can be explained as follows. When we reason defeasibly, we rely on the falsehood of certain formulas, viz. the members of Ω. Hence, whenever $\Gamma\cup\neg\Omega\Vdash_{\mathbf{L}} A$, then we have reasons to infer A from Γ. However, not every abnormality can be assumed false (if we want to avoid triviality): sometimes we know that $\Gamma\Vdash B$ for some $B\in\Omega$, or more generally, that $\Gamma\Vdash\bigvee\Theta$ where $\Theta\subseteq_f\Omega$. So in order to obtain a sensible logic, we need to distinguish between two types of abnormalities: those that can be assumed false, and those that cannot, given the premises at hand.

This is done as follows. We call an abnormality A *unreliable w.r.t.* $\langle\mathbf{L},\Omega,\Gamma\rangle$ if and only if it is a disjunct of some minimal (classical) disjunction of abnormalities that follows from Γ (by $\mathbf{L}$). In the other case, A is *reliable w.r.t.* $\langle\mathbf{L},\Omega,\Gamma\rangle$. Note that the minimality of the disjunction is required—otherwise every abnormality would be unreliable as soon as one of them is. Reliable abnormalities correspond to what we called 'safe' assumptions in the previous paragraph.

With this distinction at hand, we can now define a consequence relation for the reliability strategy. Syntactically, we say that A follows from Γ iff $\Gamma\cup\neg\Delta\Vdash_{\mathbf{L}} A$, where Δ is the set of all reliable abnormalities. Semantically, this means that reliability selects only those models of Γ that verify none of the reliable abnormalities w.r.t. $\langle\mathbf{L},\Omega,\Gamma\rangle$.

We now make this exact.

DEFINITION 2.9
$\mathbb{S}_{\mathbf{L},\Omega}(\Gamma)$ is the set of all $\Delta\subseteq_f\Omega$ such that $\Gamma\Vdash_{\mathbf{L}}\bigvee\Delta$.
$\mathbb{S}^{\mathsf{min}}_{\mathbf{L},\Omega}(\Gamma)$ is the set of all $\subset$-minimal elements of $\mathbb{S}_{\mathbf{L},\Omega}(\Gamma)$.[7]
$\mathsf{U}_{\mathbf{L},\Omega}(\Gamma)=_{\mathsf{df}}\bigcup\mathbb{S}^{\mathsf{min}}_{\mathbf{L},\Omega}(\Gamma)$.

DEFINITION 2.10
$\mathcal{M}_{\mathbf{L},\Omega,\mathsf{r}}(\Gamma)=\{M\in\mathcal{M}_{\mathbf{L}}(\Gamma)\mid\mathsf{Ab}_\Omega(M)\subseteq\mathsf{U}_{\mathbf{L},\Omega}(\Gamma)\}$.

DEFINITION 2.11
$\Gamma\Vdash_{\mathbf{L},\Omega,\mathsf{r}} A$ ($A\in\mathrm{Cn}_{\mathbf{L},\Omega,\mathsf{r}}(\Gamma)$) iff $M\models_{\mathbf{L}} A$ for every $M\in\mathcal{M}_{\mathbf{L},\Omega,\mathsf{r}}(\Gamma)$.

EXAMPLE 2.12
We consider again Γ_3 from Example 2.5. It can be easily verified that $\{\sim p\wedge\sim\sim p, t\wedge\sim t\}$ is the only member of $\mathbb{S}^{\mathsf{min}}_{\mathbf{CLuN},\Omega_c}(\Gamma_3)$, which implies that $\mathsf{U}_{\mathbf{CLuN},\Omega_c}(\Gamma_3)=\{\sim p\wedge\sim\sim p, t\wedge\sim t\}$. Hence all models $M\in\mathcal{M}_{\mathbf{CLuN},\Omega_c,\mathsf{r}}(\Gamma_3)$ are such that $\mathsf{Ab}_{\Omega_c}(M)\subseteq\{\sim p\wedge\sim\sim p, t\wedge\sim t\}$. It follows that all these models falsify $q\wedge\sim q$, whence they verify r.

Let us now see whether also s follows from Γ_3, if we use reliability. Note that there are models $M\in\mathcal{M}_{\mathbf{CLuN},\Omega_c,\mathsf{r}}(\Gamma_3)$ such that $\mathsf{Ab}_{\Omega_c}(M)=\mathsf{U}_{\mathbf{CLuN},\Omega_c}(\Gamma_3)$, and hence both $\sim p\wedge\sim\sim p\in\mathsf{Ab}_{\Omega_c}(M)$ and $t\wedge\sim t\in\mathsf{Ab}_{\Omega_c}(M)$. Among these, there are moreover models M' such that $M'\not\models_{\mathbf{CLuN}} s$. Hence, $\Gamma_3\nVdash_{\mathbf{CLuN},\Omega_c,\mathsf{r}} s$.

[7] In the AL literature, $\mathbb{S}^{\mathsf{min}}_{\mathbf{L},\Omega}(\Gamma)$ is usually denoted by $\Sigma(\Gamma)$.

A different characterization of $\mathcal{M}_{\mathbf{L},\Omega,\mathsf{r}}(\Gamma)$ can also be given, which builds on the semantics of minimal abnormality. That is,

THEOREM 2.13
For all $\Gamma \subseteq \Phi$:[8]

1. If $\mathcal{M}_{\mathbf{L}}(\Gamma) \neq \emptyset$, then $\mathsf{U}_{\mathbf{L},\Omega}(\Gamma) = \{A \in \Omega \mid A \in \mathsf{Ab}_{\Omega}(M) \text{ for an } M \in \mathcal{M}_{\mathbf{L},\Omega,\mathsf{m}}(\Gamma)\}$.
2. $\mathcal{M}_{\mathbf{L},\Omega,\mathsf{r}}(\Gamma) = \{M \in \mathcal{M}_{\mathbf{L}}(\Gamma) \mid \mathsf{Ab}_{\Omega}(M) \subseteq \bigcup_{M' \in \mathcal{M}_{\mathbf{L},\Omega,\mathsf{m}}(\Gamma)} \mathsf{Ab}_{\Omega}(M')\}$.

PROOF. *Ad 1.* Suppose $\mathcal{M}_{\mathbf{L}}(\Gamma) \neq \emptyset$. '$\subseteq$' Let $A \in \mathsf{U}_{\mathbf{L},\Omega}(\Gamma)$. Let $\Delta \subseteq_f \Omega$ be such that $\Gamma \Vdash_{\mathbf{L}} A \vee \bigvee \Delta$, $\Gamma \nVdash_{\mathbf{L}} \bigvee \Delta$. Assume that for no $M \in \mathcal{M}_{\mathbf{L},\Omega,\mathsf{m}}(\Gamma)$, $M \models A$. Hence, since $\Gamma \Vdash_{\mathbf{L}} A \vee \bigvee \Delta$, for all $M \in \mathcal{M}_{\mathbf{L},\Omega,\mathsf{m}}(\Gamma)$: $M \models B$ for a $B \in \Delta$. But then, by Theorem 2.7, for all $M' \in \mathcal{M}_{\mathbf{L}}(\Gamma)$, $M \models B$ for a $B \in \Delta$, and hence $\Gamma \Vdash_{\mathbf{L}} \bigvee \Delta$— a contradiction.

'$\supseteq$' Let $A \in \mathsf{Ab}_{\Omega}(M)$ and $M \in \mathcal{M}_{\mathbf{L},\Omega,\mathsf{m}}(\Gamma)$. Hence for no $M' \in \mathcal{M}_{\mathbf{L}}(\Gamma)$: $\mathsf{Ab}_{\Omega}(M') \subset \mathsf{Ab}_{\Omega}(M)$. Hence $\Gamma \cup {}^{\neg}(\Omega - \mathsf{Ab}_{\Omega}(M)) \cup \{\neg A\}$ is **L**-trivial. By compactness and the deduction theorem, $\Gamma \Vdash_{\mathbf{L}} A \vee \bigvee \Delta$ for a $\Delta \subseteq_f (\Omega - \mathsf{Ab}_{\Omega}(M))$. Assume now that $A \notin \mathsf{U}_{\mathbf{L},\Omega}(\Gamma)$. This means that $\Gamma \Vdash_{\mathbf{L}} \bigvee \Delta$. Then $\Gamma \cup {}^{\neg}\Delta$ is **L**-trivial, and hence also $\Gamma \cup {}^{\neg}(\Omega - \mathsf{Ab}_{\Omega}(M))$ is **L**-trivial. But this contradicts the fact that $M \in \mathcal{M}_{\mathbf{L}}(\Gamma)$.

Ad 2. Immediate in view of item 1 and Definition 2.10. ∎

By Definitions 2.10 and 2.11, we can show that the syntactic characterization of reliability, as mentioned in the third paragraph of this section, corresponds exactly to the semantic one:

THEOREM 2.14
$\mathrm{Cn}_{\mathbf{L},\Omega,\mathsf{r}}(\Gamma) = \mathrm{Cn}_{\mathbf{L}}(\Gamma \cup {}^{\neg}(\Omega - \mathsf{U}_{\mathbf{L},\Omega}(\Gamma)))$.

PROOF. $\Gamma \Vdash_{\mathbf{L},\Omega,\mathsf{r}} A$ iff [by Definition 2.11] for all $M \in \mathcal{M}_{\mathbf{L},\Omega,\mathsf{r}}(\Gamma)$, $M \models_{\mathbf{L}} A$ iff [by Definition 2.10] for all $M \in \mathcal{M}_{\mathbf{L}}(\Gamma)$ such that $\mathsf{Ab}_{\Omega}(M) \subseteq \mathsf{U}_{\mathbf{L},\Omega}(\Gamma)$, $M \models_{\mathbf{L}} A$ iff [by Fact 2.2] for all $M \in \mathcal{M}_{\mathbf{L}}(\Gamma \cup {}^{\neg}(\Omega - \mathsf{U}_{\mathbf{L},\Omega}(\Gamma)))$, $M \models_{\mathbf{L}} A$ iff [by the definition of $\Vdash_{\mathbf{L}}$], $\Gamma \cup {}^{\neg}(\Omega - \mathsf{U}_{\mathbf{L},\Omega}(\Gamma)) \Vdash_{\mathbf{L}} A$. ∎

In view of Theorem 2.14, whenever M is an **L**-model of $\mathrm{Cn}_{\mathbf{L},\Omega,\mathsf{r}}(\Gamma)$, then $\mathsf{Ab}_{\Omega}(M) \subseteq \mathsf{U}_{\mathbf{L},\Omega}(\Gamma)$, and hence $M \in \mathcal{M}_{\mathbf{L},\Omega,\mathsf{r}}(\Gamma)$. The converse also holds by Definitions 2.10 and 2.11. So we have:

COROLLARY 2.15
$\mathcal{M}_{\mathbf{L},\Omega,\mathsf{r}}(\Gamma) = \mathcal{M}_{\mathbf{L}}(\mathrm{Cn}_{\mathbf{L},\Omega,\mathsf{r}}(\Gamma))$.

As Examples 2.5 and 2.12 show, minimal abnormality sometimes yields more consequences than reliability. By Definition 2.3 and Theorem 2.13, we can derive that minimal abnormality is always at least as strong as reliability:

THEOREM 2.16 ([10], Theorem 11)
$\mathcal{M}_{\mathbf{L},\Omega,\mathsf{m}}(\Gamma) \subseteq \mathcal{M}_{\mathbf{L},\Omega,\mathsf{r}}(\Gamma)$ for all Γ. Hence, $\Vdash_{\mathbf{L},\Omega,\mathsf{r}} \subseteq \Vdash_{\mathbf{L},\Omega,\mathsf{m}}$.

By Theorem 2.16 and Corollary 2.8, also $\Vdash_{\mathbf{L},\Omega,\mathsf{r}}$ preserves consistency w.r.t. $\Vdash_{\mathbf{L}}$:

COROLLARY 2.17
For all $\Gamma \cup \{A\} \subseteq \Phi$: if $\Gamma \nVdash_{\mathbf{L}} A \wedge \neg A$, then $\Gamma \nVdash_{\mathbf{L},\Omega,\mathsf{r}} A \wedge \neg A$.

We conclude this section with a lemma that will be crucial in the next section. It states that an abnormality is reliable w.r.t. $\langle \mathbf{L}, \Omega, \Gamma \rangle$ iff its negation follows adaptively from Γ, using either of the two strategies.

[8]This property is well known, and can e.g. be derived from Lemma 5.3.2, Corollary C.1.1 and Corollary A.2 in [28]. We prove it independently for the sake of self-containedness.

Lemma 2.18
Where $\mathsf{x} \in \{\mathsf{r}, \mathsf{m}\}$, $A \in \Omega$ and $\mathcal{M}_{\mathbf{L}}(\Gamma) \neq \emptyset$: $A \notin \mathsf{U}_{\mathbf{L},\Omega}(\Gamma)$ iff $\Gamma \Vdash_{\mathbf{L},\Omega,\mathsf{x}} \neg A$.

Proof. Note first that $A \notin \mathsf{U}_{\mathbf{L},\Omega}(\Gamma)$ iff [by Theorem 2.13]

(†) $A \in \Omega$ and there is no $M \in \mathcal{M}_{\mathbf{L},\Omega,\mathsf{m}}(\Gamma)$ such that $M \models A$.

($\mathsf{x} = \mathsf{m}$) By Definition 2.4 and (C¬), (†) iff $\Gamma \Vdash_{\mathbf{L},\Omega,\mathsf{x}} \neg A$.

($\mathsf{x} = \mathsf{r}$) We have: (†) iff [by Theorem 2.13] $A \in \Omega$ and there is no $M \in \mathcal{M}_{\mathbf{L},\Omega,\mathsf{r}}(\Gamma)$ such that $M \models A$ iff [by Definition 2.11 and (C¬)] $\Gamma \Vdash_{\mathbf{L},\Omega,\mathsf{x}} \neg A$. ∎

3 Parameter 1: the monotonic core

Let a fixed set of abnormalities Ω and strategy $\mathsf{x} \in \{\mathsf{r}, \mathsf{m}\}$ be given, and consider two compact, supraclassical Tarski-logics $\mathbf{L}$ and $\mathbf{L}'$. In this section, we answer the following question: When is it the case that $\Vdash_{\mathbf{L},\Omega,\mathsf{x}}$ and $\Vdash_{\mathbf{L}',\Omega,\mathsf{x}}$ coincide? And when can it be shown that the former is a proper sub-relation of the other?

The answer consists in necessary and jointly sufficient conditions. These are expressed as a function of $\Vdash_{\mathbf{L}}$, $\Vdash_{\mathbf{L}'}$ and Ω. Our main results are spelled out in Section 3.1—see corollaries 3.6 and 3.7. Some corollaries are discussed and illustrated in Sections 3.2 and 3.3.[9]

To simplify notation, we will omit the subscript Ω throughout this section, and only refer explicitly to the strategy and monotonic core of the ALs in question. So we write $\Vdash_{\mathbf{L},\mathsf{x}}$ instead of $\Vdash_{\mathbf{L},\Omega,\mathsf{x}}$, $\mathsf{U}_{\mathbf{L}}(\Gamma)$ instead of $\mathsf{U}_{\mathbf{L},\Omega}(\Gamma)$, and so on.

In this section, we will also consider cases where $\mathbf{L}$ and $\mathbf{L}'$ are based on different languages $\mathcal{L}, \mathcal{L}'$, so that not necessarily $\Phi = \Phi'$. This way we can e.g. also cover cases where $\mathbf{L}$ is a conservative extension of $\mathbf{L}'$ (see Section 3.2). We therefore need to speak about restrictions of $\mathbf{L}$ and $\mathbf{L}'$ to a given sub-language $\mathcal{L}_{sub}$ of $\mathcal{L}$ and $\mathcal{L}'$. In the remainder, let Ψ be the set of all formulas of $\mathcal{L}_{sub}$. It is assumed throughout this section that Ψ is closed under the classical connectives $\neg, \vee, \wedge, \supset, \equiv$ and that $\Omega \subseteq \Psi$.

Notation 3.1

$\Vdash^{\Psi}_{\mathbf{L}} =_{\mathsf{df}} \Vdash_{\mathbf{L}} \cap (\wp(\Psi) \times \Psi)$

$\Vdash^{\Psi}_{\mathbf{L},\mathsf{x}} =_{\mathsf{df}} \Vdash_{\mathbf{L},\mathsf{x}} \cap (\wp(\Psi) \times \Psi)$

for all $\Gamma \subseteq \Psi$, $\mathrm{Cn}^{\Psi}_{\mathbf{L}}(\Gamma) =_{\mathsf{df}} \mathrm{Cn}_{\mathbf{L}}(\Gamma) \cap \Psi$

for all $\Gamma \subseteq \Psi$, $\mathrm{Cn}^{\Psi}_{\mathbf{L},\mathsf{x}}(\Gamma) =_{\mathsf{df}} \mathrm{Cn}_{\mathbf{L},\mathsf{x}}(\Gamma) \cap \Psi$

It can be easily verified that $\Vdash^{\Psi}_{\mathbf{L}}$ is supra-classical, compact and has the three Tarski-properties, on the supposition that these conditions hold for $\Vdash_{\mathbf{L}}$.

3.1 Reliability-conservativity

The necessary and sufficient conditions which will be considered below are spelled out in terms of a specific property, which we call *reliability-conservativity* of one logic w.r.t. another logic. The idea is that (the consequence relation of) $\mathbf{L}'$ is reliability-conservative w.r.t. $\mathbf{L}$ iff $\mathbf{L}'$ does not render any

[9] It should be noted that, although we defined $\mathrm{Cn}_{\mathbf{L},\Omega,\mathsf{x}}(\Gamma)$ in terms of $\models_{\mathbf{L}}$ and a selection of models from $\mathcal{M}_{\mathbf{L}}(\Gamma)$ in the previous section, our proofs in the current section nowhere assume any specific relation between the sets of models or the validity relation of $\mathbf{L}$ and $\mathbf{L}'$. In other words, the conditions we arrive at are spelled out purely in terms of the semantic *consequence relations* of $\mathbf{L}$ and $\mathbf{L}'$ (and the way both deal with Ω), not in terms of the internal structure of their semantics.

more abnormalities unreliable than does $\mathbf{L}$, for any premise set Γ. However, we omit the border case where Γ is $\mathbf{L}'$-trivial—note that in this case, $\mathrm{Cn}_{\mathbf{L},\mathsf{x}}(\Gamma) \subseteq \mathrm{Cn}_{\mathbf{L}',\mathsf{x}}(\Gamma) = \Phi$. If we consider this property relative to a given Ψ, we obtain:

Definition 3.2
$\Vdash_{\mathbf{L}'}$ is *reliability-conservative w.r.t.* $\langle \Vdash_{\mathbf{L}}, \Psi\rangle$ iff for all $\Gamma \subseteq \Psi$ such that $\mathcal{M}_{\mathbf{L}'}(\Gamma) \neq \emptyset$, $\mathsf{U}_{\mathbf{L}'}(\Gamma) \subseteq \mathsf{U}_{\mathbf{L}}(\Gamma)$.

Theorem 3.3
Where $\Omega \subseteq \Psi$: $\Vdash^{\Psi}_{\mathbf{L},\mathsf{x}} \subseteq \Vdash^{\Psi}_{\mathbf{L}',\mathsf{x}}$ iff both of the following hold:

1. $\Vdash^{\Psi}_{\mathbf{L}} \subseteq \Vdash^{\Psi}_{\mathbf{L}'}$
2. $\Vdash_{\mathbf{L}'}$ is reliability-conservative w.r.t. $\langle \Vdash_{\mathbf{L}}, \Psi\rangle$

Proof. ($\Rightarrow$) Ad 1. Suppose $\Vdash^{\Psi}_{\mathbf{L}} \not\subseteq \Vdash^{\Psi}_{\mathbf{L}'}$. Let $\Gamma \cup \{A\} \subseteq \Psi$ be such that (1) $\Gamma \Vdash_{\mathbf{L}} A$ and (2) $\Gamma \nVdash_{\mathbf{L}'} A$. By (2) and the supraclassicality of $\Vdash_{\mathbf{L}'}$, $\Gamma \cup \{\neg A\} \nVdash_{\mathbf{L}'} A$. By consistency preservation of $\Vdash_{\mathbf{L}',\mathsf{x}}$ w.r.t. $\Vdash_{\mathbf{L}'}$, $\Gamma \cup \{\neg A\} \nVdash_{\mathbf{L}',\mathsf{x}} A$. However, by (1) and the monotonicity of $\mathbf{L}$, $\Gamma \cup \{\neg A\} \Vdash_{\mathbf{L}} A$, and hence $\Gamma \cup \{\neg A\} \Vdash_{\mathbf{L},\mathsf{x}} A$. Since $\Gamma \cup \{A, \neg A\} \subseteq \Psi$, $\Vdash^{\Psi}_{\mathbf{L},\mathsf{x}} \not\subseteq \Vdash^{\Psi}_{\mathbf{L}',\mathsf{x}}$.

Ad 2. Suppose $\Vdash_{\mathbf{L}'}$ is not reliability-conservative w.r.t. $\langle \Vdash_{\mathbf{L}}, \Psi\rangle$. Let $\Gamma \subseteq \Psi$ be such that $\mathcal{M}_{\mathbf{L}'}(\Gamma) \neq \emptyset$ and $\mathsf{U}_{\mathbf{L}'}(\Gamma) \not\subseteq \mathsf{U}_{\mathbf{L}}(\Gamma)$. Let $A \in \mathsf{U}_{\mathbf{L}'}(\Gamma) - \mathsf{U}_{\mathbf{L}}(\Gamma)$. By Lemma 2.18, $\Gamma \Vdash_{\mathbf{L},\mathsf{x}} \neg A$, but $\Gamma \nVdash_{\mathbf{L}',\mathsf{x}} \neg A$. Since $\Gamma \cup \{\neg A\} \subseteq \Psi$, $\Vdash^{\Psi}_{\mathbf{L},\mathsf{x}} \not\subseteq \Vdash^{\Psi}_{\mathbf{L}',\mathsf{x}}$.

($\Leftarrow$) Suppose (1.) and (2.) hold. Consider an arbitrary $\Gamma \subseteq \Psi$. If $\mathcal{M}_{\mathbf{L}'}(\Gamma) = \emptyset$, it follows immediately that $\mathrm{Cn}^{\Psi}_{\mathbf{L},\mathsf{x}}(\Gamma) \subseteq \mathrm{Cn}^{\Psi}_{\mathbf{L}',\mathsf{x}}(\Gamma) = \Psi$. So suppose moreover that $\mathcal{M}_{\mathbf{L}'}(\Gamma) \neq \emptyset$.

($\mathsf{x} = \mathsf{r}$) By suppositions (1.) and (2.) and the monotonicity of $\mathbf{L}'$,

$$\mathrm{Cn}^{\Psi}_{\mathbf{L}}(\Gamma \cup {}^{\neg}(\Omega - \mathsf{U}_{\mathbf{L}}(\Gamma))) \subseteq \mathrm{Cn}^{\Psi}_{\mathbf{L}'}(\Gamma \cup {}^{\neg}(\Omega - \mathsf{U}_{\mathbf{L}'}(\Gamma)))$$

The rest is immediate in view of Theorem 2.14.

($\mathsf{x} = \mathsf{m}$) Assume that (†) $A \in \mathrm{Cn}^{\Psi}_{\mathbf{L},\mathsf{m}}(\Gamma) - \mathrm{Cn}^{\Psi}_{\mathbf{L}',\mathsf{m}}(\Gamma)$. Let $M \in \mathcal{M}_{\mathbf{L}',\mathsf{m}}(\Gamma)$ be such that $M \not\models_{\mathbf{L}'} A$. Let $\Theta = \mathsf{Ab}(M)$ and $\Theta' = \Omega - \Theta$. Hence, $\mathcal{M}_{\mathbf{L}'}(\Gamma \cup {}^{\neg}\Theta') \neq \emptyset$ and since $\Vdash^{\Psi}_{\mathbf{L}} \subseteq \Vdash^{\Psi}_{\mathbf{L}'}$, also $\mathcal{M}_{\mathbf{L}}(\Gamma \cup {}^{\neg}\Theta') \neq \emptyset$. By (†), there is an $M' \in \mathcal{M}_{\mathbf{L},\mathsf{m}}(\Gamma \cup {}^{\neg}\Theta')$ such that $\mathsf{Ab}(M') \subset \Theta$. By the minimality of M, for all $B \in \Theta - \mathsf{Ab}(M')$, $B \in \mathsf{U}_{\mathbf{L}'}(\Gamma \cup {}^{\neg}\Theta' \cup \mathsf{Ab}(M'))$, and hence by (2.), $B \in \mathsf{U}_{\mathbf{L}}(\Gamma \cup {}^{\neg}\Theta' \cup \mathsf{Ab}(M'))$. Fix an arbitrary $B \in \Theta - \mathsf{Ab}(M')$. Since $B \in \mathsf{U}_{\mathbf{L}}(\Gamma \cup {}^{\neg}\Theta' \cup \mathsf{Ab}(M'))$, there is a $\Lambda \in \mathbb{S}_{\mathbf{L}}(\Gamma \cup {}^{\neg}\Theta' \cup \mathsf{Ab}(M'))$ such that $B \in \Lambda$. Hence, ($\star$) $M' \models_{\mathbf{L}} \bigvee \Lambda$.

Let $C \in \Lambda$ be arbitrary. Assume first that $C \in \Theta'$. Then $\Gamma \cup {}^{\neg}\Theta' \cup \mathsf{Ab}(M') \Vdash_{\mathbf{L}} \neg C$ (by reflexivity), and hence $\Gamma \cup {}^{\neg}\Theta' \cup \mathsf{Ab}(M') \Vdash_{\mathbf{L}} \bigvee(\Lambda - \{C\})$. Second, assume that $C \in \mathsf{Ab}(M')$. Then $B \neq C$ and $\Gamma \cup {}^{\neg}\Theta' \cup \mathsf{Ab}(M') \Vdash_{\mathbf{L}} C$. In both cases, it follows that $\bigvee \Lambda$ is not a minimal disjunction that follows from $\Gamma \cup {}^{\neg}\Theta' \cup \mathsf{Ab}(M')$—a contradiction.

It follows that for all $C \in \Lambda$, $C \in \Theta - \mathsf{Ab}(M')$. But then $M \not\models_{\mathbf{L}} \bigvee \Lambda$, which contradicts ($\star$). ■

From Theorem 3.3, we now derive the corollaries which answer the questions posed at the start of Section 3. First of all, note that if $\Vdash^{\Psi}_{\mathbf{L}} = \Vdash^{\Psi}_{\mathbf{L}'}$ and $\Omega \subseteq \Psi$, then $\mathbb{S}^{\mathsf{min}}_{\mathbf{L}}(\Gamma) = \mathbb{S}^{\mathsf{min}}_{\mathbf{L}'}(\Gamma)$. Hence, $\mathsf{U}_{\mathbf{L}}(\Gamma) = \mathsf{U}_{\mathbf{L}'}(\Gamma)$ for all $\Gamma \subseteq \Psi$. It follows that $\Vdash_{\mathbf{L}}$ is reliability-conservative w.r.t. $\langle \Vdash_{\mathbf{L}'}, \Psi\rangle$, and that $\Vdash_{\mathbf{L}'}$ is reliability-conservative w.r.t. $\langle \Vdash_{\mathbf{L}}, \Psi\rangle$. Combining this insight with Theorem 3.3, we obtain the following:

Corollary 3.4
$\Vdash^{\Psi}_{\mathbf{L},\mathsf{x}} = \Vdash^{\Psi}_{\mathbf{L}',\mathsf{x}}$ iff $\Vdash^{\Psi}_{\mathbf{L}} = \Vdash^{\Psi}_{\mathbf{L}'}$.

Corollary 3.5
$\Vdash^{\Psi}_{\mathbf{L},\mathsf{x}} \subset \Vdash^{\Psi}_{\mathbf{L}',\mathsf{x}}$ iff $\Vdash^{\Psi}_{\mathbf{L}} \subset \Vdash^{\Psi}_{\mathbf{L}'}$ and $\Vdash_{\mathbf{L}'}$ is reliability-conservative w.r.t. $\langle \Vdash_{\mathbf{L}}, \Psi\rangle$.

For the special case when $\Psi = \Phi = \Phi'$, we have:

Corollary 3.6
$\Vdash_{\mathbf{L},\mathrm{x}} = \Vdash_{\mathbf{L}',\mathrm{x}}$ iff $\Vdash_{\mathbf{L}} = \Vdash_{\mathbf{L}'}$.

Corollary 3.7
$\Vdash_{\mathbf{L},\mathrm{x}} \subset \Vdash_{\mathbf{L}',\mathrm{x}}$ iff $\Vdash_{\mathbf{L}} \subset \Vdash_{\mathbf{L}'}$ and $\Vdash_{\mathbf{L}'}$ is reliability-conservative w.r.t. $\langle \Vdash_{\mathbf{L}}, \Psi\rangle$.

Corollary 3.7 may sound somewhat discouraging, at least in case one hopes to find a simple recipe to strengthen a given AL by adding certain axioms or rules to its underlying monotonic core. Indeed, in many concrete cases, strengthening $\mathbf{L}$ results in a logic $\mathbf{L}'$ which does not conserve reliability w.r.t. $\langle \mathbf{L}, \Omega, \Phi\rangle$. So although the resulting AL will allow for more undefeasible inferences, it will also invalidate certain defeasible inferences because their underlying assumptions are falsified by other $\mathbf{L}'$-consequences of the premise set. This is illustrated by a well-known example in Appendix A.

Nevertheless, there are cases in which the right hand side of Corollary 3.7 can easily be shown to hold. We give some examples of these in Section 3.3.

3.2 Conservative extensions

From Corollary 3.4, we can infer that whenever $\mathbf{L}'$ is a conservative extension of $\mathbf{L}$, then so is every AL based on $\mathbf{L}'$ which uses the same set of abnormalities as an AL based on $\mathbf{L}$.

Definition 3.8
Where $\Phi_1 \subset \Phi_2$, $\Vdash_1 \subseteq \wp(\Phi_1) \times \Phi_1$, and $\Vdash_2 \subseteq \wp(\Phi_2) \times \Phi_2$: $\Vdash_2$ is a *conservative extension* of $\Vdash_1$ iff $\Vdash_2^{\Phi_1} = \Vdash_1$.

Corollary 3.9
Where $\Omega \subseteq \Phi \subseteq \Phi'$, $\Vdash_{\mathbf{L}} \subseteq \wp(\Phi) \times \Phi$ and $\Vdash_{\mathbf{L}'} \subseteq \wp(\Phi') \times \Phi'$: $\Vdash_{\mathbf{L}',\mathrm{x}}$ is a conservative extension of $\Vdash_{\mathbf{L},\mathrm{x}}$ iff $\Vdash_{\mathbf{L}'}$ is a conservative extension of $\Vdash_{\mathbf{L}}$.

This result is important for various applications. For instance, suppose we want to enrich the language of a given inconsistency-adaptive logic with a knowledge operator $\Box$. In that case, we may consider using a conservative extension of our monotonic core logic which gives meaning to $\Box$, while keeping the set of abnormalities fixed. Corollary 3.9 tells us that the adaptive consequence set of the new logic may be richer, but it will not differ with respect to that part of the language that the new logic shares with the original logic.

3.3 Adding (disjunctions of) negations of abnormalities

As shown above, when two ALs use the same set of abnormalities, we can be sure that one is at least as strong as the other if the former never renders more abnormalities unreliable than the latter. As a corollary of this, we can derive that whenever $\mathbf{L}'$ can be obtained by adding to $\mathbf{L}$ the (non-defeasible) assumption that certain abnormalities are false, then the AL based on $\mathbf{L}'$ will be at least as strong as the one based on $\mathbf{L}$. Before we consider an example of this fact, let us make it formally precise.

Definition 3.10
Where $\Vdash_{\mathbf{L}} \subseteq \wp(\Phi) \times \Phi$ and where $\Gamma \cup \Delta \subseteq \Phi$: $\Gamma \Vdash_{\mathbf{L}*\Delta} A$ iff $\Gamma \cup \Delta \Vdash_{\mathbf{L}} A$.

Lemma 3.11
If Δ is an arbitrary set of formulas of the form $A_1 \vee \ldots \vee A_n$ with each $A_i \in {}^{\neg}\Omega$ $(i \leq n)$, then $\Vdash_{\mathbf{L}*\Delta}$ is reliability-conservative w.r.t. $\langle \Vdash_{\mathbf{L}}, \Phi\rangle$.

Proof. Suppose $A \in \mathsf{U}_{\mathbf{L}*\Delta}(\Gamma)$ and $\mathcal{M}_{\mathbf{L}*\Delta}(\Gamma) \neq \emptyset$. By Theorem 2.13, there is an $M \in \mathcal{M}_{\mathbf{L}*\Delta,\mathsf{m}}(\Gamma)$ such that $M \models_{\mathbf{L}} A$. Assume that $M \notin \mathcal{M}_{\mathbf{L},\mathsf{m}}(\Gamma)$. Hence there is an $M' \in \mathcal{M}_{\mathbf{L}}(\Gamma)$ such that $\mathsf{Ab}(M') \subset \mathsf{Ab}(M)$. So for all $B \in \neg\Omega$ such that $M \models_{\mathbf{L}} B$, also $M' \models_{\mathbf{L}} B$. It follows that $M' \in \mathcal{M}_{\mathbf{L}}(\Gamma \cup \Delta)$, and hence $M' \in \mathcal{M}_{\mathbf{L}*\Delta}(\Gamma)$. But this contradicts the fact that $M \in \mathcal{M}_{\mathbf{L}*\Delta,\mathsf{m}}(\Gamma)$. ■

Corollary 3.12
If Δ is an arbitrary set of formulas of the form $A_1 \vee \ldots \vee A_n$ with each $A_i \in \neg\Omega$ $(i \leq n)$, then $\Vdash_{\mathbf{L},\mathsf{x}} \subseteq \Vdash_{\mathbf{L}*\Delta,\mathsf{x}}$.

Corollary 3.13
If $\Delta \subseteq \neg\Omega$, then $\Vdash_{\mathbf{L},\mathsf{x}} \subseteq \Vdash_{\mathbf{L}*\Delta,\mathsf{x}}$.

Note that whenever some $A \in \Delta$ is not an **L**-theorem, then $\Vdash_{\mathbf{L}} \subset \Vdash_{\mathbf{L}*\Delta}$. So by Lemma 3.11 and Corollary 3.7, we can infer that whenever $\Delta \subseteq \neg\Omega$, and some members of Δ are not **L**-theorems, then $\Vdash_{\mathbf{L},\mathsf{x}} \subset \Vdash_{\mathbf{L}*\Delta,\mathsf{x}}$.

For an example, consider the system **CLuNv** from [2]. This logic is obtained by adding to **CLuN** all axioms $(A \wedge {\sim}A) \supset B$ with $A \in \Phi_{\sim} - \Sigma$. In other words, **CLuNv** trivializes all inconsistencies w.r.t. complex formulae, but it does allow for inconsistencies at the level of sentential letters. Semantically, this means that we impose the following restriction on the valuation function $v : \Phi_{\sim} \to \{0,1\}$:

(C**v**) For all $A \in \Phi_{\sim} - \Sigma$, $v({\sim}A) = v(\neg A)$

It can easily be verified that $\Vdash_{\mathbf{CLuNv}} = \Vdash_{\mathbf{CLuN}*\Theta}$, where $\Theta = \{\neg(A \wedge {\sim}A) \mid A \in \Phi_{\sim} - \Sigma\}$. In view of the preceding, $\Vdash_{\mathbf{CLuN},\mathsf{x}} \subset \Vdash_{\mathbf{CLuNv},\mathsf{x}}$.[10] So if we treat all ${\sim}$-inconsistencies as abnormal, but some as explosive, we are guaranteed to get a stronger inconsistency-adaptive logic than in the case where we allow for any type of ${\sim}$-inconsistency (in the monotonic core).

For a concrete example, consider again the premise set $\Gamma_3 = \{{\sim}p, {\sim\sim}p \vee t, q, {\sim}q \vee r, {\sim}t\}$ from Example 2.5. Since ${\sim\sim}p \Vdash_{\mathbf{CLuNv}} \neg{\sim}p$, $\Gamma_3 \Vdash_{\mathbf{CLuNv}} t \wedge {\sim}t$. Hence only t will behave inconsistently in view of Γ_3, if we take **CLuNv** as the underlying monotonic core.

Arguably, for some the road taken by **CLuNv** may appear rather extreme: if inconsistencies are to be taken seriously, how can we assume that no complex formula whatsoever behaves inconsistently? Note however that we merely used **CLuNv** as an example. One may readily think of much weaker logics, which still trivialize inconsistencies of a certain form, or with respect to certain (types of) propositional variables, etc.

An analogous point can be made about other applications of ALs. Take for instance the case of adaptive deontic logics (see e.g. [17, 27]). Many of these offer specific ways to cope with deontic conflicts. Typically, their abnormalities represent statements such as '*A* is obligatory according to some normative system, but it is not a universal obligation', '*A* is a prima facie obligation, but not an actual obligation', or '*A* and *B* are obligations, but $A \wedge B$ is not'. Here again, we may consider stronger ALs, obtained by (i) restricting the set of possibly abnormal *A* (and *B*) to a specific type of formulas, and (ii) adding axioms which enforce that all other formulas cannot behave abnormally in this sense.

Just as is the case with **CLuNv**, adding such negations of abnormalities (or disjunctions thereof) to a monotonic **L** will result in a new AL which is (in the interesting case) often stronger, but which also trivializes more premise sets than the original AL we started with. In other words, much as is the case with monotonic logics, we end up with a trade-off between inferential power on the one hand, and avoiding triviality on the other.

[10] First, note that $\Vdash_{\mathbf{CLuN}} \subset \Vdash_{\mathbf{CLuNv}}$. Second, by Lemma 3.11, $\Vdash_{\mathbf{CLuNv}}$ is reliability-conservative w.r.t. $\langle \Vdash_{\mathbf{CLuN}}, \Phi_{\sim} \rangle$. The rest is immediate in view of Corollary 3.7.

4 Parameter 2: the set of abnormalities

In this section, we hold the monotonic core **L** fixed, and ask which conditions on the sets of abnormalities Ω and Ω' warrant that $\Vdash_{\mathbf{L},\Omega,\mathsf{x}} \subseteq \Vdash_{\mathbf{L},\Omega',\mathsf{x}}$. As before, we consider this question for both $\mathsf{x}=\mathsf{r}$ and $\mathsf{x}=\mathsf{m}$.

There are at least two reasons why this question is interesting. First, it sheds new light on the concept of abnormality and its role in defeasible reasoning. What does it mean that a given formula A is an abnormality? What happens in case two abnormalities A and B are contradictory – do they cancel each other out? Or if A and B are abnormalities, then what happens if we also treat $A \wedge B$, resp. $A \vee B$ as abnormalities? More generally, what kind of logic do we obtain if we close Ω under a given (set of) connective(s)?

The second motivation for this section is pragmatic. For a given set Ω of abnormalities, one may ask whether some of its members A are redundant, in the sense that $\Vdash_{\mathbf{L},\Omega,\mathsf{x}} = \Vdash_{\mathbf{L},\Omega-\{A\},\mathsf{x}}$. If so, then one may ignore those abnormalities altogether, when one checks whether something follows from a premise set or not. We will give an example of such a case in Section 4.2 below.

The results presented in this section are significant, yet only partial. More particularly, in contrast to the previous section, we were only able to spell out conditions that are sufficient (but not necessary) for the identity or inclusion of two adaptive consequence relations.[11] We will therefore focus on concrete examples, in order to motivate further research in this direction. Our results also differ from those in the preceding section in another respect: it turns out that the conditions under which the consequence relation is preserved are different for the two strategies.

This section is organized as follows. In Section 4.1, we note some basic insights concerning pairs of adaptive logics defined from the same monotonic core but a different set of abnormalities. Next, we consider specific cases where Ω' is a superset of Ω, obtained by closing certain abnormalities under truth-functional connectives (Section 4.2). This allows us to illustrate some basic mechanisms, and the importance of this type of work for concrete applications. We will gradually work towards more generic conditions, the deepest of which are given in Section 4.3.

We assume a fixed logic **L** in this section, with $\Vdash_{\mathbf{L}} \subseteq \wp(\Phi) \times \Phi$ for a given set of formulas Φ which is closed under the classical connectives $\neg, \vee, \wedge, \supset, \equiv$. Recall that **L** is supposed to be a supra-classical, compact Tarski-logic. We will skip the subscript **L** throughout this section, and thus write e.g. $\mathcal{M}_{\Omega,\mathsf{x}}(\Gamma)$ instead of $\mathcal{M}_{\mathbf{L},\Omega,\mathsf{x}}(\Gamma)$. Sets of abnormalities are denoted by $\Omega, \Omega', \ldots$, and always assumed to be subsets of Φ.

4.1 Some preliminary insights

This section consists of some general observations concerning ALs that are based on the same **L** and two different sets of abnormalities. Some of these will be called upon in Section 4.3; others are noteworthy in their own right.

LEMMA 4.1
$\Vdash_{\Omega,\mathsf{m}} \subseteq \Vdash_{\Omega',\mathsf{m}}$ iff for all $M, M' \in \mathcal{M}$: if $\mathsf{Ab}_{\Omega}(M) \subset \mathsf{Ab}_{\Omega}(M')$, then $\mathsf{Ab}_{\Omega'}(M) \subset \mathsf{Ab}_{\Omega'}(M')$.

PROOF. ($\Rightarrow$) Suppose $M, M' \in \mathcal{M}$ are such that (1) $\mathsf{Ab}_{\Omega}(M) \subset \mathsf{Ab}_{\Omega}(M')$, yet (2) $\mathsf{Ab}_{\Omega'}(M) \not\subset \mathsf{Ab}_{\Omega'}(M')$. Let $A \in \mathsf{Ab}_{\Omega}(M') - \mathsf{Ab}_{\Omega}(M)$. Let $\Gamma = \{B \vee C \mid M \models B, M' \models C\}$. Note that each of the following holds

[11] Lemma 4.1 below does provide a necessary and sufficient condition for $\Vdash_{\mathbf{L},\Omega,\mathsf{m}} \subseteq \Vdash_{\mathbf{L},\Omega',\mathsf{m}}$, but this condition seems to be little more than a clarification of what it means that, for all Γ, $\mathcal{M}_{\mathbf{L},\Omega',\mathsf{m}}(\Gamma) \subseteq \mathcal{M}_{\mathbf{L},\Omega,\mathsf{m}}(\Gamma)$.

for all $M'' \in \mathcal{M}(\Gamma)$:

(3) $\{D \mid M'' \models D\} = \{D \mid M \models D\}$ or $\{D \mid M'' \models D\} = \{D \mid M' \models D\}$ (from the construction of Γ, relying on the classical behaviour of $\vee$ and $\neg$)
(4) $\mathsf{Ab}_\Omega(M'') = \mathsf{Ab}_\Omega(M)$ or $\mathsf{Ab}_\Omega(M'') = \mathsf{Ab}_\Omega(M')$ (from (3))
(5) $\mathsf{Ab}_{\Omega'}(M'') = \mathsf{Ab}_{\Omega'}(M)$ or $\mathsf{Ab}_{\Omega'}(M'') = \mathsf{Ab}_{\Omega'}(M')$ (from (3))

By (1) and (4), all models $M'' \in \mathcal{M}_{\Omega,\mathsf{m}}(\Gamma)$ are such that $\mathsf{Ab}_\Omega(M'') = \mathsf{Ab}_\Omega(M)$, whence they all falsify A. So $\Gamma \Vdash_{\Omega,\mathsf{m}} \neg A$.

Note that $M' \in \mathcal{M}(\Gamma)$. Moreover, by (2) and (5), there is no $M'' \in \mathcal{M}(\Gamma)$ such that $\mathsf{Ab}_{\Omega'}(M'') \subset \mathsf{Ab}_{\Omega'}(M')$. Hence $M' \in \mathcal{M}_{\Omega',\mathsf{m}}(\Gamma)$. Since $M' \models A$, $\Gamma \nVdash_{\Omega',\mathsf{m}} \neg A$. ($\Leftarrow$) Trivial. ∎

Corollary 4.2
$\Vdash_{\Omega,\mathsf{m}} = \Vdash_{\Omega',\mathsf{m}}$ iff for all $M, M' \in \mathcal{M}$: $\mathsf{Ab}_\Omega(M) \subset \mathsf{Ab}_\Omega(M')$ iff $\mathsf{Ab}_{\Omega'}(M) \subset \mathsf{Ab}_{\Omega'}(M')$.

Theorem 4.3
If $\Vdash_{\Omega,\mathsf{r}} \subseteq \Vdash_{\Omega',\mathsf{r}}$, then $\Vdash_{\Omega,\mathsf{m}} \subseteq \Vdash_{\Omega',\mathsf{m}}$.

Proof. Suppose $\Vdash_{\Omega,\mathsf{m}} \not\subseteq \Vdash_{\Omega',\mathsf{m}}$. By Lemma 4.1, there are $M, M' \in \mathcal{M}$ such that $\mathsf{Ab}_\Omega(M) \subset \mathsf{Ab}_\Omega(M')$ and $\mathsf{Ab}_{\Omega'}(M) \not\subset \mathsf{Ab}_{\Omega'}(M')$. Let Γ be constructed in the same way as in the proof of Lemma 4.1. Hence there is an A such that (1) $\Gamma \Vdash_{\Omega,\mathsf{m}} \neg A$ and (2) $\Gamma \nVdash_{\Omega',\mathsf{m}} \neg A$. By (1) and Lemma 2.18, $A \notin \mathsf{U}_\Omega(\Gamma)$ and hence again by Lemma 2.18, $\Gamma \Vdash_{\Omega,\mathsf{r}} \neg A$. By (2) and Theorem 2.16, $\Gamma \nVdash_{\Omega',\mathsf{r}} \neg A$. Hence, $\Vdash_{\Omega,\mathsf{r}} \not\subseteq \Vdash_{\Omega',\mathsf{r}}$ ∎

Corollary 4.4
If $\Vdash_{\Omega,\mathsf{r}} = \Vdash_{\Omega',\mathsf{r}}$, then $\Vdash_{\Omega,\mathsf{m}} = \Vdash_{\Omega',\mathsf{m}}$.

Corollary 4.4 shows that any sufficient condition for $\Vdash_{\Omega,\mathsf{r}} = \Vdash_{\Omega',\mathsf{r}}$ is also sufficient for $\Vdash_{\Omega,\mathsf{m}} = \Vdash_{\Omega',\mathsf{m}}$. The converse fails, in view of the examples we will give in Section 4.2.

By the definitions of $\Vdash_{\Omega,\mathsf{m}}$ and $\Vdash_{\Omega,\mathsf{r}}$, it is obvious that whenever Ω and Ω' give rise to the same selection of models for every premise set Γ, then they also define the same adaptive consequence relation. In view of the preceding, we can also show that the converse holds:

Theorem 4.5
$\Vdash_{\Omega,\mathsf{x}} \subseteq \Vdash_{\Omega',\mathsf{x}}$ iff for all $\Gamma \subseteq \Phi$, $\mathcal{M}_{\Omega',\mathsf{x}}(\Gamma) \subseteq \mathcal{M}_{\Omega,\mathsf{x}}(\Gamma)$.

Proof. ($\Leftarrow$) Immediate in view of Definitions 2.11 (for $\mathsf{x} = \mathsf{r}$) and 2.4 (for $\mathsf{x} = \mathsf{m}$). ($\Rightarrow$) ($\mathsf{x} = \mathsf{r}$) Suppose that $\Vdash_{\Omega,\mathsf{r}} \subseteq \Vdash_{\Omega',\mathsf{r}}$. Let $\Gamma \subseteq \Phi$. Hence $\mathrm{Cn}_{\Omega,\mathsf{r}}(\Gamma) \subseteq \mathrm{Cn}_{\Omega',\mathsf{r}}(\Gamma)$. The rest is immediate in view of Corollary 2.15 and the monotonicity of **L**. ($\mathsf{x} = \mathsf{m}$) Suppose $\Vdash_{\Omega,\mathsf{m}} \subseteq \Vdash_{\Omega',\mathsf{m}}$. Hence by Lemma 4.1, for all $M, M' \in \mathcal{M}$, if $\mathsf{Ab}_\Omega(M) \subset \mathsf{Ab}_\Omega(M')$, then $\mathsf{Ab}_{\Omega'}(M) \subset \mathsf{Ab}_{\Omega'}(M')$. By Definition 2.3, for all $\Gamma \subseteq \Phi$, $\mathcal{M}_{\Omega',\mathsf{m}}(\Gamma) \subseteq \mathcal{M}_{\Omega,\mathsf{m}}(\Gamma)$. ∎

The following theorem shows that, whenever $\Omega \subseteq \Omega'$, then also the converse of Theorem 4.3 holds. This will in turn simplify some proofs in the remainder.

Theorem 4.6
If $\Vdash_{\Omega,\mathsf{m}} \subseteq \Vdash_{\Omega',\mathsf{m}}$ and $\Omega \subseteq \Omega'$, then $\Vdash_{\Omega,\mathsf{r}} \subseteq \Vdash_{\Omega',\mathsf{r}}$.

Proof. Suppose the antecedent holds. We prove that for all $\Gamma \subseteq \Phi$, $\mathcal{M}_{\Omega',\mathsf{r}}(\Gamma) \subseteq \mathcal{M}_{\Omega,\mathsf{r}}(\Gamma)$; the rest is immediate in view of Theorem 4.5. The case where $\mathcal{M}_{\Omega,\mathsf{r}}(\Gamma) = \mathcal{M}(\Gamma)$ is trivial. So suppose that $\mathcal{M}_{\Omega,\mathsf{r}}(\Gamma) \neq \mathcal{M}(\Gamma)$ and consider an arbitrary $M \in \mathcal{M}(\Gamma) - \mathcal{M}_{\Omega,\mathsf{r}}(\Gamma)$. Hence there is an $A \in \mathsf{Ab}_\Omega(M) - \mathsf{U}_\Omega(\Gamma)$. By Theorem 2.13, A is false in all $M' \in \mathcal{M}_{\Omega,\mathsf{m}}(\Gamma)$. By the supposition and Theorem 4.5, A is false in all $M' \in \mathcal{M}_{\Omega',\mathsf{m}}(\Gamma)$. Since $A \in \Omega'$, $A \in \mathsf{Ab}_{\Omega'}(M) - \mathsf{U}_{\Omega'}(\Gamma)$, and hence $M \notin \mathcal{M}_{\Omega',\mathsf{r}}(\Gamma)$. ∎

THEOREM 4.7
Where $\Omega \subseteq \Phi$ and $\mathbb{A} \subseteq \wp(\Phi)$: if, for all $\Omega' \in \mathbb{A}$, $\Vdash_{\Omega,\mathsf{x}} = \Vdash_{\Omega',\mathsf{x}}$, then $\Vdash_{\Omega,\mathsf{x}} = \Vdash_{\bigcup \mathbb{A},\mathsf{x}}$.

PROOF. ($\mathsf{x} = \mathsf{m}$) Let $\mathbb{A}$ be an arbitrary set of sets Ω' such that $\Vdash_{\Omega,\mathsf{m}} = \Vdash_{\Omega',\mathsf{m}}$. Let $\Lambda = \bigcup \mathbb{A}$. Consider two models $M, M' \in \mathcal{M}$. By Lemma 4.1, we have: $\mathsf{Ab}_\Omega(M) \subset \mathsf{Ab}_\Omega(M')$ iff for all $\Omega' \in \mathbb{A}$, $\mathsf{Ab}_{\Omega'}(M) \subset \mathsf{Ab}_{\Omega'}(M')$. It follows immediately that if $\mathsf{Ab}_\Omega(M) \subset \mathsf{Ab}_\Omega(M')$, then also $\mathsf{Ab}_\Lambda(M) \subset \mathsf{Ab}_\Lambda(M')$.

Suppose now that $\mathsf{Ab}_\Lambda(M) \subset \mathsf{Ab}_\Lambda(M')$. Hence there is an $\Omega' \in \mathbb{A}$ such that $\mathsf{Ab}_{\Omega'}(M) \subset \mathsf{Ab}_{\Omega'}(M')$. By Lemma 4.1, $\mathsf{Ab}_\Omega(M) \subset \mathsf{Ab}_\Omega(M')$ as well. So we have shown that for all $M, M' \in \mathcal{M}$, $\mathsf{Ab}_\Omega(M) \subset \mathsf{Ab}_\Omega(M')$ iff $\mathsf{Ab}_\Lambda(M) \subset \mathsf{Ab}_\Lambda(M')$. By Lemma 4.1, $\Vdash_{\Omega,\mathsf{m}} = \Vdash_{\Lambda,\mathsf{m}}$.

($\mathsf{x} = \mathsf{r}$) Let $\mathbb{A}$ be an arbitrary set of sets Ω' such that $\Vdash_{\Omega,\mathsf{r}} = \Vdash_{\Omega',\mathsf{r}}$. Let again $\Lambda = \bigcup \mathbb{A}$. By Corollary 4.4, for all $\Omega' \in \mathbb{A}$, $\Vdash_{\Omega,\mathsf{m}} = \Vdash_{\Omega',\mathsf{m}}$. By Theorem 4.7 for $\mathsf{x} = \mathsf{m}$, we can infer that (i) $\Vdash_{\Omega,\mathsf{m}} = \Vdash_{\Lambda,\mathsf{m}}$ and (ii) $\Vdash_{\Omega,\mathsf{m}} = \Vdash_{\Lambda \cup \Omega,\mathsf{m}}$. Hence, (iii) $\Vdash_{\Lambda,\mathsf{m}} = \Vdash_{\Lambda \cup \Omega,\mathsf{m}}$. By Theorem 4.6 and (ii), $\Vdash_{\Omega,\mathsf{r}} = \Vdash_{\Lambda \cup \Omega,\mathsf{r}}$. By Theorem 4.6 and (iii), $\Vdash_{\Lambda,\mathsf{r}} = \Vdash_{\Lambda \cup \Omega,\mathsf{r}}$. Hence, $\Vdash_{\Omega,\mathsf{r}} = \Vdash_{\Lambda,\mathsf{r}}$. ∎

THEOREM 4.8
Where $\Omega' = \{A \in \Omega \mid \mathcal{M}(\{A\}) \neq \emptyset\}$: $\Vdash_{\Omega,\mathsf{x}} = \Vdash_{\Omega',\mathsf{x}}$.

PROOF. ($\mathsf{x} = \mathsf{m}$) Note that for all $M \in \mathcal{M}$, $\mathsf{Ab}_\Omega(M) = \mathsf{Ab}_{\Omega'}(M)$. The rest is immediate in view of Corollary 4.2. ($\mathsf{x} = \mathsf{r}$) Let $\Gamma \subseteq \Phi$ be arbitrary. In view of the case $\mathsf{x} = \mathsf{m}$ and Theorem 2.13.2, $\mathsf{U}_\Omega(\Gamma) = \mathsf{U}_{\Omega'}(\Gamma)$. Moreover, for all $A \in \Omega - \Omega'$, $\neg A \in \mathrm{Cn}(\emptyset)$. The rest follows by Theorem 2.14. ∎

4.2 Abnormalities and truth-functional connectives

In this section, we discuss some basic observations that concern cases where Ω is a superset of Ω', obtained by adding conjunctions, disjunctions or negations of abnormalities. This allows us to clarify how certain simple variations on Ω result in a stronger, identical or weaker consequence relation. Our observations also illustrate a point made at the start of this section, i.e. that with respect to variations on the set of abnormalities, the two strategies behave differently. At the end of this section, we generalize our observations to extensions by means of arbitrary truth-functional operations. All theorems in this section are corollaries of Theorem 4.23 and Fact 4.22, both of which can be found in Section 4.3.

NOTATION 4.9
Let $\Delta^\wedge$ denote the closure of Δ under conjunction, i.e. the smallest set $\Theta \supseteq \Delta$ which has the property: if $A, B \in \Theta$, then $A \wedge B \in \Theta$. Similarly, $\Delta^\vee$ denotes the closure of Δ under $\vee$.

4.2.1 Conjunction

Suppose that $A, B \in \Omega$. If we use the minimal abnormality strategy, this means that we prefer models that falsify A over those that verify A; similar for B. If we use the reliability strategy, it means that if A is not a disjunct of some minimal disjunction of abnormalities that follows from Γ, we treat it as false; similar for B. For both strategies, it seems therefore natural to also consider $A \wedge B$ as an abnormality. But what happens if we add $A \wedge B$ to Ω, resulting in a new set Ω' of abnormalities? We consider this question for each of the strategies separately.

Minimal abnormality: For minimal abnormality, adding conjunctions of abnormalities results in exactly the same consequence relation. Formally:

THEOREM 4.10
If $\Omega \subseteq \Omega' \subseteq \Omega^\wedge$, then $\Vdash_{\Omega,\mathsf{m}} = \Vdash_{\Omega',\mathsf{m}}$.

A direct proof for Theorem 4.10 is given in [29]. Here, we will see that it follows from more generic results concerning truth-functional connectives and their interplay with abnormalities.

Theorem 4.10 implies that, where each of $A, B, A \wedge B$ are in the set of abnormalities, it is safe to ignore $A \wedge B$, when trying to determine the set of minimally abnormal models of Γ. Conversely, it shows that adding conjunctions of abnormalities will not make any difference for the consequence relation of an adaptive logic that uses minimal abnormality.

Reliability: For the reliability strategy, the picture is rather different. Let us start with the positive result:

THEOREM 4.11
If $\Omega \subseteq \Omega' \subseteq \Omega^{\wedge}$, then $\Vdash_{\Omega,\mathsf{r}} \subseteq \Vdash_{\Omega',\mathsf{r}}$.

However, the antecedent of Theorem 4.11 does not imply that $\Vdash_{\Omega,\mathsf{r}} = \Vdash_{\Omega',\mathsf{r}}$. We illustrate this by means of a simple example.

EXAMPLE 4.12
We use again the adaptive logics based on **CLuN** as an illustration. Let $\Gamma_4 = \{p, q, \sim p \vee \sim q, \sim p \vee r, \sim q \vee r\}$. Since both $p \wedge \sim p$ and $q \wedge \sim q$ are unreliable abnormalities in view of Γ_4, $\Gamma_4 \nVdash_{\Omega_c,\mathsf{r}} r$.

Let $\Omega_c^{\wedge}$ denote the closure of Ω_c under conjunction. Then $(p \wedge \sim p) \wedge (q \wedge \sim q) \in \Omega_c^{\wedge}$. Note that this abnormality is false in all $M \in \mathcal{M}_{\Omega_c,\mathsf{m}}(\Gamma_4)$, and hence by Theorem 4.10, it is also false in all $M \in \mathcal{M}_{\Omega_c^{\wedge},\mathsf{m}}(\Gamma_4)$. By Theorem 2.13, $(p \wedge \sim p) \wedge (q \wedge \sim q)$ is a reliable abnormality w.r.t. $\langle \mathbf{CLuN}, \Omega_c^{\wedge}, \Gamma_4 \rangle$. It follows that all $M \in \mathcal{M}_{\Omega_c^{\wedge},\mathsf{r}}(\Gamma_4)$ falsify this abnormality, whence they verify r.

As the example illustrates, when we add conjunctions of abnormalities to Ω, the resulting logic is stronger than the one we started with. In particular, the more conjunctions of abnormalities we add, the closer—so it seems—we get to $\Vdash_{\Omega,\mathsf{m}}$. Hence we may ask whether in general, $\Vdash_{\Omega,\mathsf{m}} = \Vdash_{\Omega^{\wedge},\mathsf{r}}$. We refer the interested reader to [29] for an in-depth discussion of this matter.[12]

4.2.2 Disjunction

Adding disjunctions of abnormalities leaves the consequence relation unaltered, for both strategies:

THEOREM 4.13
Where $\mathsf{x} \in \{\mathsf{r}, \mathsf{m}\}$: if $\Omega \subseteq \Omega' \subseteq \Omega^{\vee}$, then $\Vdash_{\Omega,\mathsf{x}} = \Vdash_{\Omega',\mathsf{x}}$.

Hence, at the level of the consequence relation, not much is to be gained from closing Ω under disjunction, or from adding certain disjunctions in a more piecemeal fashion. Moreover, Theorem 4.13 implies that if certain abnormalities A in Ω are equivalent to disjunctions of other abnormalities $B_1, \ldots, B_n$, then we may safely ignore those A when checking what follows from a given premise set Γ.

A case in point are the logics $\mathbf{LI^r}$ and $\mathbf{LI^m}$ from [8, 11]. These are defined on the basis of the fragment of first order predicate logic with only unary predicates $P, Q, R, \ldots$ and without identity. Where x ranges over variables and A over formulas, they use the following set of abnormalities:

$$\Omega_{\mathbf{LI}} = \{\neg \forall x(A(x)) \mid A(.) \text{ contains no quantifiers, free variables, or constants}\}$$

[12] As shown there, for a specific class of premise sets Γ, $\mathrm{Cn}_{\Omega,\mathsf{m}}(\Gamma) = \mathrm{Cn}_{\Omega^{\wedge},\mathsf{r}}(\Gamma)$. This class can be characterized by a necessary and sufficient condition, which relates to the proof theory of ALs. However, unless Ω is finite, one may often construct premise sets that do not obey this condition.

The intuition behind these logics is that, when we try to derive a generalization from certain data, we assume every generalization to be true unless the premises prevent this.

Note that every member of $\Omega_{\mathbf{LI}}$ is equivalent to a formula in conjunctive normal form: $\neg\forall x(\bigwedge_{1\leq i\leq n}(P^i_1 x\vee\ldots\vee P^i_{m_i}x))$. This formula is in turn equivalent to $\neg\forall x(P^1_1 x\vee\ldots\vee P^1_{m_1}x)\vee\ldots\vee\neg\forall x(P^n_1 x\vee\ldots\vee P^n_{m_n}x)$, which is itself a disjunction of **LI**-abnormalities. So by Theorem 4.13, we may just restrict our attention to the following, much smaller set of abnormalities:

$$\Omega'_{\mathbf{LI}}=\{\neg\forall x(P_1 x\vee\ldots\vee P_n x)\mid n\in\mathbb{N}\}$$

4.2.3 Negation

The third and last concrete connective which we consider is classical negation. Here again, the behavior of the two strategies is essentially the same:

THEOREM 4.14
Where $\mathsf{x}\in\{\mathsf{r},\mathsf{m}\}$: if $\Omega\subseteq\Omega'\subseteq\Omega\cup\neg\Omega$, then $\Vdash_{\Omega',\mathsf{x}}\subseteq\Vdash_{\Omega,\mathsf{x}}$.

This result seems fairly intuitive: suppose that Ω_2 is obtained from Ω_1, by removing certain A from Ω_1 for which also $\neg A\in\Omega_1$. In that case, $\Vdash_{\Omega_2,\mathsf{x}}$ will be a superrelation of $\Vdash_{\Omega_1,\mathsf{x}}$. So if we make our notion of abnormality more coherent in this specific sense, we end up with a logic that is at least as strong as the one we had before. The following example shows that the set inclusion in Theorem 4.14 is sometimes proper:

EXAMPLE 4.15
Let $\Omega=\{p\}$ and $\Omega'=\{p,\neg p\}$. Clearly, $\emptyset\Vdash_{\Omega,\mathsf{x}}\neg p$, yet $\emptyset\nVdash_{\Omega',\mathsf{x}}\neg p$ for both $\mathsf{x}=\mathsf{r}$ and $\mathsf{x}=\mathsf{m}$.

In the limiting case where $\Omega'=\Omega\cup\neg\Omega$, the resulting adaptive logic is equivalent to **L** (for both strategies):

THEOREM 4.16
Where $\mathsf{x}\in\{\mathsf{r},\mathsf{m}\}$: $\Vdash_{\Omega\cup\neg\Omega,\mathsf{x}}=\Vdash$.

4.2.4 Arbitrary truth-functional operations

We now investigate truth-functional connectives more generally, for each of the two strategies.

NOTATION 4.17
Where $\mathcal{F}$ is a set of connectives, we use $\Delta^{\mathcal{F}}$ to denote the closure of Δ under all members of $\mathcal{F}$.

Minimal abnormality: In the preceding, we saw that adding conjunctions or disjunctions of abnormalities makes no difference for minimal abnormality, whereas adding negations of abnormalities results in a weaker logic. This raises the question: is there any way we may obtain a *stronger* consequence relation $\Vdash_{\Omega',\mathsf{m}}$ by adding certain truth-functions of abnormalities in Ω? The answer is simply negative. That is, let $\mathcal{T}$ be the set of all truth-functional connectives. We have:

THEOREM 4.18
If $\Omega\subseteq\Omega'\subseteq\Omega^{\mathcal{T}}$, then $\Vdash_{\Omega',\mathsf{m}}\subseteq\Vdash_{\Omega,\mathsf{m}}$.

As our example in Section 4.2.3 shows, the antecedent of Theorem 4.18 does not warrant that $\Vdash_{\Omega',\mathsf{m}}=\Vdash_{\Omega,\mathsf{m}}$. To obtain this property, we need to consider a restricted type of connectives. That is, let a truth-functional connective of arity k be *positive* iff it can be equivalently expressed by

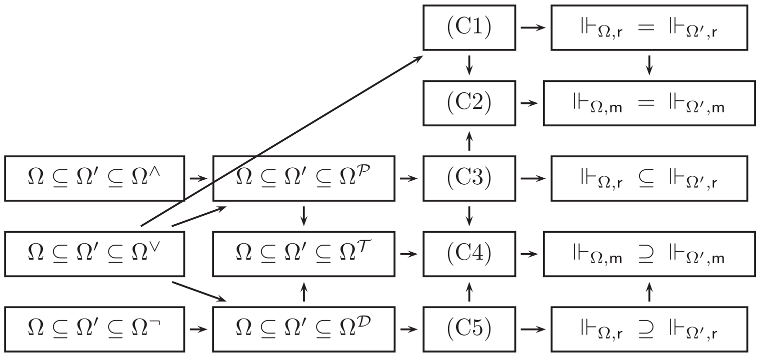

FIG. 1. Overview of the conditions from Section 4.

means of conjunction, disjunction, and $\bot$ alone. Let $\mathcal{P} \subset \mathcal{T}$ be the set of all positive truth-functional connectives. We have:

THEOREM 4.19
If $\Omega \subseteq \Omega' \subseteq \Omega^{\mathcal{P}}$, then $\Vdash_{\Omega,\mathsf{m}} = \Vdash_{\Omega',\mathsf{m}}$.

Reliability: For reliability, we can obviously not prove a counterpart of Theorem 4.18, in view of the examples from Section 4.2.1. Nevertheless, we can delineate a specific class of truth-functional connectives for which the reliability-counterpart of Theorem 4.18 holds. That is, let a truth-functional connective $\bullet$ of arity k be *disjunctive* iff for all $A_1,\ldots,A_k$, $\bullet(A_1,\ldots,A_k)$ is equivalent to $B_1 \vee \ldots \vee B_k$, where each $B_i \in \{A_i, \neg A_i, \bot\}$. Examples of disjunctive connectives are the zero-ary falsum and verum constant, negation, implication, disjunction and the *nand* connective (not … or not …). Let $\mathcal{D} \subset \mathcal{T}$ be the set of all disjunctive connectives. We have:

THEOREM 4.20
If $\Omega \subseteq \Omega' \subseteq \Omega^{\mathcal{D}}$, then $\Vdash_{\Omega',\mathsf{r}} \subseteq \Vdash_{\Omega,\mathsf{r}}$.

The antecedent of Theorem 4.20 does not imply that $\Vdash_{\Omega',\mathsf{r}} = \Vdash_{\Omega,\mathsf{r}}$; this follows from our observations concerning the addition of negations of abnormalities in Section 4.2.3. It also seems that Theorem 4.20 cannot easily be extended to (certain classes of) truth-functional connectives which are not disjunctive. For instance, it does not hold for classical equivalence.[13]

Finally, one may ask whether certain extensions in terms of truth-functional connectives will always result in a consequence relation that is at least as strong as the original one, if we use the reliability strategy. As we saw, this holds for conjunction and disjunction. The following theorem generalizes this property to all positive connectives:

THEOREM 4.21
If $\Omega \subseteq \Omega' \subseteq \Omega^{\mathcal{P}}$, then $\Vdash_{\Omega,\mathsf{r}} \subseteq \Vdash_{\Omega',\mathsf{r}}$.

4.3 Most generic conditions

In this section, we briefly outline our deepest results, which have the preceding theorems from Section 4 as corollaries. In contrast to the results from the preceding sections, the conditions used

[13] To see why, let $\mathbf{L} = \mathbf{CL}$, $\Omega = \{p, q\}$ and $\Omega' = \{p, q, p \equiv q\}$. Then $\emptyset \Vdash_{\Omega,\mathsf{r}} \neg p \wedge \neg q$, whereas $\emptyset \nVdash_{\Omega',\mathsf{r}} \neg p \wedge \neg q$. However, where $\Gamma = \{p \vee q\}$, we have $\Gamma \nVdash_{\Omega,\mathsf{r}} \neg p \vee \neg q$, wheras $\Gamma \Vdash_{\Omega',\mathsf{r}} \neg p \vee \neg q$. So in this case, $\Vdash_{\Omega,\mathsf{r}}$ and $\Vdash_{\Omega',\mathsf{r}}$ are incomparable.

here do not pose any restrictions on Ω' in terms of truth-functional or other connectives; they merely concern a relation between Ω and Ω' in terms of $\Vdash$ and the set of models $\mathcal{M}$. An overview of these conditions and their relation to those from preceding sections is given in Figure 1.

Let in the remainder $\Gamma \Vdash^{\exists}_{\mathbf{L}} A$ ($A \in \mathrm{Cn}^{\exists}(\Gamma)$) iff there is a $B \in \Gamma$ such that $\{B\} \Vdash A$. We will consider the following conditions on Ω, Ω':

(C1) For all $M \in \mathcal{M}$, $\mathrm{Cn}^{\exists}(\mathsf{Ab}_{\Omega}(M)) = \mathrm{Cn}^{\exists}(\mathsf{Ab}_{\Omega'}(M))$.
(C2) For all $M \in \mathcal{M}$, $\mathrm{Cn}(\mathsf{Ab}_{\Omega}(M)) = \mathrm{Cn}(\mathsf{Ab}_{\Omega'}(M))$
(C3) $\Omega \subseteq \Omega'$ and for all $M \in \mathcal{M}$, $\mathsf{Ab}_{\Omega'}(M) \subseteq \mathrm{Cn}(\mathsf{Ab}_{\Omega}(M))$
(C4) $\Omega \subseteq \Omega'$ and for all $M \in \mathcal{M}$, $\mathsf{Ab}_{\Omega'}(M) \subseteq \mathrm{Cn}(\mathsf{Ab}_{\Omega}(M) \cup \mathsf{Ab}_{\neg\Omega}(M))$
(C5) $\Omega \subseteq \Omega'$ and for all $M \in \mathcal{M}$, $\mathsf{Ab}_{\Omega'}(M) \subseteq \mathrm{Cn}^{\exists}(\mathsf{Ab}_{\Omega}(M) \cup \mathsf{Ab}_{\neg\Omega}(M))$

Let us briefly comment on each of these conditions. Note first that, in view of Remark 2.1 from Section 2, each of these conditions can also be stated in terms of maximal **L**-consistent sets $\Theta \subseteq \Phi$. We refer to Appendix B where this alternative formulation is spelled out.

(C1) should not be confused with the (stronger) condition that the members of Ω and Ω' are pairwise equivalent (i.e. for all $A \in \Omega$, there is a $B \in \Omega'$ such that $\Vdash A \equiv B$ and vice versa). Note for instance that (C1) holds when $\Omega' = \Omega^{\vee}$. The point is that, although $\Omega^{\vee}$ contains certain formulas that are weaker than any member of Ω, those additional abnormalities are redundant with respect to the formulas that occur in Ω.

Since **L** is a Tarski-logic, (C3) implies (C2). Whenever $\Omega \subseteq \Omega'$, it can easily be verified that (C3) and (C2) are equivalent. Both conditions state that, for all models M, the logical content of $\mathsf{Ab}_{\Omega}(M)$ equals that of $\mathsf{Ab}_{\Omega'}(M)$. This holds trivially in case $\Omega' = \Omega^{\wedge}$ or $\Omega' = \Omega^{\vee}$. More generally, it holds whenever $\Omega \subseteq \Omega' \subseteq \Omega^{\mathcal{P}}$.

This brings us to condition (C4), which can perhaps best be understood as a generalization of $\Omega \subseteq \Omega' \subseteq \Omega^{\mathcal{T}}$. The idea is that for each model M, $\mathsf{Ab}_{\Omega'}(M)$ is uniquely determined by the set of all $A \in \Omega \cup \neg\Omega$ that are valid in M. Note that (C4) does not imply (C2) even if $\Omega \subseteq \Omega'$.[14] On the other hand, if $\Omega \subseteq \Omega'$, then (C2) implies (C4).

(C5), finally, is still more stringent in that it requires that each member B of $\mathsf{Ab}_{\Omega'}(M)$ follows from a single $C \in \Omega \cup \neg\Omega$ that is valid in M. This holds e.g. when $\Omega \subseteq \Omega' \subseteq \Omega^{\mathcal{D}}$.

Fact 4.22 below summarizes the relation between the above conditions and those from Section 4.2; in view of our remarks above, its verification can be safely left to the reader.

Fact 4.22
Each of the following holds:

1. If $\Omega \subseteq \Omega' \subseteq \Omega^{\mathcal{T}}$, then (C4).
2. If $\Omega \subseteq \Omega' \subseteq \Omega^{\mathcal{P}}$, then (C3).
3. If $\Omega \subseteq \Omega' \subseteq \Omega^{\mathcal{D}}$, then (C5).
4. If $\Omega \subseteq \Omega' \subseteq \Omega^{\vee}$, then (C1).
5. If (C1), then (C2).
6. If (C3), then (C2).
7. If (C3), then (C4).
8. If (C5), then (C4).

[14] See e.g. Example 4.15, where $\mathbf{L} = \mathbf{CL}$, $\Omega = \{p\}$ and $\Omega' = \{p, \neg p\}$. Consider a model M such that $M \models \neg p$. Then $\mathrm{Cn}(\mathsf{Ab}_{\Omega}(M)) = \mathrm{Cn}(\emptyset) \subset \mathrm{Cn}(\mathsf{Ab}_{\Omega'}(M)) = \mathrm{Cn}(\{\neg p\})$.

9. If $\Omega \subseteq \Omega'$ and (C2), then (C3).
10. If $\Omega \subseteq \Omega'$ and (C2), then (C4).

THEOREM 4.23
Each of the following holds:

1. If (C2), then $\Vdash_{\Omega,\mathsf{m}} = \Vdash_{\Omega',\mathsf{m}}$.
2. If (C1), then $\Vdash_{\Omega,\mathsf{m}} = \Vdash_{\Omega',\mathsf{m}}$
3. If (C3), then $\Vdash_{\Omega,\mathsf{m}} = \Vdash_{\Omega',\mathsf{m}}$.
4. If (C4), then $\Vdash_{\Omega',\mathsf{m}} \subseteq \Vdash_{\Omega,\mathsf{m}}$.
5. If (C1), then $\Vdash_{\Omega,\mathsf{r}} = \Vdash_{\Omega',\mathsf{r}}$.
6. If (C3), then $\Vdash_{\Omega,\mathsf{r}} \subseteq \Vdash_{\Omega',\mathsf{r}}$.
7. If (C5), then $\Vdash_{\Omega',\mathsf{r}} \subseteq \Vdash_{\Omega,\mathsf{r}}$.
8. If (C5), then $\Vdash_{\Omega',\mathsf{m}} \subseteq \Vdash_{\Omega,\mathsf{m}}$.

PROOF. *Ad 1.* Suppose (C2) holds. Let $M, M' \in \mathcal{M}$ be arbitrary. Suppose $\mathsf{Ab}_\Omega(M) \subset \mathsf{Ab}_\Omega(M')$. Note first that, by (C2) and the monotonicity of **L**, $\mathsf{Ab}_{\Omega'}(M) = \mathrm{Cn}(\mathsf{Ab}_\Omega(M)) \cap \Omega' \subseteq \mathrm{Cn}(\mathsf{Ab}_\Omega(M')) \cap \Omega' = \mathsf{Ab}_{\Omega'}(M)$. Hence, (a) $\mathsf{Ab}_{\Omega'}(M) \subseteq \mathsf{Ab}_{\Omega'}(M')$.

Assume now that $\mathsf{Ab}_{\Omega'}(M) = \mathsf{Ab}_{\Omega'}(M')$. Then by (C2), $\mathsf{Ab}_\Omega(M) = \mathrm{Cn}(\mathsf{Ab}_{\Omega'}(M)) \cap \Omega = \mathrm{Cn}(\mathsf{Ab}_{\Omega'}(M')) \cap \Omega = \mathsf{Ab}_\Omega(M')$, and hence $\mathsf{Ab}_\Omega(M) = \mathsf{Ab}_\Omega(M')$. But this contradicts the supposition. Hence (b) $\mathsf{Ab}_{\Omega'}(M) \neq \mathsf{Ab}_{\Omega'}(M')$. By (a) and (b), $\mathsf{Ab}_{\Omega'}(M) \subset \mathsf{Ab}_{\Omega'}(M')$. Since (C2) is symmetric w.r.t. Ω and Ω', we have shown that for all $M, M' \in \mathcal{M}$, $\mathsf{Ab}_\Omega(M) \subset \mathsf{Ab}_\Omega(M')$ iff $\mathsf{Ab}_{\Omega'}(M) \subset \mathsf{Ab}_{\Omega'}(M')$. The rest follows by Lemma 4.1.

Ad 2 and 3. From item 1, relying on Facts 4.22.5-6.

Ad 4. Let $M, M' \in \mathcal{M}$ be such that ($\star$) $\mathsf{Ab}_{\Omega'}(M') \subset \mathsf{Ab}_{\Omega'}(M)$. By Lemma 4.1, it suffices to show that $\mathsf{Ab}_\Omega(M') \subset \mathsf{Ab}_\Omega(M)$.

Consider an arbitrary $A \in \mathsf{Ab}_\Omega(M')$. By (C4), $A \in \mathsf{Ab}_{\Omega'}(M')$, and hence by ($\star$), $A \in \mathsf{Ab}_{\Omega'}(M)$. Since $A \in \Omega$, also $A \in \mathsf{Ab}_\Omega(M)$. So (a) $\mathsf{Ab}_\Omega(M') \subseteq \mathsf{Ab}_\Omega(M)$.

Let now $B \in \mathsf{Ab}_{\Omega'}(M) - \mathsf{Ab}_{\Omega'}(M')$. By (C4), $B \in Cn(\mathsf{Ab}_\Omega(M) \cup \mathsf{Ab}_{\neg\Omega}(M))$. If $\mathsf{Ab}_\Omega(M') = \mathsf{Ab}_\Omega(M)$, then $\mathsf{Ab}_{\neg\Omega}(M) = \mathsf{Ab}_{\neg\Omega}(M')$, whence $B \in Cn(\mathsf{Ab}_\Omega(M') \cup \mathsf{Ab}_{\neg\Omega}(M'))$. It follows that $M' \models B$, and hence $B \in \mathsf{Ab}_{\Omega'}(M')$ — a contradiction. Hence (b) $\mathsf{Ab}_\Omega(M') \neq \mathsf{Ab}_\Omega(M)$. By (a) and (b), $\mathsf{Ab}_\Omega(M') \subset \mathsf{Ab}_\Omega(M)$.

Ad 5. Suppose (C1) holds and let $\Gamma \subseteq \Phi$ be arbitrary. Assume $M \in \mathcal{M}_{\Omega,\mathsf{r}}(\Gamma) - \mathcal{M}_{\Omega',\mathsf{r}}(\Gamma)$. Let $A \in \mathsf{Ab}_{\Omega'}(M) - \mathsf{U}_{\Omega'}(\Gamma)$. By (C1), there is a $B \in \mathsf{Ab}_\Omega(M)$ such that $\{B\} \Vdash A$. Since $M \in \mathcal{M}_{\Omega,\mathsf{r}}(\Gamma)$ and by Theorem 2.13, there is an $M' \in \mathcal{M}_{\Omega,\mathsf{m}}(\Gamma)$ such that $B \in \mathsf{Ab}_\Omega(M')$, and hence also $M' \models A$. By (C1), item 2 and Theorem 4.5, $M' \in \mathcal{M}_{\Omega',\mathsf{m}}(\Gamma)$. But then, since $A \in \Omega'$, $A \in \mathsf{U}_{\Omega'}(\Gamma)$ — a contradiction.

So we have shown that $\mathcal{M}_{\Omega,\mathsf{r}}(\Gamma) \subseteq \mathcal{M}_{\Omega',\mathsf{r}}(\Gamma)$ for all $\Gamma \subseteq \Phi$. Since (C1) is symmetric w.r.t. Ω and Ω', also $\mathcal{M}_{\Omega,\mathsf{r}}(\Gamma) \supseteq \mathcal{M}_{\Omega',\mathsf{r}}(\Gamma)$ for all $\Gamma \subseteq \Phi$. By Theorem 4.5, $\Vdash_{\Omega,\mathsf{r}} = \Vdash_{\Omega',\mathsf{r}}$.

Ad 6. Let $\Gamma \subseteq \Phi$. Suppose that $M \in \mathcal{M}(\Gamma) - \mathcal{M}_{\Omega,\mathsf{r}}(\Gamma)$. Let $A \in \mathsf{Ab}_\Omega(M) - \mathsf{U}_\Omega(\Gamma)$. By (C3), $A \in \Omega'$. By Theorem 2.13, for no $M' \in \mathcal{M}_{\Omega,\mathsf{m}}(\Gamma)$, $M' \models A$. By item 3 and (C3), there is no $M' \in \mathcal{M}_{\Omega',\mathsf{m}}(\Gamma)$ such that $M' \models A$. By Theorem 2.13, $M \notin \mathcal{M}_{\Omega',\mathsf{r}}(\Gamma)$. Hence, $\mathcal{M}_{\Omega',\mathsf{r}}(\Gamma) \subseteq \mathcal{M}_{\Omega,\mathsf{r}}(\Gamma)$ for all $\Gamma \subseteq \Phi$. By Theorem 4.5, $\Vdash_{\Omega,\mathsf{r}} \subseteq \Vdash_{\Omega',\mathsf{r}}$.

Ad 7. Let $\Gamma \subseteq \Phi$ be arbitrary. Note that

(†) if $B \in \Omega$ and $M \models B$ for every $M \in \mathcal{M}_{\Omega,\mathsf{m}}(\Gamma)$, then $\Gamma \Vdash B$.

That is, suppose the antecedent of (†) holds. Let $M' \in \mathcal{M}(\Gamma)$ be arbitrary. By Theorem 2.7, there is an $M'' \in \mathcal{M}_{\Omega,\mathsf{m}}(\Gamma)$ with $\mathsf{Ab}_\Omega(M'') \subseteq \mathsf{Ab}_\Omega(M')$. Since $M'' \models B$, also $M' \models B$.

Suppose now that (C5) holds and that $M \in \mathcal{M}(\Gamma) - \mathcal{M}_{\Omega',\mathsf{r}}(\Gamma)$. Let $A \in \mathsf{Ab}_{\Omega'}(M) - \mathsf{U}_{\Omega'}(\Gamma)$. By (C5), there is a $B \in \mathsf{Ab}_{\Omega}(M) \cup \mathsf{Ab}_{\neg\Omega}(M)$ such that $\{B\} \Vdash A$. By Theorem 2.13, A is false in every model $M' \in \mathcal{M}_{\Omega',\mathsf{m}}(\Gamma)$, and hence so is B. By item 4 and Fact 4.22.8, B is also false in every $M' \in \mathcal{M}_{\Omega,\mathsf{m}}(\Gamma)$.

Case 1: $B \in \mathsf{Ab}_{\neg\Omega}(M)$. Hence $B = \neg C$ with $C \in \Omega$. It follows that C is true in every $M \in \mathcal{M}_{\Omega,\mathsf{m}}(\Gamma)$. By (†), $\Gamma \Vdash C$. But this contradicts the fact that $M \models B$ and $M \in \mathcal{M}(\Gamma)$.

Case 2: $B \in \mathsf{Ab}_{\Omega}(M)$. It follows that $B \in \Omega - \mathsf{U}_{\Omega}(\Gamma)$, whence $M \notin \mathcal{M}_{\Omega,\mathsf{r}}(\Gamma)$.

So altogether, we have shown that $\mathcal{M}_{\Omega,\mathsf{r}}(\Gamma) \subseteq \mathcal{M}_{\Omega',\mathsf{r}}(\Gamma)$ for all $\Gamma \subseteq \Phi$. Hence, by Theorem 4.5, $\Vdash_{\Omega,\mathsf{r}} \supseteq \Vdash_{\Omega',\mathsf{r}}$.

Ad 8. Immediate in view of item 7 and Theorem 4.3. ∎

This theorem deserves some further comments. First of all, using the examples from Section 4.2, it can be easily verified that the set inclusion in the consequent of items 4 and 6–8 is sometimes proper. For items 4, 7 and 8, this follows by Example 4.15. For item 6, it follows by Example 4.12. The latter example also illustrates why (C2) does not imply that $\Vdash_{\Omega,\mathsf{r}} = \Vdash_{\Omega',\mathsf{r}}$.

Second, one may wonder whether the conditions (C1)-(C5) are not just sufficient, but also necessary for their respective consequents. We answer this question in the negative in Appendix B.

5 From abnormalities to expectations

In his [20, Chapter 2], David Makinson discusses so-called default assumption consequence relations (henceforth DACRs). This is a restricted version of the expectation-based inference relations studied in [16]. We restrict ourselves to DACRs here, leaving the study of the more general format for a later occasion. However, we generalize the account from [20], replacing classical logic with the compact supraclassical logic **L** that was used in the previous sections. Finally, to avoid confusion with our informal use of the term ‘assumption’ in preceding sections, we shall use the term ‘expectations’ to denote the specific type of default knowledge used in the DACR framework.

DACRs: Every DACR is defined on the basis of **L** and a set of formulas $\Delta \subseteq \Phi$. The members of Δ are called *expectations*. The idea is that these expectations are taken to be true whenever possible, and hence that we can treat them as additional premises. However, if they are incompatible with our premise set Γ, we need to reject some of our expectations.

It is well known from the literature on belief revision and nonmonotonic logic that in such cases, there are often several options—some expectations may be in themselves compatible with the premises, but not jointly. In the DACR framework this problem is tackled as follows: we consider the set $\mathbb{C}^{\mathsf{max}}_{\mathbf{L},\Delta}(\Gamma)$ of all $\subset$-maximal $\Theta \subseteq \Delta$, such that $\mathrm{Cn}_{\mathbf{L}}(\Gamma \cup \Theta) \neq \Phi$.[15] A formula A is a default assumption consequence of Γ modulo the set of expectations Δ, $\Gamma \Vdash_{\mathbf{L},\Delta,\mathsf{d}} A$, iff for every $\Theta \in \mathbb{C}^{\mathsf{max}}_{\mathbf{L},\Delta}(\Gamma)$, $\Gamma \cup \Theta \Vdash_{\mathbf{L}} A$. We write $\mathrm{Cn}_{\mathbf{L},\Delta,\mathsf{d}}(\Gamma)$ to denote the set of all default assumption consequences of Γ.

For several reasons, DACRs take up a notorious place in the field of non-monotonic logic and belief revision. First, they can be seen as a generalization of the so-called *Strong Rescher-Manor consequence relation* from [25]. This relation is restricted to the case where $\Gamma = \emptyset$ and **L** is propositional classical logic. Second, DACRs are a specific, very well-behaved type of Poole default systems, i.e. those for which the set of constraints is empty—see [19] for the details. Third, they can be used to characterize the operation of so-called *full meet revision* $\oplus$, putting $\Delta \oplus A = \mathrm{Cn}_{\mathbf{L},\Delta,\mathsf{d}}(\{A\})$,

[15] In the cited literature, this condition is expressed by means of a bottom constant, but this is obviously equivalent in the presence of a classical negation.

where Δ is the original belief set and A is the incoming information – see [21] for a detailed study of this correspondence.

DACRs and minimal abnormality: Now, suppose we translate every abnormality A into an expectation $\neg A$, and every expectation A into an abnormality $\neg A$. Then minimizing abnormalities—as specified in the AL framework, using the minimal abnormality strategy—corresponds exactly to maximizing the associated expectations—as specified in the DACR framework—and vice versa:[16]

THEOREM 5.1
Each of the following holds:

1. $\{\mathsf{Ab}_{\neg\Omega}(M) \mid M \in \mathcal{M}_{\mathbf{L},\Omega,\mathsf{m}}(\Gamma)\} = \mathbb{C}^{\mathsf{max}}_{\mathbf{L},\neg\Omega}(\Gamma)$
2. $\mathbb{C}^{\mathsf{max}}_{\mathbf{L},\Delta}(\Gamma) = \{\mathsf{Ab}_{\Delta}(M) \mid M \in \mathcal{M}_{\mathbf{L},\neg\Delta,\mathsf{m}}(\Gamma)\}$
3. $\Vdash_{\mathbf{L},\Omega,\mathsf{m}} = \Vdash_{\mathbf{L},\neg\Omega,\mathsf{d}}$
4. $\Vdash_{\mathbf{L},\Delta,\mathsf{d}} = \Vdash_{\mathbf{L},\neg\Delta,\mathsf{m}}$

PROOF. *Ad 1.* '$\subseteq$' Suppose $\Theta = \mathsf{Ab}_{\neg\Omega}(M)$ for an $M \in \mathcal{M}_{\mathbf{L},\Omega,\mathsf{m}}(\Gamma)$. Since M is a model of Γ, $\Gamma \cup \Theta$ is $\mathbf{L}$-satisfiable. Suppose now that $\Theta \notin \mathbb{C}^{\mathsf{max}}_{\mathbf{L},\neg\Omega}(\Gamma)$. Hence there is a Θ' such that $\Theta \subset \Theta' \subseteq \neg\Omega$ such that $\mathrm{Cn}_{\mathbf{L}}(\Gamma \cup \Theta') \neq \Phi$. Let M' be an $\mathbf{L}$-model of $\Gamma \cup \Theta'$. It follows that $\mathsf{Ab}_{\Omega}(M') \subset \mathsf{Ab}_{\Omega}(M)$, which contradicts the fact that $M \in \mathcal{M}_{\mathbf{L},\Omega,\mathsf{m}}(\Gamma)$.

'$\supseteq$' Let $\Theta \in \mathbb{C}^{\mathsf{max}}_{\mathbf{L},\neg\Omega}(\Gamma)$. Hence $\Gamma \cup \Theta$ is $\mathbf{L}$-satisfiable. Let $M \in \mathcal{M}_{\mathbf{L}}(\Gamma \cup \Theta)$. Assume that $M \notin \mathcal{M}_{\mathbf{L},\Omega,\mathsf{m}}(\Gamma)$. Hence there is an $M' \in \mathcal{M}_{\mathbf{L}}(\Gamma)$ such that $\mathsf{Ab}_{\Omega}(M') \subset \mathsf{Ab}_{\Omega}(M)$. Let $\Theta' = \neg(\Omega - \mathsf{Ab}_{\Omega}(M'))$. Note that $\Theta' \supset \Theta$ and that $\Gamma \cup \Theta'$ is $\mathbf{L}$-satisfiable. But then $\Theta \notin \mathbb{C}^{\mathsf{max}}_{\mathbf{L},\neg\Omega}(\Gamma)$—a contradiction.

Ad 2. Analogous to item 1, safely left to the reader.

Ad 3. $\Gamma \Vdash_{\mathbf{L},\neg\Omega,\mathsf{d}} A$ iff [by the definition of $\Vdash_{\mathbf{L},\neg\Omega,\mathsf{d}}$] $A \in \bigcap_{\Theta \in \mathbb{C}^{\mathsf{max}}_{\mathbf{L},\neg\Omega}(\Gamma)} \mathrm{Cn}_{\mathbf{L}}(\Gamma \cup \Theta)$ iff [by the definition of $\Vdash_{\mathbf{L}}$] for every $\Theta \in \mathbb{C}^{\mathsf{max}}_{\mathbf{L},\neg\Omega}(\Gamma)$, for every $M \in \mathcal{M}_{\mathbf{L}}(\Gamma \cup \Theta)$, $M \models A$ iff [by item 1] for every $M \in \mathcal{M}_{\mathbf{L},\Omega,\mathsf{m}}(\Gamma)$, $M \models A$ iff [by Definition 2.4] $\Gamma \Vdash_{\mathbf{L},\Omega,\mathsf{m}} A$.

Ad 4. Similar to item 3, relying on item 2 instead of item 1. ■

Before we discuss some consequences of this correspondence, it should be stressed that there is an important difference between the way ALs and DACRs have been developed and presented in the literature. Whereas ALs are proposed as *formal* logics and hence it is required that Ω is characterized in terms of one or more logical forms, no such restriction is imposed on the Δ of a DACR. Hence, Theorem 5.1 applies only if we are willing to remove this restriction.[17]

This point also relates to a difference in the way both formats are usually presented. On the one hand, in most work on ALs, the focus is on one specific application which requires a certain $\mathbf{L}$ and set of abnormalities. This $\mathbf{L}$ usually has significantly more expressive power than (propositional) classical logic, and in many applications it will also have certain non-standard features (e.g. a paraconsistent negation, a non-normal modal operator, or a very weak conditional). As a result, there is room for questions such as 'which inference *schemas* should we make defeasible in this context', or 'what *types* of formulas would constitute an abnormality for this application'.

On the other hand, the DACR-format is usually only considered for the case where $\mathbf{L}$ is propositional classical logic. Here, Δ is treated as a variable, and no specific set of expectations is thought

[16] Items 3 and 4 of Theorem 5.1 were proven in [30], for the syntactic characterization of ALs. Here, we follow the semantic route. Items 1 and 2 are mainly auxiliary, and will become important once we consider a reliability-variant of DACRs—see below.

[17] Nevertheless, if we stick to this restriction, we can still characterize DACRs in terms of ALs, by using a translation along the lines of [23]. This option would take us too far astray for present concerns.

of as privileged. Applications of this proposal are relatively scarce, and the focus is rather on the metatheory and extensions of this format, e.g. to include priorities or constraints—see [20, Chapter 2] for a survey of this work.

As a further result of this difference, the standard format only arose relatively late (around 2000), as a proposal to unify a wide range of very divergent systems. In contrast, the DACR-format was there much earlier (see in particular [16]), and was presented as a direct link between the logic of belief revision [1] and non-monotonic reasoning.

Putting these differences aside, it should be noted that Theorem 5.1 has several interesting implications. Here, we just mention some of the most salient ones.[18]

First, by Theorem 5.1, and relying on Corollaries 3.6 and 3.7 from Section 3, we have:[19]

Corollary 5.2
$\Vdash_{\mathbf{L},\Delta,\mathsf{d}} = \Vdash_{\mathbf{L}',\Delta,\mathsf{d}}$ iff $\Vdash_{\mathbf{L}} = \Vdash_{\mathbf{L}'}$.

Corollary 5.3
$\Vdash_{\mathbf{L},\Delta,\mathsf{d}} \subset \Vdash_{\mathbf{L}',\Delta,\mathsf{d}}$ iff $\Vdash_{\mathbf{L}} \subset \Vdash_{\mathbf{L}'}$ and $\mathbf{L}'$ is reliability-conservative w.r.t. $\langle \mathbf{L}, {}^{\neg}\Delta, \Phi\rangle$.

Second, we can infer from Corollary 3.12 that whenever we add certain (disjunctions of) expectations as axioms to $\mathbf{L}$, then the resulting DACR will always be at least as strong as the one we started with:

Corollary 5.4
Where $\Theta \subseteq \Delta^{\vee}$: $\Vdash_{\mathbf{L},\Delta,\mathsf{d}} \subseteq \Vdash_{\mathbf{L}*\Theta,\Delta,\mathsf{d}}$.

Let us now consider what happens if we change the set of expectations Δ, a question which runs parallel to our investigations in Section 4. In view of Theorem 5.1, we can easily translate each of the conditions from Section 4.3 (or their syntactic counterparts in Appendix B) to the DACR-framework. It suffices to replace each Ω and Ω' by ${}^{\neg}\Delta$, respectively ${}^{\neg}(\Delta')$. From the conditions in terms of truth-functional connectives from Section 4.2, we obtain the following:[20]

Corollary 5.5
Each of the following holds.

1. Where $\Delta \subseteq \Delta' \subseteq \Delta^{\mathcal{T}}$: $\Vdash_{\mathbf{L},\Delta',\mathsf{d}} \subseteq \Vdash_{\mathbf{L},\Delta,\mathsf{d}}$
2. Where $\Delta \subseteq \Delta' \subseteq \Delta^{\mathcal{P}}$: $\Vdash_{\mathbf{L},\Delta',\mathsf{d}} = \Vdash_{\mathbf{L},\Delta,\mathsf{d}}$

Some readers might think that Corollary 5.5.2 can be further strengthened, so that $\Vdash_{\mathbf{L},\Delta,\mathsf{d}} = \Vdash_{\mathbf{L},\Delta',\mathsf{d}}$ whenever $\Delta \subseteq \Delta' \subseteq \mathrm{Cn}_{\mathbf{L}}(\Delta)$. However, this fails in view of a well-known result from the study of DACRs:

Theorem 5.6 ([20], Th. 2.7)
If $\Delta' = \mathrm{Cn}_{\mathbf{L}}(\Delta)$, then $\mathrm{Cn}_{\mathbf{L},\Delta',\mathsf{d}}(\Gamma) = \mathrm{Cn}_{\mathbf{L}}(\Gamma)$ whenever $\Gamma \cup \Delta$ is $\mathbf{L}$-trivial.

Theorem 5.6 implies that, if we take as our set of expectations the closure of some Δ under $\mathrm{Cn}_{\mathbf{L}}$, then the resulting DACR reduces to $\mathbf{L}$ for all the interesting cases, i.e. whenever there are conflicts

[18] As argued in [30], Theorem 5.1 has several other interesting consequences. For instance, complexity results for ALs—see e.g. [24] for a recent overview—can be translated into complexity results for DACRs, and every DACR can be characterized in terms of an AL, and hence we may use the adaptive proof theory to explicate the internal dynamics of a DACR.

[19] Note that, in the DACR-terminology, $\mathbf{L}'$ is reliability-conservative w.r.t. $\langle \mathbf{L}, {}^{\neg}\Delta, \Phi\rangle$ means that for all $\Gamma \subseteq \Phi$ such that $\mathrm{Cn}_{\mathbf{L}}(\Gamma) \neq \Phi$, $\Delta - \bigcup \mathbb{C}^{\mathsf{max}}_{\mathbf{L}',\Delta}(\Gamma) \subseteq \Delta - \bigcup \mathbb{C}^{\mathsf{max}}_{\mathbf{L},\Delta}(\Gamma)$.

[20] Item 2 of this corollary has been shown in [15], for the more restricted case where $\mathbf{L} = \mathbf{CL}$ and for finite languages.

between the premise set and the set of expectations. We refer to [20, Chapter 2] where this property of DACRs is explained and discussed in detail.

A reliability-variant of DACRs: Naturally, one may ask whether it is possible to define a reliability-variant of the DACR-format, and what it looks like. In fact, such a variant is already implicit in the way we introduced the reliability strategy. Let us now turn this into an explicit definition.

First, we call an expectation $A\in\Delta$ *safe* w.r.t. $\langle \mathbf{L},\Delta,\Gamma\rangle$ iff A is a member of every $\Theta\in\mathbb{C}^{\mathsf{max}}_{\mathbf{L},\Delta}(\Gamma)$. Equivalently, iff $A\in\bigcap\mathbb{C}^{\mathsf{max}}_{\mathbf{L},\Delta}(\Gamma)$. Second, we define a *safe assumption consequence relation* from $\mathbf{L}$ and Δ as follows: $\Gamma\Vdash_{\mathbf{L},\Delta,\mathsf{s}}A$ iff A follows from Γ *together with* all the assumptions that are safe w.r.t. $\langle \mathbf{L},\Delta,\Gamma\rangle$.

In view of Theorems 2.13 and 5.1.1, $A\in\Omega$ is a reliable abnormality w.r.t. $\langle \mathbf{L},\Omega,\Gamma\rangle$ iff the assumption $\neg A$ is a member of *every* $\Theta\in\mathbb{C}^{\mathsf{max}}_{\mathbf{L},\neg\Omega}(\Gamma)$. Conversely, A is a safe assumption w.r.t. $\langle \mathbf{L},\Delta,\Gamma\rangle$ iff $\neg A$ is a reliable abnormality w.r.t. $\langle \mathbf{L},{}^{\neg}\Delta,\Gamma\rangle$. Putting this together with Theorem 2.14, we have:

Corollary 5.7
Each of the following holds:

1. $\Vdash_{\mathbf{L},\Delta,\mathsf{s}}=\Vdash_{\mathbf{L},\neg\Delta,\mathsf{r}}$
2. $\Vdash_{\mathbf{L},\Omega,\mathsf{r}}=\Vdash_{\mathbf{L},\neg\Omega,\mathsf{s}}$

Again, for this variant we obtain various interesting corollaries, on the basis of the core results from this paper and the simple translation from abnormalities to expectations and back. First, Corollaries 5.2–5.4 also apply when we replace the subcript d with s everywhere. Second, we can translate the results from Section 4 to the setting with default expectations. This requires some preparation.

Call a truth-functional connective $\bullet$ *conjunctive* iff for all $A_1,\ldots,A_k$, $\bullet(A_1,\ldots,A_k)$ is equivalent to $B_1\wedge\ldots\wedge B_k$, where each $B_i\in\{A_i,\neg A_i,\top,\bot\}$. Let $\mathcal{C}\subset\mathcal{T}$ be the set of all conjunctive connectives. We have:

Theorem 5.8
Each of the following holds:

1. Where $\Delta\subseteq\Delta'\subseteq\Delta^{\mathcal{C}}$, $\Vdash_{\mathbf{L},\Delta',\mathsf{s}}\subseteq\Vdash_{\mathbf{L},\Delta,\mathsf{s}}$.
2. Where $\Delta\subseteq\Delta'\subseteq\Delta^{\mathcal{P}}$, $\Vdash_{\mathbf{L},\Delta,\mathsf{s}}\subseteq\Vdash_{\mathbf{L},\Delta',\mathsf{s}}$.

6 Conclusion

In this article, we have investigated the standard format of ALs as a parametric framework for nonmonotonic logics. In particular, we considered pairs of ALs, asking under which conditions one of them is stronger than the other and when they are equivalent. Our main results can be summarized as follows:

(i) If both ALs use the same set of abnormalities, then (a) they are equivalent iff their underlying monotonic cores are equivalent, and (b) one is stronger than the other iff the monotonic core of the former is stronger than *and* reliability-conservative (see Definition 3.2) w.r.t. the monotonic core of the latter.

(ii) If both ALs use the same underlying logic, then there are various generic conditions on their sets of abnormalities which warrant that they are equivalent, or that one is at least as strong as the other. These conditions are different for the two strategies.

The properties in (i) and (ii) were shown for all ALs in standard format. Moreover, they were shown to be easily translatable to the framework of DACRs, letting expectations play the role of negated abnormalities and vice versa.

Future work in this area may take on several forms. First, there is the obvious question whether one may spell out conditions that subsume those mentioned in (ii) and are not just sufficient but also necessary. Second, one may consider more complex comparisons of two ALs, where they use both a different underlying monotonic core *and* a different set of abnormalities. Third, one may try to generalize these results to more generic frameworks which have the standard format as a special case; examples are the format from [24] which does not assume supraclassicality of **L**, the format of lexicographic ALs from [31] in which abnormalities can have various priority degrees, and the format of [28, Chapter 5] which generalizes the notion of a strategy using so-called threshold functions. Our current results will be useful for all three types of investigation.

A Appendix to Section 3

In this appendix we give a concrete example of two logics **L** and **L**′ (based on the same underlying language) and a set of abnormalities Ω, where $\Vdash_{\mathbf{L}} \subseteq \Vdash_{\mathbf{L}'}$, but **L**′ is not reliability-conservative w.r.t. **L** and Ω. We show that as a result, also $\Vdash_{\mathbf{L},\Omega,\mathsf{x}} \not\subseteq \Vdash_{\mathbf{L}',\Omega,\mathsf{x}}$, for none of the two strategies.

For **L**, we use again the logic **CLuN** which was introduced before. For **L**′, we use the logic **CLuNs** from [13], which is a monotonic (proper) extension of **CLuN**. Semantically, it can be characterized by the following additional restrictions on the valuation functions $v: \Phi_{\sim} \to \{1,0\}$ of **CLuN**:

(S1) $v(\sim\sim A) = v(A)$
(S2) $v(\sim(A \supset B)) = v(A \wedge \sim B)$
(S3) $v(\sim(A \wedge B)) = v(\sim A \vee \sim B)$
(S4) $v(\sim(A \vee B)) = v(\sim A \wedge \sim B)$
(S5) $v(\sim(A \equiv B)) = v(\sim((A \supset B) \wedge (B \supset A)))$
(S6) $v(\sim\neg A) = v(A)$

This means that in **CLuNs**, we can analyse (paraconsistent) negations of complex formulas (e.g. $\sim(p \vee q) \Vdash_{\mathbf{CLuNs}} \sim p$) and derive negations of complex formulas from simpler formulas (e.g. $p, \sim q \Vdash_{\mathbf{CLuNs}} \sim(p \supset q)$).

Consider now again $\Gamma_3 = \{\sim p, \sim\sim p \vee t, q, \sim q \vee r, \sim t, \sim\sim p \vee s, t \vee s\}$ from Example 2.5. Recall that according to **CLuN**, $q \wedge \sim q$ is not unreliable w.r.t. Γ_3. However, the following minimal disjunction of abnormalities is **CLuNs**-derivable from Γ_3:

$$(q \wedge \sim q) \vee ((\sim p \wedge r) \wedge \sim(\sim p \wedge r)) \vee (t \wedge \sim t)$$

As a result, $q \wedge \sim q$ is an unreliable abnormality w.r.t. $\langle \mathbf{CLuNs}, \Omega_c, \Gamma_3 \rangle$. So **CLuNs** does not conserve reliability w.r.t. $\langle \mathbf{CLuN}, \Omega_c, \Phi_{\sim} \rangle$. Since there are models $M \in \mathcal{M}_{CLuNs,\Omega_c,\mathsf{m}}(\Gamma_3)$ which verify $q \wedge \sim q$ and falsify r, $\Gamma_3 \nVdash_{\mathbf{CLuNs},\Omega_c,\mathsf{m}} r$. By Theorem 2.16, also $\Gamma_3 \nVdash_{\mathbf{CLuNs},\Omega_c,\mathsf{r}} r$.

As a matter of fact, the ALs based on **CLuNs** and Ω_c are not the standard **CLuNs**-based adaptive logics that appear in the literature. That is, it is shown in [12, Chapter 7, Section 3] that whenever Γ is $\sim$-inconsistent but not $\neg$-inconsistent, $\mathrm{Cn}_{\mathbf{CLuNs},\Omega_c,\mathsf{r}}(\Gamma) = \mathrm{Cn}_{\mathbf{CLuNs},\Omega_c,\mathsf{m}}(\Gamma) = \mathrm{Cn}_{\mathbf{CLuNs}}(\Gamma)$. So the ALs defined from **CLuNs** and Ω_c reduce to their monotonic core in all interesting cases. This shortcoming can be solved by using a different, restricted set of abnormalities, i.e. one that contains only inconsistencies w.r.t. propositional variables—see [12, Chapter 7, Section 3] for the details.

B Appendix to Section 4

Alternative formulation of the conditions: Let in the remainder $\mathbb{M}_{\mathbf{L}}$ denote the set of all sets Θ such that (i) $\mathrm{Cn}_{\mathbf{L}}(\Theta)\subset\Phi$ and (ii) there is no Θ' with $\Theta\subset\Theta'\subseteq\Phi$ such that $\mathrm{Cn}_{\mathbf{L}}(\Theta')\subset\Phi$. In view of Remark 2.1, the conditions (C1)-(C5) can be rephrased as follows:

(C1') For all $\Theta\in\mathbb{M}_{\mathbf{L}}$, $\mathrm{Cn}^{\exists}_{\mathbf{L}}(\Theta\cap\Omega)=\mathrm{Cn}^{\exists}_{\mathbf{L}}(\Theta\cap\Omega')$.
(C2') For all $\Theta\in\mathbb{M}_{\mathbf{L}}$, $\mathrm{Cn}_{\mathbf{L}}(\Theta\cap\Omega)=\mathrm{Cn}_{\mathbf{L}}(\Theta\cap\Omega')$
(C3') $\Omega\subseteq\Omega'$ and for all $\Theta\in\mathbb{M}_{\mathbf{L}}$, $\Theta\cap\Omega'\subseteq\mathrm{Cn}_{\mathbf{L}}(\Theta\cap\Omega)$
(C4') $\Omega\subseteq\Omega'$ and for all $\Theta\in\mathbb{M}_{\mathbf{L}}$, $\Theta\cap\Omega'\subseteq\mathrm{Cn}_{\mathbf{L}}(\Theta\cap(\Omega\cup{}^{\neg}\Omega))$
(C5') $\Omega\subseteq\Omega'$ and for all $\Theta\in\mathbb{M}_{\mathbf{L}}$, $\Theta\cap\Omega'\subseteq\mathrm{Cn}^{\exists}_{\mathbf{L}}(\Theta\cap(\Omega\cup{}^{\neg}\Omega))$

Counterexample to the necessity of the conditions: In the remainder, we show that none of the conditions (C1)–(C5) are necessary for the identity of $\Vdash_{\mathbf{L},\Omega,\mathsf{x}}$ and $\Vdash_{\mathbf{L},\Omega',\mathsf{x}}$, where either $\mathsf{x}=\mathsf{r}$ or $\mathsf{x}=\mathsf{m}$. In other words, we will give two (very simple) sets Ω and Ω' which do not satisfy any of these conditions, but for which nevertheless $\Vdash_{\mathbf{L},\Omega,\mathsf{r}}=\Vdash_{\mathbf{L},\Omega',\mathsf{r}}$ and $\Vdash_{\mathbf{L},\Omega,\mathsf{m}}=\Vdash_{\mathbf{L},\Omega',\mathsf{m}}$. Our example uses two very simple sets of abnormalities and is based on propositional classical logic. As in Section 4, we skip **CL** from our usual notations.

Let $\Omega=\{p\}$ and $\Omega'=\{p,p\wedge q,p\wedge\neg q\}$. It can be easily verified that none of the conditions (C1)-(C5) are satisfied for this example. That is, consider an $M\in\mathcal{M}_{\mathbf{CL}}$ such that $M\models p\wedge q$. Then (C2) fails since $\mathrm{Cn}_{\mathbf{L}}(\mathsf{Ab}_{\Omega}(M))=\mathrm{Cn}_{\mathbf{L}}(\{p\})\subset\mathrm{Cn}_{\mathbf{L}}(\mathsf{Ab}_{\Omega'}(M))=\mathrm{Cn}_{\mathbf{L}}(\{p,p\wedge q\})$. By Facts 4.22.5 and 4.22.6, also (C1) and (C3) fail. For similar reasons, (C4) fails, and hence by Fact 4.22.8 also (C5) fails.

We now show that Ω and Ω' yield exactly the same consequence relation, for both strategies.

Note first that (1) for all $M\in\mathcal{M}_{\mathbf{CL}}$, $\mathsf{Ab}_{\Omega}(M)=\emptyset$ or $\mathsf{Ab}_{\Omega'}(M)=\{p\}$. Also, (2) for all $M\in\mathcal{M}_{\mathbf{CL}}$, $\mathsf{Ab}_{\Omega'}(M)=\emptyset$ or $\mathsf{Ab}_{\Omega'}(M)=\{p,p\wedge q\}$ or $\mathsf{Ab}_{\Omega'}(M)=\{p,p\wedge\neg q\}$. From this, we can derive that, for all $M,M'\in\mathcal{M}_{\mathbf{CL}}$: $\mathsf{Ab}_{\Omega}(M)\subset\mathsf{Ab}_{\Omega}(M')$ iff [by (1)] $\mathsf{Ab}_{\Omega}(M)=\emptyset$ and $\mathsf{Ab}_{\Omega}(M')=\{p\}$ iff [by **CL**-properties] $\mathsf{Ab}_{\Omega'}(M)=\emptyset$ and ($\mathsf{Ab}_{\Omega'}(M')=\{p,p\wedge q\}$ or $\mathsf{Ab}_{\Omega'}(M')=\{p,p\wedge\neg q\}$) iff [by (2)] $\mathsf{Ab}_{\Omega'}(M)\subset\mathsf{Ab}_{\Omega'}(M')$. So by Lemma 4.1, we immediately have:

$$\Vdash_{\mathbf{CL},\Omega,\mathsf{m}}=\Vdash_{\mathbf{CL},\Omega',\mathsf{m}} \tag{1}$$

Since $\Omega\subseteq\Omega'$, we can derive by Theorem 4.6 that $\Vdash_{\mathbf{CL},\Omega,\mathsf{r}}\subseteq\Vdash_{\mathbf{CL},\Omega',\mathsf{r}}$.

Assume now that, for some $\Gamma\subseteq\Phi$ and $M\in\mathcal{M}_{\mathbf{CL}}$, $M\in\mathcal{M}_{\mathbf{CL},\Omega,\mathsf{r}}(\Gamma)-\mathcal{M}_{\mathbf{CL},\Omega',\mathsf{r}}(\Gamma)$. Hence by Theorem 2.16, also $M\notin\mathcal{M}_{\mathbf{CL},\Omega',\mathsf{m}}(\Gamma)$. So let $M'\in\mathcal{M}_{\mathbf{CL}}(\Gamma)$ be such that $\mathsf{Ab}_{\Omega'}(M')\subset\mathsf{Ab}_{\Omega'}(M)$. Note that it is not possible that $\mathsf{Ab}_{\Omega'}(M')=\{p,p\wedge q\}$ or that $\mathsf{Ab}_{\Omega'}(M')=\{p,p\wedge\neg q\}$. Thus, $\mathsf{Ab}_{\Omega'}(M')=\mathsf{Ab}_{\Omega}(M')=\emptyset$ and $\mathsf{Ab}_{\Omega}(M)=\{p\}$. It follows that $\mathbb{S}_{\mathbf{CL},\Omega}(\Gamma)=\emptyset$ and hence also $\mathsf{U}_{\mathbf{CL},\Omega}(\Gamma)=\emptyset$. But then, since $\mathsf{Ab}_{\Omega}(M)\neq\emptyset$, $M\notin\mathcal{M}_{\mathbf{CL},\Omega,\mathsf{r}}(\Gamma)$ — again a contradiction.

So we have shown that for all $\Gamma\subseteq\Phi$, $\mathcal{M}_{\mathbf{CL},\Omega,\mathsf{r}}(\Gamma)\subseteq\mathcal{M}_{\mathbf{CL},\Omega',\mathsf{r}}(\Gamma)$, and hence by Theorem 4.5, $\Vdash_{\mathbf{CL},\Omega,\mathsf{r}}\supseteq\Vdash_{\mathbf{CL},\Omega',\mathsf{r}}$.

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