Five-fold symmetry in fractal atom hydrogen probed with accurate 1S-nS terms

G. Van Hooydonk,
Ghent University, Faculty of Sciences, Ghent, Belgium

Abstract. We probe Penrose’s five-fold symmetry and fractal behavior for atom H. With radius $r_H$ derived from H mass $m_H$, H symmetry is governed by Euclid’s golden ratio $\phi = \frac{\sqrt{5} - 1}{2}$, as proved with accurate H terms. Our prediction for H 1S-3S, to be measured soon, is 2 922 743 278 654 kHz.

I. Introduction
Euclid-Phidias numbers appear in fundamental and applied sciences, in arts… [1-3], and for chaotic or fractal behavior (Mandelbrot [4], Gutzwiller [5]) and Penrose’s 5-fold symmetry [6]. With $(1-x)/x = x/1$ for complementary parts $+x$, $1-x$ of composite units, Euclidean harmony is $x = \phi = -\frac{1}{2}(1 \pm \sqrt{5})$.

The simplest, smallest but most abundant neutral unit in the Universe [7], composite H has electron (mass $m_e$) and proton (mass $m_p$) as complementary parts: $m_H = m_e + m_p = m_e + (m_H - m_e)$ or $1 = x + (1-x)$, if $x = m_e/m_H$. If $\phi$ applied to H, it must show in its spectrum. By its compact nature, Bohr theory fails on $\phi$-symmetry, also invisible in bound state QED [8]. We probe $\phi$ for H using mass $m_H$ and radius $r_H$ related by $m_H = \frac{4\pi}{3} \gamma r_H^3$. Scaling H levels by virial $\frac{1}{2}e^2/r_H$ gives away $\phi$ and fractal behavior. This is in line with Rydberg’s original formula [9] and confirmed with accurate H 1S-nS terms [10]. We predict a value of 2922743278654 kHz for H 1S-3S, to be measured in the near future [11].

II. Rydberg equation and fractal behavior of atom H

II.1 Chaotic/fractal interpretation of the Rydberg formula for composite H

With constant $a$ in Å and line number $n$, the original Rydberg formula [9] for H terms

$$T_n = an^2/(n^2-1) \text{ Å or } T_n/(an) = n/(n^2-1) = 1/(n-1/n)$$

suggests that H may well exhibit fractal or chaotic behavior [4,5]. Bohr energy differences

$$\Delta E_n = 1/T_n = (n^2-1)10^9/(an^2) = R_H (1-1/n^2) = R_H - E_{n妄}$$

with Rydberg $R_H = 10^9$ cm$^{-1}$, give fractal behavior (1) in linear form

$$n\Delta E_n/R_H = (n-1)(n+1)/n = n-1/n$$

With $E_n$ [12] instead of $\Delta E_n$, plots of $nE_n$ versus $n$ and $1/n$ give power laws

$$E_n(n) = E_n(1/n) = 109679,223605211n^{1.00004252339} = 109679,223605211(1/n)^{1.00004252339}$$

Linear $n$ and inverse $1/n$ views suggest fractal H (3) within 0,007 cm$^{-1}$, while Bohr $1/n^2$ theory has errors of 0,0126 cm$^{-1}$ (a power fit in $1/n^2$ has its exponent shifted by 1). The greatest difference with Bohr theory and QED is asymptote 109679,2236 cm$^{-1}$ in (4), larger than $-E_1 = 109678,77304$ cm$^{-1}$ in [12]. Since $1/n$ secures convergence, a 4th order fit in $1/n$

$$nE_n = 0.00689343262/n^4 - 3.757565800476/n^3 + 5.5580713748932/n^2 + 109677,585385323000/n (5)$$
is accurate within $10^{-8}$ cm$^{-1}$ or 0.45 kHz (less precise data [13] behave similarly). By its precision, (5) for fractal H must be important for metrology [10, 14-15], as we discuss further below.

II.2 Generalizing Bohr H theory and reduced mass: opening for $\varphi$

To not to interrupt the argument on $\varphi$, we compare H theories in Appendix A. With (A1)-(A2), Bohr’s integer quantum number $n$ and Rydberg $R_H$ give rotational level energies

$$E_n = -\frac{R_H}{n^2} = -\frac{\hbar^2}{2\mu c^2}/n^2 = -\frac{\hbar^2}{2\mu c^2}/n^2$$

(6)
Here, $r_0$ is Bohr radius $r_0 = \hbar^2/(m_\epsilon e^2)$, corrected for reduced electron mass, according to

$$\mu = m_e m_p/(m_e + m_p) = m_e m_p/m_H = m_e/(1 + m_e/m_p) \equiv m_e(1 - m_e/m_H)$$

(7)

Generalizing (6) with a critical $n_c$ for another H radius $r_H$ by means of

$$r_H = n_c^2 r_0$$

(8)

$$E_n = -(R_H/n_c^2)(\sqrt{n_c}/n)^2 = -\frac{\hbar^2}{2\mu c^2}r_0^2/n_c^2$$

(9)

allows an infinite number of solutions, trivial or not.

(i) Any $n_c$ (except 0) will lead to the same accuracy as (6). A relation between $n_c$ and $\varphi$ like

$$n_c = A \varphi^m$$

(10)

for (9) may probe Euclidean symmetry, but only if an alternative $r_H$ existed (see Section II.3).

(ii) Detecting internal $\varphi$-effects in H depends on specific $\varphi$-relations [2-5] like

$$\varphi^{m+2} + \varphi^{m+1} = \varphi^m; 1/\varphi - \varphi^2; \varphi^2 - \varphi + 1 = 0 \text{ and } \varphi(\varphi+1) = 1$$

(11)

Internal $\varphi$-symmetries (11) are available from (7) in dimensionless form. With $m_H$, this gives product

$$q_H = \mu/m_H = (m_e/m_H)(1 - m_e/m_H) = x(1 - x)$$

(12)

for parts ($dq/dx = 1 - 2x = 0$ gives $q_{max} = 1/4$ when $x = 1/2$ or parts are equal; $q = x(1-x)$ or $x^2 - x + q = 0$ gives $x_+ = 1/2[1 \pm \sqrt{(1 - 4q)}]$; a center between parts leads to $-x_-, (1-x)$, $x^2 - x + q = 0$ and $x_+ = 1/2[1 \pm \sqrt{(1 + 4q)}]$).

With only part ratios, all symmetries in (ii) are Euclidean (see Introduction).

By virtue of (10)-(12), reduced mass for H (12) implies Euclidean harmony between parts, obeying

$$q_H = x(1-x) - A\varphi^m(1-A\varphi^m)$$

(13)

If valid, these are only small corrections to $E_n$, since $\mu/m_e$ (7) is 1837 times larger than $\mu/m_H$ (12). The fate of H symmetries (9)-(13) depends solely on the existence of a valid alternative radius $r_H$.

II.3 Alternative classical H radius $r_H$

Apart from [16], a first principles alternative quantum radius for H, other than Bohr length $r_0$, does not exist. Only a classical 19th century macroscopic view on spherical H can give $r_H$ using

$$m_H = (4\pi/3)\gamma r_H^3 \text{ and } r_H = [(3/4\pi\gamma)m_H]^{1/3}$$

(14)

where $4\pi/3$ is the form factor for a sphere and $\gamma$ in g/cm$^3$ is H density.

With $m_H = m_e + m_p = 9.10938215.10^{-28} + 1.672621637.10^{-24}$ g [10] and $\gamma = 1$ g/cm$^3$ for H, the result is

$$r_H = 7.36515437.10^{-9} \text{ cm} = 0.736515437 \text{ Å}$$

(15)
This is the only real, theoretically possible alternative to Bohr length $r_B=0.529177209 \, \text{Å}$ [16]. Apart from form factor and $\gamma$, its accuracy relies on the precision for $m_e$ and $m_p$ [10].

In (6), H radius $r_0$ is Bohr length $r_B$, corrected for recoil (7) or

$$r_0=\frac{\hbar^2/(m_e^2)}{(1+m_e/m_p)}=0.5294654075 \, \text{Å} \quad (16)$$

The ratio of classical natural H radius $r_H$ in (15) and Bohr's $r_0$ in (16) is

$$x=r_H/r_0=1.391054876... \quad (17)$$

(without recoil, $r_H/r_B=1.391812469...$).

The natural virial Coulomb energy $-\frac{1}{2}e^2/r_H$ for any two charge-conjugated parts amounts to

$$\frac{1}{2}e^2/r_H=78844.900590508 \, \text{cm}^{-1}=2363710654879.4 \, \text{kHz} \quad (18)$$

When multiplied by (18), the conventional H asymptote ($n=\infty$) is $xe^2/r_H=109677.583516024 \, \text{cm}^{-1}$.

### III. Scaling $E_n$ by $\frac{1}{2}e^2/r_H$: probing Penrose's five-fold or $\varphi$-symmetry in atom H

Scaling $E_n$ by natural H asymptote (18) gives numbers

$$N_n=E_n/(\frac{1}{2}e^2/r_H) \quad \text{or} \quad nN_n=nE_n/(\frac{1}{2}e^2/r_H) \quad (19)$$

Due to (18), plots of $nN_n$ versus $1/n$ and $(1-1/n)$ in Fig. 1 give 4th order fits (with 5 decimals)

$$N_n=-0.00006/n^4+0.00007/n^3+1.39105/n^2 \quad (20)$$

$$N_n=-0.000056(1-1/n)^4+0.00015(1-1/n)^3+1.39093(1-1/n)^2-2.78210(1-1/n)+1.39107 \quad (21)$$

With $(1-1/n)$, typical for molecular potentials [16], (20)-(21) reveal the effect of odd powers in $1/n$.

In (A16)-(A17), we prove that the H force constant $k_n$, away from critical configuration $n_c$, varies with

$$1.5/n \quad (22a)$$

$$N_n=-0.00001(1.5/n)^4+0.00002(1.5/n)^3+0.61825(1.5/n)^2 \quad (22b)$$

Coefficients of $(1.5/n)^2$ in (22a) and $(1-1.5/n)^2$ in (22b) are close to Euclid or Phidias number (10)

$$\varphi=\frac{\sqrt{5}-1}{2}=1/\varphi=\Phi-1=0.618034 \ \ldots \quad (23)$$

Correction factor $f_\varphi$ for $\varphi$-symmetry and $f_r$ for recoil

$$f_\varphi=0.618247/0.618034=1.000344; f_r=m_e/m_p=1/1836,15267247=0.000545 \quad (24)$$

shows that $f_\varphi$ is smaller than $f_r$ by 40 %. Difference $\delta$ for $\varphi$-symmetry is 0.02 %, i.e.

$$\delta=0.618247-0.618034=0.000213 \quad (25)$$

In terms of ratio $m_e/m_p=1/1837,15267247$ in (7), difference (25)

$$(m_e/m_p)0.000213=0.390635\approx(9/4)-1=3/2(\sqrt{5}-17/18) \quad (26)$$

reflects the importance of Euclid's golden ratio for H. Combining coefficient for $1.5/n$ (22a) and asymptotes $0.618247$ in (22a-b) gives a 9-decimal result, close to ratio $x$ in (18), since

$$x=(9/4). 0.618246619=(3/2)^2=1.391054894=r_H/r_0 \quad (27a)$$

Using (9), the Euclidean H variable $x_E$ must obey
\[ x_e = a\phi^{1/2}/n \tag{27b} \]

Results (21)-(27) probe Penrose’s five-fold or Euclid’s \( \varphi \)-symmetry in H, due to alternative classical radius \( r_H \) (15). For internal \( \varphi \)-symmetry in H according to (13), (27) prescribes Euclidean variable

\[ X_E \sim x_e(1-x_e) \sim (a\phi^{1/2}/n)(a\phi^{1/2}/n-1) \tag{28} \]

Given their smallness, of order recoil (13), only precise H terms [17-21] can provide with evidence for internal five-fold H symmetry (28).

IV. Putting \( \varphi \) to the test in H with accurate H intervals (prediction of H 1S-3S) 

The precision needed to validate (28) requires an upgrade of \( E_n \) [12]. Table 1 shows precise H terms available. Its 4 precisely known intervals A, B, D and E give 2 derived intervals C and F. Since only B and F are void of 1S, the immeasurable series limit or \( -E_1 \), B and F allow multiplicative scaling. Precision at this level requires many significant digits. A fit of \( E_n \) [12] to 4th order in \( 1/n \) through the origin generates these digits and allows a test with terms in Table 1. Slope 1-1,79201817.10^{-8} and intercept 26940,95752/29979245,8=0,00008965361 cm^{-1} give the terms in kHz in Table 2. The conversion corresponds with a change of Erickson’s 1977 R=109737,3177±0,00083 cm^{-1} [12].

Table 1 reveals that A, B and C are exactly reproduced. The small discrepancies for D, E and F are much lower than experimental uncertainties, 10 kHz for D and 21 for E in [20-21]. With the small error of 1,74 kHz for F removed, the error reappears for D and E (1,71 kHz). The small difference of 1,26 kHz for all terms caused by this correction justifies their omission in Table 2.

With ongoing experiments [11] in mind, we safely conclude that our predicted H 1S-3S interval (G in Table 1 and also in Table 2) is correct within 1,74 kHz, i.e. the largest error in Table 2.

Table 1 Observed [10] and intervals from this work in kHz (with errors \( \delta \)). Prediction of H 1S-3S

<table>
<thead>
<tr>
<th>Intervals</th>
<th>Observed</th>
<th>This work</th>
<th>( \delta )(kHz)</th>
<th>Ref.</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. 1S-2S</td>
<td>2466061413187,07</td>
<td>2466061413187,07</td>
<td>0,00</td>
<td>[17,18]</td>
</tr>
<tr>
<td>B. 2S-8S</td>
<td>770649350012,00</td>
<td>770649350012,00</td>
<td>0,00</td>
<td>[19]</td>
</tr>
<tr>
<td>C. [1S-8S]</td>
<td>3236710763199,07</td>
<td>3236710763199,07</td>
<td>0,00</td>
<td></td>
</tr>
<tr>
<td>D. 2S-4S</td>
<td>4797338</td>
<td>4797334,20</td>
<td>-3,80</td>
<td>[20]</td>
</tr>
<tr>
<td>E. 2S-6S</td>
<td>4197604</td>
<td>4197601,94</td>
<td>-2,06</td>
<td>[21]</td>
</tr>
<tr>
<td>F. [6S-2S]</td>
<td>599734</td>
<td>599732,26</td>
<td>-1,74</td>
<td></td>
</tr>
<tr>
<td>G. 1S-3S predicted$^d$</td>
<td>to be measured</td>
<td>2922743278654,37</td>
<td></td>
<td>[11]</td>
</tr>
</tbody>
</table>

$^a$ only B and derived F do not depend on 1S  
$^b$ derived values between square brackets result from C=A+B and F=D-E  
$^c$ the four intervals A,B,D,E are used for metrology in [10]  
$^d$ by the same argument, all other intervals nS in Table 1 are predicted with the relative accuracy to reference term B [19]  

Surprisingly, 4th order is still sufficient to fit all data accurately, when 15 significant digits are used.

\[ N_n = E_n/(\sqrt{\gamma^2/r_H}), \]

\( x_e = 2^{-1}(-0,000028651871617x_e^4 + 0,0000429685424202x_e^3 + 1,000344034289810x_e^2 - 0,00000000165642x_e) \tag{29} \]

G. Van Hooydonk, Five-fold symmetry in fractal atom H ....1st version 8 Dec 2008
For 19 terms 2S to 20S in Table 2, average errors of 0.11 kHz give a precision of 1.610^{-12}%. Small deviations ε_n nevertheless increase with increasing n (which we discuss elsewhere).

H terms in Table 2 allow a check of Euclidean variable $X_{E}$ (28) for internal Euclidean φ-symmetry.

Table 2 H 1S-nS: original $E_n$ [12] and converted $E'_n$ in cm^{-1}, terms $T_n$ in kHz and deviations $ε_n$ with fitting to 4th order (29)

<table>
<thead>
<tr>
<th>n</th>
<th>$E_n$ (cm^{-1})</th>
<th>$E'_n$ (cm^{-1})</th>
<th>$T_n$ (kHz)</th>
<th>$ε_n$ (kHz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>109678,773704000</td>
<td>109678,77174307900</td>
<td>2466061413187,07</td>
<td>1,706</td>
</tr>
<tr>
<td>2</td>
<td>27419,81734379700</td>
<td>27419,8173213227</td>
<td>2922743278654,37</td>
<td>0,139</td>
</tr>
<tr>
<td>3</td>
<td>12186,5501899660</td>
<td>8691281950291,07</td>
<td>5082581563818,04</td>
<td>0,078</td>
</tr>
<tr>
<td>4</td>
<td>4387,1408889000</td>
<td>3156563684658,80</td>
<td>3156563684658,80</td>
<td>0,097</td>
</tr>
<tr>
<td>5</td>
<td>3046,6219504000</td>
<td>3196751430452,60</td>
<td>3196751430452,60</td>
<td>0,083</td>
</tr>
<tr>
<td>6</td>
<td>2238,3324135261</td>
<td>3220983339585,82</td>
<td>3220983339585,82</td>
<td>0,065</td>
</tr>
<tr>
<td>7</td>
<td>1713,7220591500</td>
<td>3236710763199,07</td>
<td>3236710763199,07</td>
<td>0,050</td>
</tr>
<tr>
<td>8</td>
<td>1354,0512214300</td>
<td>3247493423457,69</td>
<td>3247493423457,69</td>
<td>0,038</td>
</tr>
<tr>
<td>9</td>
<td>1096,7809744200</td>
<td>3255206191292,99</td>
<td>3255206191292,99</td>
<td>0,029</td>
</tr>
<tr>
<td>10</td>
<td>906,4302025300</td>
<td>3260912763770,46</td>
<td>3260912763770,46</td>
<td>0,022</td>
</tr>
<tr>
<td>11</td>
<td>761,6529039900</td>
<td>3265253077913,06</td>
<td>3265253077913,06</td>
<td>0,016</td>
</tr>
<tr>
<td>12</td>
<td>648,9821718400</td>
<td>3268806142732,32</td>
<td>3268806142732,32</td>
<td>0,012</td>
</tr>
<tr>
<td>13</td>
<td>559,5814289180</td>
<td>327131028226,93</td>
<td>327131028226,93</td>
<td>0,008</td>
</tr>
<tr>
<td>14</td>
<td>487,4574954570</td>
<td>3273473249318,27</td>
<td>3273473249318,27</td>
<td>0,005</td>
</tr>
<tr>
<td>15</td>
<td>428,4293581010</td>
<td>3275248852326,18</td>
<td>3275248852326,18</td>
<td>0,003</td>
</tr>
<tr>
<td>16</td>
<td>379,5082947800</td>
<td>327670948828,61</td>
<td>327670948828,61</td>
<td>0,001</td>
</tr>
<tr>
<td>17</td>
<td>338,5117735350</td>
<td>3277938658687,20</td>
<td>3277938658687,20</td>
<td>0,001</td>
</tr>
<tr>
<td>18</td>
<td>303,8168027570</td>
<td>3278966709043,60</td>
<td>3278966709043,60</td>
<td>0,002</td>
</tr>
<tr>
<td>19</td>
<td>274,1946308760</td>
<td>3278978658687,20</td>
<td>3278978658687,20</td>
<td>0,002</td>
</tr>
<tr>
<td>20</td>
<td>274,1946308760</td>
<td>3278978658687,20</td>
<td>3278978658687,20</td>
<td>0,002</td>
</tr>
<tr>
<td>average</td>
<td>0,124</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

V. Beyond Bohr H 1/n^2 theory: probing internal φ-symmetry for fractal H

A 4th order fit of accurate $E'_n$ data in Table 2 exposes the contribution of Bohr’s 1/n^2 theory

$$E'_n=-4,368336200714/n^{4}+5,555412530899/n^{3}+109677,583783388/n^{2}-0,000015348196n$$ (30)

Apart from small 1/n, subtracting term 1/n^2 discloses accurate symmetry bound energy differences

$$\Delta E'_n=(4,368336200714/n^{2}-5,555412530899/n)/n^{2} \text{ cm}^{-1}$$ (31)

Series limit $E_1$ in Table 2 gives $\Delta E'_n$, shifted by 1,18…/n^2. Coefficients in (31) reveal a parabola, obtained by adding (√5,5554/√4,3683)^2=1,32901²=1,766268. This hidden term in 1/n^2 provides with a harmonic Rydberg $R_{\text{harm}}$, larger than $R_\infty$ and $R_1$, in line with power fit (4) and is equal to [22]

$$R_{\text{harm}}=109677,5837838+1,766268=109679,350051 \text{ cm}^{-1}$$ (32)

H symmetry equation (31) with $R_{\text{harm}}$ now becomes a perfect Mexican hat curve, i.e. quartic [23]

$$\Delta_{\text{harm}}=(4,368336/n^{2}-5,555413/n+1,766268)/n^{2} \text{ cm}^{-1}=1,766268(1-1,572642/n)^2/n^2$$ (33)

which is critical at $n=2.1,572642/n\approx\pi\approx4\phi^{1/2}$ [23]. Fig. 2 gives quartics for $R_{\text{harm}}$, $R_n$ and $E_1$ versus $4\phi^{1/2}/n$. The more symmetrical Hund-type Mexican hat curve with $R_{\text{harm}}$ (32) is an undeniable signature for left-right H behavior [23] but is usually, and unjustly, disregarded.
Using $R_\infty$ to disclose internal H symmetries as in QED creates large energy differences (see Fig. 2). With (33) accurate to order kHz, Euclidean symmetry for fractal H is obvious. In fact, all numbers in (33) are sufficiently close to Euclidean variables (27)-(28), i.e.

$$9\phi^{\frac{1}{4}} = 1,768840600$$

$$2\phi^{\frac{1}{2}} = 1,572302756$$

transforming (33) in $9\phi^{\frac{1}{4}}/(1-2\phi^{\frac{1}{2}}/n)^2/n^2$ and (31) in $9\phi^{\frac{1}{4}}/(1-(1-2\phi^{\frac{1}{2}}/n)^2)/n^2$. For internal symmetry (35), the parts’ ratio, the difference is only 0.000338763, just like 0.000344 in (24). This proves that internal H symmetry stems from chaotic/fractal behavior [4-5], Euclid’s golden number [1-3] or Penrose’s 5-fold symmetry [6], the most important, almost divine symmetry in nature [1-3].

VI. Discussion

(i) Spectral H data are accurately matched with a closed form quartic in 1/n. Unless for Lamb shifts, odd 1/n powers are absent in 1/n² and QED theories. If observed data [13] had 5 decimals, QED data in [12] could have been avoided, since all main intervals in Table 1 are also available from [13]. Only the smaller intervals remain with an error (for F in Table 1, a persisting error of only 100 kHz suggests Kelly data have a wrong 4th decimal for 4S and/or 6S).

(ii) Euclidean H harmony rests on algebra, overlooked for recoil [16], see Section II.2. We agree with Cagnac et al. [14] that reduced mass as used in relativistic theories, see Section III, does not make sense. Using reduced mass instead of mass at $n=\infty$ creates a huge error of about 60 cm⁻¹.

(iii) In the H₂ spectrum, natural asymptote $\frac{1}{2}e^2/r_{H}\approx78844.9$ cm⁻¹ shows as ionic energy $D_{\text{ion}}=e^2/r_{H}$ [16]: $r_{H}$ is close to observed separation 0.74 Å in H₂ [24] and gives fundamental H₂ frequency of 4410 cm⁻¹ [24]. With $r_{H}$ and $\varphi$, molecular H₂ and atomic H spectra are intimately linked [16].

(iv) Incidentally, an angle of 30°, typical for Euclid’s $\varphi$, also appears in the SM [25] as mixing angle for perpendicular interactions.

(v) Higher order terms in $\xi=a/n$ or (1-$\xi$) brings H theory in line with Kratzer-type expansions like

$$E_n=a_n\xi^2(1+a_1\xi^2+a_2\xi^4+a_3\xi^6+...$$

formally similar to but different than the more familiar Dunham expansion [26-30].

(vi) Results for D and the H nP series are given elsewhere. With a Sommerfeld-Dirac fine structure formula [31], the internal variable for H nP is $1.5/n$, rather than (35) for nS, which is responsible for the observed standard Lamb shift [22, 31]. Euclidean H-symmetry, brought about by natural radius $r_{H}$ is in line with Rydberg’s (1) and connects H terms with its most important property, mass $m_H$. H is not only prototypical for atomic and molecular physics [16]; it is prototypical for fractal behavior, in line with Mandelbrot [4]. We do not elaborate on discrete Euclidean geometries for composite H, conforming to Penrose 5-fold
symmetry [6]. Bohr’s model for composite H may well have to be refined on the basis of classical physics as suggested in Appendix A.

VII. Conclusion

Euclidean H symmetry only shows when H mass is directly linked to the H spectrum by virtue of its natural, classical radius $r_H$. Questions on conceptual, theoretical and practical (metrological) issues are outside the scope of this work. Definite conclusions depend on the observation of H 1S-3S [11].

References

[31] G. Van Hooydonk, physics/0612141
Fig. 1 $n\Delta_n$ versus $1/n$ (Δ), $1-1/n$ (□) (solid lines), $1,5/n$ (+) and $1-1,5/n$ (x) (dashes).

Fig. 2 Symmetry breaking curves in Euclidean H: $E_{n-differences}$ (31)-(33) in cm$^{-1}$ versus the appropriate Euclidean variable, see text): exact Mexican hat curve with $R_{\text{harm}}$ (full-line □), $E_1$ (short dashes o) and NIST's $R_\infty$ (broken dashes +).
Appendix A Comparison of classical and Bohr H theories

This self-explanatory table contains all formulae for a stable charge-conjugated two particle Coulomb system, subject to periodic motion. Main results and differences are in bold.

<table>
<thead>
<tr>
<th>Description</th>
<th>Classical H theory</th>
<th>Bohr H theory</th>
<th>#</th>
</tr>
</thead>
<tbody>
<tr>
<td>Energy E=T+V</td>
<td>E=½μv²-c²/r</td>
<td>idem</td>
<td>A1</td>
</tr>
<tr>
<td>Hamiltonian</td>
<td>E=½p²/μ-c²/r</td>
<td>idem</td>
<td>A2</td>
</tr>
<tr>
<td>Periodic motion</td>
<td>E=½μω²r²-c²/r</td>
<td>idem</td>
<td>A3</td>
</tr>
<tr>
<td>Repulsive force d/dr</td>
<td>μω²=½(½p²/μ)</td>
<td>idem</td>
<td>A4</td>
</tr>
<tr>
<td>Attractive force d/dr</td>
<td>e²/r²</td>
<td>idem</td>
<td>A5</td>
</tr>
<tr>
<td>Equal forces (Newton)</td>
<td>μv²r²=e²</td>
<td>idem</td>
<td>A6</td>
</tr>
<tr>
<td>Equal forces (Kepler, HO)</td>
<td>μv²=½(e²/r); μω²=e²/r²; ω²=e²/μr²; ω=√k/μ</td>
<td>vibrator or HO not considered</td>
<td>A7</td>
</tr>
<tr>
<td>Force constant k at rₖ</td>
<td>k=ε²/rₖ³</td>
<td>absent</td>
<td>A8</td>
</tr>
<tr>
<td>Constant periodicity dE/dω</td>
<td>μω²=μv²pr=C</td>
<td>μο²=μv²pr=nh</td>
<td>A9</td>
</tr>
<tr>
<td>Moment</td>
<td>p=C/r</td>
<td>p=nh/r</td>
<td>A10</td>
</tr>
<tr>
<td>Ratio A6/A9</td>
<td>v²=C²/μ</td>
<td>v²=(nh)/v</td>
<td>A11</td>
</tr>
<tr>
<td>H radius</td>
<td>r=C/(μv²-C²/μe²)</td>
<td>r=nh/(μv²/n²)</td>
<td>A12</td>
</tr>
<tr>
<td>Feedback of A10 in E (A1)</td>
<td>½p²/μ-e²/r=½μv²-μe²/C=½(μ²r²)-μe²/C²/(μ²r²)-e²/r²</td>
<td>½μv²-e²/r²=½μe²/(n²h²)-μe²/(n²h²)=-(R_i/n²)</td>
<td>A13</td>
</tr>
<tr>
<td>Feedback to dE/dr=0 at r₀</td>
<td>-C²/(μ²r²)+e²/r² or C²/μ=e²r₀</td>
<td>absent</td>
<td>A14</td>
</tr>
<tr>
<td>Feedback to E (A13)</td>
<td>E=½μc²/r₀²-e²/r²=½(2ε²/r₀)²-2ε²/r²</td>
<td>absent</td>
<td>A15</td>
</tr>
<tr>
<td>Feedback to d²E/dr²=k</td>
<td>k=3C²/(μ²r²)-2ε²/r²=3ε²/r₀²-2ε²/r²</td>
<td>absent</td>
<td>A16</td>
</tr>
<tr>
<td>Classical r definition using n</td>
<td>r²=πr₀</td>
<td>absent, replaced by A12 or r²=n²r₀</td>
<td>A17</td>
</tr>
<tr>
<td>Plugging (A17) in k (A16)</td>
<td>kₖ=k(1/n³)(1,5/n-1); k₁=e²/r₀³</td>
<td>absent</td>
<td>A18</td>
</tr>
<tr>
<td>Plugging (A17) in E (A15)</td>
<td>E=½ε²/(n²/2)</td>
<td>absent</td>
<td>A19</td>
</tr>
<tr>
<td>Adding E₀=½ε²/r₀ to (A19)</td>
<td>E=En+½ε²/r₀/2n²</td>
<td>absent</td>
<td>A20</td>
</tr>
<tr>
<td>Replacing 1/n by (1-1/n)</td>
<td>E=En[1-(1-1/n)]²-E₀/n²</td>
<td>see result A13</td>
<td>A21</td>
</tr>
<tr>
<td>Energy difference, terms Tₙ</td>
<td>Tₙ=Eₙ-E₀/n²/E₀[1-(1-1/n)]²</td>
<td>Tₙ=Rᵣ/2Rᵣ/n²=Rᵣ(1-1/n²)</td>
<td>A22</td>
</tr>
<tr>
<td>Identical T formulæ</td>
<td>n defined classically in (A17)</td>
<td>n in Bohr quantum hypothesis (A9)</td>
<td>A23</td>
</tr>
</tbody>
</table>

*HO is the classical Harmonic Oscillator

Force constant equations (A16)

A switch to complementary variable (A21) is a switch from (i) energy V=-e²/r in (A1) to energy difference ΔV=-e²/r+c²/n₀ and (ii) of moment p=C/r in (A10) to moment difference Δp=C(1/r-1/n₀). Kinetic and potential differences give ΔE=½(e²/n₀)²(1/n⁻²)(1-1/n)²=½ε²/n² (A21).

The usefulness of complementary variable 1-1/n in (A21), usually not considered for H theory, is illustrated by respective 4th order fits (2 digit version) of Eₙ in Table 2

| 1/n:                         | Eₙ=4,37/n²+5,55/n³+109677,59/n²+0,00/n+0,00 cm⁻¹ |
| (1-1/n):                     | Eₙ=4,37(1-1/n)²+11,91(1-1/n)³+109668,05(1-1/n)²-219354,37(1-1/n)+109678,77 cm⁻¹ |

Reducing H size classically in (A17) without a quantum theory gives the same results as Bohr’s quantum hypothesis for angular momentum (A9).