Spin-embeddings, two-intersection sets and two-weight codes

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Abstract

Let Δ be one of the dual polar spaces $DQ(8,q)$, $DQ^-(7,q)$, and let $e : \Delta \to \Sigma$ denote the spin-embedding of Δ. We show that $e(\Delta)$ is a two-intersection set of the projective space Σ. Moreover, if $\Delta \cong DQ^-(7,q)$, then $e(\Delta)$ is a (q^3+1)-tight set of a nonsingular hyperbolic quadric $Q^+(7,q^2)$ of $\Sigma \cong \mathrm{PG}(7,q^2)$. This (q^3+1)-tight set gives rise to more examples of (q^3+1)-tight sets of hyperbolic quadrics by a procedure called field-reduction. All the above examples of two-intersection sets and (q^3+1)-tight sets give rise to two-weight codes and strongly regular graphs.

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1 Introduction

1.1 Two-intersection sets, two-weight codes and strongly regular graphs

A simple undirected graph G without loops is called a *strongly regular graph with parameters* (v, K, λ, μ) if G is a connected graph of diameter 2 having precisely v vertices, K vertices adjacent to any given vertex, λ vertices adjacent to any two given adjacent vertices and μ vertices adjacent to any two given nonadjacent vertices.

Let q be a prime power and $k, n \in \mathbb{N}$ with $n \geq k$. An $[n,k]_q$-code is a k-dimensional subspace $\mathcal{C}$ of the n-dimensional vector space $\mathbb{F}_q^n$. The elements of $\mathcal{C}$ are called *codewords.* We will denote the elements of $\mathbb{F}_q^n$ by row vectors. The *weight* of an element of $\mathbb{F}_q^n$ is the number of nonzero coordinates. $\mathcal{C}$ is called a *two-weight code* if there exist $w_1, w_2 \in \{1, \ldots, n\}$

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such that every nonzero codeword of $\mathcal{C}$ has weight either w_1 or w_2. In this case, the numbers w_1 and w_2 are called the *weights* of the two-weight code.

A two-weight $[n,k]_q$-code $\mathcal{C}$ is generated by k row vectors. We can use these k row vectors to build a $(k \times n)$-matrix. The column vectors of this matrix define a set of n not necessarily distinct points in $\mathrm{PG}(k-1,q)$. If all these n points are distinct, then the two-weight code is called *projective*. Two distinct generating sets of k row vectors of a projective two-weight $[n,k]_q$-code $\mathcal{C}$ will give rise to two sets of n points in $\mathrm{PG}(k-1,q)$ which are projectively equivalent. It makes therefore sense to denote any of these sets by $X_{\mathcal{C}}$.

A set X of points of $\mathrm{PG}(k-1,q)$ is called a *two-intersection set* with *intersection numbers* h_1 and h_2 if every hyperplane of $\mathrm{PG}(k-1,q)$ intersects X in either h_1 or h_2 points. We can embed $\mathrm{PG}(k-1,q)$ as a hyperplane in $\mathrm{PG}(k,q)$ and define the following graph G_X. The vertices of G_X are the points of $\mathrm{PG}(k,q) \setminus \mathrm{PG}(k-1,q)$ and two distinct vertices x_1 and x_2 are adjacent whenever the line x_1x_2 of $\mathrm{PG}(k,q)$ contains a point of X.

Delsarte ([9], [10], [11], [12]) was the first to investigate the relationships between projective two weight codes, two-intersection sets of projective spaces and strongly regular graphs, see Calderbank and Kantor [3] for a nice survey. We collect the basic relationships in the following proposition. For a proof of this proposition, we refer to Calderbank and Kantor [3, Theorem 3.2].

Proposition 1.1 *Let X be a proper set of n points of $\mathrm{PG}(k-1,q)$ generating $\mathrm{PG}(k-1,q)$. Then the following are equivalent:*

(1) *X is a two-intersection set;*

(2) *X is projectively equivalent to a set $X_{\mathcal{C}}$ where $\mathcal{C}$ is some projective two weight $[n,k]_q$-code;*

(3) *G_X is a strongly regular graph.*

There exist specific relationships between the parameters h_1 and h_2 of the two-intersection set, the parameters w_1 and w_2 of the associated two-weight code and the parameters v, K, λ and μ of the corresponding distance-regular graph. These are as follows (up to transposition of w_1 and w_2), see e.g. Calderbank and Kantor [3, Corollary 3.7]:

$$w_1 = n - h_1,\ w_2 = n - h_2,$$

$$v = q^k,\ K = n(q-1),\ \mu = w_1 w_2 q^{2-k},$$

$$\lambda = K^2 + 3K - q(w_1 + w_2) - Kq(w_1 + w_2) + q^2 w_1 w_2.$$

1.2 i-tight sets of polar spaces and two-intersection sets

Let P be a finite polar space of rank $r \geq 2$ with $q+1 \geq 3$ points on each line. Then by Tits [20], P is one of the following polar spaces:

(1) a generalized quadrangle $\mathrm{GQ}(q,t)$ of order (q,t), $t \geq 1$;

(2) the polar space $W(2r-1,q)$ of the subspaces of $\mathrm{PG}(2r-1,q)$ which are totally isotropic with respect to a given symplectic polarity of $\mathrm{PG}(2r-1,q)$;

(3) the polar space $Q(2r,q)$ of the subspaces of $\mathrm{PG}(2r,q)$ which lie on a given nonsingular parabolic quadric of $\mathrm{PG}(2r,q)$;

(4) the polar space $Q^+(2r-1,q)$ of the subspaces of $\mathrm{PG}(2r-1,q)$ which lie on a given nonsingular hyperbolic quadric of $\mathrm{PG}(2r-1,q)$;

(5) the polar space $Q^-(2r+1,q)$ of the subspaces of $\mathrm{PG}(2r+1,q)$ which lie on a given nonsingular elliptic quadric of $\mathrm{PG}(2r+1,q)$;

(6) the polar space $H(2r-1,q)$ (q square) of the subspaces of $\mathrm{PG}(2r-1,q)$ which lie on a given nonsingular Hermitian variety of $\mathrm{PG}(2r-1,q)$;

(7) the polar space $H(2r,q)$ (q square) of the subspaces of $\mathrm{PG}(2r,q)$ which lie on a given nonsingular Hermitian variety of $\mathrm{PG}(2r,q)$.

If X is a set of points of P, then by Drudge [13] the number of ordered pairs of distinct collinear points of X is bounded above by

$$(q^{r-1}-1)\cdot|X|\cdot\Big(\frac{|X|}{q^r-1}+1\Big). \tag{1}$$

If equality holds, then X is called *i-tight*, where $i := \frac{|X|\cdot(q-1)}{q^r-1}$. In case of equality, $i \in \mathbb{N}$. Moreover, every point x of X is collinear with precisely $(i+q-1)\frac{q^{r-1}-1}{q-1}$ points of $X \setminus \{x\}$ and every point y outside X is collinear with precisely $i\frac{q^{r-1}-1}{q-1}$ points of X. We call a set of points of P *tight* if it is i-tight for some $i \in \mathbb{N}$. Tight sets were introduced by Payne [15] for generalized quadrangles and by Drudge [13] for arbitrary polar spaces. We refer to these references for proofs of the above-mentioned facts. We take the following proposition from Bamberg et al. [1, Theorem 12].

Proposition 1.2 ([1]) *Let P be one of the polar spaces $W(2r-1,q)$, $Q^+(2r-1,q)$, $H(2r-1,q)$ and let X be a nonempty tight set of P. Then X is a two-intersection set of the ambient projective space of P.*

1.3 Dual polar spaces and embeddings

Let $\Delta = (\mathcal{P},\mathcal{L},\mathrm{I})$, $\mathrm{I} \subseteq \mathcal{P}\times\mathcal{L}$, be a point-line geometry. The distance between two points of Δ will be measured in the collinearity graph of Δ.

If x is a point of Δ and $i \in \mathbb{N}$, then $\Delta_i(x)$ denotes the set of points at distance i from x. A *hyperplane* of Δ is a proper subset of $\mathcal{P}$ intersecting each line in either a singleton or the whole line.

A *full (projective) embedding* of Δ is an injective mapping e from $\mathcal{P}$ to the point-set of a projective space Σ satisfying: (E1) the image $e(\Delta) := e(\mathcal{P})$ of e spans Σ; (E2) for every line L of Δ, $e(L)$ is a line of Σ. If $e : \Delta \rightarrow \Sigma$ is a full embedding of Δ, then for every hyperplane α of Σ, $e^{-1}(e(\mathcal{P}) \cap \alpha)$ is a hyperplane of Δ. We say that the hyperplane $e^{-1}(e(\mathcal{P}) \cap \alpha)$ *arises from the embedding* e.

With every polar space P of rank $r \geq 2$, there is associated a *dual polar space* Δ *of rank* r, see Shult and Yanushka [19] or Cameron [4]. Δ is the point-line geometry whose points and lines are the maximal and next-to-maximal singular subspaces of P, with reverse containment as incidence relation. For every singular subspace α of P, we denote by F_α the set of all maximal singular subspaces of P containing α. The points and lines contained in F_α define a dual polar space of rank $n - 1 - \dim(\alpha)$. The set F_α is called a *quad*, respectively a *max*, of Δ if $\dim(\alpha) = n-3$, respectively $\dim(\alpha) = 0$. The points and lines contained in a quad define a generalized quadrangle. The set of points of Δ at non-maximal distance from a given point x of Δ is a hyperplane of Δ, called *the singular hyperplane of* Δ *with deepest point* x. A hyperplane H of Δ is called *locally singular* if for every quad Q of Δ, $Q \cap H$ is either Q or a singular hyperplane of the generalized quadrangle associated with Q.

Let $Q^+(2n+1,q)$, $n \geq 2$, denote a nonsingular hyperbolic quadric in $\mathrm{PG}(2n+1,q)$. The set of generators (= maximal singular subspaces) of $Q^+(2n+1,q)$ can be divided into two families $\mathcal{M}^+$ and $\mathcal{M}^-$ such that two generators of the same family intersect in a subspace of even co-dimension. For every $\epsilon \in \{+,-\}$, let $\mathcal{S}^\epsilon$ denote the point-line geometry whose point-set is equal to $\mathcal{M}^\epsilon$ and whose line-set coincides with the set of all $(n-2)$-dimensional subspaces of $Q^+(2n+1,q)$ (natural incidence). The isomorphic geometries $\mathcal{S}^+$ and $\mathcal{S}^-$ are called the *half-spin geometries* for $Q^+(2n+1,q)$. The half-spin geometry $\mathcal{S}^\epsilon$, $\epsilon \in \{+,-\}$, admits a nice full embedding into $\mathrm{PG}(2^n-1,q)$ which is called the *spin-embedding* of $\mathcal{S}^\epsilon$. We refer to Chevalley [6] or Buekenhout and Cameron [2] for a description of this embedding. For $n = 3$, this embedding has the following nice description. Let θ be a triality of $Q^+(7,q)$ mapping $\mathcal{M}^+$ to the point-set of $Q^+(7,q)$, the point-set of $Q^+(7,q)$ to $\mathcal{M}^-$ and $\mathcal{M}^-$ to $\mathcal{M}^+$. Then θ realizes the spin-embedding of $\mathcal{S}^+$ into $\mathrm{PG}(7,q)$. From this argument it is also clear that the half-spin geometries for $Q^+(7,q)$ are isomorphic to the point-line system of $Q^+(7,q)$.

Now, consider the embedding $Q(2n,q) \subseteq Q^+(2n+1,q)$. Every generator M of $Q(2n,q)$ is contained in a unique element $\phi(M)$ of $\mathcal{M}^+$. If e denotes the spin-embedding of $\mathcal{S}^+$, then $e \circ \phi$ defines a full embedding of the dual polar space $DQ(2n,q)$ associated with $Q(2n,q)$ into the projective space

$\mathrm{PG}(2^n-1,q)$. This embedding is called the *spin-embedding of* $DQ(2n,q)$. The spin-embedding of $DQ(4,q)$ is isomorphic to the natural embedding of $DQ(4,q) \cong W(3,q)$ into $\mathrm{PG}(3,q)$.

Now, suppose q is a square and consider the inclusion $Q^-(2n+1,\sqrt{q}) \subseteq Q^+(2n+1,q)$ defined by a quadratic form of Witt-index n over $\mathbb{F}_{\sqrt{q}}$ which becomes a quadratic form of Witt-index $n+1$ when regarded over the quadratic extension $\mathbb{F}_q$ of $\mathbb{F}_{\sqrt{q}}$. For every generator α of $Q^-(2n+1,\sqrt{q})$, let $\phi'(\alpha)$ denote the unique element of $\mathcal{M}^+$ containing α. If e again denotes the spin-embedding of $\mathcal{S}^+$, then $e \circ \phi'$ defines a full embedding of the dual polar space $DQ^-(2n+1,\sqrt{q})$ associated with $Q^-(2n+1,\sqrt{q})$ into the projective space $\mathrm{PG}(2^n-1,q)$. This embedding is called the *spin-embedding of* $DQ^-(2n+1,\sqrt{q})$. The construction of this embedding is due to Cooperstein and Shult [7].

1.4 The Main Theorem

We will prove the following:

Main Theorem. (1) *If* $e : \Delta \rightarrow \Sigma$ *is the spin-embedding of the dual polar space* $\Delta = DQ(8,q)$*, then* $e(\Delta)$ *is a two-intersection set of* $\Sigma \cong \mathrm{PG}(15,q)$.

(2) *If* $e : \Delta \rightarrow \Sigma$ *is the spin-embedding of the dual polar space* $\Delta = DQ^-(7,q)$*, then* $e(\Delta)$ *is a two-intersection set of* $\Sigma \cong \mathrm{PG}(7,q^2)$*. Moreover,* $e(\Delta)$ *is a* (q^3+1)*-tight set of a nonsingular hyperbolic quadric* $Q^+(7,q^2)$ *of* Σ.

The parameters of the two-intersection sets $e(\Delta)$ and their corresponding two-weight codes and strongly regular graphs are listed in the following table.

Δ	$DQ(8,q)$	$DQ^-(7,q)$
$e(\Delta)$	$(q+1)(q^2+1)(q^3+1)(q^4+1)$	$(q^2+1)(q^3+1)(q^4+1)$
Σ	$\mathrm{PG}(15,q)$	$\mathrm{PG}(7,q^2)$
w_1	q^{10}	q^9
w_2	$q^{10}+q^7$	q^9+q^6
v	q^{16}	q^{16}
K	$(q^8-1)(q^3+1)$	$(q^8-1)(q^3+1)$
λ	$q^8+q^6-q^3-2$	$q^8+q^6-q^3-2$
μ	$q^3(q^3+1)$	$q^3(q^3+1)$

We cannot rule out that the two-intersection set $e(\Delta)$ ($\Delta = DQ(8,q)$ or $\Delta = DQ^-(7,q)$) is nonisomorphic to any of the many two-intersection sets described in the literature. However, even if the two-intersection set $e(\Delta)$ would not be new, we still would have a nice alternative description of this special set of points.

Another problem which remains open is whether the two-intersection sets of $\mathrm{PG}(15,q)$ related to the spin-embedding of $DQ(8,q)$ can be obtained from the two-intersection sets of $\mathrm{PG}(7,q^2)$ arising from the spin-embedding of $DQ^-(7,q)$ by applying a change of the underlying field as described in Section 6 of Calderbank and Kantor [3].

The (q^3+1)-tight sets of $Q^+(7,q^2)$ arising from the spin-embedding of $DQ^-(7,q)$ have not been described before in the literature. A construction for these tight sets can be given which does not refer any more to any particular embedding. As before, consider an inclusion $Q^-(7,q) \subseteq Q^+(7,q^2)$, let $\mathcal{M}^+$ and $\mathcal{M}^-$ denote the two families of generators of $Q^+(7,q^2)$ and let θ be a triality of $Q^+(7,q^2)$ which maps $\mathcal{M}^+$ to the point-set of $Q^+(7,q^2)$, the point-set of $Q^+(7,q^2)$ to $\mathcal{M}^-$ and $\mathcal{M}^-$ to $\mathcal{M}^+$. If U denotes the set of generators of $Q^-(7,q)$ and V denotes the set of generators of $\mathcal{M}^+$ containing an element of U, then $\theta(V)$ is a (q^3+1)-tight set of points of $Q^+(7,q^2)$.

Using a procedure referred to as field reduction in [14], one can construct i-tight sets of $Q^+(2er-1,q)$ from i-tight sets of $Q^+(2r-1,q^e)$ by constructing a copy of $Q^+(2r-1,q^e)$ inside $Q^+(2er-1,q)$. So, a (q^3+1)-tight set of $Q^+(7,q^2)$ will give rise to a (q^3+1)-tight set of $Q^+(15,q)$ and even to more (q^3+1)-tight sets of hyperbolic quadrics if q is not prime. By Propositions 1.1 and 1.2, also these (q^3+1)-tight sets will give rise to two-intersection sets, two-weight codes and strongly regular graphs.

Remark. Suppose $e : \Delta \to \Sigma$ is a full projective embedding of a point-line geometry $\Delta = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ and $h_1, h_2 \in \mathbb{N} \setminus \{0\}$ such that

(*) $|H| \in \{h_1, h_2\}$ for any hyperplane H of Δ arising from the embedding e.

Then $e(\mathcal{P})$ is a two-intersection set of Σ. Many point-line geometries (e.g., generalized quadrangles, polar spaces, the dual polar space $DQ(6,q)$) have a projective embedding e for which (*) holds. However, for almost all these examples the corresponding two-intersection sets are well-known. We have therefore restricted our discussion to the dual polar spaces $DQ(8,q)$ and $DQ^-(7,q)$ since for these geometries we have found no description of the corresponding two-intersection sets in the literature.

2 A two-intersection set arising from the spin-embedding of $DQ(8,q)$

Let $e : \Delta \to \Sigma$ denote the spin-embedding of $\Delta = DQ(8,q)$ into $\Sigma = \mathrm{PG}(15,q)$. By De Bruyn [8] (see also Shult and Thas [18] for q odd), the hyperplanes of $DQ(8,q)$ which arise from e are precisely the locally singular hyperplanes of $DQ(8,q)$. By Cardinali, De Bruyn and Pasini [5], there are three types of locally singular hyperplanes in $DQ(8,q)$: the singular hyperplanes, the extensions of the hexagonal hyperplanes and the so-called $Q^+(7,q)$-hyperplanes.

(1) If H is the singular hyperplane of $DQ(8,q)$ with deepest point x, then $|H| = |\Delta_0(x)| + |\Delta_1(x)| + |\Delta_2(x)| + |\Delta_3(x)| = 1 + q(q^3+q^2+q+1) + (q^2+1)(q^2+q+1)q^3 + (q^3+q^2+q+1)q^6 = (q^5+q^3+1)(q^4+q^3+q^2+q+1)$.

(2) Suppose H is the extension of a hexagonal hyperplane. Then there exists a max $M \cong DQ(6,q)$ in $DQ(8,q)$ and a hexagonal hyperplane A in M such that $H = M \cup (\Delta_1(A) \setminus M)$. [A hyperplane of $DQ(6,q)$ is called *hexagonal* (Shult [17]) if the points and lines contained in it define a split-Cayley hexagon $H(q)$.] Since every point of $\Delta \setminus M$ is collinear with a unique point of M, $|H| = |M| + |A| \cdot q^4 = (q+1)(q^2+1)(q^3+1) + (q^3+1)(q^2+q+1)q^4 = (q^3+1)(q^6+q^5+q^4+q^3+q^2+q+1)$.

(3) Suppose now that H is a $Q^+(7,q)$-hyperplane of $DQ(8,q)$, i.e. a hyperplane which can be constructed in the way as described now. Let $Q(8,q)$ be the nonsingular parabolic quadric of $\mathrm{PG}(8,q)$ associated with the dual polar space $DQ(8,q)$. Intersecting $Q(8,q)$ with a suitable hyperplane of $\mathrm{PG}(8,q)$ we obtain a $Q^+(7,q) \subset Q(8,q)$. Let $\mathcal{M}^+$ and $\mathcal{M}^-$ denote the two families of generators of $Q^+(7,q)$ and let $\mathcal{S}^+$ denote the half-spin geometry for $Q^+(7,q)$ defined on the set $\mathcal{M}^+$. $\mathcal{S}^+$ is isomorphic to the point-line system of $Q^+(7,q)$ and hence has a hyperplane A which carries the structure of a $Q(6,q)$. Let B denote the set of all generators π of $Q(8,q)$ not contained in $Q^+(7,q)$ such that the unique element of $\mathcal{M}^+$ through $\pi \cap Q^+(7,q)$ belongs to A. Then $H := A \cup \mathcal{M}^- \cup B$ is a locally singular hyperplane of $DQ(8,q)$. Any such hyperplane is called a $Q^+(7,q)$-*hyperplane* of $DQ(8,q)$. These hyperplanes were introduced in Cardinali, De Bruyn and Pasini [5].

Every max M of $DQ(8,q)$ corresponds with a point x_M of $Q(8,q)$. If $x_M \in Q^+(7,q)$, then by [5], $M \cap H$ is a singular hyperplane of M and hence contains precisely $q^5+q^4+2q^3+q^2+q+1$ points. If $x_M \in Q(8,q) \setminus Q^+(7,q)$, then by [5], $M \cap H$ is a hexagonal hyperplane of M and hence contains precisely $(q^3+1)(q^2+q+1)$ points. Since every point of Δ is contained in precisely q^3+q^2+q+1 maxes, the number of points of H is equal to $(q^3+q^2+q+1)^{-1}\Big(|Q^+(7,q)| \cdot (q^5+q^4+2q^3+q^2+q+1) + (|Q(8,q)| - |Q^+(7,q)|) \cdot (q^3+1)(q^2+q+1)\Big) = (q^3+1)(q^6+q^5+q^4+q^3+q^2+q+1)$.

By (1), (2) and (3) above, it follows that every hyperplane of Σ intersects $e(\Delta)$ in either $(q^4+q^3+q^2+q+1)(q^5+q^3+1)$ or $(q^3+1)(q^6+q^5+q^4+q^3+q^2+q+1)$ points. So, $e(\Delta)$ is indeed a two-intersection set of $\mathrm{PG}(15,q)$.

The parameters of this two-intersection set are listed in the table given in Section 1.4.

3 A two-intersection set arising from the spin-embedding of $DQ^-(7, q)$

Let $e : \Delta \to \Sigma$ denote the spin-embedding of $\Delta = DQ^-(7, q)$ into $\Sigma = \mathrm{PG}(7, q^2)$. De Bruyn [8] classified all hyperplanes of Δ which arise from e. There are three types: the singular hyperplanes, the extensions of the classical ovoids in the quads and the so-called hexagonal hyperplanes.

(1) Suppose H is the singular hyperplane of Δ with deepest point x. Then $|H| = |\Delta_0(x)|+|\Delta_1(x)|+|\Delta_2(x)| = 1+q^2(1+q+q^2)+q^5(q^2+q+1) = q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + 1$.

(2) Suppose H is the extension of a classical ovoid O in a quad $Q \cong DQ^-(5, q) \cong H(3, q^2)$, i.e. $H = Q \cup (\Gamma_1(O) \setminus Q)$. [An ovoid of $H(3, q^2)$ is called *classical* if it is obtained by intersecting $H(3, q^2)$ with a nontangent plane.] Then $|H| = |Q| + |O| \cdot q^4 = (q^2 + 1)(q^3 + 1) + (q^3 + 1)q^4 = q^7 + q^5 + q^4 + q^3 + q^2 + 1$.

(3) Suppose H is a hexagonal hyperplane of $DQ^-(7, q)$. Then H is obtained in the way as described now. Let $Q^-(7, q)$ denote the nonsingular elliptic quadric of $\mathrm{PG}(7, q)$ associated with $DQ^-(7, q)$ and let $Q(6, q)$ be a nonsingular parabolic quadric obtained by intersecting $Q^-(7, q)$ with a nontangent hyperplane.

Let $\mathcal{G}$ denote a set of generators of $Q(6, q)$ defining a hexagonal hyperplane of the dual polar space $DQ(6, q)$ associated with $Q(6, q)$ and let $\mathcal{L}$ denote the set of lines L of $Q(6, q)$ with the property that every generator of $Q(6, q)$ through L belongs to $\mathcal{G}$. Then by Pralle [16], the set H of generators of $Q^-(7, q)$ containing at least one element of $\mathcal{L}$ is a hyperplane of $DQ^-(7, q)$. We call any hyperplane which can be obtained in this way a *hexagonal hyperplane* of $DQ^-(7, q)$. The number $|\mathcal{L}|$ is the number of lines of $DQ(6, q)$ contained in a hexagonal hyperplane and is equal to $\frac{q^6-1}{q-1}$. Each element of $\mathcal{L}$ is contained in $q + 1$ generators of $Q^-(7, q)$ which are contained in $Q(6, q)$ and $q^2 - q$ generators of $Q^-(7, q)$ which are not contained in $Q(6, q)$. Hence, $|H| = |\mathcal{G}| + (q^2 - q)|\mathcal{L}| = q^7 + q^5 + q^4 + q^3 + q^2 + 1$.

By (1), (2) and (3) above, it follows that every hyperplane of Σ intersects $e(\Delta)$ in either $q^7+q^6+q^5+q^4+q^3+q^2+1$ or $q^7+q^5+q^4+q^3+q^2+1$ points. So, $e(\Delta)$ is indeed a two-intersection set of $\mathrm{PG}(7, q^2)$. The parameters of this two-intersection set are listed in the table given in Section 1.4.

4 A (q^3+1)-tight set arising from the spin-embedding of $DQ^-(7,q)$

Again, let $e : \Delta \to \Sigma$ denote the spin-embedding of $\Delta = DQ^-(7,q)$ into $\Sigma = \mathrm{PG}(7,q^2)$. We show that $e(\Delta)$ is a (q^3+1)-tight set of a nonsingular hyperbolic quadric $Q^+(7,q^2)$ of $\mathrm{PG}(7,q^2)$. We recall the construction of the spin-embedding of $\Delta = DQ^-(7,q)$. Let $Q^-(7,q)$ be the nonsingular elliptic quadric associated with $DQ^-(7,q)$, and consider the inclusion $Q^-(7,q) \subseteq Q^+(7,q^2)$. Let $\mathcal{M}^+$ and $\mathcal{M}^-$ denote the two families of generators of $Q^+(7,q^2)$ and let θ be a triality of $Q^+(7,q^2)$ mapping $\mathcal{M}^+$ to the point-set of $Q^+(7,q^2)$, the point-set of $Q^+(7,q^2)$ to $\mathcal{M}^-$ and $\mathcal{M}^-$ to $\mathcal{M}^+$. For every generator M of $Q^-(7,q)$, let $\phi'(M)$ denote the unique generator of $\mathcal{M}^+$ containing M. Then $\theta \circ \phi'$ is the spin-embedding e of $DQ^-(7,q)$. Obviously, $e(\Delta)$ is a set of points of $Q^+(7,q^2)$.

Lemma 4.1 (a) *If M_1 and M_2 are two generators of $Q^-(7,q)$ which meet each other, then $e(M_1)$ and $e(M_2)$ are collinear points of $Q^+(7,q^2)$.*

(b) *If M_1 and M_2 are two disjoint generators of $Q^-(7,q)$, then $e(M_1)$ and $e(M_2)$ are noncollinear points of $Q^+(7,q^2)$.*

Proof. (a) Suppose M_1 and M_2 are two generators of $Q^-(7,q)$ which have a point x in common. Then the points $e(M_1)$ and $e(M_2)$ of $Q^+(7,q^2)$ are contained in the generator $\theta(x) \in \mathcal{M}^-$ of $Q^+(7,q^2)$. Hence, $e(M_1)$ and $e(M_2)$ are collinear on $Q^+(7,q^2)$.

(b) Suppose that M_1 and M_2 are two disjoint generators of $Q^-(7,q)$. Let $\overline{M_i}$, $i \in \{1,2\}$, denote the 2-space of $Q^+(7,q^2)$ containing M_i, Then $\overline{M_1}$ and $\overline{M_2}$ are disjoint. Since $\phi'(M_1)$ and $\phi'(M_2)$ belong to the same family of generators of $Q^+(7,q^2)$, they intersect in either the empty set or a line. But since $\overline{M_1} \cap \overline{M_2} = \emptyset$, they must intersect in the empty set. Then $e(M_1) = \theta \circ \phi'(M_1)$ and $e(M_2) = \theta \circ \phi'(M_2)$ are not collinear on $Q^+(7,q^2)$. ∎

Now, let N_1 denote the total number of ordered pairs of distinct points of $e(\Delta)$ which are collinear on $Q^+(7,q^2)$. By Lemma 4.1,

$$N_1 \quad = \quad |\Delta| \cdot \Big(|\Delta_1(x)| + |\Delta_2(x)|\Big), \tag{2}$$

where $|\Delta| = (q^2+1)(q^3+1)(q^4+1)$ denotes the total number of points of Δ and x denotes an arbitrary point of Δ. So, $N_1 = (q^2+1)(q^3+1)(q^4+1)\Big(q^2(q^2+q+1)+q^5(q^2+q+1)\Big) = (q^2+1)(q^3+1)(q^4+1)q^2(q^3+1)(q^2+q+1)$. Calculating expression (1) of Section 1.2, we find

$$(q^6-1)\cdot(q^2+1)(q^3+1)(q^4+1)\cdot\Big(\frac{(q^2+1)(q^3+1)(q^4+1)}{q^8-1}+1\Big)$$

$$= q^2(q^2+1)(q^3+1)^2(q^4+1)(q^2+q+1).$$

Since the expressions (1) and (2) are equal, $e(\Delta)$ is a tight set of points of $Q^+(7, q^2)$. The set $e(\Delta)$ is i-tight where

$$i = \frac{|\Delta| \cdot (q^2-1)}{q^8-1} = q^3+1.$$

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