

A non-existence result on Cameron-Liebler line classes

J. De Beule, A. Hallez* and L. Storme

April 30, 2007

Abstract

Cameron-Liebler line classes are sets of lines in $\text{PG}(3, q)$ that contain a fixed number x of lines of every spread. Cameron and Liebler classified Cameron-Liebler line classes for $x \in \{0, 1, 2, q^2 - 1, q^2, q^2 + 1\}$ and conjectured that no others exist. This conjecture was disproven by Drudge for $q = 3$ [8] and his counterexample was generalised to a counterexample for any odd q by Bruen and Drudge [4]. A counterexample for q even was found by Govaerts and Penttinen [9]. Non-existence results on Cameron-Liebler line classes were found for different values of x . In this paper, we improve the non-existence results on Cameron-Liebler line classes of Govaerts and Storme [11], for q not a prime. We prove the non-existence of Cameron-Liebler line classes for $3 \leq x < \frac{q}{2}$.

1 Introduction

Cameron-Liebler line classes were introduced by Cameron and Liebler [5] in an attempt to classify collineation groups of $\text{PG}(n, q)$ that have equally many point orbits and line orbits. In their paper, they conjectured which groups these are. It is now known [2] that the conjecture is true when the group is irreducible, but there is no classification yet of Cameron-Liebler line classes. In this paper, new non-existence results are presented.

There are many equivalent definitions for Cameron-Liebler line classes. Following Penttinen [15], a *clique* in $\text{PG}(3, q)$ is either the set of all lines through a point P , denoted by $\text{star}(P)$, or dually the set of all lines in a plane π , denoted by $\text{line}(\pi)$. The planar pencil of lines in a plane π through a point P is denoted by $\text{pen}(P, \pi)$.

Definition 1.1 (Cameron and Liebler [5], Penttinen [15]) Let \mathcal{L} be a set of lines in $\text{PG}(3, q)$ and let $\chi_{\mathcal{L}}$ be its characteristic function. Then \mathcal{L} is called a *Cameron-Liebler line class* if one of the following equivalent conditions is satisfied.

*The research of the second author is supported by a research grant of the Research council of Ghent University.

1. There exists an integer x such that $|\mathcal{L} \cap \mathcal{S}| = x$ for all spreads \mathcal{S} .
2. There exists an integer x such that for every incident point-plane pair (P, π)

$$|\text{star}(P) \cap \mathcal{L}| + |\text{line}(\pi) \cap \mathcal{L}| = x + (q + 1)|\text{pen}(P, \pi) \cap \mathcal{L}|. \quad (1)$$

3. There exists an integer x such that for every line l of $\text{PG}(3, q)$

$$|\{m \in \mathcal{L} : m \text{ meets } l, m \neq l\}| = (q + 1)x + (q^2 - 1)\chi_{\mathcal{L}}(l). \quad (2)$$

The parameter x is called the *parameter* of the Cameron-Liebler line class. We note that the first definition implies that $x \in \{0, 1, 2, \dots, q^2 + 1\}$. Cameron and Liebler [5] showed that a Cameron-Liebler line class of parameter x consists of $x(q^2 + q + 1)$ lines and that the only Cameron-Liebler line classes for $x = 1$ are the cliques, i.e., all lines through a point or all lines in a plane, and for $x = 2$ the unions of two disjoint cliques. They also noted that the complement of a Cameron-Liebler line class with parameter x is a Cameron-Liebler line class with parameter $q^2 + 1 - x$. So, it suffices to study Cameron-Liebler line classes with parameter $x \leq \lfloor (q^2 + 1)/2 \rfloor$. Thus, the case $q = 2$ was immediately solved. In their paper, Cameron and Liebler conjectured that no other Cameron-Liebler line classes exist.

Penttila [15] shows that for $q \neq 2$ there exist no Cameron-Liebler line classes with parameter $x = 3$ or $x = 4$, with possible exception of the cases $(x, q) \in \{(4, 3), (4, 4)\}$. Bruen and Drudge [3] prove the non-existence of Cameron-Liebler line classes with parameter $2 < x \leq \sqrt{q}$. Drudge [8] excludes the existence of a Cameron-Liebler line class with parameter $x = 4$ in $\text{PG}(3, 3)$, and proves that for $q \neq 2$ there exist no Cameron-Liebler line classes with parameter $2 < x \leq \epsilon$, where $q + 1 + \epsilon$ denotes the size of the smallest nontrivial blocking sets in $\text{PG}(2, q)$. He also gives a counterexample to the conjecture of Cameron and Liebler: a Cameron-Liebler line class with parameter $x = 5$ in $\text{PG}(3, 3)$, in this way settling the case $q = 3$. Bruen and Drudge [4] then construct a Cameron-Liebler line class with parameter $x = (q^2 + 1)/2$ for any odd q . In [9], Govaerts and Penttila completed the study of the case $x = 4$ by showing that there exists no Cameron-Liebler line class with parameter $x = 4$ in $\text{PG}(3, 4)$. In [9], Govaerts and Penttila also disproved the conjecture of Cameron and Liebler for q even by showing the existence of a Cameron-Liebler line class with parameter $x = 7$ in $\text{PG}(3, 4)$.

In this paper, new bounds on x for the non-existence of Cameron-Liebler line classes with parameter x are obtained. We improve the results of Govaerts and Storme for q not prime. They proved the following two theorems and corollary [11].

Theorem 1.2 *In $\text{PG}(3, q)$, q prime, $q > 2$, there exist no Cameron-Liebler line classes with parameter $2 < x \leq q$.*

Theorem 1.3 (1) *In $\text{PG}(3, q)$, q square, there exist no Cameron-Liebler line classes with parameter $2 < x \leq \min(\epsilon', q^{3/4})$, where $q + 1 + \epsilon'$ denotes the size of the smallest nontrivial blocking sets in $\text{PG}(2, q)$ not containing a Baer subplane.*

(2) *Let $q = p^{3h}$, $p \geq 7$ prime, $h \geq 1$ odd, and let $q + 1 + \epsilon''$ denote the size of the smallest nontrivial blocking sets in $\text{PG}(2, q)$ containing neither a minimal blocking set of*

size $q+p^{2h}+1$, nor one of size $q+p^{2h}+p^h+1$. In $\text{PG}(3, q)$, there exist no Cameron-Liebler line classes with parameter $2 < x \leq \min(\epsilon'', q^{5/6})$.

(3) Let $q = p^{3h}$, $p \geq 7$ prime, $h > 1$ even, and let $q + 1 + \epsilon''$ denote the size of the smallest nontrivial blocking sets in $\text{PG}(2, q)$ containing neither a Baer subplane, nor a minimal blocking set of size $q + p^{2h} + 1$, nor one of size $q + p^{2h} + p^h + 1$. In $\text{PG}(3, q)$, there exist no Cameron-Liebler line classes with parameter $2 < x \leq \min(\epsilon'', q^{3/4})$.

Corollary 1.4 (1) Let q be a square, $q = p^h$, p prime.

1. If $q > 16$, then there exist no Cameron-Liebler line classes in $\text{PG}(3, q)$ with parameter $2 < x \leq c_p q^{2/3}$, where c_p equals $2^{-1/3}$ when $p \in \{2, 3\}$ and 1 when $p \geq 5$.

2. If $p > 3$ and $h = 2$, then there exist no Cameron-Liebler line classes in $\text{PG}(3, q)$ with parameter $2 < x \leq q^{3/4}$.

(2) Let $q = p^3$, $p \geq 7$ prime, then there exist no Cameron-Liebler line classes in $\text{PG}(3, q)$ with parameter $2 < x \leq q^{5/6}$.

(3) Let $q = p^6$, $p \geq 7$ prime, then there exist no Cameron-Liebler line classes in $\text{PG}(3, q)$ with parameter $2 < x \leq q^{3/4}$.

We improve these results for q not prime. Theorem 4.2 gives a new improved bound for general $q \neq 2$, q not prime.

This theorem will be proven by studying how the lines of the Cameron-Liebler line class with parameter x correspond with x -tight sets on $Q^+(5, q)$ and $\{x(q^2+q+1), x(q+1); 5, q\}$ -minihypers contained in the Klein quadric $Q^+(5, q)$. Using properties of the associated $\{x(q^2+q+1), x(q+1); 5, q\}$ -minihyper combined with the fact that this minihyper lives on $Q^+(5, q)$, gives us new non-existence results on Cameron-Liebler line classes.

2 Definitions and preliminary results

Let $v_{n+1} = (q^{n+1} - 1)/(q - 1)$ denote the number of points of $\text{PG}(n, q)$.

An i -tight set of a finite generalised quadrangle was introduced by Payne [13, 14] and was generalised to polar spaces of higher rank by Drudge [7].

Definition 2.1 A set of points \mathcal{T} of a finite polar space of rank $r \geq 2$ over a finite field of order q is i -tight if

$$|P^\perp \cap \mathcal{T}| = \begin{cases} i \frac{q^{r-1}-1}{q-1} + q^{r-1} & \text{if } P \in \mathcal{T} \\ i \frac{q^{r-1}-1}{q-1} & \text{if } P \notin \mathcal{T}. \end{cases}$$

This definition poses restrictions on the intersection of a hyperplane with a point set. This has a lot in common with the concept of the minihypers.

Definition 2.2 An $\{f, m; n, q\}$ -minihyper is a pair (F, w) , where F is a subset of the point set of $\text{PG}(n, q)$ and w is a weight function $w : \text{PG}(n, q) \rightarrow \mathbb{N} : P \mapsto w(P)$, satisfying

1. $w(P) > 0 \Leftrightarrow P \in F$,
2. $\sum_{P \in F} w(P) = f$, and

3. $\min\{\sum_{P \in H} w(P) : H \text{ is a hyperplane}\} = m.$

The weight function w determines the set F completely. When this function has only the values 0 and 1, then (F, w) is determined completely by the set F . In this paper, this will always be the case, so we will not make any further reference to the weight function w .

In this paper, we are interested in the $\{x(q^2+q+1), x(q+1); 5, q\}$ -minihypers contained in the Klein quadric $Q^+(5, q)$, and associated with the Cameron-Liebler line classes with parameter x . The following results discuss the intersections of subspaces with these minihypers. They will be very crucial to prove the improved results on the non-existence of Cameron-Liebler line classes. The first theorem is stated as a corollary in [6].

Theorem 2.3 *Let F be a $\{\sum_{i=0}^{n-1} \epsilon_i v_{i+1}, \sum_{i=1}^{n-1} \epsilon_i v_i; n, q\}$ -minihyper, where $q > h, 0 \leq \epsilon_i \leq q-1, 0 \leq i \leq n-1, \sum_{i=0}^{n-1} \epsilon_i = h$.*

Then a plane of $PG(n, q)$ is either contained in F or intersects it in an $\{m_1(q+1) + m_0, m_1; 2, q\}$ -minihyper, where $m_1 + m_0 \leq h$.

Theorem 2.4 (Hamada [12]) *Let F be a $\{\sum_{i=0}^{n-1} \epsilon_i v_{i+1}, \sum_{i=1}^{n-1} \epsilon_i v_i; n, q\}$ -minihyper, where $0 \leq \epsilon_i \leq q-1, i = 0, \dots, n-1$. Then $|F \cap \Delta| \geq \sum_{i=1}^{n-1} \epsilon_i v_{i-1}$ for any $(n-2)$ -space Δ in $PG(n, q)$ and $|F \cap G| = \sum_{i=1}^{n-1} \epsilon_i v_{i-1}$ for some $(n-2)$ -spaces G in $PG(n, q)$.*

Let $H_j, j = 1, 2, \dots, q+1$, be the $q+1$ hyperplanes in $PG(n, q)$ that pass through an $(n-2)$ -space G intersecting F in $\sum_{i=1}^{n-1} \epsilon_i v_{i-1}$ points. Then $F \cap H_j$ is a

$$\{\delta_j + \sum_{i=1}^{n-1} \epsilon_i v_i, \sum_{i=1}^{n-1} \epsilon_i v_{i-1}; n-1, q\}\text{-minihyper}$$

in H_j for $j = 1, 2, \dots, q+1$, where the δ_j are some non-negative integers such that $\sum_{j=1}^{q+1} \delta_j = \epsilon_0$.

In the case of a $\{\delta v_{\mu+1}, \delta v_\mu; n, q\}$ -minihyper, the parameters in Hamada's theorem become very nice. In the remainder of this article, we will only consider minihypers of this form. The next result of [10] is fundamental for the induction arguments used in the lemmas and theorem which follow.

Lemma 2.5 (Govaerts and Storme [10]) *Let (F, w) be a $\{\delta v_{\mu+1}, \delta v_\mu; n, q\}$ -minihyper satisfying $0 \leq \delta \leq (q+1)/2, 0 \leq \mu \leq n-1$, and containing a μ -space π_μ . Then the minihyper (F', w') defined by the weight function w' , where*

- $w'(p) = w(p) - 1$, for $p \in \pi_\mu$, and
- $w'(p) = w(p)$, for $p \in PG(n, q) \setminus \pi_\mu$,

is a $\{(\delta-1)v_{\mu+1}, (\delta-1)v_\mu; n, q\}$ -minihyper.

It is easy to see that minihypers are closely related to blocking sets. A $\{\delta v_{\mu+1}, \delta v_\mu; n, q\}$ -minihyper is a δv_μ -fold blocking set. We state some useful definitions on blocking sets.

Definition 2.6 A k -fold blocking set in $\text{PG}(n, q)$ is a set of points that intersects every hyperplane in at least k points.

A k -fold blocking set is called *minimal* if no proper subset is a k -fold blocking set.

A 1-fold blocking set is simply called a *blocking set*. It is called *trivial* if it contains a line.

Theorem 2.7 • (Szőnyi [16]) A 1-fold blocking set B in $\text{PG}(2, q)$, of size $|B| < q + \frac{q+3}{2}$, where $q = p^h$, p prime, $h \geq 1$, is uniquely reducible to a minimal blocking set B' intersecting every line in $1 \pmod{p}$ points.

• (Szőnyi and Weiner [17]) A minimal 1-fold blocking set B in $\text{PG}(n, q)$, $n \geq 3$, $q = p^h$, $p > 2$ prime, $h \geq 1$, of size $|B| < q + \frac{q}{2}$, intersects every line in zero points or in $1 \pmod{p}$ points.

3 Minihypers on the Klein quadric

It is our intention to prove the non-existence of Cameron-Liebler line classes of parameter $2 < x < \frac{q}{2}$ in $\text{PG}(3, q)$ by using $\{x(q^2 + q + 1), x(q + 1); 5, q\}$ -minihypers F contained in the Klein quadric $\text{Q}^+(5, q)$.

Consider an $\{x(q^2 + q + 1), x(q + 1); 5, q\}$ -minihyper F , with $x < \frac{q}{2}$, on $\text{Q}^+(5, q)$. We know that a hyperplane H intersects $\text{Q}^+(5, q)$ in either a parabolic quadric $\text{Q}(4, q)$ or in a tangent cone $\langle R, \text{Q}^+(3, q) \rangle$ with vertex R in $\text{Q}^+(5, q)$ and base a 3-dimensional hyperbolic quadric $\text{Q}^+(3, q)$.

Lemma 3.1 Let F be an $\{x(q^2 + q + 1), x(q + 1); 5, q\}$ -minihyper, with $x < \frac{q}{2}$, contained in the Klein quadric $\text{Q}^+(5, q)$, and let H_0 be a hyperplane in $\text{PG}(5, q)$ such that $H_0 \cap \text{Q}^+(5, q) = \langle R, \text{Q}^+(3, q) \rangle$ and such that $H_0 \cap F$ is an $\{x(q + 1), x; 4, q\}$ -minihyper. Then there exists a solid in H_0 , not containing R , intersecting F in exactly x points.

Proof First of all, $|H_0 \cap F| = x(q + 1) < \frac{q^2 + q}{2}$. Consider a point R' of $\text{Q}^+(5, q) \cap H_0$ with $R' \notin F$, $R' \neq R$. There are $q^3 + q^2 + q + 1$ lines in H_0 through R' . At most $\frac{q^2 + q}{2}$ of them can contain a point of F , so there exists a line l through R' having an empty intersection with F and not containing R . Similarly, we can find a plane π through l having an empty intersection with F . The $q + 1$ solids through π together contain $x(q + 1)$ points of F and each one of them contains at least x points of F (Theorem 2.4). This means that every solid through π contains exactly x points of F . Choose one of those solids, not containing R , and this is the desired solid. \square

Lemma 3.2 Let F' be an $\{x(q + 1), x; 4, q\}$ -minihyper, $x < \frac{q}{2}$, contained in $\text{Q}(4, q)$. Then F' is the union of x pairwise disjoint lines.

Proof For every point $R \in F'$, we find a plane π through R only intersecting F' in R . Then consider all solids through π , they all contain at least $x - 1$ other points of F' , since every solid contains at least x points of F' . There remain $x(q + 1) - 1 - (q + 1)(x - 1) = q$ other points of F' . So some hyperplane K_0 through π contains more than x points of F' .

By [10, Corollary 2], $K_0 \cap F'$ is a blocking set with respect to the planes of K_0 .

Consider the minimal blocking set B inside $K_0 \cap F'$. Suppose that B is not a line.

Take three non-collinear points $R_1, R_2, R_3 \in B$. Every line intersects B in zero or in $1 \pmod{p}$ points (Theorem 2.7). The line $l_1 = \langle R_1, R_2 \rangle$ already contains two points of B , so must contain at least $1 + p \geq 3$ points of B . A line containing more than two points of a quadric lies on that quadric. Similarly, the lines $l_2 = \langle R_1, R_3 \rangle$ and $l_3 = \langle R_2, R_3 \rangle$ are lines of $Q(4, q)$. Consider the plane π spanned by l_1, l_2 and l_3 . Since these three lines are lines of $Q(4, q)$, π is contained in $Q(4, q)$, which is impossible.

Thus the minimal blocking set B is a line, hence the minihyper F' contains a line l . By Lemma 2.5, we have that $F' \setminus l$ is an $\{(x-1)(q+1), x-1; 4, q\}$ -minihyper. Repeating the previous arguments x times gives us that F' is the union of x pairwise disjoint lines. \square

Lemma 3.3 *Suppose that F is an $\{x(q^2 + q + 1), x(q + 1); 5, q\}$ -minihyper, with $x < \frac{q}{2}$. Suppose that P is a point of F lying on two lines l_1, l_2 , completely contained in F . Then the plane $\langle l_1, l_2 \rangle$ is completely contained in F .*

Proof Suppose that the plane $\langle l_1, l_2 \rangle \not\subseteq F$, then $F \cap \langle l_1, l_2 \rangle$ is an $\{m_1(q+1) + m_0, m_1; 2, q\}$ -minihyper F' , where $m_1 + m_0 \leq x < \frac{q}{2}$ (Theorem 2.3). Furthermore, $l_1 \cup l_2 \subseteq F$, implying that $|\langle l_1, l_2 \rangle \cap F| \geq 2q + 1$, which implies $m_1 \geq 2$. So $\langle l_1, l_2 \rangle \cap F$ is a t -fold blocking set, with $m_1 = t \geq 2$. Assume now that $|\langle l_1, l_2 \rangle \cap F| = tq + a$, with $a = m_0 + m_1 \leq x$.

Considering the lines l_1 and l_2 , and the other $q - 1$ lines of $\langle l_1, l_2 \rangle$ on P , we find that $|\langle l_1, l_2 \rangle \cap F| \geq 2q + 1 + (q - 1)(t - 1) = (t + 1)q - t + 2$. Hence, $|\langle l_1, l_2 \rangle \cap F| = tq + a \geq (t + 1)q - t + 2$, implying $a \geq q - t + 2$. Now $\langle l_1, l_2 \rangle \cap F$ is a t -fold blocking set of size $tq + a$. Note that $a \leq x < \frac{q}{2}$, giving $t \geq \frac{q}{2} + 2$, a contradiction since $t < \frac{q}{2}$. We conclude that $\langle l_1, l_2 \rangle \subseteq F$. \square

4 Cameron-Liebler line classes and minihypers

We can now prove the following theorem.

Theorem 4.1 *An $\{x(q^2 + q + 1), x(q + 1); 5, q\}$ -minihyper, with $x < \frac{q}{2}$, contained in $Q^+(5, q)$ is the union of x pairwise disjoint planes. So for $x \geq 3$, such a minihyper does not exist.*

Proof From Theorem 2.4, we can find a solid Δ which intersects F in x points, and such that the $q + 1$ hyperplanes through Δ intersect F in an $\{x(q + 1), x; 4, q\}$ -minihyper F' . These $q + 1$ hyperplanes intersect $Q^+(5, q)$ in either a tangent cone or in a non-singular parabolic quadric $Q(4, q)$.

We can make sure that at least $q - 1$ hyperplanes through Δ intersect $Q^+(5, q)$ in non-singular parabolic quadrics. If at least one of them intersects $Q^+(5, q)$ in a tangent cone $\langle R, Q^+(3, q) \rangle$, Lemma 3.1 says that we can choose Δ in this hyperplane in such a way that Δ intersects $Q^+(5, q)$ in a 3-dimensional hyperbolic quadric. The polarity of the Klein quadric then implies that only two hyperplanes through Δ intersect $Q^+(5, q)$ in tangent cones.

The $\{x(q+1), x; 4, q\}$ -minihypers F' which are the intersection of the other $q-1$ hyperplanes H_1, \dots, H_{q-1} through Δ with F are contained in non-singular parabolic quadrics and so are the union of x pairwise disjoint lines (Lemma 3.2). Each line of the minihyper $H_i \cap F$ intersects Δ in a point. Suppose that P is a point of $\Delta \cap F$. Then P lies on one line of each minihyper $H_i \cap F$, so P lies on at least two lines of the minihyper F . From Lemma 3.3, we know that the plane π spanned by these lines is completely contained in F . Using Lemma 2.5, we have that $F \setminus \pi$ is an $\{(x-1)(q^2+q+1), (x-1)(q+1); 5, q\}$ -minihyper. With $x' = x-1 < \frac{q}{2}$, we can repeat the previous arguments.

Doing this x times gives us that F is the union of x pairwise disjoint planes. But three planes cannot be pairwise disjoint in $Q^+(5, q)$. So this minihyper does not exist when $x \geq 3$. \square

We now state the new non-existence results on Cameron-Liebler line classes.

Theorem 4.2 *In $PG(3, q)$, $q \geq 3$, there exist no Cameron-Liebler line classes with parameter $2 < x < \frac{q}{2}$.*

Proof Let \mathcal{L} be a Cameron-Liebler line class with parameter x . A line l intersects $x(q+1)$ lines of \mathcal{L} if $l \notin \mathcal{L}$ and l intersects $(q+1)x + q^2$ lines of \mathcal{L} , including l , if $l \in \mathcal{L}$ (Definition 1.1).

Translated via the Klein correspondence, \mathcal{L} defines a set \mathcal{T} on $Q^+(5, q)$ such that

$$|P^\perp \cap \mathcal{T}| = \begin{cases} x(q+1) + q^2 & \text{if } P \in \mathcal{T} \\ x(q+1) & \text{if } P \notin \mathcal{T}, P \in Q^+(5, q). \end{cases}$$

So \mathcal{T} defines an x -tight set on $Q^+(5, q)$, with $|\mathcal{L}| = |\mathcal{T}| = x(q^2 + q + 1)$. So [1, Theorem 12] implies that \mathcal{T} defines an $\{x(q^2 + q + 1), x(q+1); 5, q\}$ -minihyper F on $Q^+(5, q)$. We only need to check that \mathcal{T} generates $PG(5, q)$.

Since $|\mathcal{T}| \geq 3(q^2 + q + 1)$, $\dim \langle \mathcal{T} \rangle \geq 4$. If $\dim \langle \mathcal{T} \rangle = 4$, then $\langle \mathcal{T} \rangle \cap Q^+(5, q) = Q(4, q)$ since \mathcal{T} is not contained in a tangent hyperplane to $Q^+(5, q)$.

Since $|\mathcal{T}| < |Q(4, q)|$, let $R \in Q(4, q) \setminus \mathcal{T}$. Consider in $T_R(Q(4, q))$ a plane only intersecting $Q(4, q)$ in R . This plane then lies in the tangent hyperplane $T_R(Q(4, q))$ and in q hyperplanes sharing an elliptic quadric $Q^-(3, q)$ with $Q(4, q)$.

These elliptic quadrics $Q^-(3, q)$ define via the Klein correspondence regular spreads of $PG(3, q)$ sharing x lines with \mathcal{L} (Definition 1.1), so these elliptic quadrics contain x points of \mathcal{T} . Since R^\perp contains $x(q+1)$ points of \mathcal{T} , we find that, in total, \mathcal{T} would contain $x(q+1) + xq = 2xq + x$ points. But this is false, since $|\mathcal{T}| = x(q^2 + q + 1)$.

So, it is indeed true that \mathcal{T} defines an $\{x(q^2 + q + 1), x(q+1); 5, q\}$ -minihyper F on $Q^+(5, q)$. But Theorem 4.1 states that this minihyper does not exist, so we conclude that the Cameron-Liebler line classes with parameter $3 \leq x < \frac{q}{2}$ do not exist. \square

References

- [1] J. Bamberg, S. Kelly, M. Law, and T. Penttila. Tight sets and m -ovoids of finite polar spaces. *J. Combin. Theory, Ser. A*, to appear.

- [2] J. Bamberg and T. Penttila. Overgroups of cyclic Sylow subgroups of linear groups. *Communications in Algebra*, submitted.
- [3] A. A. Bruen and K. Drudge. On the non-existence of certain Cameron-Liebler line classes in $\text{PG}(3, q)$. *Des. Codes Cryptogr.*, 14(2):127–132, 1998.
- [4] A. A. Bruen and K. Drudge. The construction of Cameron-Liebler line classes in $\text{PG}(3, q)$. *Finite Fields Appl.*, 5(1):35–45, 1999.
- [5] P. J. Cameron and R. A. Liebler. Tactical decompositions and orbits of projective groups. *Linear Algebra Appl.*, 46:91–102, 1982.
- [6] J. De Beule, K. Metsch, and L. Storme. Characterization results on arbitrary non-weighted minihypers and on linear codes meeting the Griesmer bound. *Des. Codes Cryptogr.*, submitted.
- [7] K. Drudge. *Extremal sets in projective and polar spaces*. PhD Thesis, The University of Western Ontario, 1998.
- [8] K. Drudge. On a conjecture of Cameron and Liebler. *European J. Combin.*, 20(4):263–269, 1999.
- [9] P. Govaerts and T. Penttila. Cameron-Liebler line classes in $\text{PG}(3, 4)$. *Bull. Belgian Math. Soc., Simon Stevin*, 12:793–804, 2005.
- [10] P. Govaerts and L. Storme. On a particular class of minihypers and its applications. I: The result for general q . *Des. Codes Cryptogr.*, 28:51–63, 2003.
- [11] P. Govaerts and L. Storme. On Cameron-Liebler line classes. *Adv. Geom.*, 4:279–286, 2004.
- [12] N. Hamada. A characterization of some $[n, k, d; q]$ -codes meeting the Griesmer bound using a minihyper in a finite projective geometry. *Discrete Math.*, 116(1-3):229–268, 1993.
- [13] S. E. Payne. Tight pointsets in finite generalized quadrangles. *Congr. Numer.*, 60:243–260, 1987.
- [14] S. E. Payne. Tight pointsets in finite generalized quadrangles. II. *Congr. Numer.*, 77:31–41, 1990.
- [15] T. Penttila. Cameron-Liebler line classes in $\text{PG}(3, q)$. *Geom. Dedicata*, 37(3):245–252, 1991.
- [16] T. Szőnyi. Blocking sets in Desarguesian affine and projective planes. *Finite Fields Appl.*, 3(3):187–202, 1997.
- [17] T. Szőnyi and Zs. Weiner. Small blocking sets in higher dimensions. *J. Combin. Theory, Ser. A*, 95:88–101, 2001.

Address of the authors: Department of pure mathematics and computer algebra,
Ghent University, Krijgslaan 281-S22, 9000 Ghent, Belgium.

J. De Beule: jdebeule@cage.ugent.be, <http://cage.ugent.be/~jdebeule>

A. Hallez: athallez@cage.ugent.be

L. Storme: ls@cage.ugent.be, <http://cage.ugent.be/~ls>