Accurate 2.5-D Boundary Element Method for Conductive Media

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The solution of the time-harmonic Maxwell equations using a boundary element method, for 2-D geometries illuminated by arbitrary 3-D excitations, gives rise to numerical difficulties if highly conductive media are present. In particular, the interaction integrals arising in the method of moments involve kernels that strongly oscillate in space and, at the same time, decay exponentially. We present an accurate method to tackle these issues over a very broad conductivity range (from lossy dielectric to conductor skin-effect regime), for both magnetic and non-magnetic conductors. Important applications are the modal analysis of waveguides with non-perfect conductors, scattering problems and shielding problems with enclosures with arbitrary permeability and conductivity and 3-D noise sources.
1. Introduction

Boundary element methods (BEMs) provide a powerful framework to solve the time-harmonic Maxwell equations numerically. If the problem domain consists of homogeneous material regions, a BEM generally requires fewer unknowns than a volumetric discretization technique. This paper considers two-dimensional geometries with conductive material regions, which can be magnetic, illuminated by arbitrary three-dimensional sources, leading to a so-called 2.5-D boundary element method.

Important applications of this class of problems are propagation in uniform waveguides with non-perfect conductors [Coluccini et al., 2013; Tong et al., 2005; Dobbelaere et al., 2013a], scattering problems [Murphy et al., 1991] and shielding problems [Dobbelaere et al., 2013b].

Interaction integrals appearing in the method of moments (MoM), with the scalar Green’s function and its normal derivatives as kernels, are numerically challenging due to two specific and interplaying aspects. First, the kernels in good conductors are strongly oscillating and exponentially decaying in space, due to the large magnitude and imaginary part of the conductor’s wave number w.r.t. the free space wave number. Second, the kernels are singular, or nearly singular, in those regions of the integration domain where the test and basis functions’ supports overlap or lie close to each other, respectively. This behavior requires special care for an accurate numerical evaluation. Moreover, the combination of the two aspects, i.e. interaction integrals with both (nearly) singular and oscillating as well as exponentially damped integrands, poses
further difficulties. In this paper we present an accurate method to handle both
problems.

A large amount of literature is available concerning the numerical evaluation of
MoM interaction integrals in low-loss dielectric media. Integrals with singular or
nearly singular integrands are usually evaluated with a singularity extraction [Wilton
et al., 1984; Yla-Oijala and Taskinen, 2003; Graglia, 1993] or cancellation technique
[Khayat and Wilton, 2005; Graglia and Lombardi, 2008; Polimeridis and Mosig, 2010].
In 3-D, Chakraborty and Jandhyala [2004] use singularity cancellation with RWG
basis functions [Rao et al., 1982] to evaluate the interaction integrals in conductive
media more accurately. A good overview of the additional problems that arise in
conductive media can be found in Peeters et al. [2012], together with a singularity
cancellation technique for the three-dimensional case.

To the authors’ best knowledge, no accurate method for handling interaction in-
tegrals in conductive media for the 2.5-D case has been presented yet. This work
proposes a new method specifically tailored to the properties of the 2.5-D Green’s
function in conductive media. It is shown, both theoretically and through corrob-
orating examples, that the method accurately evaluates interaction integrals for a
wide range of electrical conductivities (low-loss dielectric to highly conductive) and
frequencies, and allows media with arbitrary permeability. In addition to the earlier
mentioned fields of application, the new method is highly relevant to the accurate
analysis of state-of-the-art multiconductor transmission lines and enclosures.
The structure of this paper is as follows: in Sections 2 and 3 we briefly outline the employed integral equations and the interaction integrals appearing in the MoM. The problems encountered in evaluating the integrals in conductive media are elaborated in Section 4, followed by our new method in Section 5. Finally, the numerical examples in Section 6 testify to the accuracy and applicability of the method, and clearly demonstrate the advantages over existing methods. Conclusions are formulated in Section 7.

2. Geometry and Boundary Integral Equations

Consider a 2-D geometry consisting of isotropic homogeneous material regions $\Omega_i$, with permittivity $\epsilon_i \in \mathbb{C}$, permeability $\mu_i \in \mathbb{C}$ and boundary $C_i$ (Fig. 1). Assume that all sources and fields have a common time and longitudinal dependence $e^{j(\omega t - \beta z)}$ ($\beta \in \mathbb{C}$), which is omitted for notational convenience. A general 3-D excitation can be expanded into sources of this kind via Fourier transformation in the $z$ direction.

The unknowns of the problem are the tangential electric and magnetic boundary fields, given by $\hat{n} \times \mathbf{E} \times \hat{n} = E_t\hat{t} + E_z\hat{z}$ and $\hat{n} \times \mathbf{H} \times \hat{n} = H_t\hat{t} + H_z\hat{z}$, with $E_t$ and $H_t$ the transverse tangential components, and $E_z$ and $H_z$ the longitudinal components. The following representation formulas hold \cite{Olyslager et al., 1993}, with $E_t^{(i)}\hat{t} + E_z^{(i)}\hat{z}$ the incoming tangential electric field generated by sources in $\Omega_i$, $\mathbf{r} = x\hat{x} + y\hat{y}$, and
\[ \gamma_i = \sqrt{\omega^2 \epsilon_i \mu_i - \beta^2}; \]

\[ E_z(r) = E_z^{(i)}(r) + \oint_{C_i} \left[ E_z(r') \frac{\partial G_i(r|r')}{\partial n'} - \left( \frac{j \gamma_i^2}{\omega \epsilon_i} H_i(r') - \frac{\beta}{\omega \epsilon_i} \frac{\partial H_z(r')}{\partial t'} \right) \right] \, dc', \]  

(1)

\[ E_i(r) = E_i^{(i)}(r) + \oint_{C_i} \left[ \frac{j \omega \mu_i}{\gamma_i^2} H_i(r') \frac{\partial^2 G_i(r|r')}{\partial n \partial n'} - j \frac{\beta}{\gamma_i^2} E_z(r') \frac{\partial^2 G_i(r|r')}{\partial t \partial n'} \right. \]

\[ + j \frac{\beta}{\gamma_i^2} \left( \frac{j \gamma_i^2}{\omega \epsilon_i} H_i(r') - \frac{\beta}{\omega \epsilon_i} \frac{\partial H_z(r')}{\partial t'} \right) \left( \frac{\partial G_i(r|r')}{\partial t} \right) \right] \, dc'. \]  

(2)

Similar expressions for the magnetic field components are found with the duality substitutions \( E \rightarrow H, H \rightarrow -E, \epsilon_i \rightarrow \mu_i \) and \( \mu_i \rightarrow \epsilon_i \) in (1) and (2). With the choice \( G_i(r|r') = \frac{j}{4} H_0^{(2)}(\gamma_i |r - r'|) \), the Green’s function satisfies the Sommerfeld radiation condition at infinity, provided the branch cuts of \( \gamma_i \) are chosen such that \( \Im \gamma_i \leq 0 \).

A system of coupled integral equations is obtained after imposing continuity of the tangential fields at the boundaries, yielding a 2.5-D version of the PMCHWT (Poggio-Miller-Chang-Harrington-Wu-Tsai) operator [Poggio and Miller, 1973; Chang and Harrington, 1977; Wu and Tsai, 1977]. A finite-dimensional linear system is obtained with the MoM.

### 3. MoM Interaction Integrals

Before presenting our new theory from Section 4 onwards, we briefly recall which type of interaction integrals occur in the MoM of the 2.5-D PMCHWT boundary integral equation [Olyslager et al., 1993; Fostier et al., 2011]. The boundaries \( C_i \) are meshed into a union of segments \( S_j \) with length \( l_j \), separated by nodes \( r_k \) (Fig. 2).
The transverse tangential components $E_t$ and $H_t$ are expanded in terms of pulse functions $p_j(r)$, with support over segment $S_j$, whereas the longitudinal components $E_z$ and $H_z$ are expanded into triangular functions $t_k(r)$, with support over segments that share a node $r_k$ [Olyslager et al., 1993; Fostier et al., 2011]:

$$
p_j(r) = 1 \quad r \in S_j,
$$

$$
t_k(r) = 1 - |r - r_k| \quad r, r_k \in S_j. \quad (3)
$$

The continuity equations for $E_z$ and $H_z$ are tested with pulse functions, whereas the equations for $E_t$ and $H_t$ are tested with triangular functions. To calculate the elements in the MoM system matrix, the interaction integrals (4)-(6) below need to be evaluated numerically for basis and test functions with support over segments that have $\Omega_i$ as a neighboring medium. This can easily be seen by inspecting (1) and (2). The tangential derivatives of the Green’s function can be transferred to the test function using integration by parts such that only three types of interaction integrals remain:

$$
I_{jk}^{(1)} = \int_{C_i} p_j(r) dc \int_{C_i} G_i(r|r') p_k(r') dc',
$$

$$
I_{jk}^{(2)} = \int_{C_i} p_j(r) dc \int_{C_i} \frac{\partial G_i(r|r')}{\partial n'} t_k(r') dc', \quad (5)
$$

$$
I_{jk}^{(3)} = \int_{C_i} t_j(r) dc \int_{C_i} \frac{\partial^2 G_i(r|r')}{\partial n \partial n'} t_k(r') dc'. \quad (6)
$$

4. Difficulties in Conductive Media

Consider a conductive region $\Omega$, with conductivity $\sigma$, complex permittivity $\epsilon = \epsilon_0 - j\frac{\sigma}{\omega}$ and permeability $\mu$. The transversal wave number $\gamma$ can be written as a function of the skin depth $\delta = \sqrt{2/(\omega \mu \sigma)}$ for moderate to high conductivity values,
as follows:

\[ \gamma = \sqrt{\omega^2 \mu (\epsilon_0 - j\sigma / \omega) - \beta^2} \]  

\[ \approx \frac{1 - j}{\sigma \gg \omega \epsilon_0} \]  

(7)  

(8)

The particular form of this wave number is responsible for the difficulties that arise in evaluating the interaction integrals (4)-(6) in a highly conductive medium. The Green’s function in the conductor reduces to \( \frac{j}{4} H_0^{(2)}((1 - j)r/\delta) \), with \( r = |r - r'| \), while its normal derivatives are expressible in terms of the zeroth, first and second order Hankel functions of the second kind (see appendix A). For large \( |\gamma r| \), the Hankel function of the second kind and order \( \nu \) behaves as \( [Watson, 1995] \)

\[ H_\nu^{(2)}(\gamma r) \sim \left( \frac{2}{\pi \gamma r} \right)^{1/2} e^{-j\gamma r + j\pi(2\nu + 1)} \]  

\[ (|\arg \gamma r| < \pi). \]  

(9)

In a highly conductive medium, the large imaginary part of the wave number causes a strong exponential decay of the Green’s function and its derivatives. Moreover, the wavelength \( \lambda = 2\pi \delta \) is small w.r.t. the free space wavelength, which leads to a spatially strong oscillation of the Green’s function and its derivatives. If \( S_j \) is a segment on the interface between the conductive region \( \Omega \) and a dielectric region \( \Omega_d \), it is sufficient to choose the segment length to be a fraction of the wavelength \( \lambda_d \) in the dielectric, say \( l_j = \frac{\lambda_d}{10} = \frac{2\pi}{10\omega \sqrt{\epsilon_d \mu_d}} \), in order to capture the varying field behavior at the interface. This is because the tangential fields at the interface can only vary at a pace on the order of \( \lambda_d \) in the dielectric and remain continuous at the interface (there are no surface currents). Typically \( \delta \ll l_j \) and accordingly a lot of oscillations occur along one segment and standard quadrature techniques fail.
to correctly evaluate the interaction integrals in the conductive region. Choosing $l_j = \frac{\lambda}{10}$ (with $\lambda$ corresponding to the wavelength in the conductor) to try to tackle this problem is unnecessary to capture the field behavior and would lead to a very large increase of the number of unknowns.

5. Accurate Evaluation of MoM Interaction Integrals

This section proposes a new method to accurately evaluate the MoM integration integrals in conductive media, with a relatively low quadrature order. The method reduces to the traditional approach in Fostier et al. [2011] for $\sigma \ll \omega \varepsilon_0$ (low-loss dielectric case), and is therefore applicable to arbitrary conductivities $\sigma \in [0, \infty]$, as shown in this section and corroborated by the numerical examples in Section 6.

5.1. Cutoff Distance

The key to accurately integrate the strongly oscillating and exponentially decaying integrands in conductive media, for a fixed number of quadrature points, is to distribute those points over the integration domain where the Green’s function has a non-negligible value. Because the magnitude of the Green’s function decays exponentially in a good conductor, it can be approximated as $\frac{j}{4} H_0^{(2)}(\gamma r) \mathbb{H}(r_{\text{cut}} - r)$, neglecting its tail, with $\mathbb{H}$ the Heaviside step function and $r_{\text{cut}}$ the cutoff distance.

The cutoff distance is the distance above which the asymptotic Green’s function (using (9)) becomes smaller in magnitude than a threshold $\Delta_{\text{cut}}$. It can be written in terms of the principal branch of the Lambert W function, denoted $W(z)$. 
Definition 1 (Cutoff distance $r_{\text{cut}}$).

\[ r_{\text{cut}} = -\frac{1}{2\mathfrak{m}^2} \mathcal{W} \left( \frac{-4\mathfrak{m}^2}{\pi \Delta_{\text{cut}}^2} \right) \approx \frac{\delta}{2} \mathcal{W} \left( \frac{2\sqrt{2}}{\pi \Delta_{\text{cut}}^2} \right) \]  

(10)

An upper bound on the Green’s function itself is given in Theorem 5.1. For sufficiently small $\Delta_{\text{cut}}$, the asymptotic expansion (9) is a good approximation and $C \approx 1$.

Theorem 5.1. For $r \geq r_{\text{cut}}$ the following inequality holds: $|G(r)| \leq C \Delta_{\text{cut}} e^{(r-r_{\text{cut}})\mathfrak{m}}$ with $C = 1 + \frac{1}{8\mathfrak{m}r_{\text{cut}}}$. 

Proof. Note that

\[ H_0^{(2)}(z) = \sqrt{\frac{2}{\pi z}} e^{-j(z-\frac{\pi}{4})} \left( 1 - \frac{\theta_2(z)}{8jz} \right), \text{ with } |\theta_2(z)| < 1 \text{ if } \mathfrak{m}z < 0 \]  

[Gradshteyn and Ryzhik, 2007]. For $r \geq r_{\text{cut}}$, this leads to

\[ \left| \int \frac{j}{4} H_0^{(2)}(\mathfrak{m}r) \right| \leq \frac{1}{4\pi \mathfrak{m}r_{\text{cut}}} e^{r_{\text{cut}}\mathfrak{m}} e^{(r-r_{\text{cut}})\mathfrak{m}} \left( 1 + \frac{1}{8\mathfrak{m}r_{\text{cut}}} \right) \]  

(11)

\[ = C \Delta_{\text{cut}} e^{(r-r_{\text{cut}})\mathfrak{m}}, \]  

(12)

where the last step follows from Definition 1. \( \square \)

To illustrate the use of the cutoff distance in the calculation of the interaction integrals, consider $I_{jk}^{(1)}$ in a conductive medium:

\[ I_{jk}^{(1)} \approx \int_{S_j} p_j(r)dr \int_{S_k} G(r|r')H(r_{\text{cut}} - |r - r'|)p_k(r')dr'. \]  

(13)

The boundaries of the test integral over test segment $S_j$ follow from the intersection of $S_j$ with the set of points that are closer than the cutoff distance from the source segment $S_k$ (region $\Upsilon_k$ in Fig. 3). Because $\Upsilon_k$ is convex, either $S_j \cap \Upsilon_k = \emptyset$ (no interaction) or $S_j \cap \Upsilon_k$ is a subsegment ($AB$ in Fig. 3). For each test point $r \in (S_j \cap \Upsilon_k)$, the basis integration interval is a subsegment of $S_k$ ($CD$ in Fig. 4).

In this way, interactions between points that are separated further than $r_{\text{cut}}$ are neglected and the quadrature points are distributed over the region where the in-
tegrand is non-negligible, which alleviates the problem of the exponential damping of the integrand. At the same time, the number of oscillations of the integrands in the interaction integrals is small, independent of the conductivity, allowing a relatively low quadrature order. To show this, consider an interface between free space (wavelength $\lambda_0$) and a conductive region (conductivity $\sigma$, wave number $\gamma = \sqrt{\omega^2\mu_0(\epsilon_0 - j\sigma/\omega) - \beta^2}$), with boundary segment length equal to $l = \lambda_0/10$.

The integrands of the interaction integrals (4)-(6) can be expressed in terms of Hankel functions of the second kind, as shown in (15). For $\sigma \gg \omega\epsilon_0$, it is evident from (10) that $r_{\text{cut}} \sim \delta$, implying that the number of oscillations of $H^{(2)}_\eta(\gamma r)$ in $r \in [0, r_{\text{cut}}]$ is bounded for high conductivities. For $\sigma \ll \omega\epsilon_0$, $r_{\text{cut}} > l$ and the number of oscillations of $H^{(2)}_\eta(\gamma r)$ for $r \in [0, l]$ in a dielectric region is also bounded. A measure for the maximum number of oscillations of the integrands is given by

$$Z = \max_{\sigma \in [0, \infty]} z(\mathcal{P}H^{(2)}_\eta(\gamma r), [0, \min(r_{\text{cut}}, l)]),$$

where $z(f(r), \mathcal{A})$ denotes the number of zero-crossings of $f(r)$ in $r \in \mathcal{A}$. It can be easily verified that $Z = 2, 4$ and $6$ if $\Delta_{\text{cut}} = 10^{-3}, 10^{-6}$ and $10^{-9}$, respectively, which shows that the number of oscillations increases if a higher accuracy is required (larger $r_{\text{cut}}$), but remains small, allowing a low quadrature order, independent of $\sigma$.

In conclusion, the cutoff distance alleviates both problems of exponentially damped and highly oscillatory kernels in conductive media. This approach is an extension of the traditional method in Fostier et al. [2011], to accurately evaluate the interaction integrals in media with arbitrary conductivity.
5.2. Singularity Extraction

The three types of interaction integrals (4)-(6) can be written as

$$I^{(l)}_{jk} = \int_{C_i} w_j^{(l)}(r) \int_{C_i} b_k^{(l)}(r') \mathbb{H}(r_{\text{cut}} - r) \sum_{\eta=0}^2 f_{\eta}^{(l)}(r, r') H_{\eta}^{(2)}(\gamma r) dc' dc.$$  \hspace{1cm} (15)

The test and basis functions are given by

$$w_j^{(1)}(r) = w_j^{(2)}(r) = p_j(r),$$ \hspace{1cm} (16)

$$w_j^{(3)}(r) = t_j(r),$$ \hspace{1cm} (17)

$$b_k^{(1)}(r') = p_k(r'),$$ \hspace{1cm} (18)

$$b_k^{(2)}(r') = b_k^{(3)}(r') = t_k(r').$$ \hspace{1cm} (19)

As shown in appendix A, the functions $f_{\eta}^{(l)}(r, r')$ that are not identically zero are given by

$$f_0^{(1)} = \frac{j}{4},$$ \hspace{1cm} (20)

$$f_1^{(2)} = \frac{j \gamma}{4} (\mathbf{n}' \cdot \mathbf{r}),$$ \hspace{1cm} (21)

$$f_0^{(3)} = \frac{j \gamma^2}{8} (\mathbf{n} \cdot \mathbf{n}'),$$ \hspace{1cm} (22)

$$f_2^{(3)} = \frac{j \gamma^2}{8} \left( \mathbf{n} \cdot \mathbf{n}' - 2(\mathbf{n} \cdot \mathbf{r})(\mathbf{n}' \cdot \mathbf{r}) \right).$$ \hspace{1cm} (23)

If the test and basis functions’ supports overlap or lie next to each other, the integrands in (15) have a singularity in the integration domain. We employ a singularity extraction technique with an extracted singular part that is also limited by the cutoff distance, given by

$$I^{(l)}_{jk, \text{sing}} = \int_{C_i} w_j^{(l)}(r) \int_{C_i} b_k^{(l)}(r') \mathbb{H}(r_{\text{cut}} - r) \sum_{\eta=0}^2 f_{\eta}^{(l)}(r, r') S_{\eta}(\gamma r) dc' dc.$$  \hspace{1cm} (24)
The functions $S_\eta$ are given by

$$S_0(\gamma r) = -\frac{2j}{\pi} \log r,$$

$$S_1(\gamma r) = \frac{2j}{\pi \gamma r},$$

$$S_2(\gamma r) = \frac{4j}{\pi \gamma^2 r^2}.$$  \hspace{1cm} (25) \hspace{1cm} (26) \hspace{1cm} (27)

The integrals of the limited singular parts are known in closed-form. For example, the self-patch term of the first type is given by

$$I^{(1)}_{j,j,\text{sing}} = -\frac{j}{\pi} a((4l_j - 2a) \log a - 4l_j + a),$$

with $a = \min(r_{\text{cut}}, l_j)$.

6. Numerical Results

Plane wave scattering at a conductive cylinder is used to validate the accuracy of the proposed method as a function of the accuracy parameter $\Delta_{\text{cut}}$, for a wide range of electrical conductivities in the general case of oblique incidence ($\beta \neq 0$), and to compare it with existing methods. The next examples are practically relevant shielding problems, in which a conductive and (non-)magnetic enclosure with apertures is used to shield the interior from the fields generated by an exterior electric current source. The new method is able to accurately calculate the shielding performance over a broad frequency range, and outperforms existing methods in terms of accuracy and simulation time.
6.1. Scattering at a Conductive Cylinder

To validate the accuracy of the proposed 2.5-D BEM for conductive media and compare it with existing numerical methods, we consider the problem of plane wave scattering at a conductive cylinder (diameter $d$, finite conductivity $\sigma$ and permittivity $\varepsilon_0 - j\sigma/\omega$), embedded in free space (Fig. 5). An analytical expression of the solution can be obtained via separation of variables [Van Bladel, 2007]. The accuracy of the proposed method is compared with the traditional method without cutoff distance, and with a surface impedance approximation, over a wide conductivity range, from the low-loss dielectric ($\omega\varepsilon_0 \gg \sigma$) to the conductive region ($\omega\varepsilon_0 \ll \sigma$).

The numerically obtained radar cross section (RCS), denoted $S_n(\phi)$, is compared with the analytical solution, denoted $S_a(\phi)$. The relative error between these cross sections is defined by

$$E = \sqrt{\sum_{k=1}^{K} |S_n(\phi_k) - S_a(\phi_k)|^2 / \sum_{k=1}^{K} |S_a(\phi_k)|^2},$$  \hspace{1cm} (29)

with $\phi_k = 2\pi k/K$ and $K = 100$. Figures 6-7 show the relative error as a function of the skin depth for the two polarizations (VV and VH) of oblique plane wave incidence ($\alpha = 45^\circ$). The skin depth $\delta = \sqrt{2/(\omega\mu_0\sigma)}$ ranges from $10^{-5}$ m to 10 m, covering the region between a good conductor with conductivity $\sigma = 10^7$ S/m and a low-loss dielectric with relative dielectric constant $1 - 5 \cdot 10^{-4} j$. Observe that, in general, the error decreases if the accuracy threshold $\Delta_{cut}$ becomes smaller. The relative error saturates around five significant digits for small and large skin depths, but this lower bound is determined mostly by the boundary meshing of the circular cross section into straight segments. The asymptotic value for the cutoff distance in (10), in case of...
high conductivity, is shown in the legend. For $r_{\text{cut}} > d$, no interactions are neglected, and the numerical solution becomes independent of $\Delta_{\text{cut}}$, as can be seen for high $\delta/d$ values.

The inability of the traditional method without cutoff distance (i.e. the proposed method with $r_{\text{cut}} = \infty$) to accurately evaluate the interaction integrals in conductive media, for a fixed quadrature order and constant number of boundary segments, is clearly demonstrated in Fig. 7. The problems mentioned in Section 4, i.e. the exponential decay combined with strong oscillation of the integrands, render the traditional method inaccurate or useless for low values of $\delta/d$. The proposed method focuses the quadrature points in the region where the integrands are non-negligible, by introducing the cutoff distance (10) and a singularity extraction with limited extracted part, which in turn limits the number of oscillations. This leads to a near constant accuracy over the considered conductivity range (if enough oscillations are taken into account, i.e. for sufficiently low $\Delta_{\text{cut}}$). For $r_{\text{cut}} > d$, or equivalently for high $\delta/d$ values, our new method reduces to the traditional one, and the numerical solution is the same for both methods.

Another approach to incorporate good conductors in a BEM is the use of a surface impedance approximation, by imposing the condition $\mathbf{E} \times \mathbf{n} = Z_s (\mathbf{n} \times \mathbf{H} \times \mathbf{n})$ on the conductor boundary, with $Z_s = (1 + j) \sqrt{\frac{\mu_0}{\sigma}}$ and $\mathbf{n}$ the outward normal to the conductive region. Figures 6-7 show that this is a good approximation for low values of $\delta/d$, i.e. in the conductor skin-effect regime (note that the error does not saturate around 100 dB because the analytical solution with surface impedance approximation
is considered). It is apparent from Fig. 7 that the proposed method (with $\Delta_{\text{cut}} = 10^{-9}$) outperforms the surface impedance approximation and traditional method in terms of accuracy, in the transition region between low-loss dielectric and skin-effect regime.

6.2. Slotted Coaxial Shield

In this example, we investigate the shielding performance of a coaxial enclosure with one or two slots at angles $\alpha_1$ and $\alpha_2$ (Fig. 8). The coaxial enclosure is illuminated by an electric line current $I_0\delta(r - r_0)\hat{z}$ (hence $\beta = 0$), which induces an unwanted noise current $I_1$ in the enclosed copper signal conductor. Remark that, in addition to our MoM integral equation technique, scattering at a concentrically loaded cylindrical shield with $n - 1$ apertures can be solved by reducing an $n$-series problem to an equivalent Riemann-Hilbert problem [Ziolkowski, 1985; Ziolkowski and Grant, 1987]. A similar radial mode matching technique has been employed for multi-slotted shields with finite thickness [Lee et al., 2012]. We consider three enclosure materials: copper ($\sigma = 5.8 \cdot 10^7$ S/m, $\mu_r = 1$), a magnetic conductor with the same skin depth ($\sigma = 5.8 \cdot 10^4$ S/m, $\mu_r = 1000$), and a perfect electric conductor ($\sigma = \infty$). The configurations with one and two slots are described by $\alpha_1 = 90^\circ$ and $(\alpha_1, \alpha_2) = (60^\circ, 120^\circ)$, respectively.

Figure 9 shows the relative noise current amplitude $|I_1/I_0|$ of the copper and perfect electric conducting (PEC) enclosure, over a broad frequency range (from 100 Hz to 1 GHz). Observe that the analytical solution for the closed coaxial enclosure (no slots) coincides with the numerical solution. At low frequencies, there is leakage...
through the copper enclosure, as the skin depth is comparable to the thickness, and
the presence of slots does not deteriorate the shielding performance significantly. At
high frequencies, the predominant leakage mechanism is diffraction of the fields at
the slots, and the copper and PEC shields exhibit the same behavior. For the given
position of the slots and line current, the noise current for two slots is about 15 dB
higher than for one slot.

Figure 10 shows the relative noise current amplitude for the magnetic conducting
enclosure. For the configuration without slots, the numerical and analytical solution
coincide. Compared to the copper enclosure, at low frequencies, the presence of slots
now has a larger influence. This is due to a different shielding mechanism in the
magnetic conductor, adding to the effect of the conductivity. If $\mu_r > 1$, the magnetic
induction produced by the source is diverted into the enclosure, then shunted within
the material in a direction nearly parallel to its surface, and finally released back into
free space [Celozzi et al., 2008]. The presence of slots disturbs the flux shunting, and
negatively affects the shielding performance.

It is interesting to compare our new method with the traditional method ($r_{\text{cut}} = \infty$)
in terms of accuracy and simulation time. Fig. 11 shows the calculated shielding
performance as a function of the quadrature order $Q$, for the copper shield with two
slots. For $\Delta_{\text{cut}} = 10^{-9}$, the new method already converges to the solution for $Q = 10$,
compared to $Q = 80$ for the traditional method. For the same quadrature order $Q =
10$, the traditional method fails to accurately predict the shielding performance, due
to the problems mentioned in Section 4. Evidently, the need for a smaller quadrature
order to obtain the same accuracy leads to a decrease in simulation time (Table 1). Even for the same quadrature order \((Q = 10)\), our method is faster than the traditional method because interactions between segments separated by at least the cutoff distance are not taken into account.

6.3. Cable Tray Shield

The geometry of a metal cable tray with polygonal cross section (Fig. 12) is similar to the previous example, but arguably more interesting from a practical perspective. In this case, no closed-form analytical solution is available for the closed cable tray \((g = 0)\). The enclosure is illuminated by an electric line current \(I_0 \delta(r - r_0)\hat{z}\) (hence \(\beta = 0)\), which induces unwanted noise currents \(I_i\) in the three copper signal conductors. Figure 13 shows the relative current magnitude \(|I_2/I_0|\) in the middle conductor, for an open and closed cable tray (aperture length \(g = 5.5\) mm and \(g = 0\), respectively). We consider three enclosure materials: copper \((\sigma = 5.8 \cdot 10^7\) S/m, \(\mu_r = 1)\), a magnetic conductor with the same skin depth \((\sigma = 5.8 \cdot 10^4\) S/m, \(\mu_r = 1000)\), and a perfect electric conductor \((\sigma = \infty)\).

At low frequencies (up to \(10^5\) Hz), we notice that the influence of the apertures can be neglected, as the open and closed cable tray yield approximately the same shielding performance, for both copper and the magnetic conductor. In this region, the skin depth is comparable to the thickness, allowing the fields to penetrate the enclosure. At high frequencies, the copper and perfectly conducting open cable tray behave in the same way, indicating that diffraction of the fields through the aperture
is the predominant leakage mechanism. Observe that the magnetic conductor exhibits a worse shielding performance than copper, for all considered frequencies.

7. Conclusions

This paper presents a novel method to accurately and efficiently calculate 2.5-D MoM interactions integrals in conductive media, with arbitrary permeability. The resulting BEM is practically relevant to a large number of application domains, including modal analysis of waveguides with non-perfect conductors, scattering problems, and shielding problems with general three-dimensional sources.
Appendix A: Derivation of the Green’s function’s normal derivatives

Using the notation of Figure 14, the gradients of the Green’s function \( G(\mathbf{r}|\mathbf{r}') = \frac{i}{4} H^{(2)}_0(\gamma r) \) w.r.t. the observation point \( \mathbf{r} \) and source point \( \mathbf{r}' \) are given by

\[
\nabla G(\mathbf{r}|\mathbf{r}') = -\nabla' G(\mathbf{r}|\mathbf{r}') = -\frac{j\gamma}{4} H^{(2)}_1(\gamma r)\mathbf{\hat{r}}.
\]

(A1)

From these expressions, the normal derivatives of the Green’s function follow immediately:

\[
\frac{\partial G}{\partial n}(\mathbf{r}|\mathbf{r}') = -\frac{j\gamma}{4} H^{(2)}_1(\gamma r)(\mathbf{\hat{n}} \cdot \mathbf{\hat{r}}),
\]

(A2)

\[
\frac{\partial G}{\partial n'}(\mathbf{r}|\mathbf{r}') = \frac{j\gamma}{4} H^{(2)}_1(\gamma r)(\mathbf{\hat{n}}' \cdot \mathbf{\hat{r}}).
\]

(A3)

Taking the gradient w.r.t. \( \mathbf{r}' \) of (A1) leads to the following dyadic, with the dot representing the derivative of a holomorphic function:

\[
\nabla' \nabla G(\mathbf{r}|\mathbf{r}') = \nabla' \left( \frac{j\gamma}{4} H^{(2)}_0(\gamma r) \right) \mathbf{\hat{r}} + \frac{j\gamma}{4} \mathbf{\hat{H}}_0^{(2)}(\gamma r) \nabla' \mathbf{\hat{r}}
\]

\[
= -\frac{j\gamma^2}{4} \mathbf{\hat{H}}_0^{(2)}(\gamma r) \mathbf{\hat{r}} - \frac{j\gamma}{4r} \mathbf{\hat{H}}_0^{(2)}(\gamma r) \mathbf{\hat{\phi}}. \tag{A4}
\]

After some manipulations, the second order normal derivative of the Green’s function can finally be written as

\[
\frac{\partial^2 G}{\partial n \partial n'}(\mathbf{r}|\mathbf{r}') = \mathbf{\hat{n}}' \cdot \nabla' \nabla G \cdot \mathbf{\hat{n}}
\]

\[
= \frac{j\gamma^2}{8} \left( H_0^{(2)}(\gamma r) + H_2^{(2)}(\gamma r) \right) \mathbf{\hat{n}} \cdot \mathbf{\hat{n}}' - \frac{j\gamma^2}{4} H_2^{(2)}(\gamma r)(\mathbf{\hat{n}} \cdot \mathbf{\hat{r}})(\mathbf{\hat{n}}' \cdot \mathbf{\hat{r}}). \tag{A5}
\]

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author. The authors are grateful to the editor and the anonymous reviewers for their valuable comments.

References


Dobbelaere, D., H. Rogier, and D. De Zutter (2013b), Accurate 2D MoM technique for arbitrary dielectric, magnetic and conducting media applied to shielding prob-


Figure 1. General isotropic piecewise-homogeneous 2-D geometry with a 3-D excitation.
Figure 2. The boundaries are approximated with straight segments along which triangular and pulse functions are defined.

Figure 3. Test integration interval AB, where the interactions from segment $S_k$ are non-negligible.
Figure 4. Basis integration interval CD, for test point $r$.

Figure 5. Plane wave scattering at a conductive cylinder with conductivity $\sigma$, permittivity $\varepsilon_0 - j\sigma/\omega$ and diameter $d = 1$ m. The cylinder is illuminated by a linearly polarized plane wave, with free space wavelength $\lambda_0 = 1$ m, impinging at an angle $\alpha$ w.r.t. the $(x,y)$ plane.
Figure 6. Relative error of the co-polarization RCS (VV) as a function of the skin depth $\delta$ for oblique incidence ($\alpha = 45^\circ$) for the proposed method, the traditional method without cutoff distance and limited extracted part, and a surface impedance approximation. The quadrature order of the interaction integrals ($Q = 32$) and number of boundary segments ($N = 630$) are constant.
\[ \Delta_{\text{cut}} = 10^{-3} \quad \left( r_{\text{cut}} \sim 4.4\delta \right) \]
\[ \Delta_{\text{cut}} = 10^{-6} \quad \left( r_{\text{cut}} \sim 11\delta \right) \]
\[ \Delta_{\text{cut}} = 10^{-9} \quad \left( r_{\text{cut}} \sim 18\delta \right) \]

- Traditional method \( (r_{\text{cut}} = \infty) \)
- Surface impedance approximation

Figure 7. Relative error of the cross-polarization RCS (VH) as a function of the skin depth \( \delta \) for oblique incidence \( (\alpha = 45^\circ) \) for the proposed method, the traditional method without cutoff distance and limited extracted part, and a surface impedance approximation. The quadrature order of the interaction integrals \( (Q = 32) \) and number of boundary segments \( (N = 630) \) are constant.
Figure 8. Cross section of a coaxial enclosure with conductivity \( \sigma \) and relative permeability \( \mu_r \), illuminated by an electric line current \( I_0 \), and enclosing a copper signal conductor with induced noise current \( I_1 \). There are one or two slots present at angles \( \alpha_1 \) and \( \alpha_2 \).
Figure 9. Shielding performance of the coaxial enclosure as a function of frequency, for a copper and perfect electric conducting (PEC) enclosure, with a varying number of slots.
Figure 10. Shielding performance of the coaxial enclosure as a function of frequency, for the magnetic conductor, with a varying number of slots.
Figure 11. Comparison between the traditional method (with $r_{cut} = \infty$) and the new method in this work (with $\Delta_{cut} = 10^{-9}$), of the calculated shielding performance of the copper coaxial enclosure with two slots, for a varying quadrature order $Q$. 
Table 1. Simulation time of the new method (top) and the traditional method (bottom), versus the quadrature order $Q$ (see Fig. 11).

<table>
<thead>
<tr>
<th>$Q$</th>
<th>Time (s)</th>
</tr>
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<tbody>
<tr>
<td>10</td>
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</tr>
<tr>
<td>10</td>
<td>17</td>
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<td>20</td>
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<td>150</td>
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<tr>
<td>80</td>
<td>575</td>
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</table>

Figure 12. Cross section of an open cable tray with conductivity $\sigma$ and relative permeability $\mu_r$, illuminated by an electric line current $I_0$, and enclosing three copper signal conductors with induced noise currents $I_1$ to $I_3$. The geometry is symmetrical w.r.t. a vertical line through the center of the middle conductor.
Figure 13. Shielding performance of the open \((g = 5.5 \text{ mm})\) and closed \((g = 0)\) cable tray as a function of frequency, for various shielding materials.

Figure 14. Relevant to the derivation of the normal derivatives.