# The metaplectic Howe duality and polynomial solutions for the symplectic Dirac operator

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**Abstract**

We study various aspects of the metaplectic Howe duality realized by Fischer decomposition for the metaplectic representation space of polynomials on $\mathbb{R}^{2n}$ valued in the Segal-Shale-Weil representation. As a consequence, we determine symplectic monogenics, i.e. the space of polynomial solutions of the symplectic Dirac operator $D_s$.

**Key words:** Symplectic Dirac operator, symplectic monogenics, Fischer decomposition, Howe duality.

**MSC classification:** 30G35, 37J05, 37J05, 17B10.

## 1 Introduction

The classical topic of separation of variables, realized by a Howe dual pair acting on the representation of interest, is one of the cornerstones in harmonic analysis. The general abstract classification scheme of the Howe correspondence is formulated in [14], [15]. One of the geometrically most interesting classical examples leading to the notion of spherical harmonic is the Howe dual pair $O(n) \times sl(2, \mathbb{R})$, realized by orthogonal Lie group $O(n)$ acting on the space of polynomials on $\mathbb{R}^n$. The refinement given by a "double cover" of this dual pair is associated to the case of spinor-valued polynomials, realized by $Pin(n) \times osp(1|2)$ and known in classical Clifford analysis as the Fischer decomposition for the space of spinor-valued polynomials, see [2] for more details. A similar situation appears also in the case of spinor-valued forms, where $(Pin(n), sl(2, \mathbb{R}))$ can be regarded as a "double cover" of the classical Howe dual pair $(O(n), o(3))$ used to decompose the space of spinor valued differential forms on a $Spin(n)$-manifold into irreducible subbundles, [19]. Recently, a lot of attention (see e.g. [6, 7, 8, 9]) was devoted to the theory of harmonic and Clifford analysis on superspace, leading to dual pairs $Sp(2n, \mathbb{R}) \times sl(2, \mathbb{R})$ and $Sp(2n, \mathbb{R}) \times osp(1|2)$. However, in these papers, the Fischer decomposition of polynomials with values in a suitable representation was not addressed. This issue was later settled in [3], using the dual pair of superalgebras $(osp(m|2n), osp(1|2))$

In the present article we start with the polynomial algebra on an even dimensional symplectic vector space $(\mathbb{R}^{2n}, \omega)$, $\omega \in \wedge^2(\mathbb{R}^{2n})^\star$. The analogue of the spinor representation in this situation was described many years ago by B. Kostant, who introduced for the purposes of representation theory and geometric quantization a symplectic analogue of the Dirac operator called the symplectic Dirac operator $D_s$, see [17]. The symplectic Dirac operator was studied mainly from the geometrical point of view, see [13] and references therein, and also as an invariant differential operator in [16]. The spectral properties of symplectic Dirac operator are difficult to obtain and as for its kernel, basically nothing is known up to now.

On the other hand, the general abstract algebraic classification scheme of reductive dual pairs [15] shows the existence of the dual pair $sp(2n,\mathbb{R})\times sl(2,\mathbb{R})\subset sp(6n,\mathbb{R})$, responsible for the multiplicity free decomposition of the space of polynomial symplectic spinors. However, its natural geometrical model comprising the intrinsic action of the metaplectic lift of this dual pair was not constructed. Let us emphasize that some of the generators of the Howe dual partner (in our case, the symplectic Dirac operator) are the starting point in the geometric analysis of geometrical structures on manifolds (in our case, the space of symplectic spinors on the symplectic space $(\mathbb{R}^{2n},\omega)$.)

The aim of our paper is to fill this gap by describing the full analogue of the Fischer decomposition for the polynomials on the symplectic vector space with values in the vector space of Kostant's spinors, naturally including the symplectic Dirac operator and symplectic Clifford algebras. Inspired by the terminology used for the Dirac operator in *Spin*-geometry, we call symplectic monogenics particular irreducible pieces in the solution space of the symplectic Dirac operator. In addition, our geometric realization leads to the dual pair $(Mp(2n,\mathbb{R}),sl(2,\mathbb{R}))$ as a "double cover" of the classical Howe dual pair $(Sp(2n,\mathbb{R}),so(2,1))$ in the space of endomorphisms of polynomials on $\mathbb{R}^{2n}$ valued in the Segal-Shale-Weil representation of the metaplectic Lie group $Mp(2n,\mathbb{R})$. Following general principles of Howe dual pairs, this is the underlying structure responsible for the Fischer decomposition mentioned above.

It follows from general abstract principles that the existence of the Howe dual pair $G_1\times\mathfrak{g}_2$ allows to separate variables (in certain specific situations termed Fischer decomposition). This amounts to express the elements of the representation space as $\sum_{i\in I}R_i\otimes S_i$, where $I$ is an index set depending on the representation and $R_i$ resp. $S_i$ is an irreducible representation of $G_1$ resp. $\mathfrak{g}_2$. In our specific case $G_1=Mp(2n,\mathbb{R})$, $\mathfrak{g}_2=sl(2,\mathbb{R})$ and the representation on polynomials on the symplectic vector space valued in the Segal-Shale-Weil representation decomposes into non-isomorphic irreducible infinite dimensional highest weight Verma modules for $\mathfrak{g}_2=sl(2,\mathbb{R})$.

The paper is organized as follows. In Section 2 and 3 we repeat some well-known facts on symplectic Lie algebras and their finite and at the same time highest weight infinite dimensional representations. In Section 4 we define the symplectic Dirac operator and show how it appears in the realization of the dual partner $sl(2,\mathbb{R})$. In Section 5 we obtain the Fischer decomposition and construct explicit projection operators on all summands together with the consequences for the kernel of the symplectic Dirac operator on $(\mathbb{R}^{2n},\omega)$. We end with some conclusions and an outlook for further research.

## 2 Symplectic Lie algebra, symplectic Clifford algebra and simple highest weight modules for $sp(2n, \mathbb{R})$

In this section we review several basic facts related to the structure of the simple Lie algebra $sp(2n, \mathbb{R})$ and its representation theory, see e.g. [12], and also symplectic Clifford algebras, see e.g. [4], [13], [16].

Let $\epsilon_1, \dots, \epsilon_n$ be the vectors of the canonical basis of $\mathbb{C}^n$. Identify $\mathbb{C}^n$ with the dual of the Cartan subalgebra $\mathfrak{h}^\star$ such that the root system of $sp(2n, \mathbb{R})$ is

$$\{\pm(\epsilon_i \pm \epsilon_j) : 1 \le i < j \le n\} \cup \{\pm 2\epsilon_i : 1 \le i \le n\},$$

with the set of simple roots

$$\triangle = \{\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_3, \dots, \alpha_{n-1} = \epsilon_{n-1} - \epsilon_n, \alpha_n = 2\epsilon_n\}$$

and fundamental weights $\omega_1, \dots, \omega_n$.

Let us consider the symplectic vector space $(\mathbb{R}^{2n}, \omega)$ and a symplectic basis $e_1, \dots, e_n$, $f_1, \dots, f_n$ with respect to the non-degenerate two form $\omega$ on $\mathbb{R}^{2n}$. Let $E_{i,j}$ be the $2n \times 2n$ matrix with 1 on the intersection of the $i$-th row and $j$-th column, and zero otherwise. Then the symplectic Lie algebra $sp(2n, \mathbb{R})$ is generated by

$$X_{ij} = E_{i,j} - E_{n+j,n+i}, \ Y_{ij} = E_{i,n+j} + E_{j,n+i}, \ Z_{ij} = E_{n+i,j} + E_{n+j,i}$$

for $i, j \in \{1, \dots, n\}$.

The metaplectic Lie algebra $mp(2n, \mathbb{R})$ is a Lie algebra attached to the twofold covering $\rho : Mp(2n, \mathbb{R}) \to Sp(2n, \mathbb{R})$ of the symplectic Lie group $Sp(2n, \mathbb{R})$. It can be realized by homogeneity two elements in the symplectic Clifford algebra $Cl_s(\mathbb{R}^{2n}, \omega)$, where the homomorphism $\rho_\star : mp(2n, \mathbb{R}) \to sp(2n, \mathbb{R})$ is given by

$$\begin{aligned} &\rho_\star(e_i e_j) = -Y_{ij}, \\ &\rho_\star(f_i f_j) = Z_{ij}, \\ &\rho_\star(e_i f_j + f_j e_i) = 2X_{ij} \end{aligned} \tag{2.1}$$

for $i, j \in \{1, \dots, n\}$. Recall that $Cl_s(\mathbb{R}^{2n}, \omega)$ is an associative unital algebra, realized as a quotient of the tensor algebra $T(e_1, \dots, e_n, f_1, \dots, f_n)$ by a two-sided ideal $I \subset T(e_1, \dots, e_n, f_1, \dots, f_n)$ generated by

$$v_i \cdot v_j - v_j \cdot v_i = -2\omega(v_i, v_j)$$

for all $v_i, v_j \in \mathbb{R}^{2n}$.

There is another useful realization of the symplectic Lie algebra as a subalgebra of the Weyl algebra $W_n$ of rank $n$. Let $x_i$, $i \in \{1, \dots, n\}$, be the generators of the algebra of polynomials. The Weyl algebra is an associative algebra generated by $\{x_i, \partial_i\}$, $i \in \{1, \dots, n\}$, the multiplication operators $x_i$ and partial differentiation with respect to $x_i$, acting on polynomials on $\mathbb{R}^n$. The root spaces of $sp(2n, \mathbb{R})$ corresponding to simple positive roots $\alpha_i$ are spanned by $x_{i+1}\partial_i$, $i \in \{1, \dots, n-1\}$ and $\alpha_n$ is spanned by $-\frac{1}{2}\partial_n^2$.

The Segal-Shale-Weil representation $\mathcal{C}$ is the highest weight unitary representation of $Mp(2n,\mathbb{R})$ on the vector space $L^2(\mathbb{R}^n, d\mu)$, where $d\mu = \exp^{-||x||^2} dx_{\mathbb{R}^n}$ with $dx_{\mathbb{R}^n}$ the Lebesgue measure on $\mathbb{R}^n$. We take for the basis of $L^2(\mathbb{R}^n, d\mu)$ the space of polynomials on a maximally isotropic subspace $\mathbb{R}^n \subset \mathbb{R}^{2n}$. The differential $L_\star : mp(2n,\mathbb{R}) \to End(Pol(\mathbb{R}^n))$ of the action on the Segal-Shale-Weil representation is

$$\begin{aligned} L_\star(e_ie_j) &= ix_ix_j, \\ L_\star(f_if_j) &= -i\partial_i\partial_j, \\ L_\star(e_if_j + f_je_i) &= x_i\partial_j + x_j\partial_i \end{aligned} \tag{2.2}$$

for $i,j \in \{1,\dots,n\}$. Notice that we will term by Segal-Shale-Weil representation the reducible representation, given by the direct sum of two irreducible highest weight representations described in the next section.

The class of finite dimensional irreducible representations of $sp(2n,\mathbb{R})$ coming out of the decomposition of $Pol(\mathbb{R}^{2n})$ is given by symmetric powers $S^i(\mathbb{C}^{2n}), i \in \mathbb{N}$, of the complexification of the fundamental vector representation $\mathbb{R}^{2n}$. In particular, the $sp(2n,\mathbb{R})$-module $S^i(\mathbb{C}^{2n})$ is irreducible with highest weights $i\epsilon_1$, $i \in \mathbb{N}$.

# 3 Decomposition of tensor products of finite dimensional representations and the Segal-Shale-Weil representation

Let $\mathfrak{g}$ be a finite dimensional simple complex Lie algebra over $\mathbb{C}$, $\mathfrak{h}$ its Cartan subalgebra and $M$ a simple $\mathfrak{h}$-diagonalizable $\mathfrak{g}$-module having a weight space decomposition $M = \bigoplus_{\mu \in Weight(M)} M_\mu$ with $Weight(M) \subset \mathfrak{h}^*$ denoting the set of weights of $M$. The module $M$ is said to have bounded multiplicities provided there is a natural number $c \in \mathbb{N}$ such that $dim(M_\mu) \leq c$ for all $\mu \in Weight(M)$. In this case, the minimal $c$ is called the degree of the module $M$. Modules of degree 1 are called completely pointed.

In this section we will make explicit, for the purposes of our article, several results in [1] on the decomposition of the tensor product of completely pointed highest weight modules with a certain class of finite dimensional representations. In particular, we will consider the two irreducible components of the Segal-Shale-Weil representation and symmetric powers of the fundamental vector representation $\mathbb{C}^{2n}$ of $\mathfrak{g} = sp(2n,\mathbb{R})$.

Throughout the article, $V(\mu)$ denotes the Verma module of highest weight $\mu$ and $L(\mu)$ denotes the simple module of highest weight $\mu$, i.e. the quotient of $V(\mu)$ by its unique maximal submodule $I(\mu) \subset V(\mu)$.

Let us introduce the set

$$\tau^i = \{\sum_{j=1}^n d_j\epsilon_j \,|\, (d_j + \delta_{1,i}\delta_{n,j}) \in \mathbb{N}_+, \sum_{j=1}^n d_j = 0 \ mod\ 2\}.$$

Here $\mathbb{N}$ denotes the set of natural numbers including 0, $d_j \in \mathbb{N}$ and $i = 0$ resp. $i = 1$ for $L(-\frac{1}{2}\omega_n)$ resp. $L(\omega_{n-1} - \frac{3}{2}\omega_n)$. These sets are in bijective correspondence with the set of weights of the two irreducible parts of the Segal-Shale-Weil representation.

Let $\lambda = \sum_{i=1}^{n} \lambda_i \omega_i$ be a dominant integral weight written in the basis of fundamental weights. Define the set of weights

$$\begin{aligned} \tau_\lambda^i &= \{\mu | \lambda - \mu = \sum_{j=1}^{n} d_j \epsilon_j \in \tau^i, \\ & \quad 0 \leq d_j \leq \lambda_j \, (j = 1, \ldots, n-1), 0 \leq d_n + \delta_{1,i} \leq 2\lambda_n + 1\}. \end{aligned}$$

Let us recall an important result of [1]:

**Theorem 3.1.** *Let $L(-\frac{1}{2}\omega_n)$ resp. $L(\omega_{n-1} - \frac{3}{2}\omega_n)$ denote the simple highest weight modules corresponding to the two irreducible components of the Segal-Shale-Weil representation. Then for any finite dimensional irreducible representation $F(\lambda)$ with highest weight $\lambda$, we have*

$$L(-\frac{1}{2}\omega_n) \otimes F(\lambda) \simeq \bigoplus_{\mu \in \tau_\lambda^0} L(-\frac{1}{2}\omega_n + \mu) \tag{3.3}$$

*and*

$$L(\omega_{n-1} - \frac{3}{2}\omega_n) \otimes F(\lambda) \simeq \bigoplus_{\mu \in \tau_\lambda^1} L(\omega_{n-1} - \frac{3}{2}\omega_n + \mu). \tag{3.4}$$

*In particular, the tensor product is completely reducible and the decomposition is direct.*

Recall that the simple modules appearing in this theorem are irreducible and have no other singular vectors other than the highest weight ones.

The consequence of this result is the decomposition of the tensor product of $L(-\frac{1}{2}\omega_n)$ resp. $L(\omega_{n-1} - \frac{3}{2}\omega_n)$ with symmetric powers $S^k(\mathbb{C}^{2n})$, $k \in \mathbb{N}$, of the fundamental vector representation $\mathbb{C}^{2n}$ of $sp(2n, \mathbb{R})$. Note that these are irreducible representations, as mentioned in Section 2.

**Corollary 3.2.** *We have for $L(-\frac{1}{2}\omega_n)$*

1. *In the even case $k = 2l$ ($2l + 1$ terms on the right-hand side):*

$$\begin{aligned} L(-\frac{1}{2}\omega_n) \otimes S^k(\mathbb{C}^{2n}) \quad &\simeq \quad L(-\frac{1}{2}\omega_n) \oplus L(\omega_1 + \omega_{n-1} - \frac{3}{2}\omega_n) \\ & \quad \oplus L(2\omega_1 - \frac{1}{2}\omega_n) \oplus L(3\omega_1 + \omega_{n-1} - \frac{3}{2}\omega_n) \oplus \ldots \\ & \quad \oplus L((2l-1)\omega_1 + \omega_{n-1} - \frac{3}{2}\omega_n) \oplus L(2l\omega_1 - \frac{1}{2}\omega_n), \end{aligned}$$

2. *In the odd case $k = 2l + 1$ ($2l + 2$ terms on the right-hand side):*

$$\begin{aligned} L(-\frac{1}{2}\omega_n) \otimes S^k(\mathbb{C}^{2n}) \quad &\simeq \quad L(\omega_{n-1} - \frac{3}{2}\omega_n) \oplus L(\omega_1 - \frac{1}{2}\omega_n) \\ & \quad \oplus L(2\omega_1 + \omega_{n-1} - \frac{3}{2}\omega_n) \oplus L(3\omega_1 - \frac{1}{2}\omega_n) \oplus \ldots \\ & \quad \oplus L(2l\omega_1 + \omega_{n-1} - \frac{3}{2}\omega_n) \oplus L((2l+1)\omega_1 - \frac{1}{2}\omega_n), \end{aligned}$$

*We have for* $L(\omega_{n-1}-\frac{3}{2}\omega_n)$

1. *In the even case* $k=2l$ *(*$2l+1$ *terms on the right-hand side):*

$$\begin{aligned} L(\omega_{n-1}-\frac{3}{2}\omega_n)\otimes S^k(\mathbb{C}^{2n}) \quad &\simeq \quad L(\omega_{n-1}-\frac{3}{2}\omega_n)\oplus L(\omega_1-\frac{1}{2}\omega_n)\\ &\oplus L(2\omega_1+\omega_{n-1}-\frac{3}{2}\omega_n)\oplus\dots\\ &\oplus L((2l-1)\omega_1-\frac{1}{2}\omega_n)\oplus L(2l\omega_1+\omega_{n-1}-\frac{3}{2}\omega_n), \end{aligned}$$

2. *In the odd case* $k=2l+1$ *(*$2l+2$ *terms on the right-hand side):*

$$\begin{aligned} L(\omega_{n-1}-\frac{3}{2}\omega_n)\otimes S^k(\mathbb{C}^{2n}) \quad &\simeq \quad L(-\frac{1}{2}\omega_n)\oplus L(\omega_1+\omega_{n-1}-\frac{3}{2}\omega_n)\oplus\dots\\ &\oplus L(2l\omega_1-\frac{1}{2}\omega_n)\oplus L((2l+1)\omega_1+\omega_{n-1}-\frac{3}{2}\omega_n). \end{aligned}$$

As we shall see, a geometrical reformulation of this Corollary in the language of differentials operators leads to Theorem 5.4.

# 4 Generators of the Howe dual Lie algebra $sl(2,\mathbb{R})$

Let $(\mathbb{R}^{2n},\omega)$ be the symplectic vector space with coordinates $x_1,\dots,x_{2n}$, coordinate vector fields $\partial_1,\dots,\partial_{2n}$ and symplectic basis $e_1,f_1,\dots,e_n,f_n$, i.e.

$$\omega(e_i,e_j)=0,\qquad \omega(f_i,f_j)=0,\qquad \omega(e_i,f_j)=\delta_{ij}$$

for all $i,j=1,\dots,n$. It follows from the action of $sp(2n,\mathbb{R})$ on these vectors that

$$\begin{aligned} X_s &:= \sum_{j=1}^{n}(x_{2j-1}f_j+x_{2j}e_j),\\ D_s &:= \sum_{j=1}^{n}(\partial_{x_{2j-1}}e_j-\partial_{x_{2j}}f_j),\\ \mathbb{E} &:= \sum_{j=1}^{2n}x_j\partial_{x_j} \end{aligned}$$

are invariant and so will be used as linear maps intertwining the $sp(2n,\mathbb{R})$ action on the space $\mathcal{P}\otimes\mathcal{C}$ of polynomials on $\mathbb{C}^{2n}$ valued in the Segal-Shale-Weil representation $\mathcal{C}$, i.e. $\mathcal{P}:=Pol(\mathbb{C}^{2n})$ and $\mathcal{C}:=L(-\frac{1}{2}\omega_n)\oplus L(\omega_{n-1}-\frac{3}{2}\omega_n)$. The space of homogeneous polynomials of degree $k$ will be denoted by $\mathcal{P}_k$. The operator $D_s$ is crucial for the sequel and we call it the symplectic Dirac operator.

It is easy to verify that these operators fulfill $sl(2,\mathbb{R})$ commutation relations:

$$\begin{aligned} [\mathbb{E}+n,D_s]&=-D_s,\\ [\mathbb{E}+n,X_s]&=X_s,\\ [D_s,X_s]&=\mathbb{E}+n. \end{aligned} \tag{4.5}$$

The action of $sp(2n,\mathbb{R}) \times sl(2,\mathbb{R})$ will generate the multiplicity free decomposition of the representation of interest.

Further we introduce the operator

$$\Gamma_s = X_s D_s - \frac{1}{2}\mathbb{E}(2n-1+\mathbb{E}), \tag{4.6}$$

which is the Casimir operator in $sl(2,\mathbb{R})$. Using the formulas (4.5) it is easy to check that $\Gamma_s$ commutes with both $X_s$ and $D_s$.

# 5 Fischer decomposition and homomorphisms of $sp(2n,\mathbb{R})$-modules appearing in the decomposition of polynomial symplectic spinors

Before introducing the scheme in full generality, we start with a few explicit remarks concerning the homogeneity zero and one parts in the decomposition. The tensor product $L(-\frac{1}{2}\omega_n) \otimes \mathbb{C}^{2n}$ (analogously, one can consider $L(\omega_{n-1} - \frac{3}{2}\omega_n) \otimes \mathbb{C}^{2n}$) decomposes as a direct sum $V_1 \oplus V_2$ of two invariant subspaces, given by

$$V_1 := \{\sum_{i=1}^{n} e_i s \otimes f_i - \sum_{i=1}^{n} f_i s \otimes e_i |\ s \in L(-\frac{1}{2}\omega_n)\},$$
$$V_2 := \{\sum_{i=1}^{n} s_i \otimes e_i + \sum_{j=1}^{n} s_j \otimes f_j;\ s_i, s_j \in L(-\frac{1}{2}\omega_n)|\ \sum_{i=1}^{n} e_i s_i + \sum_{j=1}^{n} f_j s_j = 0\}.$$

The map

$$\begin{aligned} i: \qquad & L(\omega_{n-1} - \frac{3}{2}\omega_n) \to L(-\frac{1}{2}\omega_n) \otimes \mathbb{C}^n, \\ & s \mapsto \sum_{i=1}^{n} e_i s \otimes f_i - \sum_{i=1}^{n} f_i s \otimes e_i, \end{aligned} \tag{5.7}$$

(resp. $L(-\frac{1}{2}\omega_n) \to L(\omega_{n-1} - \frac{3}{2}\omega_n) \otimes \mathbb{C}^n$) is injective and onto $V_1$. The reason is that injectivity $i(s) = 0$ is equivalent to $e_i s = 0$ resp. $f_i s = 0$ for all $i \in \{1, \ldots, n\}$. Now the symplectic Clifford algebra relation $e_i f_j - f_j e_i = \delta_{ij}$ implies $s = 0$ and the result follows. In other words, the action of $X_s$ induces an isomorphism between two irreducible submodules in homogeneity zero and homogeneity one.

Let us consider an application of the tool in representation theory called the infinitesimal character. The sum of fundamental weights (or half of the sum of positive roots) for $sp(2n,\mathbb{R})$ is $\delta = (n, n-1, \ldots, 2, 1)$. The highest weights of irreducible simple $sp(2n,\mathbb{R})$-modules, coming from the decomposition of tensor products of central interest, were determined for each homogeneity $k \in \mathbb{N}$ in Corollary 3.2. The multiplication by $X_s$ gives an intertwining map between neighboring columns, say the $k$-th and $(k+1)$-th. Let us determine possible target modules when restricting the action of $X_s$ to a given simple irreducible $sp(2n,\mathbb{R})$-module $L(a\omega_1 - \frac{1}{2}\omega_n)$

with highest weight $a\omega_1 - \frac{1}{2}\omega_n$ for some $a \leq k$ (the case of $L(b\omega_1 + \omega_{n-1} - \frac{3}{2}\omega_n)$ being completely analogous). The comparison of infinitesimal characters of the collection of weights $\{\mu_a = a\omega_1 - \frac{1}{2}\omega_n, \nu_b = b\omega_1 + \omega_{n-1} - \frac{3}{2}\omega_n\}$ $(a, b \in \mathbb{N})$ yields that

$$||\mu_a + \delta||^2 = ||\nu_b + \delta||^2$$

if and only if either

1. $2a + n - \frac{1}{2} = 2b + n - \frac{1}{2}$, which implies $a = b$, or
2. $2a + n - \frac{1}{2} = -(2b + n - \frac{1}{2})$, i.e. $a + b = -n + \frac{1}{2}$ and there is no solution in this case.

It remains to prove that the image of $X_s$, when restricted to an irreducible simple module in the $k$-th column, is nonzero (or, as follows from the irreducibility, is the irreducible simple module in $(k+1)$-th column with the same infinitesimal character.)

To complete this line of reasoning, we employ the Lie algebra $sl(2, \mathbb{R})$ from Section 4. To illustrate its impact explicitly, we start in homogeneity zero with the simple module $L(-\frac{1}{2}\omega_n)$ and assume that it is $Ker(X_s)$. Because it is in $Ker(D_s)$, it is in the kernel of the commutator $[D_s, X_s] = \mathbb{E} + n$. However, $\mathbb{E} + n$ acts in homogeneity zero by $n$, which is the required contradiction and so $X_s$ acts as an isomorphism $L(-\frac{1}{2}\omega_n) \to L(\omega_{n-1} - \frac{1}{2}\omega_n)$. Let us now consider the action of $X_s$ on $L(\omega_{n-1} - \frac{1}{2}\omega_n)$ sitting in the homogeneity one part and assume it acts trivially. Then due to the previous isomorphism, this kernel is $Ker(X_s^2)$ when $X_s^2$ is acting on $L(-\frac{1}{2}\omega_n)$ in homogeneity zero. As before, the commutator $[X_s^2, D_s]$ acts by zero. Because it is equal to $-X_s(\mathbb{E} + n) - (\mathbb{E} + n)X_s$, it acts on homogeneity zero elements by $-(2n+1)X_s$ and due to the fact that $X_s$ is isomorphism, it is nonzero and so yields the contradiction. In conclusion, $X_s : L(\omega_{n-1} - \frac{1}{2}\omega_n) \to L(-\frac{1}{2}\omega_n)$ acting between homogeneity one and two is an isomorphism. Clearly, one can iterate the procedure further using the subsequent Lemmas 5.2 and 5.3 on $sp(2n, \mathbb{R})$-invariant intertwining operators acting on the direct sum of simple highest weight $sp(2n, \mathbb{R})$-modules. An analogous induction procedure can be used to prove the isomorphic action of $D_s$.

In what follows, we turn the previous qualitative observation into a more quantitative statement. Following the decomposition of $\mathcal{P}_l \otimes \mathcal{C}$ in Corollary 3.2, we first introduce the concept of symplectic monogenic polynomial:

**Definition 5.1.** *Denote by $\mathcal{M}_l^+$ resp. $\mathcal{M}_l^-$ the (irreducible) simple $sp(2n, \mathbb{R})$-modules with highest weight $L(l\omega_1 - \frac{1}{2}\omega_n)$ resp. $L(l\omega_1 + \omega_{n-1} - \frac{3}{2}\omega_n))$, and call them* **symplectic monogenics** *of degree $l$ (or $l$-homogeneous symplectic monogenics).*

We put $\mathcal{M}_l := \mathcal{M}_l^+ \oplus \mathcal{M}_l^-$. Based on our previous discussion, this space is characterized by

$$\mathcal{M}_l = \ker D_s \cap (\mathcal{P}_l \otimes \mathcal{C})$$

and the name symplectic monogenic is hence justified by analogy with the orthogonal case (see e.g. [2, 18]).

We then obtain two auxiliary lemmas.

**Lemma 5.2.** *Suppose $M_\ell \in \mathcal{M}_\ell$ is a symplectic monogenic of degree $\ell$. Then*

$$D_s(X_s^k M_\ell) = \frac{1}{2}k(2n + 2l + k - 1)X_s^{k-1}M_\ell.$$

**Proof:** A straightforward proof follows by induction.

$\square$

**Lemma 5.3.** *Suppose $M_\ell \in \mathcal{M}_\ell$ is a symplectic monogenic of degree $\ell$. Then*

$$D_s^j(X_s^k M_\ell) = c_{j,k,\ell} X_s^{k-j} M_\ell$$

*with*

$$c_{j,k,l} = \begin{cases} \dfrac{1}{2^j}\dfrac{k!}{(k-j)!}\dfrac{(2n+2l+k-1)!}{(2n+2l+k-j-1)!} & j \leq k, \\ 0 & j > k. \end{cases}$$

*Proof.* The lemma follows from $j$ iterations of Lemma 5.2. $\square$

The previous considerations can be summarized in the symplectic analog of the classical theorem on separation of variables in the orthogonal case, see for example [18] and the references therein.

**Theorem 5.4.** *The space $\mathcal{P} \otimes \mathcal{C}$ decomposes under the action of $sl(2,\mathbb{R})$ into the direct sum of simple highest weight $sp(2n,\mathbb{R})$-modules*

$$\bigoplus_{l=0}^{\infty}\bigoplus_{j=0}^{\infty} X_s^j \mathcal{M}_l,$$

*where we used the notation $\mathcal{M}_l := \mathcal{M}_l^+ \oplus \mathcal{M}_l^-$. The decomposition takes the form of an infinite triangle*

$$\begin{tikzcd}
\mathcal{P}_0 \otimes \mathcal{C} & \mathcal{P}_1 \otimes \mathcal{C} & \mathcal{P}_2 \otimes \mathcal{C} & \mathcal{P}_3 \otimes \mathcal{C} \arrow[d, equal] & \mathcal{P}_4 \otimes \mathcal{C} & \mathcal{P}_5 \otimes \mathcal{C} & \dots \\
\mathcal{M}_0 \arrow[r] & X_s\mathcal{M}_0 \arrow[r] & X_s^2\mathcal{M}_0 \arrow[r] & X_s^3\mathcal{M}_0 \arrow[r] & X_s^4\mathcal{M}_0 \arrow[r] & X_s^5\mathcal{M}_0 & \dots \\
 & & & \oplus & & & \\
 & \mathcal{M}_1 \arrow[r] & X_s\mathcal{M}_1 \arrow[r] & X_s^2\mathcal{M}_1 \arrow[r] & X_s^3\mathcal{M}_1 \arrow[r] & X_s^4\mathcal{M}_1 & \dots \\
 & & & \oplus & & & \\
 & & \mathcal{M}_2 \arrow[r] & X_s\mathcal{M}_2 \arrow[r] & X_s^2\mathcal{M}_2 \arrow[r] & X_s^3\mathcal{M}_2 & \dots \\
 & & & \oplus & & & \\
 & & & \mathcal{M}_3 \arrow[r] & X_s\mathcal{M}_3 \arrow[r] & X_s^2\mathcal{M}_3 & \dots \\
 & & & & \mathcal{M}_4 \arrow[r] & X_s\mathcal{M}_4 & \dots \\
 & & & & & \mathcal{M}_5 & \dots
\end{tikzcd}$$

*where all summands are simple highest weight $sp(2n,\mathbb{R})$-modules. The $k$-th column gives the decomposition of homogeneous polynomials of degree $k$ taking values in $\mathcal{C} = L(-\frac{1}{2}\omega_n) \oplus L(\omega_{n-1} - \frac{3}{2}\omega_n)$. The $l$-th row forms a highest weight $sl(2,\mathbb{R})$-module $\oplus_{j=0}^{\infty} X_s^j \mathcal{M}_l$ generated by the space of symplectic monogenics $\mathcal{M}_l$.*

One immediate Corollary is the structure of polynomial solutions of the symplectic Dirac operator on $\mathbb{R}^{2n}$. The statement is given for both symplectic spin modules $L(-\frac{1}{2}\omega_n)$ and $L(\omega_{n-1} - \frac{3}{2}\omega_n)$ separately.

**Corollary 5.5.** *The kernel of (half of) the symplectic Dirac operator $D_s$ acting on $L(-\frac{1}{2}\omega_n)$-valued polynomials is*

$$Ker^+(D_s) \simeq \bigoplus_{l\in\mathbb{N}_0} \left( L(2l\omega_1 - \frac{1}{2}\omega_n) \oplus L((2l+1)\omega_1 - \frac{1}{2}\omega_n) \right).$$

*The kernel of (half of) the symplectic Dirac operator $D_s$ acting on $L(\omega_{n-1} - \frac{3}{2}\omega_n)$-valued polynomials is*

$$Ker^-(D_s) \simeq \bigoplus_{l\in\mathbb{N}_0} \left( L(2l\omega_1 + \omega_{n-1} - \frac{3}{2}\omega_n) \oplus L((2l+1)\omega_1 + \omega_{n-1} - \frac{3}{2}\omega_n) \right).$$

Every homogeneous polynomial of degree $k$, taking values in $\mathcal{C}$, can now be decomposed into monogenic components as follows.

**Theorem 5.6.** *Let $p \in \mathcal{P}_k \otimes \mathcal{C}$. Then there exists a unique representation of $p$ as*

$$p = \sum_{i=0}^{k} p_i,$$

*where $p_i = X_s^{k-i} m_i$ and $m_i \in \mathcal{M}_i$.*

We now proceed to construct projection operators that allow to explicitly compute the representation given in Theorem 5.6. They are given in the following theorem.

**Theorem 5.7.** *The operators*

$$\pi_i^k = \sum_{j=0}^{k-i} a_j^{i,k} X_s^{i+j} D_s^{i+j} \tag{5.8}$$

*with*

$$a_j^{i,k} = (-1)^j (2n + 2k - 2i - 1) \frac{2^{i+j}}{i!j!} \frac{(2n + 2k - 2i - j - 2)!}{(2n + 2k - i - 1)}$$

*and $i = 0, \ldots, k$ satisfy*

$$\pi_i^k (X_s^j \mathcal{M}_{k-j}) = \delta_{ij} X_s^i \mathcal{M}_{k-i}.$$

*Proof.* Using Lemma 5.3 it is easy to see that $\pi_i^k(X_s^j \mathcal{M}_{k-j}) = 0$ for all $j < i$. The coefficients $a_j^{i,k}$, for fixed $i$ and $k$, can now be determined iteratively. First of all, expressing $\pi_i^k(X_s^i \mathcal{M}_{k-i}) = X_s^i \mathcal{M}_{k-i}$ yields

$$a_0^{i,k} = \frac{1}{c_{i,i,k-i}} = \frac{2^i}{i!} \frac{(2n + 2k - 2i - 1)!}{(2n + 2k - i - 1)!}.$$

Similarly, expressing $\pi_i^k(X_s^{i+1} \mathcal{M}_{k-i-1}) = 0$ then yields

$$a_1^{i,k} = -\frac{c_{i,i+1,k-i-1}}{c_{i+1,i+1,k-i-1}} a_0^{i,k} = -\frac{1}{n + k - i - 1} a_0^{i,k}.$$

Thus continuing we arrive at the hypothesis

$$a_j^{i,k} = (-1)^j \frac{2^j}{j!} \frac{(2n+2k-2i-j-2)!}{(2n+2k-2i-2)!} a_0^{i,k},$$

which can be proven using induction. Indeed, suppose that the statement holds for $a_j^{i,k}$, $j \leq l$, then we prove that it also holds for $a_{l+1}^{i,k}$. This last coefficient has to satisfy

$$\sum_{j=0}^{l+1} a_j^{i,k} c_{i+j,i+l+1,k-i-l-1} = 0.$$

Substituting the known expressions we obtain

$$\begin{aligned}
a_{l+1}^{i,k} &= -\sum_{j=0}^{l} a_j^{i,k} \frac{c_{i+j,i+l+1,k-i-l-1}}{c_{i+l+1,i+l+1,k-i-l-1}} \\
&= -\sum_{j=0}^{l} a_j^{i,k} \frac{2^{l+1-j}}{(l+1-j)!} \frac{(\alpha-2l-1)!}{(\alpha-l-j)} \\
&= -\frac{2^{l+1}}{(l+1)!} \frac{(\alpha-2l-1)!}{\alpha!} a_0^{i,k} \sum_{j=0}^{l} (-1)^j \binom{l+1}{j} \frac{(\alpha-j)!}{(\alpha-l-j)!},
\end{aligned}$$

where we have put $\alpha = 2n+2k-2i-2$. The proof is now complete by remarking that

$$\sum_{j=0}^{l+1} (-1)^j \binom{l+1}{j} \frac{(\alpha-j)!}{(\alpha-l-j)!} = 0.$$

This can either be obtained directly (see e.g. Lemma 5 in [5]) or as a consequence of Gauss's hypergeometric theorem, expressing ${}_2F_1(a,b;c;1)$ in terms of a product of Gamma functions. □

Note that there exists another way of computing the projection operators on irreducible summands, namely using the Casimir operator of $sl(2,\mathbb{R})$. First observe that

$$\Gamma_s \mathcal{M}_k = -\frac{1}{2} k(2n-1+k) \mathcal{M}_k.$$

It is then clear that the operators

$$\mathbb{P}_i^k = \prod_{j=0, j\neq i}^{k} \frac{2\Gamma_s + j(2n-1+j)}{j(2n-1+j) - i(2n-1+i)}, \quad i = 0, \ldots, k$$

defined on the space $\mathcal{P}_k \otimes \mathcal{C}$ satisfy

$$\mathbb{P}_i^k (X_s^{k-j} \mathcal{M}_j) = \delta_{ij} X_s^{k-i} \mathcal{M}_i.$$

# 6 Open questions and unresolved problems

In [16], the symplectic Dirac operator $D_s$ on $\mathbb{R}^{2n} \hookrightarrow \mathbb{R}^{2n+1}$ was studied as an $sp(2n+2)$-invariant differential operator in the context of contact parabolic geometry on the big open cell of a homogeneous space $\mathbb{R}^{2n+1} \hookrightarrow Sp(2n+2,\mathbb{R})/P$ for a maximal parabolic subgroup $P \subset Sp(2n+2,\mathbb{R})$, where the nilradical of the parabolic subalgebra is isomorphic to the Heisenberg algebra. As a consequence, the kernel of $D_s$ has the structure of an $sp(2n+2,\mathbb{R})$-module. In fact $Ker(D_s)$ is, as a vector space, isomorphic to $Pol(\mathbb{R}^{2n+2})$ (see Corollary 5.5) and we leave the question of its representation theoretic content open.

In [18], the authors studied the specific deformation of Howe duality and Fischer decomposition for the Dirac operator acting on spinor valued polynomials, coming from the Dunkl deformation of the Dirac operator. It is an interesting question to develop the Dunkl version of the symplectic Dirac operator in the context of symplectic reflection algebras (see [11]).

In the context of super mathematics, the symplectic Dirac operator may be viewed as an operator that combines with the Dirac operator of [19] to form a "twisted" super Dirac operator (see bottom half of Figure 1 of [3]). A detailed representation theoretic study of this operator would complete the picture of [3].

We should also remark that analytic properties of symplectic monogenics were not studied at all, and it is challenging to employ the techniques of e.g., the symplectic Fourier transform, to understand the properties of the symplectic Dirac operator $D_s$ in detail.

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