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FACULTY OF ECONOMICS AND BUSINESS ADMINISTRATION

ESSAYS ON THE PARTNERSHIP PROBLEM

JOOST VANDENBOSSCHE

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SUPERVISOR: PROF. DR. DIRK VAN DE GAER (GHENT UNIVERSITY)

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Doctoral committee: Prof. Dr. Marc De Clercq (Ghent University, Dean)
Prof. Dr. Patrick Van Kenhove (Ghent University, Academic Secretary)
Prof. Dr. Dirk Van de gaer (Ghent University)
Dr. Thomas Demuynck (KU Leuven - KULAK and Ghent University)
Prof. Dr. Vincent Vannetelbosch (CORE (UCL) and FUSL)
Prof. Dr. Ana Mauleon (CORE (UCL) and FUSL)
Prof. Dr. Tamás Fleiner (EGRES (ELTE))
Prof. Dr. Freddy Heylen (Ghent University)
Prof. Dr. Eddy Omey (Ghent University)

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1
Introduction

Since the seminal Gale and Shapley (1962) paper, the field of matching theory has seen a lot of research on theoretical and empirical topics by economist, mathematicians and computer scientist alike. This dissertation aims to shed light on a relatively little discussed matching problem, the *partnership problem*, which is a very general type of matching problem that allows for a wide range of preference structures. In its current form, as it is used in this dissertation, it was first introduced by Fleiner (2010).

There are two main contributions of this dissertation to the current state of the literature. First, in the second chapter, we look at an application of a partnership problem, a network formation model with a heterogeneous cost function. We show the existence of a unique stable outcome for this application and give an illustration of its relevance in predicting real life matching outcomes. Besides contributing to the matching theory literature, it also contributes to the network formation literature by introducing a cost function which allows for a high degree of heterogeneity. Second, in the third and fourth chapter, we consider known results from simpler matching problems and extend them to the partnership problem. As such, the main take away from these last two chapters is that, while the partnership problem is a more complicated problem with a much richer preference structure, structurally it is still very similar to the more simple matching problems. In essence, we are dealing with complex matching problems which have a surprisingly simple basic structure.

This introduction serves two purposes. First, we introduce the concept of a matching problem by looking at some examples and we define the partnership problem, highlighting the key ways in which it is more general than most known matching problems. Second, we give an overview of the main results and contributions of this dissertation which can be found in Chapters 2, 3 and 4.

1.1 Matching problems

A matching problem is a resource allocation game where agents have to decide with whom to form meaningful relationships, with the restriction that both parties have to agree on forming a relationship. Gale and Shapley (1962) describe some examples of a matching problem:

> [**Marriage.**] A certain community consists of n men and n women. Each person ranks those of the opposite sex in accordance with his or her preferences for a marriage partner. We seek a satisfactory way of marrying off all members of the community (Gale and Shapley, 1962, p.11).
>
> [**Roommates.**] An even number of roommates wishes to divide up into pairs of roommates (Gale and Shapley, 1962, p. 12).
>
> [**College admission.**] A set of n applicants is to be assigned among m colleges where q_i is the quota of the i-th college. Each applicant ranks the colleges in the order of his preferences.[...] Each college similarly ranks the students who have applied to it in order of preferences. [...] We wish to determine an assignment of applicants

> to colleges in accordance with some agreed-upon criterion of fairness (Gale and Shapley, 1962, p. 9).

An example that will be discussed in the second chapter of this dissertation is the following.

> [**Risk sharing.**] A certain community consists of n households. In order to mitigate the adverse effects of downward risk, households will try to form one or more bilateral relationships to informally share risk.

Now, any of those matching problems can be defined by a combination of three elements. First, there is the group of agents. For example in the marriage problem, this group of agents is a group of men and women. In this dissertation, we denote the group of agents by the set V. Second, there is the set of potential relationships, E. If, say, agents u and v have an interest in forming the relationship e between them, then e is a potential relationship; e has the potential to be formed. Returning to the example of the marriage problem, if u and v are both male then e will not be a potential relationship, as u and v are looking for a mate of the other sex. Visually, the set of agents and the set of potential relationships can be represented by an undirected graph (V,E), with V the vertices and E the edges between the vertices. A matching is then a subset of E, a collection of relationships. The third element is the set of preferences of each agent v in V over the set of potential relationships that involve this agent v, $E(v)$. These preferences can be modeled through a choice function $C_v(.)$, a function that maps an option set – the set of potential relationships – to a choice set – a set of chosen relationships. By using any possible subset of $E(v)$ for this function, we can map out the complete preference structure of v. Using this notation, an undirected graph (V,E) and a set of choice functions $C = \{C_1,\dots,C_{|V|}\}$, we can define all matching problems by imposing restrictions on E and C.

Stability In this dissertation, we assume that agents are myopic and that they do not act strategically with respect to the revelation of their preferences. The first implies that agents are not able to foresee the reactions of other agents on their actions – they are short-sighted. The second implies that all agents will always act according to their true preferences, that is, no agent will try to strategically manipulate the problem by misrepresenting her preferences. As such, a key concept, which will be the focus of much of this dissertation, is the notion of a *stable matching*. A stable matching is a matching which fulfills two conditions. First, it is individually rational in the sense that no agent has a relationship that it does not want to maintain, given the other relationships the agent is involved in. Second, there does not exist a blocking relationship, a relationship that is not part of the matching, but that both agents involved wish to form given the relationships in the matching. Hence, a stable matching can be considered an equilibrium situation, as no agent is capable of improving her current situation.

1.1.1 Types of problems

The example problems given in the first paragraph of this section are different from each other in two respects (see table 1.1.).

	two-sided	one-sided
one-to-one	Marriage	Roommate
many-to-many	College admissions	Risk sharing

Table 1.1: Examples of different types of matching problem.

First, there is the distinction between two-sided and one-sided matching problems. In a two-sided matching problem, V can be split up in two subsets such that E contains only relationships between agents of a different subset. In a one-sided problem, this division cannot be made as there are no a priori restrictions on the set of potential relationships. The stable marriage problem and the college admissions problem are two examples of a two-sided problem as men (women) cannot be matched with other men (women) and colleges (applicants) cannot be matched with other colleges (applicants). In contrast, the stable roommate problem and the example of the informal risk sharing are one-sided in that anyone can form a relationship with anyone. The distinction between one-sided and two-sided problems is important as Gale and Shapley (1962) showed that it defines the boundary between matching problems for which a stable matching will always exist – a two-sided problem – and problems for which a stable matching might not exist – a one-sided problem. Thus, for one-sided problems we always have to take into account that a stable matching might not exist.

A second feature of a matching problem is how many different relationships an agent is willing to be involved in. For example, in the marriage and roommate problem all agents only want to be paired up with one other agent – there is no polygamy. These problems are examples of one-to-one matching problems. If agents want to form multiple relationships we are in a many-to-many matching problem. The college admissions problem and the risk sharing example are examples of a many-to-many matching problem. Other than in the stable marriage/roommate problem, colleges are now willing to take in more than one student at a time,[1] and households may want to share risk with more than one other household.

The distinction between one-to-one and many-to-many problems is important because of the possible preference structures. Let us assume that all preferences have to satisfy the *weak axiom of revealed preferences* (WARP). In the context of choice sets, WARP means that if X is the set of chosen options while a set Z is also available, then Z is never the set of chosen options whenever X is also available. Formally, if $C_v(Y) = X$, such that $X \subset Y$ and $Z \subset Y$, then there does not exist another option set Y' such that $X \subset Y'$ and $Z \subset Y'$ while $C_v(Y') = Z$. Then, for one-to-one problems, if WARP has to hold, all choice functions C_v can be characterized by a strict ordering of edges in $E(v)$. The choice of v for a subset X of $E(v)$ is then the first element of X in the ordering of $E(v)$. For many-to-many problems, this strict ordering of edges does not cover all possible preference structures that satisfy WARP. Hence, within the class of many-to-many matching problems, there are different types of problems depending on the assumptions on the preference structure. The many-to-many matching problem closest in nature to a one-to-one problem is the *stable b-matching problem*. In this problem, it is assumed that agents have a capacity b and a strict preference ordering of the potential relationships they are involved in. Agents then choose the b first relationships in their ordering. An example of such a problem is the college-admissions problem and the one-sided stable b-matching problem (Cechlarova and

[1] Technically this is a many-to-one matching problem as students can only attend one college.

Fleiner, 2003). Another problem, which will be discussed in chapter 3, is *the partnership problem with linear preferences* which maintains the strict preference ordering but is more general than the stable b-matching problem as the capacity for each agent is not fixed. When we drop the assumption of a strict preference ordering, we are dealing with the *partnership problem*, the most general of all problems discussed in this chapter.

1.1.2 The Partnership problem.

A partnership problem (Fleiner, 2010) is a one-sided, many-to-many matching problem where the choice functions have two restrictions, substitutability and increasingness.

Substitutability The first restriction, substitutability, means that when a relationship e is chosen from an option set $X(v)$, then v will still choose option e when some other options are no longer available in $X(v)$. Formally,

SUB A choice function, C_v, is *substitutable* if, for $e \neq f$ and $e \in C_v(X(v))$, then $e \in C_v(X(v) \setminus f)$.

Increasingness The second restriction, increasingness, means that if k relationships are chosen from a set, then in a superset of that set, at least k relationships will be chosen.

INCR A choice function, C_v, is *increasing* if $Y \subseteq X$ implies $|C_v(Y)| \leq |C_v(X)|$.

While the restriction of substitutability is straightforward when we assume that relationships are substitutes – rather than compliments, the restriction of increasingness is less straightforward. There are, however, two arguments for imposing increasingness. First, if we again assume that all preferences have to satisfy WARP, then substitutability alone does not suffice. For example, if $E(v) = \{e_1, e_2, e_3\}$ then $C_v(e_1, e_2, e_3) = e_1$, $C_v(e_1, e_3) = \{e_1, e_3\}$ and $C_v(e_1, e_2) = \{e_1, e_2\}$ implies a substitutable choice function – for which increasingness is violated – that does not satisfy WARP. In contrast, increasingness together with substitutability implies WARP. Increasingness guarantees that removing an unchosen option from an option set has no impact on the choice set, thereby guaranteeing WARP. The second argument is of a more technical nature. The restriction of increasingness is necessary to prove existence of a stable matching in a two-sided problem (Fleiner, 2003)[2] or the existence of a stable half-matching – a structure that generalizes the notion of a stable matching (Fleiner, 2010).

A special case of the partnership problem is the partnership problem with linear preferences which has, besides substitutability and increasingness, a third restriction on the choice functions, linearity. Linearity states that if an option e is chosen from an option set X while another available option f is not, then it can never be that e and f are in an option set Y and f is chosen while e is not.

LIN A choice function C_v is linear if, for $e \neq f$, there exists an $X(v)$ containing both e and f such that $e \in C_v(X(v))$ and $f \notin C_v(X(v))$, then there does not exist a $Z(v)$ containing both e and f, such that $e \notin C_v(Z(v))$ while $f \in C_v(Z(v))$.

Any choice function $C_v(.)$ satisfying substitutability, increasingness and linearity implies a strict preference ordering of the set of potential relationships $E(v)$.

1.2 Results and contributions.

We will now briefly touch on the content of the following chapters, discussing the main results and their contribution to the literature.

The **second chapter** looks at a particular partnership problem with a cardinal utility framework as the payoffs that agents can derive from forming relationships are assumed to be measurable. This payoff is a concave function of the number of relationships that an agent forms. Hence, new relationships increase the payoff for the agent, but do so at a decreasing rate. In addition, the formation of these relationships is costly, as time and effort has to be invested to create and monitor the relationship. The cost function is driven by two factors. The first cost factor is *distance* which can be interpreted as social distance. As the social distance between two agents

[2] Fleiner (2003) used a weaker restriction than increasingness (if $C_v(X) \subseteq Y \subseteq X$ then $C_v(X) = C_v(Y)$) to prove existence.

increases, the costs to create and monitor the relationship increase. The second cost factor is *agent quality* and relates to the attractiveness of the potential partner, which is assumed to be objectively measurable, i.e. every agent has the same perceived quality of another agent. A higher quality of the potential partner decreases the cost for the agent. To sum up, every agent v has a payoff function of the following form given a matching M:

$$v(|M(v)|) - \sum_{uv \in M(v)} d(u,v) - \theta(u),$$

with $v(.)$ a concave function, $d(u,v)$ the distance between u and v and $\theta(u)$ the quality of agent u.

As a result, we get a preference structure where agents can rank other agents in terms of distance and quality. The number of relationships an agent wishes to form is determined by the marginal benefit of adding an extra agent versus the cost that has to be invested to form a relationship with this agent.

The chapter contributes to two different fields. First, it contributes to the matching literature by providing an application of a partnership problem with linear preferences. Second, it contributes to the network formation literature as the model presented in the chapter can also be viewed as a network formation model where the payoffs of each agent are only determined by her direct network, the links an agent has. The innovation in the model is the introduction of a high degree of heterogeneity in the cost function. Earlier network formation models have very restricted cost functions, which do not allow for a lot of heterogeneity over agents and potential partners. The network stability concept used in the chapter is strong pairwise stability which, for this particular model, is equivalent to the concept of a stable matching.

There are two sets of results. First, it is shown that the model has a unique stable matching. This unique stable matching implies a hierarchy over agents. The set of relationships of an agent in the stable matching is his desired set of relationships when taking into account the choices of the agents above her in the hierarchy. For example, for the agent at the top of the hierarchy, her stable matching relationships will be exactly her desired set of relationships. This knowledge is then used to devise an algorithm that produces the unique stable matching.

Second, we consider an *informal risk sharing network* as an illustration of the model. In a risk sharing network, households are looking for partners to informally share risk with. Results from the empirical literature on these networks suggest that they are well suited examples for the model discussed in this chapter. We then illustrate the relevance of this model by using data from a risk sharing network in Nyakatoke, Tanzania. First, we show that our model, and the underlying formation structure, has some relevance in predicting the existence of risk sharing relationships. Second, our model is also able to capture most features of risk sharing networks observed in real life.

The **third chapter** investigates a very simple idea. When looking at matching problems in practice we see that in many applications centralized matching institutions are present that ensure that a stable outcome is reached. For those applications that lack a centralized institution, Roth and Vande Vate (1990) observed however that in many cases a stable outcome was also reached; the myopic actions of the individual agents involved created a decentralized matching process that eventually converged to stability. The question is now for what type of matching problems we can guarantee the convergence of a decentralized process to stability. In this dissertation we show convergence for a partnership problem with linear preferences.

This result – for a partnership problem with linear preferences, a decentralized matching process will converge to a stable outcome with probability one – is an extension of the result by Roth and Vande Vate (1990) for the stable marriage problem and the result by Diamantoudi et al. (2004) for the stable roommate problem. Another study, Kojima and Ünver (2008), shows convergence for a two-sided matching problem with similar restrictions on the choice functions. Our matching problem and Kojima and Ünver (2008)'s matching problem are non-nested matching problems, such that they both hold as most general matching problems for which convergence was shown.

As we mentioned above, for one-sided problems, a stable matching might not exist. For these problems with no stable matchings, we can ask where the decentralized matching process leads to. As a second result, we extend a result by Iñarra et al. (2008) for the stable roommate problem and show that any decentralized matching process eventually leads to a set of matchings that are related to a stable half-matching, a generalized notion of a stable matching. However, this set of matchings is not absorbing in the sense that, when in this set, it is possible that a decentralized matching process leads to a matching not in this set. Hence, whether we can extend Iñarra et al. (2010)'s result on absorbing sets for the stable roommate problem remains an open question.

The **fourth chapter** digs deeper into the structure of the partnership problem and the structure of a stable matching in particular. Extending results from Gusfield (1988) and Borbelova and Cechlarova (2010), we expose the common structure of all stable matchings in a given problem. The results can be summarized as follows.

As a first result, we show that Fleiner (2010)'s extension of Irving (1985)'s algorithm that produces a stable matching for the partnership problem when one exists, can be used to find all stable matchings for a given problem. This already hints that stable matchings share a common structure. The second set of results shows what this structure is.

The second result shows that any stable matching can be linked to a set of rotations. A rotation is what Irving (1985) dubbed an all-or-nothing cycle. It is a looping sequence of relationships such that either none of the odd-labeled elements are part of a stable matching or all of them are part of some stable matching but then there exists a stable matching such that all even-labeled elements of the sequence are part of it. Hence, by deleting the odd-labeled elements of a rotation that is exposed at some step in Fleiner (2010)'s algorithm, we are sure to get closer to a stable matching. This process is denoted the *elimination of a rotation*. Now, this chapter shows that if we take account of all rotations that are eliminated en route to a stable matching, then there is a one-one correspondence between this set of rotations and this stable matching. One particular stable matching can only be produced by eliminating one particular set of rotations and no set of rotations can produce two different stable matchings. Hence, any stable matching is linked to a unique set of rotations. Now, as a second part of this second result it is shown that these sets of rotations are very similar. There exist rotations which are part of all sets of rotations – they are always eliminated – and if a rotation π is part of one set A but not of another B, then B contains a rotation π' that is a dual of π, a rotation that contains the same elements of π but deletes another set of relationships when eliminated. Hence, the common structure of a stable matching can be linked to (1) the fact that these stable matchings can all be linked to some set of rotations and (2) the fact that these sets of rotations are very similar to each other as the only difference between them is which type of a rotation is eliminated.

The third set of results then shows the implications of this common structure on the level of the relationships. As there are rotations that are always eliminated, there are relationships that are always part of a stable matching and these will be the relationships that are part of these rotations but not deleted – the even-labeled elements of the rotation. Relationships that are part of some stable matchings but not all, are part of rotations that are sometimes eliminated but not always. Hence, the structure of a stable matching mimics the structure of the eliminated rotations.

In sum, this dissertation explores a very general matching problem by taking three different routes. The fourth chapter is the widest in scope, aiming to shed some light on the structure of stable matchings in a given partnership problem, uncovering the similarities between them. The third chapter zooms in to one particular question and tries to answer if and how these stable matchings will be reached in the absence of a centralized matching institution. The second chapter is most focused and looks at one particular application of a partnership problem, using a cardinal utility framework to model the preference structure.

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2

Network formation with heterogeneous agents and absolute friction[1]

2.1 Introduction and motivation

We present a model of strategic network formation with absolute friction and heterogeneity among agents. We show that our model always has a unique network that is strong pairwise stable. We provide an algorithm to compute this network and we illustrate its use by applying it to a data set of a risk sharing network in small village in Tanzania.

[1]This chapter is the result of joint work with Thomas Demuynck. Another version of this chapter is forthcoming in the Journal of Computational Economics (published online in january 2012).

Overview Social networks are among the most valuable contributors to apprehend modern human life. They determine, among others, communication, friendship, trust, marriage, trade, risk sharing and wealth. In order to understand these phenomena it is crucial to study how these networks are formed and what a stable network looks like when agents are able to form and/or cut links. This insight has lead to an enormous growth of scientific research analyzing the development, stability and empirical regularities of social networks.

In this research we present a model of strategic network formation. Our model has three important features. First, the model assumes that link formation is only possible under mutual consent. This means that in order to establish a link, two agents have to agree to form this link, while every agent can unilaterally decide to cut a pre-existing link (see Jackson and Wolinsky (1996)). This feature distinguishes our research from models of unilateral link formation.[2] In models of unilateral link formation, any agent can unilaterally decide to link with another agent. In addition, our model does not allow for transfers between agents, where it is possible that one agent compensates another agent for the formation a link between them.[3] Second, our model allows for a wide range of heterogeneity in the determination of the link-costs. The link-cost is the cost that each agent incurs when adding a certain link. Third, we restrict ourselves to networks with absolute friction. This means that the benefits of a certain link only attribute to the two agents that form this link, excluding any spillover to other agents. In this section, we discuss each of these three features in more detail. This will allow us to position and distinguish our research from other research in the network formation literature.

Bilateral link formation We assume that links can only be formed under mutual consent. In particular, no agent can form a link with another agent without the agreement of this agent but any agent can unilaterally decide to cut an existing link (see Myerson, 1977, 1991; Jackson and Wolinsky, 1996). We say that a network is stable if there is no *coalition* of agents that

[2]See, among others, Bala and Goyal (1997); Dutta and Jackson (2000); Bala and Goyal (2000a,b); Haller and Sarangi (2005); Galeotti, Goyal, and Kamphorst (2006); Feri (2007); Feri and Meléndez-Jiméz (2009); Hojman and Szeidl (2008).

[3]See, among others, Dutta and Mutuswami (1997); Dutta, van den Nouweland, and Tijs (1998); Slikker and van den Nouweland (2000, 2001); Jackson (2005); Bloch and Jackson (2007).

can *alter* the network in such a way that at least one agent in the coalition benefits from the change without any agent in the coalition losing from the change.[4] This definition hinges on two further specifications. First, we need to specify which coalitions are allowed to alter the existing network, and second, we need to specify in which manner these coalitions may alter the current network.

Depending on the specifications of coalition size and alterations, different stability concepts can be established. At the one extreme we may allow for coalitions of arbitrary size that can modify a given network by creating any link between two agents within the coalition and cut any number of links that involves at least one agent from the coalition. This corresponds to the notion of strong stability as introduced by Dutta and Mutuswami (1997) and Jackson and van den Nouweland (2005). At the other end of the spectrum we may consider the pairwise stability concept where deviating coalitions h"ave size at most two and these coalitions can change the network by either creating the link between the two agents in the coalition or either cut a single link involving an agent in this coalition (see Jackson and Wolinsky (1996); Jackson and Watts (2001)). An intermediary stability concept which we will discuss, is the concept of strong pairwise stability, introduced by Belleflamme and Bloch (2004). In this case, the maximal coalition size is two, as for the pairwise stability concept, but all types of alterations available to the agents in the coalition are allowed, as for the strong stability concept. Hence, a pair of agents may cut as many links as they like and simultaneously form a link between them. We show that our model always has a unique strongly (pairwise) stable network.

Heterogeneity in the payoff structure Within this model of network formation under mutual consent, we define the payoff structure for the various agents. The payoff from being part of a given network consists of a benefit part and a link-cost part, reflecting the idea that agents have

[4]This definition abstains from various forms of farsightedness (see for example Watts (2002); Deroïan (2003); Dutta, Ghosal, and Ray (2005); Page, Wooders, and Kamat (2005); Page and Wooders (2009b); Herings, Mauleon, and Vannetelbosch (2009) or evolutionary dynamics (see, for instance, Watts (2001); Jackson and Watts (2002); Dutta et al. (2005); Tercieux and Vannetelbosch (2006); Carayol and Roux (2006); Feri (2007); Page and Wooders (2009a)).

to incur costs to establish a link. Agents have to invest time and effort to build a relationship and agree on the conditions of the link. We assume that this link-cost can be split into two parts. The first part is a pairwise component which we will call *distance* d and is symmetric — i.e. for all agents i, j in a population N the distance from i to j is equal to the distance from j to i. Distance reflects the idea that the better the ex-ante relationship between agents is, the less time and effort needs to be invested, because of a pre-established level of trust and understanding or better monitoring possibilities. The second part is an agent-specific, idiosyncratic, component which we will call *quality* θ. The assumption is made that increased quality of the linking partner reduces link-costs.[5]

All other studies of network models with bilateral link formation use more restrictive forms for the cost topology. Among others, Jackson and Wolinsky (1996); Goyal and Joshi (2006); Calvó-Armengol (2004) analyze models with homogeneous costs, i.e. the setting where linking costs are constant for all pairs of agents. Johnson and Gilles (2000) assume that costs are described by a line cost topology. In this model, agents are situated on a line and the linking-cost between two agents is determined by the Euclidean distance between them. Carayol and Roux (2006) uses a circle-cost topology, where agents are situated on a circle and the linking-cost between two agents is determined by their shortest distance. Finally, Jackson and Rogers (2005) impose an island cost topology. In this setting, agents are allocated to a finite number of islands and the linking-cost between two agents may take on one of two values conditional on the requirement that the link-cost between two agents on the same island is less costly than the link-cost between agents on distinct islands.

More general models of heterogeneity were investigated for models with non-bilateral link formation. In particular, Galeotti, Goyal, and Kamphorst (2006) investigate a model of one sided-link formation where costs can vary freely. Finally, Brueckner (2006) and Haller and Sarangi (2005) use the same heterogeneity concepts as we do — distance and quality — in a network formation model with probabilistic graphs. In probabilistic graph models, link formation may fail with a certain probability(Bala and Goyal, 2000b). More specifically, Brueckner (2006) assumes

[5] This impact may be indirect in that a higher quality may be reflected in an extra benefit, which increases the willingness to link, effectively reducing link-cost in an additive framework.

that this probability depends on the effort — costs — made by the agents involved. Hence, a major contribution of our model is that it is the first model which combines bilateral link formation with such general forms of heterogeneity.

Absolute friction In addition, and contrary to previously mentioned studies of bilateral link formation, our model assumes absolute friction. Absolute friction states that the benefits from the network to a certain agent only depend on the agents that are directly linked to this agent. A model with less than absolute friction assumes that the benefits are decreasing in the length of the path while a model with no friction assumes that the length of the path does not matter in defining the benefits. The assumption of absolute friction is necessary to keep our model tractable. Indeed, the restrictive nature of the cost heterogeneity used in the literature with models of bilateral link formation(see above) strongly indicates that models with moderate levels of friction are only tractable by imposing more stringent restrictions on the cost topology.

The assumption of absolute friction is defendable for networks of informal insurance, trust or trade. Fafchamps and Lund (2003) show that risk sharing is not frictionless due to transaction costs, imperfect commitment, asymmetric information or other processes that limit exchange. This finding is corroborated by De Weerdt and Dercon (2006) who indicate that direct network partners may be most relevant for understanding risk sharing. In addition, we could also assume that agents are shortsighted because they do not know the complete structure of the network and only take into account the benefits from their direct partners, while assuming other benefits to be non-existent. In this case, the network that emerges looks like the network that would emerge in the case of absolute friction, so that the absolute friction assumption is valid in this setting.
When we do not assume shortsightedness the assumption of absolute friction may be more difficult to maintain. For example in communications networks, an agent with many connections may be appealing to link to because she may pass on to you the information she got from her connections. Likewise, in a collaboration network an agent with many connections may be less attractive since his time is taken by many agents, leaving less time to work together.[6] In these

[6]See, among others, the connections and co-author models in Jackson and Wolinsky (1996), the job contact network in Calvó-Armengol (2004) and the local spillover game in Goyal and Joshi (2006).

cases, the indirect network matters. Thus, we conclude that the validity of our assumption of absolute friction depends on the context and the additional assumptions made.

Main results This paper contributes to the present literature by providing a tractable model of two-sided network formation with few restrictions on linking costs, while considering a wide range of stability concepts. We show that for our model, there always exists a unique stable network which is both strong pairwise and strongly stable.

Besides the existence and uniqueness result, we provide an algorithm that reproduces this stable network. In this algorithm, starting from the complete network, we let a sequence of agents cut the links they do not want to maintain, given the cutting actions of the agents before them in the sequence. The stability of the outcome of the algorithm hinges on the order of the agents in the sequence. We can show that for the resulting network, in every iteration of the algorithm there will always be at least one agent whose set of links she would like to maintain in that network will be equal to her set of links in the unique stable network. Hence, the sequence of agents is such that in every iteration exactly those agents are selected. Naturally, an agent cannot be selected twice and is deleted from the selection pool once selected. This algorithm is one of the key innovations in this paper. Once we know the structure of the cost functions, and the form for the benefit functions, we can compute the particular sequence of agents and the emerging stable network, observe its general features and relate these features to these structures.

We illustrate the use of our algorithm by applying it to a data set on an informal risk sharing network drawn from the village of Nyakatoke in rural Tanzania.[7] Although our approach is rather crude in the measurement of the cost structure, implying that our exercise can be considered as an illustration at best, we find that our algorithm outperforms a random network formation model where probabilities were fitted from a logit regression which uses the same data.

Section 2 introduces the model and provides the definition of a stable network. Section 3 contains and discusses the main result relating to the uniqueness and existence of the stable network. Section 4 provides the empirical illustration and section 5 concludes.

[7] This data set is also used in the research of Comola (2008); De Weerdt (2004); De Weerdt and Dercon (2006); Comola and Fafchamps (2009).

2.2 The model

Models of network formation Consider a set of agents $N = \{1, \dots, n\}$. An *undirected network* g is given by a collection of two-element sets $\{i, j\}$ where $i, j \in N$ and $i \neq j$. For ease of notation, we will write ij instead of $\{i, j\}$. If $ij \in g$, we say that agents i and j are linked in the network g. The set of links adjacent to i in the network g, will be denoted the *direct network*,

$$g(i) = \{hj \in g | j \in N, h = i\}.$$

We denote the *complete network* —the network where every agent is linked to every other agent— by g_N, i.e. $g_N = \{ij | i, j \in N, i \neq j\}$. We denote by $\mathcal{G}_n$ the collection of all networks on a set of n agents. To each agent $i \in N$, we endow a *payoff function* π_i from the set $\mathcal{G}_n$ to the set $\mathbb{R}$ defining the payoff, $\pi_i(g)$ that agent i receives when the network g is established. A *payoff structure* $\{\pi_i\}_{i \leq n}$ consists of a finite number, n, of agents and payoff functions π_i for every agent $i \leq n$. We denote by $\mathcal{S}$ the set of all payoff structures.

Stability concepts A *stability concept*, Π, is a correspondence from the set of payoff structures, $\mathcal{S}$, to the set of all networks $\bigcup_n \mathcal{G}_n$ such that for all payoff structures, $\{\pi_i\}_{i \leq n} \in \mathcal{S}$,

$$\Pi(\{\pi_i\}_{i \leq n}) \subseteq \mathcal{G}_n.$$

The set $\Pi(\{\pi_i\}_{i \leq n})$ determines the set of stable networks corresponding to the payoff structure $\{\pi_i\}_{i \leq n}$ and stability concept Π.

We begin by defining two particular stability concepts. In order to model these concepts, we borrow the idea of a *linking game* as introduced by Myerson (1991). For the moment let us fix a payoff structure $\{\pi_i\}_{i \leq n}$ and the set of agents $N = \{1, \dots, n\}$. For an agent, $i \in N$, his/her strategy set S_i consists of all subsets of $N - \{i\}$. This definition includes the empty set, $\emptyset$. The interpretation is that $j \in s_i$ if i proposes to j to form a link and $j \notin s_i$ if i is not willing to link with j. For a given strategy profile, $(s_1, \dots, s_n)$, a link between i and j will be formed if $j \in s_i$ and $i \in s_j$. In other words, i and j will be linked if i proposes to form a link to j and j proposes to form a link to i. We denote by $g(s_1, \dots, s_n)$ the network that is formed when agent $i \in N$ chooses strategy

$s_i \in S_i$. In particular, we will have that $ij \in g(s_1,\ldots,s_n)$ if and only if $j \in s_i$ and $i \in s_j$. Observe that, in general, there are several strategy profiles that may lead to the same network, i.e. there will exist distinct profiles $(s_1,\ldots,s_n)$ and $(s'_1,\ldots,s'_n)$ such that $g(s_1,\ldots,s_n) = g(s'_1,\ldots,s'_n)$. One notable exception is the complete network, g_N, where $g(s_1,\ldots,s_n) = g_N$ if and only if $s_i = N - \{i\}$ for all $i \in N$. For every coalition $C \subseteq N$ and strategy profile $s \in S$, we also write $s = (s_C, s_{N-C})$, where s_C is the strategy profile s restricted to agents within the coalition C. We call s_C a coalition strategy profile.

The first stability concept is strong stability as developed in Dutta and Mutuswami (1997) and Jackson and van den Nouweland (2005). A network is strongly stable if for all possible coalitions $C \subseteq N$ of agents there does not exist a deviation of the strategy profile by members of C such that at least one agent within the coalition strictly gains from the deviation, and no agent within the coalition loses from this rearrangement. Formally,[8]

Strong Stability A network g is strongly stable if there exists a strategy profile $(s_1^*,\ldots,s_n^*)$ such that $g = g(s_1^*,\ldots,s_n^*)$ and for all nonempty coalitions C, there does not exist a coalition strategy profile s_C such that for all $i \in C$, $s_i \in S_i$ and

$$\pi_i(g(s_C, s_{N-C}^*) \geq \pi_i(g(s_C^*, s_{N-C}^*)),$$

with at least one inequality being strict.

For a given payoff structure $\{\pi_i\}_{i\leq n}$, we denote by $\Pi_{ss}(\{\pi_i\}_{i\leq n})$ the (possibly empty) collection of strongly stable networks.

For our purposes, we also consider a second stability concept, strong pairwise stability as defined in Belleflamme and Bloch (2004), which restricts the size of deviating coalitions to at most two.

Strong Pairwise Stability A network g is strongly pairwise stable if there exists a strategy profile $(s_1^*,\ldots,s_n^*)$ such that $g = g(s_1^*,\ldots,s_n^*)$ and for all nonempty coalitions C of size less than or

[8]We follow the definition as stated in Jackson and van den Nouweland (2005), since this definition is compatible with pairwise stability.

equal to 2, there does not exist a coalition strategy profile s_C such that for all $i \in C$, $s_i \in S_i$ and

$$\pi_i(g(s_C, s^*_{N-C})) \geq \pi_i(g(s^*_C, s^*_{N-C})),$$

with at least one inequality being strict.

For a given payoff structure $\{\pi_i\}_{i\leq n}$, we denote by $\Pi_{sps}(\{\pi_i\}_{i\leq n})$ the (possibly empty) collection of strongly pairwise stable networks. In the following section we will show that our model allows for a unique strong pairwise stable network that is also strongly stable.

For completeness we also give the definition of pairwise stability and we denote for a given payoff structure $\{\pi_i\}_{i\leq n}$, the collection of pairwise stable networks by $\Pi_{ps}(\{\pi_i\}_{i\leq n})$.

Pairwise stability A network g with strategy profile s is pairwise stable if

- for all $i, j \in N$ such that $j \in s_i$ and $i \in s_j$ we have that $\pi_i(g(s_i/j, s_{-i})) \leq \pi_i(g(s))$ and $\pi_j(g(s_j/i, s_{-j})) \leq \pi_j(g(s))$.
- for all $i, j \in N$ such that $j \notin s_i$ or $i \notin s_j$ we have that if $\pi_i(g \cup \{ij\}) > \pi_i(g)$ then $\pi_j(g \cup \{ij\}) < \pi_j(g)$.

The main difference between strong pairwise stability and pairwise stability lies in the fact that the former allows for simultaneous cutting and forming of a link while the latter does not.

Model specification We are now ready to define a specific functional structure of the payoff functions π_i. We assume that the payoff for i of a network g, $\pi_i(g)$, can be decomposed in two parts. A first part provides the benefit that i obtains from the network g and a second part provides the costs that i incurs from the formation of the network g.

For the benefit part, we impose the assumption of absolute friction, meaning that agents do not have any advantage or disadvantage of the presence of indirectly connected agents. In particular, each agent $i \in N$ is endowed with an agent specific utility function $v_i : \mathbb{N} \to \mathbb{R}$ representing the utility that i derives from her number of direct links $n_i(g)$,

$$n_i(g) = |g(i)|.$$

By making the assumption that only the number of direct links matter, we impose homogeneity on the benefit side, i.e. all potential partners are deemed to be equal in terms of linking gains. Furthermore, we assume that the change in utility is strictly decreasing in the number of links and that the additional benefit of adding an extra link becomes arbitrarily small or negative if the number of links is large enough. Formally, for all $l \in \mathbb{N}$:

a) $v_i(l+2) - v_i(l+1) < v_i(l+1) - v_i(l)$ and,

b) for all $\varepsilon > 0$, there exists a number $l(\varepsilon) \in \mathbb{N}$ such that for all $l \geq l(\varepsilon), v_i(l+1) - v_i(l) \leq \varepsilon$.

Establishing and maintaining a link is costly. We introduce the notion of a link-cost function to model these linking costs. As mentioned in the introduction, we assume that this link-cost function depends on two factors: a distance factor and a quality factor. We can think of social networks where, if the social distance between agents is large, they will have to incur more costs to overcome this distance and establish a link. For every pair of agents i and j, we define a number $d(i, j)$ representing the distance between i and j. We assume that this function is symmetric. Formally,

c) for all $i, j \in N$: $d(i, j) = d(j,i)$.

Besides this distance function, we assume that the cost of link formation is also determined by a component idiosyncratic of the partner agent which we call the quality of the agent. For each agent we consider a number $\theta(i)$ that represents this quality. The larger $\theta(i)$ the higher the quality of agent i. For i to link to j, the quality of agent j, $\theta(j)$, has a decreasing impact on link-costs. Hence, we assume that the cost of i from linking with j, $c(i, j)$, can be written as a difference between these two parts:

$$c(i, j) = d(i, j) - \theta(j).$$

At the end of section 3, we show that our model can be adapted to include alternative forms for this cost function.

Finally, in order to obtain our uniqueness result, we impose two additional assumptions:

d) for all i, j and $k \in N$: $c(i, j) \neq c(i,k)$ and,

e) for all $l \in \mathbb{N}$ and all i and $j \in N$, $v(l+1) - v(l) \neq c(i,j)$.

The first assumption states that every agent can perfectly order the other agents in terms of linking costs, without there being any indifference between potential linking partners. The second assumption implies that no agent can ever be indifferent between having a certain link or not. These requirements are generic in the sense that any violation of them will be undone for a small perturbation in the distance or quality functions. No other assumptions are made with respect to the cost structure. Note that this makes it possible that $c(i,j) < 0$.

Bringing together the cost and benefit sides, we get that the payoff of i derived from the network g is given by,

$$\begin{aligned}\pi_i(g) &= v_i\left(n_i(g)\right) - \sum_{j:ij \in g(i)} c(i,j) \\ &= v_i\left(n_i(g)\right) - \sum_{j:ij \in g(i)} \left(d(i,j) - \theta(j)\right).\end{aligned}$$

2.3 Stable networks

Having outlined the various concepts of stability and the specific payoff structure of our model, we proceed by showing existence and uniqueness of the strong pairwise stable network. We also give some additional insights into the configuration of this stable network. In particular, we provide an algorithm that allows the computation of the stable network from information on the individual benefit functions $v_i(.)$ and cost structure $c(i,.)$. In this section, we restrict ourselves to providing the necessary definitions and concepts to understand our main results and the main idea behind the algorithm. We illustrate this algorithm by an example. The formal proof of our result can be found in the appendix.

Consider four agents, $N = \{1,2,3,4\}$. The distance, quality and cost functions are given below in table 2.1 together with the benefit function v_i which is assumed to be identical for all $i = 1,\ldots,4$. Observe that the cost function, $c(i,j)$, is constructed using the formula, $c(i,j) = d(i,j) - \theta(j)$. The algorithm, given in figure 1, selects for each iteration $t = 0,1,2,\ldots$ a set of agents $A_t \subseteq N_t$ – with N_t a subpopulation of N– who are allowed to change their strategy. This change in strategy

$d(i,j)$					$\theta(i)$		$c(i,j)$					$v(n)$	
$i \backslash j$	1	2	3	4	i		$i \backslash j$	1	2	3	4	n	
1	/	49	76	23	1	10	1	/	44	56	8	0	0
2	49	/	58	16	2	5	2	39	/	38	1	1	100
3	76	58	/	62	3	20	3	66	53	/	47	2	150
4	23	16	62	/	4	15	4	13	11	42	/	3	180

Table 2.1: Overview

changes a given network g_t to the network g_{t+1}. More specific, we allow the agents in A_t to delete the links in network g_t they do not want to maintain. We start with the complete network $g_0 = g_N$ and the grand population $N_0 = N$. The algorithm guarantees that for each iteration $t = 1,2,\ldots,$ $g_t \subseteq g_{t-1}$, $N_t = N_{t-1} - A_t \subseteq N_{t-1}$ and $N_t \neq N_{t-1}$. We end the algorithm when $N_t = \emptyset$, lets say at iteration T, which is guaranteed to happen in finite time.

We now discuss each step in the algorithm, for a certain iteration t, and we relate it to our numerical example above.

Assume that we arrive at step t of the algorithm with network g_t and subpopulation $N_t \subseteq N$. In order to determine A_t we take the following steps.

In a first step, we compute for every $i \in N_t$ and $j \in N$ for which $ij \in g_t$, the number of agents, k, in N such that $ik \in g_t$ and $c(i,k) \leq c(i,j)$. We denote this number by $r(i,j,g_t)$.

Definition For all $i,j \in N$ and $g \in \mathcal{G}_n$,

$$r(i,j,g) = |\{k \in N | ik \in g \text{ and } c(i,k) \leq c(i,j)\}|.$$

Considering our numerical example, we have that, for example, $r(1,3,g_0) = 3$: there are three agents, namely 2, 3 and 4, who are linked to 1 in g_0 and have linking-costs less than or equal to $c(1,3) = 56$.

In a second step, we compute for each $i \in N_t$ and $j \in N$ with $ij \in g_t$ the maximal number $t(i,j)$

I. Initiate $t = 0$, $N_0 = N$ and let $(s_1^0, \ldots, s_n^0)$ be such that $g_0 = g_N = g(s_1^0, \ldots, s_n^0)$.

1. Let $A_t = \{i \in N_t | \forall j \in N_t : \rho_i(g_t) - \theta(i) \leq \rho_j(g_t) - \theta(j)\}$
2. For all $i \in A_t$, define s_i^{t+1} by:

$$j \in s_i^{t+1} \text{ if and only if } c(i,j) \leq \rho_i(g_t).$$

   For all $j \notin A_t$ define $s_j^{t+1} = s_j^t$.

3. Set $N_{t+1} = N_t - A_t$ and $g_{t+1} = g(s_1^{t+1}, \ldots, s_n^{t+1})$.
4. If $N_{t+1} = \emptyset$ go to step II, else, increase t by one ($t := t+1$) and go to step I–1.

II. Set $g_A = g(s_1^{t+1}, \ldots, s_n^{t+1})$ and end the algorithm.

Figure 2.1: Stable network algorithm.

such that $v_i(t(i,j)) - v_i(t(i,j)-1) \geq c(i,j)$ if it exists. If it does not exist (i.e. $v_i(l) - v_i(l-1) < c(i,j)$ for all $l = 1, 2, \ldots$), we set $t(i,j) = 0$.

Definition For all $i, j \in N$,

$$t(i,j) = \max\{\{l \in \mathbb{N} | v_i(l) - v_i(l-1) \geq c(i,j)\}; 0\}.$$

The existence of $t(i,j)$ is guaranteed by assumption *b*. For our example, we have $t(1,3) = 1$. Given the first and second step above, we can form, for each pair of agents (i,j) ($i \in N_t, j \in N$) the pair $(r(i,j,g_t), t(i,j))$. These pairs are shown in the first four columns of table 2.2 below.
In a third step, we look for all pairs of agents (i,j) with $i \in N_t$ and $j \in N$ with $ij \in g_t$ for which $r(i,j,g_t) \leq t(i,j)$. These are indicated in bold in the first four columns of table 2.2. The set of all these agents $j \in N$ will be equal to the set of 'acceptable' agents for i in network g_t. We collect these in the set $\Lambda_i(g_t)$. This set of acceptable agents for agent i can be interpreted as the best response for agent i in the network g_t: the set $\Lambda_i(g_t)$ gives the agents that are linked to i in network g_t and with whom i does not want to cut his link.

	$(r(i,j,g_0),t(i,j))$				$\Lambda_i(g_0)$	$\rho_i(g_0)$	$\rho_i(g_0)-\theta(i)$
	1	2	3	4			
1	/	**(2,2)**	(3,1)	**(1,3)**	$\{4,2\}$	44	34
2	(3,2)	/	**(2,2)**	**(1,4)**	$\{4,3\}$	38	33
3	(3,1)	(2,1)	/	**(1,2)**	$\{4\}$	47	27
4	**(2,3)**	**(1,3)**	(3,2)	/	$\{2,1\}$	13	**-2**

Table 2.2: Summary table for t=0

Definition For all $i \in N$ and $g \in \Lambda_n$,

$$\Lambda_i(g) = \{j \in N | ij \in g_t \text{ and } r(i,j,g_t) \leq t(i,j)\}.$$

For the example, both agents 2 and 4 are acceptable for agent 1 in network g_0. Hence, $\Lambda_1(g_0) = \{4,2\}$. Observe that it is possible that $\Lambda_i(g)$ is empty for some i and g. For our example, the set of acceptable agents in the first iteration is given in table 2.2 (second column).

Given that we have constructed the best responses for each agent i in N_t. What is left to determine now is the set of agents A_t who will change their strategy to their best responses strategy, i.e. who will be the agents in N_t that will effectively delete all links $ij \in g_t$ for which $j \notin \Lambda_i(g_t)$ while keeping all other links of i in g_t.

Therefore, we consider in the fourth step, for each $i \in N_t$ and each $j \in \Lambda_i(g_t)$, the cost $c(i,j)$ and we retain the highest value. We call this the critical cost for i in g_t and we denote it by $\rho_i(g_t)$. If $\Lambda_i(g_t)$ is empty, we set $\rho_i(g_t) = 0$. For our example, we have that $c(1,2) > c(1,4)$, hence $\rho_1(g_0) = c(1,2) = 44$ is the critical cost for agent 1 in g_0.

Definition For all $i, j \in N$ and all $g \in \mathcal{G}_n$, the critical cost for i in g is given by,

$$\rho_i(g) = \max\{\{c(i,j) | j \in \Lambda_i(g)\}\}, \text{ if } \Lambda_i(g) \neq \emptyset \text{ and } \rho_i(g) = 0 \text{ otherwise.}$$

Now, we compute for all agents $i \in N_t$ the value $\rho_i(g_t) - \theta(i)$ and we identify the collection of agents $i \in N_t$ for which $\rho_i(g_t) - \theta(i)$ is smallest. This collection determines A_t. For our numerical example, we obtain $A_0 = \{4\}$.

The intuition behind the particular choice of A_t is the following. We know that any agent $i \in N_t$ would like to cut all links $ij \in g_t$ for which $c(i,j) > \rho_i(g_t)$ (or equivalently, $j \notin \Lambda_i(g_t)$) and keep all links with $c(i,j) \leq \rho_i(g_t)$ (or equivalently $j \in \Lambda_i(g_t)$). Now consider an agent $i \in A_t$ and let $c(i,j) \leq \rho_i(g_t)$. Then,

$$\begin{aligned} c(j,i) &= d(i,j) - \theta(i) \\ &= c(i,j) - \theta(i) + \theta(j) \\ &\leq \rho_i(g_t) - \theta(i) + \theta(j) \\ &\leq \rho_j(g_t). \end{aligned}$$

The first and second equalities follow from the definition of $c(j,i)$ and $c(i,j)$. The first inequality follows from the fact that $c(i,j) \leq \rho_i(g_t)$ while the last inequality follows from the definition of A_t. As such, we see that $c(j,i) \leq \rho_j(g_t)$ or equivalently $i \in \Lambda_j(g_t)$, i.e. i is an acceptable agent for j in g_t. As such, j will not have an incentive to cut his link with i in network g_t or in any other network g_s with $s > t$, as is shown in the proof of theorem 1.

We are now ready to update the algorithm to the next step. To construct g_{t+1} from g_t we take the collection of agents obtained from the previous step, A_t, and we let them cut the links with all agents j not in $\Lambda_i(g_t)$. Considering our particular example, we have that $A_0 = \{4\}$ and $\Lambda_4(g_0) = \{2,1\}$, hence, $g_1 = g_0/\{\{3,4\}\}$. Next, we construct the set $N_{t+1} = N_t - A_t$, giving $N_1 = \{1,2,3\}$. Finally, we go back to step 1 of the next iteration.

For our example, we see that in the next iteration, $A_1 = \{2,3\}$ and that g_2 is formed by deleting the link between 1 and 2 and between 1 and 3 (see table 2.3). Note that since agent 4 has deleted his link with agent 3, agent 2 became a member of the set of acceptable agents of agent 3, $\Lambda_3(g_1)$. Finally, we end with $N_2 = \{1\}$ in round 2, see table 2.4. No link is deleted in round 2. This is a general feature of our algorithm: any link that is not deleted in a certain iteration t will not be deleted further in the algorithm (see lemma 2.6.2 in the appendix). As such, we may end the algorithm as soon as N_t is a singleton. We see that g_3 is the final network in the algorithm

	$(s(i,j,g_1),k(i,j))$ 1	2	3	4	$\Lambda_i(g_1)$	$\rho_i(g_1)$	$\rho_i(g_1)-\theta(i)$
1	/	**(2,2)**	(3,1)	**(1,3)**	$\{4,2\}$	44	34
2	(3,2)	/	**(2,2)**	**(1,4)**	$\{4,3\}$	38	33
3	(2,1)	**(1,1)**	/	/	$\{2\}$	53	33

Table 2.3: Summary table for t=1

	$(s(i,j,g_2),k(i,j))$ 1	2	3	4	$\Lambda_i(g_2)$	$\rho_i(g_2)$	$\rho_i(g_2)-\theta(i)$
1	/	/	/	**(1,3)**	$\{x_4\}$	8	-2

Table 2.4: Summary table for t=2

because $N_3 = \emptyset$. We obtain $g_3 = \{\{1,4\},\{2,3\},\{2,4\}\}$. The full algorithm is summarized in figure 1.

The following theorem shows that a strong pairwise stable network always exists, in the form of the output of figure 2.1, and that it is always unique. The proof can be found in the appendix.

Theorem 2.3.1 *The set of strong pairwise stable networks $\Pi_{sps}(\{\pi_i\}_{i\leq n})$ is a singleton. The unique element of $\Pi_{sps}(\{\pi_i\}_{i\leq n})$, g_{sps}, is reproduced by the output of the algorithm in figure 2.1.*

Allowing for the simultaneous formation and cutting of a link, as presented in the definition of a strong pairwise stable network, is essential for uniqueness of the stable network. Indeed, it is possible that there are multiple pairwise stable networks (see definition 2.2). To see this, consider an example with three agents i, j and k and assume that both j and k are willing to link with i but not with each other. Further assume that i would benefit from linking to either j and k, but not to both. This happens if, for example,

$$c(j,i) < v_j(1) - v_j(0) < c(j,k), \qquad c(k,i) < v_k(1) - v_k(0) < c(k,j),$$
$$v_i(1) - v_i(0) > c(i,k), c(i,j) \quad \text{and} \quad v_i(2) - v_i(1) < c(i,k), c(i,j).$$

Assume that $c(i,j) < c(i,k)$ and let $g = \{ik\}$. Since i and k have no other links, neither of them will have an incentive to cut the link ik. Agent j may propose to form a link with i. In the pairwise stability framework, where i cannot cut and form a link at the same time, i will decline j's proposal although i would like to cut his link with k and link with j because this would give him a positive net benefit of $c(i,k) - c(i,j)$. In this setting, the concept of pairwise stability does not lead to a unique stable network as both networks $g = \{ij\}$ and $g = \{ik\}$ are pairwise stable. On the other hand, if we impose the strong pairwise stability concept, we see that, in this case, i will cut his link with k and accept the proposal to link with j at the same time leaving only $g = \{ij\}$ to be strong pairwise stable.

As for the strongly stable network, we show in the following proposition that the unique strong pairwise stable network is also strongly stable. This implies that if we would impose no restrictions on the actions available to the agents and the size of the deviating coalition the strong pairwise stable network would still be stable.

Proposition 2.3.2 *The unique strong pairwise stable network g_{sps} is strongly stable:*

$$\Pi_{ss}(\{\pi_i\}_{i \le n}) = \Pi_{sps}(\{\pi_i\}_{i \le n})$$

The uniqueness of the weakly and strongly stable network hinges on the following property of the payoff structure: if $c(i,j) < c(i,k)$ and $c(j,k) < c(j,i)$, then $c(k,j) < c(k,i)$ — if i prefers j to k and j prefers k to i, then k will prefer j to i. Hence the preferences of k are such that the objective of two players (j and k) are aligned. This rules out any cycles in the preference structure and uniqueness ensues.

We finalize this section by giving an alternative cost structure for which above existence and uniqueness result remains to hold. In the basic model, we assumed that $c(i,j)$ could be written as the difference between a symmetric distance function $d(i,j)$ and an agent specific quality function $\theta(j)$. On the other hand, we could also specify $c(i,j)$ as the ratio of a symmetric distance function $d(i,j)$ and an agent specific (non-zero) quality function $\theta(j) : N \to \mathbb{R}_{++}$:

$$c(i,j) = \frac{d(i,j)}{\theta(j)}.$$

All the results of this section are valid for this cost function with a minor adjustment in the definition of the set, A_t, which is used in step I–1 in the algorithm of figure 2.1, which changes to,

$$A_t \in \left\{ i \in N_t \,\middle|\, \forall j \in N_t : \frac{\rho_i(g_t)}{\theta(i)} \leq \frac{\rho_j(g_t)}{\theta(j)} \right\}.$$

2.4 Empirical Illustration

Using data from a risk sharing network in Nyakatoke, Tanzania, we verify whether our model is able to predict the existence of such risk sharing links and whether it can capture the features of risk sharing networks observed in real life. Informal risk sharing networks — networks where linking partners have an informal mutual agreement to help each other when one partner incurs a negative shock — are particularly well suited examples of the payoff structure under consideration.

In the literature on informal risk-sharing networks, for example, De Weerdt (2004) finds that kinship, geographical proximity, clan membership and religious affiliation are strong determinants in the formation of risk-sharing networks. This result is corroborated by the results of Fafchamps and Gubert (2007) who find that geographic proximity, kith and kin relationships are strong determinants of link formation within an informal insurance network in the rural Philippines. Hence we can think of the pairwise component $d(i,j)$ as a notion of social and geographical distance. Households who live close by or know each other well through friendship or kinship ties can link to each other at lower costs.

In addition, empirical studies on risk-sharing networks reveal that some agents are more desirable linking partners than others because they have more wealth or status. For example, Comola (2008) and De Weerdt (2004), find that the probability of being linked to a household rises with the wealth of this household, that the rich have a denser network than the poor and that a pair with a rich person in it has more chance of being linked. Hence, households are more willing to link to a household of higher wealth which is incorporated in our model by the agent-specific component $\theta(.)$.

We keep this illustration as simple as possible, using the geographical distance between two households as a proxy for our distance measure $d(i,j)$, while the value of livestock owned by a household serves as a proxy for our quality measure $\theta(i)$. Various empirical studies [9] on risk sharing networks identify geographical proximity as one of the important determinants of risk sharing links. For the case of livestock, Comola (2008) argues that livestock is a real wealth dimension, more so than land ownership.

At the time of the survey in 2000, Nyakatoke consisted of 119 households, 116 for which data are fully available. All households were asked the following question: *Can you give a list of people [...], who you can personally rely on for help and/or that can rely on you for help in cash, kind or labour?* The 116 households mentioned a total of 960 intra-village network partners, which amounts to 480 risk sharing links. However, the degree (i.e. the number of network partners) is unevenly distributed, ranging from 1 to 32 partners. Furthermore, the Nyakatoke risk sharing network has features corresponding to other empirical regularities of large social networks, as noted by Comola (2008): there is only one component which has an average path length of 2.5 steps, clustering is higher than would be expected if the formation process would be totally random and clustering tends to be negatively correlated with degree.[10] The key characteristics of the Nyakatoke network are shown in the first column of table 2.6 below.

As a measure of predictive power we look at the fraction of existing links in the Nyakatoke network that also exist in the stable network produced by the algorithm, as well as the fraction of non-existing links in the Nyakatoke network that do not exist in the stable network. In order to get a sense of how accurate the predictions based on the model are, we compare our results with predictions based on using fitted link probabilities from a logit regression model using geograph-

[9] See Fafchamps and Lund (2003), Fafchamps and Gubert (2007), De Weerdt (2004) and Comola (2008).

[10] Another empirical regularity is a positive correlation in degree of connected households for which we could not find evidence in the Nyakatoke risk sharing network.

ical distance and/or livestock values. Since both predictions rely on the same set of information — geographical distance and livestock value — we are able to filter out the contribution of our network formation model by comparing the two prediction fractions. We fully realize we use very crude measures of distance and quality and in this respect this exercise should not be viewed as an empirical application but rather as a way of showing that the network formation process presented in this paper might have some validity when it comes to predicting the existence of links.

First, we focus on distances only, simply inputting geographical distances between any pair while assuming $\theta(i) = 0$ for all $i \in N$. The payoff function is of the form

$$\pi_i(g) = \alpha * n_i(g)^{2/3} - \sum_{j:ij\in g} d(i,j),$$

with the scale parameter α chosen such that the total number of links in the network is as close to 480 (the total number of links in the real-life network) as possible. For this exercise, $\alpha = 0,454$, with a total number of 479 links. The exponent for the benefit function is fixed at $\frac{2}{3}$. Simulations with other values within the range of $[\frac{2}{5}, \frac{4}{5}]$ yielded similar results.

The prediction results are given in the first row of table 2.5. Keeping in mind that our handling of the data is rather raw, the results are mixed. Somewhat less than one third of all existing links is predicted in our model. Earlier simulations — not shown here — show that if we randomly select 480 pairs to be linked, we get a prediction rate for linked pairs between 3 and 13% and between 92,5 and 93,2% for unlinked pairs. Hence, we get prediction rates well above what we would expect from a non-informative model.

In order to filter out the contribution of our model to predicting the Nyakatoke network, we compare the results from the first row of table 2.5 with the prediction results from the fitted probability model as shown in the second row. In this model, we fitted the following logit regression with standard errors in parentheses:

$$\ln\left(\frac{Pr(ij\in g)}{1-Pr(ij\in g)}\right) = \underset{(0,09)}{-1,3} - \underset{(0,22)}{2,9} * \text{distance}_{ij}$$

Then, networks are simulated where the probability that a link exists between a pair is equal to the fitted probability from the logit regression. The numbers shown in the table are the [5-

Fraction of Correct predictions	Linked	Unlinked
Model (only distance)	0,313	0,947
Fitted Probability (only distance)	[0,088 - 0,133]	[0,926 - 0,936]
Model (distance and quality)	0,333	0,948
Fitted Probability (distance and quality)	[0,098 - 0,146]	[0,927 - 0,937]

Table 2.5: Prediction results

95] percentiles, obtained over 100 000 simulations. The prediction rates for our model are above these intervals, indicating that the network formation process as described in this paper is relevant to explaining the existence or absence of certain links in the Nyakatoke risk sharing network.

The simulation with the quality measure is a bit more complex. The quality measure $\theta(i)$ is the value of livestock of i. First, since the livestock value is of a different magnitude than the geographical distance measure,[11] we scale it down using the ratio of coefficients from the following logit regression:

$$\ln\left(\frac{Pr(ij \in g)}{1 - Pr(ij \in g)}\right) = \underset{(0,09)}{-1,4} - \underset{(0,22)}{3} * \text{distance}_{ij} + \underset{(1,3*10^{-7})}{1 * 10^{-6}} * \sum_{k=i,j} \text{livestock}_k$$

which turns out to be $\frac{3}{1000000}$. Second, adding quality adds a weighting parameter β to the net benefit function:

$$\pi_i(g) = \alpha * n_i(g)^{2/3} - \sum_{j:ij \in g} ((\beta * d(i,j) - (1-\beta) * \theta(j)).$$

For different values of β — between 0 and 1 with steps of 0,02 — we calibrate α such that the total number of links is approximately 480. The third row of table 2.5 gives the prediction results for the $\{\beta, \alpha\}$-combination which yielded the best prediction results, $\{\beta = 0,48, \alpha = 2,49\}$. Again, comparing these results with the results from the fitted probability model with distance

[11]Distance is expressed in kilometers with values ranging from close to 0 to 1,8, while livestock can take on values to over 1,5 million.

and quality as shown in the fourth row indicates that our model can help explain the existence of links in the Nyakatoke network, although prediction fractions do not improve markedly when adding quality.

	Nyakatoke	Model (distance)	Model (distance and quality)
[Min Median Max]	[1 7 32]	[1 8 16]	[3 8 10]
Char. Path length	2,54	4,23	3,64
Clustering	0,19	0,62	0,52
Corr. Clust. - Degree	-0,21	-0,043	-0,39
Corr. Degree linked agents	0,04	0,70	0,49

Table 2.6: Key statistics of networks: The clustering coefficient used is the overall clustering coefficient (see Jackson (2008)).

As far as the features of the simulated networks, shown in table 2.6, are concerned, we can say that they match at some points the features of networks observed in real life — high degree of clustering, negative correlation between clustering and degree and positive correlation in degree of connected agents. At other points — the uneven distribution of degree as well as the low characteristic path length — the model fails to replicate, which can be attributed to the usage of the raw distance and quality measures. For example, geographical distance is a measure which is a metric, satisfying triangle inequality[12], hence producing high characteristic path lengths. If we would introduce more accurate social distance measures, which would not necessarily satisfy triangle inequality, characteristic path length may be lower. The inaccuracy of the data may also explain the fact that we did not obtain an uneven degree distribution. Additionally, we assumed identical benefit functions for every household. Introducing some heterogeneity on the benefit side may lead to a more uneven degree distribution.

[12] $d(i,j)+d(j,k) \geq d(i,k)$

2.5 Concluding remarks

In an endogenous network formation setting where only the direct network matters for utility, this paper introduced general forms of heterogeneity on the cost side. The costs are characterized by the distance between agents and individual quality. The only restrictions we impose is that the distance is symmetric, that each agent has a perfect ordering of linking costs without indifference and that agents are never indifferent between maintaining and cutting a link. Using an algorithm we were able to identify the unique strong pairwise stable network. Agents will try to link with agents with the lowest linking costs so as to maximize the net gains from the direct network. Not everyone will succeed in doing this however, as some desirable network partners may refuse to form a link with them.

The algorithm provides an elegant way to determine the emerging unique stable network. Different structures will give rise to different configurations and hence, using the framework, it might become possible to relate differences in features of networks to differences in the underlying cost structure.

The uniqueness result of the stable network can be attributed to two factors. Firstly there are the assumptions made regarding the cost function — perfect preference ordering and symmetric distance function — as well as on the actions available to the agents — the fact that agents can simultaneously cut and form a link. Relaxing the assumptions would invalidate the uniqueness result by including some history dependence in the model. However, the basic configuration of the emerging stable networks will be little affected by this relaxation, so that we can consider this kind of randomness as irrelevant to the model.

A second set of factors which leads to uniqueness of the stable network are the specific structure of the link-costs — with a pairwise and a agent-specific component — which precludes preference cycles and the assumptions made with respect to the utility function — absolute friction and concavity. Hence, relaxing the assumptions with respect to the link-costs even further as well as imposing more intermediate forms of friction may be an important direction for future research.

2.6 Appendix

2.6.1 Proofs

Preliminary results

Before proving theorem 2.3.1, we introduce some lemmata.

Lemma 2.6.1 *If $i \in A_t$, $ij \in g_{t+1}$ and $s \geq t$, then $ij \in g_{s+1}$.*

Proof Assume that $i \in A_t$ and $ij \in g_{t+1}$. Further, let $j \in A_p$. If $p < t$, we have, by $g_{t+1} \subseteq g_{p+1}$, that $ij \in g_{p+1}$. As there is no iteration between t and s where ij could be cut, we conclude that $ij \in g_{s+1}$
Therefore assume that $p \geq t$. Observe that it suffices to show that $c(j,i) \leq \rho_j(g_p)$ because this guarantees that j will not cut the link ij in iteration p of our algorithm. From $i \in A_t$ and $ij \in g_{t+1}$, we conclude that,

$$c(j,i) = c(i,j) - \theta(i) + \theta(j) \leq \rho_i(g_t) - \theta(i) + \theta(j) \leq \rho_j(g_t).$$

This implies,

$$r(j,i,g_t) \leq t(j,i),$$

and from $g_p \subseteq g_t$, we derive,

$$r(j,i,g_p) \leq r(j,i,g_t) \leq t(j,i),$$

Hence, $c(j,i) \leq \rho_j(g_p)$.

An important corollary of lemma 2.6.1 is that for all iterations t of the algorithm and $i \in A_t$, if $ij \in g_{t+1}$, then $ij \in g_A$ (here g_A is the output of the algorithm).

Lemma 2.6.2 *If the network g is strong pairwise stable and $g \subseteq g'$, then for all $ij \in g'$, if $c(i,j) \leq \rho_i(g')$ and $c(j,i) \leq \rho_j(g')$, then $ij \in g$.*

Proof Assume that $c(i,j) \leq \rho_i(g')$ and $c(j,i) \leq \rho_j(g')$. If $ij \notin g$, then by strong pairwise stability of g it follows that (using assumption e),

$$v_i(n_i(g)+1) - v_i(n_i(g)) < c(i,j) \qquad \text{or} \qquad v_j(n_j(g)+1) - v_j(n_j(g)) < c(j,i).$$

Otherwise we would have that both i and j would benefit from creating the link ij. This shows that either $t(i,j) < n_i(g)+1$ or $t(j,i) < n_j(g)+1$. Assume without loss of generality that the first of the two holds. From $c(i,j) \leq \rho_i(g')$, it follows that

$$r(i,j,g') \leq t(i,j) \leq n_i(g).$$

As $g \subseteq g'$, $ij \in g'$ and $ij \notin g$, it follows that there must be an agent, k, which is linked to i in g and for which $c(i,k) > c(i,j)$. This shows that i would benefit by cutting the link with k and creating the link with j. g is strong pairwise stable, hence, it must be that this is not beneficial for j, i.e.

$$v_j(n_j(g)+1) - v_j(n_j(g)) < c(j,i).$$

This implies $t(j,i) < n_j(g)+1$. Parallel to the previous case, we conclude that there must be an agent h such that $jh \in g$ and $c(j,h) > c(j,i)$.
Conclude that both i and j can benefit from linking with each other and cutting their link with respectively k and h. Therefore, g cannot be strong pairwise stable.

Proof of theorem 2.3.1

We only show uniqueness. The property that the output of the algorithm g_A is strong pairwise stable follows from the proof of proposition 2.3.2 as every strongly stable network is also strong pairwise stable.
Therefore, assume that there exists a strong pairwise stable network g_{sps}. We need to show that $g_A = g_{sps}$. We work by induction on the iterations in the algorithm. In particular we show that for all $t = 0, 1, \ldots, T-1$,

1. if $i \in A_t$ and $ij \in g_{t+1}$, then $ij \in g_{sps}$,

2. $g_{sps} \subseteq g_{t+1}$.

If these two conditions hold, then from lemma 2.6.1 it follows that $g_{sps} = g_T = g_A$.
First of all, observe that $g_{sps} \subseteq g_0 = g_N$. Consider the case $t = 0$. Now, let $i \in A_0$ and $ij \in g_1$. It follows that $c(i, j) \leq \rho_i(g_0)$. As such,

$$c(j,i) = c(i, j) - \theta(i) + \theta(j) \leq \rho_i(g_0) - \theta(i) + \theta(j) \leq \rho_j(g_0).$$

It follows, from lemma 2.6.2 that $ij \in g_{sps}$. In addition, consider $i \in A_0$ and $ih \notin g_1$. It follows that $c(i,h) > \rho_i(g_0)$ or equivalently

$$v_i(n_i(g_1)) - v_i(n_i(g_1) - 1) < c(i,h).$$

Assume that $ih \in g_{sps}$. Since all $ij \in g_{t+1}$ are part of g_{sps}, it must be that $n_i(g_1) < n_i(g_{sps})$. Then,

$$v_i(n_i(g_{sps}) - v_i(n_i(g_{sps}) - 1) < c(i,h).$$

As such, i will benefit from cutting his link with h in g_{sps}. Hence, g_{sps} cannot be a stable network. We have that $ih \notin g_{sps}$ and $g_{sps} \subseteq g_1$.
Assume that the induction holds for all iterations up to $t-1$ and consider iteration t. Let $i \in A_t$ and $ij \in g_{t+1}$, then

$$c(j,i) = c(i, j) - \theta(i) + \theta(j) \leq \rho_i(g_t) - \theta(i) + \theta(j) \leq \rho_j(g_t).$$

Again, using lemma 2.6.2, it follows that $ij \in g_{sps}$. This shows the first requirement. We still need to show the second requirement. Towards this end, assume on the contrary that for some $i \in A_t$, $n_i(g_t) \geq n_i(g_{sps}) > n_i(g_{t+1})$. From lemma 2.6.1 one can easily deduce that for all $i \in A_t$, $\rho_i(g_t) = \rho_i(g_{t+1})$. Hence for $ij \in g_{sps}, g_t$ and $ij \notin g_{t+1}$,

$$c(i, j) > \rho_i(g_{t+1}) \qquad \text{or equivalently} \qquad v_i(n_i(g_{t+1})) - v_i(n_i(g_{t+1}) - 1) < c(i, j).$$

From $n_i(g_{sps}) > n_i(g_{t+1})$, we deduce that,

$$v_i(n_i(g_{sps}) - v_i(n_i(g_{sps}) - 1) < c(i, j).$$

As such, i will benefit from cutting his link with j in g_{sps} and g_{sps} cannot be a stable network. Now, we have that $g_{sps} \subseteq g_t$, $n_i(g_{sps}) \leq n_i(g_{t+1})$ for all $i \in A_t$ and $ij \in (g_t - g_{t+1})$ only if i or $j \in A_t$. Conclude that $g_{sps} \subseteq g_{t+1}$.

Proof of proposition 2.3.2

Take any $C \subseteq N$ and let g_A be the output of the algorithm. We prove that there will always be at least one agent $i \in C$ who loses by any rearrangement of the linking pattern among the members of C, under the condition that links are formed under mutual consent. Consider an agent, i, defined by:

$$i \in \{j \in C | j \in A_t \text{ and } \forall k \in C \text{ if } k \in A_p, \text{ then } t \leq p\},$$

Agent i is a the member of C who, in our algorithm, cuts links first.

Let g_A be the output of the algorithm and let g' be a network where only links within C are rearranged and perhaps links with at least one agent in the coalition are cut. Further, assume on the contrary, that all agents within C benefit from changing from g_A to g'. Let $A = g_A(i) \backslash g'(i)$ and $B = g'(i) \backslash g_A(i)$. A is the set of agents who are linked to i in network g_A but not in g' and B is the set of agents who are linked to i in g' but not in g_A. We assume that at least one of the sets A or B is nonempty. Otherwise, i is indifferent between network g_A and g'. In that case, we may exclude i from the set C and consider the smaller set $C - \{i\}$. Now, for all $j \in N$, if $ij \in g_{t+1}$, then it follows (from lemma 2.6.1) that $ij \in g_A$. This implies that whenever $j \in A$ then $c(i,j) \leq \rho_i(g_t)$. On the other hand, because i is the first agent to choose in the algorithm, it must be that for all $j \in B$, $c(i,j) > \rho_i(g_t)$. Conclude that for all $j \in A$ and $k \in B$: $c(i,j) < c(i,k)$. We have that:

$$\pi_i(g_A) - \pi_i(g') = v_i(n_i(g_A)) - v_i(n_i(g')) - \sum_{j \in A} c(i,j) + \sum_{j \in B} c(i,j).$$

If $|A| = |B|$, the proof follows from the facts that $c(i,j) < c(i,k)$ for all $j \in A$ and $k \in B$ and $n_i(g_A) = n_i(g')$. We distinguish two other cases.

(1) If $|B| > |A|$, we can write:

$$\pi_i(g_A) - \pi_i(g') = \sum_{k=0}^{|B|-|A|-1} (v_i(n_i(g_A)+k) - v_i(n_i(g_A)+k+1)) - \sum_{j \in A} c(i,j) + \sum_{j \in B} c(i,j).$$

Now, select $|A|$ agents from B and call the remaining set B', then:

$$\pi_i(g_A) - \pi_i(g') > \sum_{k=0}^{|B|-|A|-1} (v_i(n_i(g_A)+k) - v_i(n_i(g_A)+k+1)) + \sum_{j \in B'} c(i,j).$$

For all $k \geq 0$ and $j \in B$, it is also the case that:

$$v_i(n_i(g_A)+k+1) - v_i(n_i(g_A)+k) < c(i,j).$$

Hence,

$$\pi_i(g_A) - \pi_i(g') > -\sum_{j \in B'} c(i,j) + \sum_{j \in B'} c(i,j) = 0.$$

(2) Now, consider the case where $|A| > |B|$. Then we can write:

$$\pi_i(g_A) - \pi_i(g') = \sum_{k=0}^{|A|-|B|+1} (v_i(n_i(g_A)-k) - v_i(n_i(g_A)-k-1)) - \sum_{j \in A} c(i,j) + \sum_{j \in B} c(i,j).$$

Select $|B|$ agents from A and call the remaining set A'. This gives:

$$\pi_i(g_A) - \pi_i(g') > \sum_{k=0}^{|A|-|B|-1} (v_i(n_i(g_A)-k) - v_i(n_i(g_A)-k-1)) - \sum_{j \in A'} c(i,j).$$

For all $k \geq 0$ and $j \in A$, it is also the case that:

$$v_i(n_i(g_A)-k) - v_i(n_i(g_A)-k-1) > c(i,j).$$

Hence,

$$\pi_i(g_A) - \pi_i(g') > \sum_{j \in A'} c(i,j) - \sum_{j \in A'} c(i,j) = 0.$$

Conclude that i has higher payoffs in g_A than in g' contradicting the assumption that g_A is not strongly stable.

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3

Paths to stability in the partnership problem

3.1 Introduction

We study a decentralized matching process for partnership problems with linear preferences, which is a one-sided many-to-many matching problem. It is shown that such a process converges to a stable matching with probability one, by establishing that for every unstable matching there is a sequence of myopic blockings that leads to a stable matching.

In a matching problem, agents within a group are faced with the decision to form bilateral relationships with other agents in that group, based on the preferences that they might have over other agents or over subsets of agents. Depending on the restrictions imposed on the possible bilateral relationships and the preference structure, different matching problems exist. Two famous examples (both by Gale and Shapley (1962)) of matching problems are the stable marriage problem where a group of men and a group of women are looking for a mate of the other sex and the stable roommate problem, where a group of students is looking for someone to share a dorm room with. The former problem is an example of a two-sided problem; the population is partitioned into two groups and there is no within-group relationship possible. The latter example is one-sided, without any such restriction on the set of possible relationships. Both problems are examples of one-to-one matching, where agents do not wish to be involved in multiple relationships – there is no desire for polygamy and dorm rooms have only two beds. The matching problem that will be discussed in this paper, the partnership problem (due to Fleiner (2010)), is more general than those two simple problems in that agents may want to be involved in multiple relationships; it is a many-to-many matching problem. In addition, they may want to be involved in multiple relationships with the same agent.[1] Examples of one-sided many-to-many matching problems include peer-to-peer networks (Lebedev et al., 2006); the stable crews problem (Cechlarova and Ferkova, 2004), where a pilot and co-pilot are assigned to a flight; informal risk sharing networks (Vandenbossche and Demuynck, 2012), where households have informal bilateral agreements to help each other out in bad times; or the scheduling of non-conference games in US college sports.

Now, a collection of bilateral relationships constitutes a matching. A matching is stable when no agent wishes to disband a relationship and when no pair of agents wishes to form a previously non-existing relationship between them. One of the many great contributions of matching theory to the more applied side of economics has been the design and study of various algorithms that produce a stable matching when one exists, for a certain type of matching problem.[2] For example, these algorithms were implemented as centralized market clearing institutions in entry level labor markets in medicine, psychology, law and business with considerable success (Blum et al., 1997). In some matching markets, however, a centralized organization of the market may

[1] That is why we will not speak of a *match* between two agents, as it implies that there can be only one such match for every pair of agents.

[2] Examples are the *deferred acceptance* algorithm (Gale and Shapley, 1962) for the stable marriage problem, Irving (1985)'s algorithm and Tan and Hsueh (1995)'s algorithm for the stable roommate problem and Fleiner (2010)'s extension of Irving's algorithm for the partnership problem.

not be feasible or desirable. Blum et al. (1997) point out that “centralized market institutions will turn out to be rarer in senior labor markets than in entry level markets." Moreover, as Roth and Vande Vate (1990) point out, a matching market often does not need a central institution to obtain a stable outcome, indicating that some decentralized matching process – like an invisible hand – drives the market towards its stable outcome. The question then becomes for what types of matching problems convergence towards stability can be assured. This paper contributes to the literature by showing that, when a stable matching exists, convergence is the case for partnership problems with linear preferences, a one-sided many-to-many matching problem that is more general than the stable marriage or roommate problem.

We follow the approach taken first by Roth and Vande Vate (1990) and consider the following decentralized matching process: a pair of myopic agents is selected at random, with positive probability for each possible pair. If the pair is a blocking pair — they are both willing to form a relationship between them, possibly giving up on previously existing relationships — the new matching will result by satisfying the blocking pair. If the pair is not a blocking pair, then the new matching is identical to the old matching. In this way, a path of blocking pairs – an improving path – develops. If convergence is assured and a stable matching exists, eventually this process should end in a stable matching, rather than going on indefinitely in a cycle. To prove convergence, it suffices to show that, starting from any unstable matching, at least one improving path exists such that it ends in a stable matching. If one such path exists for any unstable matching, then as each pair is selected with positive probability, the process must eventually converge towards a stable matching; in the long run the probability that such an improving path is taken tends to one. Hence, the approach for the proof is to construct an improving path for which it could be proven that it ends in a stable matching. For the stable marriage problem, Roth and Vande Vate (1990) constructed one such improving path such that the matching at the end of the path was stable. In later years, several studies extended the results from Roth and Vande Vate (1990). For a two-sided, many-to-many matching problem, Kojima and Ünver (2008) showed that a path to stability exists when one side has substitutable preferences and the other side has responsive preferences. For the roommate problem, Diamantoudi et al. (2004) showed that for all instances of the roommate problem where a stable matching exists, “the process of myopic blockings leads to a stable matching," hence generalizing the Roth and Vande Vate (1990) result to one-sided matching problems.

The main contribution of this paper extends the findings of Roth and Vande Vate (1990) and Diamantoudi et al. (2004) and shows the existence of an improving path towards a stable matching for any unstable matching in the partnership problem with linear preferences. As mentioned before, the partnership problem is one-sided, implying no restrictions on the possible bilateral relationships, as opposed to the two-sided nature of the stable marriage problem. In addition, the preference structure is such that agents may want to have more than one relationship, as opposed to the one-to-one nature of the stable roommate problem. The preference structure, which reveals the preference of one subset of relationships over another subset, is modeled by means of a choice function, a function that maps an option set – a set of possible relationships – to a choice set – a set of chosen relationships. These choice functions are assumed to satisfy three restrictions. First, there is *substitutability*; if a relationship is chosen from a set, then it is also chosen from a subset of that set. Second, we have *increasingness*; if k relationshisp are chosen from a set, then in a superset of that set at least k relationships will be chosen. Finally, we also assume *linearity*; if a relationship is chosen from a set while another relationship is not, then there does not exist a set where the reverse holds. Defined like this, the partnership problem with linear preferences is more restrictive than the partnership problem as defined by Fleiner (2010), which does not assume linearity, but less restrictive than the stable roommate problem and the stable b-matching problem (Irving and Scott, 2007), a many-to-many matching problem where each agent has a fixed capacity, b, and choses the first b elements in a linear preference list. In our problem, the capacity may vary over different option sets, within the limits of the restriction of increasingness. Note that the partnership problem with linear preferences and the two-sided many-to-many matching problem as presented in Kojima and Ünver (2008) are non-nested matching problems.[3]

[3]Recall that Kojima and Ünver (2008) assumes substitutable preferences on one side and responsive preferences on the other. Our problem is more general in the sense that it is one-sided and that responsive preferences as defined by Kojima and Ünver (2008) imply substitutability, increasingness and linearity while the reverse does not hold. Kojima and Ünver (2008)'s problem is more general in the sense that substitutability alone is more general than substitutability, together with increasingness and linearity.

The partnership problem with linear preferences along with the key concepts of the model are defined in more detail in section 3.2. Section 3.3 provides us with the main result, the convergence towards a stable outcome if a stable outcome exists, and gives an outline of the proof. The proof of the main theorem builds on Fleiner (2010)'s stable partnership algorithm. As we will show, this algorithm, which is a generalization of Irving's algorithm (Irving, 1985), gives us a lot of intuition as to what an improving path towards stability should look like. The proof itself is given in the appendix.

The main theorem, Theorem 3.3.1, presented in section 3.3 implies convergence towards stability for problems that are solvable, i.e. problems for which a stable matching exists. However, Gale and Shapley (1962) pointed out that, for one-sided matching problems, a stable matching might not exist. In section 3.4, we provide a discussion on convergence for non-solvable problems. In particular, we extend the results from Iñarra et al. (2010) and Biro and Norman (2012) and show that from an unstable matching there is an improving path towards a set of matchings, denoted instances, that can be associated to a stable half-matching, which is a structure that generalizes the notion of a stable matching (see Tan (1991) for the roommate problem and Fleiner (2010) for the partnership problem.) Section 3.5 presents some concluding remarks.

3.2 The model

We will introduce a matching problem, a partnership problem with linear preferences, which is a special version of a partnership problem (Fleiner, 2010). Any matching problem can be defined by a set of possible relationships and preferences over that set of relationships for each agent, where preferences are given by a choice function. The set of possible relationships and the restrictions on the choice function determine the type of matching problem. Further in this section, we define the concept of stability.

Potential relationships. Let V be a set of agents. Agents in V wish to establish bilateral relationships with other agents in V, given certain preferences. A set of relationships among agents in V will be denoted a *partnership*. The problem can be represented by means of a finite

undirected graph (V,E) with V the set of vertices - agents - and E the set of edges. A relationship between an agent u and an agent v will be defined by an edge $e = uv$.[4] The set E contains all *potential edges*, where an edge uv is potential if there exists a situation – a partnership – where both u and v are willing to form this edge. A partnership M is then a subset of the set of potential edges E, $M \subseteq E$. Further, for a subset X of E and vertex v in V let us denote by $X(v)$ the set of edges that are incident with v, the set of relationships in which v is involved. Finally, note that we are not restricting the model to simple graphs[5]; it may be that edges e_1 and e_2 in E involve the same pair of agents. In the context of a matching problem this means that multiple distinct relationships are possible between the same pair of agents. For example, if u and v in V are scientists and an edge stands for *writing a paper together*, then $e_a = uv_a$ could be *writing a paper on subject a*, while $e_b = uv_b$ could be *writing a paper on subject b*. Hence, in non-simple graphs different types of relationships are possible within the same matching problem; the problem could be *how does a group of friends spend the weekend together* but the (bilateral) relationships could be *play a game of chess* or *have dinner together*. As such, the maximal size of the set of potential edges is essentially determined by the number of agents and the number of different types of relationships, whereas for simple graphs the maximal size would be $\frac{|V|*|V-1|}{2}$. Therefore, it is important to note that the set of potential edges will be determined by which edges agents find desirable; some agents may not know the rules of chess and the edge *play a game of chess* will therefore never be adjacent to those agents.
The example given in Figure 3.1 illustrates the preceding paragraph. For example, the set of potential edges adjacent to agent u consists of uv, ux and two edges between agent u and agent w, labeled uw_a and uw_b.

Preferences. The preferences of every agent v are given by a choice function $C_v : 2^{E(v)} \to 2^{E(v)}$ that maps any subset $X(v)$ of edges incident with v – the option set – to a subset of $X(v)$ that v chooses from $X(v)$ – the choice set. The edges in $X(v)$ that are not chosen will be denoted by $\overline{C}_v(X(v)) = X(v) \backslash C_v(X(v))$. The *inverted choice set* for an agent v is the set of edges incident to

[4] Note that an edge is a set of pairs $\{u,v\} = \{v,u\}$ but for notational simplicity, we write it as uv.

[5] See footnote 6 for a problem with the interpretation of stability when dealing with non-simple graphs.

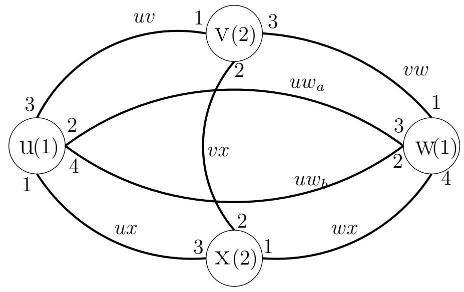

Figure 3.1: A four-agent example. This particular example is a stable b-matching problem: the capacity for each agent is given between parentheses, the preference ordering is indicated by the numbers on the edges close to the vertices, with 1 representing the edge that is first in the preference order, 2 representing the edge that is second and so on.

v that are in the choice set of others for a subset X of E. Formally,

$$C_v^-(X) = \{uv \in X(v) | uv \in C_u(X(u))\}.$$

For an edge $e \in C_v(X(v))$, the $X(v)$-*replacement* of e is the set $C_v(X(v) \backslash e) \backslash C_v(X(v)$; if an option e is in the choice set of an option set $X(v)$ incident to v and is deleted from that option set, then the new elements in the choice set without e will be the $X(v)$-replacements of e; they *replace e* in the choice set.

For example in Figure 3.1, the choice function is given by a preference list (the numbers on the edges) and a capacity (the number between parentheses). If $X = \{uv, uw_a, ux, vw, vx, wx\}$, then $C_u(X(u)) = \{ux\}$ and $\overline{C}_u(X(u)) = \{uw_a, uv\}$. The edge uw_a is the $X(u)$-replacement of ux. The inverted choice set of u for X, $C_u^-(X)$, contains only uv.

With this notation, every matching problem can be represented by a finite graph (V,E) with $V = \{1,\ldots,|V|\}$ the set of agents (vertices) and E the set of potential relationships (edges) and a set of choice functions $C = \big(C_1(.),\ldots,C_{|V|}(.)\big)$. We write $\{(V,E),C\}$. We say that a partnership M is *contained* in the partnership problem $\{(V,E),C\}$, or equivalently, M is *contained* in E, if $M \subseteq E$.

The partnership problem with linear preferences. We discuss a one-sided matching problem where agents may want to form multiple relationships. The latter implies that choice sets may contain more than one element. One-sidedness implies that there are no restrictions on the set of potential edges E. As mentioned in the previous paragraph, the preferences over $E(v)$ for each agent v are modeled by the choice function C_v, which is assumed to satisfy three restrictions: *substitutability* (SUB), *increasingness* (INCR) and *linearity* (LIN). First, substitutability means that when e is chosen from an option set $X(v)$, then v will still choose option e when some options are no longer available in $X(v)$. In other words, if e is in the choice set of an option set $X(v)$, then e is in the choice set of any option set that is a subset of $X(v)$.

SUB A choice function, C_v, is *substitutable* if, for $e \neq f$ and $e \in C_v(X(v))$, then $e \in C_v(X(v) \backslash f)$.

Fleiner (2010, Theorem 2.1) finds that a choice function C_v on a finite groundset is substitutable if and only if $\overline{C}_v$ is monotonous; if an option e is not chosen from an option set $Y(v)$, then it will not be chosen either when the option set is expanded:

MON The function of non-chosen options $\overline{C}_v$, is *monotonous* if $Y(v) \subseteq X(v)$ implies that $\overline{C_v}(Y(v)) \subseteq \overline{C_v}(X(v))$.

Another corollary of SUB is that if an edge $uv \in E$ — that is, uv is a potential edge — then $uv \in C_u(uv) \cap C_v(uv)$; if an edge is potential, then it is always chosen by both agents involved, when no other options are available.

Second, increasingness means that if extra options are added to an option set $Y(v)$, then v chooses at least as many options in the expanded option set as in $Y(v)$; if more options are available, then more options will be chosen. Formally,

INCR A choice function, C_v, is *increasing* if $Y \subseteq X$ implies $|C_v(Y)| \leq |C_v(X)|$.

INCR, together with SUB, guarantees that the $X(v)$-replacement of an edge $e \in C_v(X(v))$ will be at most one element, which will be important for the rest of the analysis.

Third, a choice function satisfies linearity if for any pair of edges e and f such that there is an option set $X(v)$ incident to v where e is chosen, while f is not, it holds that that whenever f is part of a choice set for an option set $Z(v)$, then e will also be chosen if e is added to that option set $Z(v)$. If an option e is chosen while another available option f is not chosen, then there does not exist an option set where the reverse holds. Formally,

LIN A choice function C_v is *linear* if for $e \neq f$, there exists an $X(v)$ such that $e \in C_v(X(v))$ and $f \in \overline{C}_v(X(v))$, then there does not exist an $Z(v)$ containing both e and f, such that $e \in \overline{C}_v(Z(v))$ while $f \in C_v(Z(v))$.

Note that the restriction of linearity, together with INCR and SUB, implies an ordering of the edges in $E(v)$ for each v – hence the term *linearity*. If an option e is chosen while another available option f is not chosen, then e can be assumed to be placed before f in this implicit ordering.

We can then define the partnership problem with linear preferences as follows:

$\mathcal{P}$ A partnership problem with linear preferences $\mathcal{P}$ is a matching problem $\{(V,E),C\}$ such that any $C_v \in C = (C_1,\ldots,C_{|V|})$ satisfies SUB, INCR and LIN.

This matching problem is related to two other one-sided many-to-many matching problems. First, a more restrictive version of the partnership problem with linear preferences is the *stable b-matching problem.* In the stable b-matching problem, every agent v has a strict preference list of $E(v)$, along with a fixed capacity $b(v)$, indicating the number of relationships she is willing to form. The matching problem discussed here is similar in the sense that, as shown above, LIN and INCR, also imply a preference ordering. However, it is more general than a stable b-matching problem as the capacity is not fixed but governed by the assumption of increasingness. For example, it is possible that for an option set $\{uv,uw,ux\}$, the choice set is $\{uv,uw\}$, containing two edges, but for an option set $\{uv,ux\}$, the choice set may be $\{uv\}$ containing only one edge. Hence, in this example, the capacity depends on the option set considered, which is not possible in a stable b-matching problem. The examples that will be discussed in this paper will be examples of stable b-matching problems to make the illustration easier. Second, a more general version is the partnership problem as defined by Fleiner (2010), which also assumes INCR together with substitutability (SUB) but does not impose linearity. In the remainder of this paper, we will speak of the *partnership problem* meaning the *partnership problem with linear preferences* unless otherwise mentioned.

Stability For a partnership problem $\mathcal{P} = \{(V,E),C\}$, a partnership M is *individually rational* if no agent $v \in V$ has an edge $uv \in M$ such that $uv \in \overline{C}_v(M(v))$; no agent wishes to cut a relationship it has in M. In addition, a *blocking edge* uv of M is such that $uv \notin M$ and $uv \in C_u(M(u) \cup uv) \cap C_v(M(v) \cup uv)$; both u and v wish to form uv in M. For example, in Figure 3.1, if $M = \{uw_b, wx\}$, then M is not individually rational as $wx \in \overline{C}_w(M(w))$. An example of a blocking edge for M would be uv. A partnership that is individually rational and has no blocking edge is denoted a *stable partnership*. Formally,

Stable partnerships A partnership S is *stable* for the problem $\mathcal{P} = \{(V,E),C\}$ if S satisfies:

- **Individual rationality:** for any agent v we have $C_v(S(v)) = S(v)$, and
- **Absence of a blocking edge:** There does not exist an edge $e = uv \notin S$ such that both $e \in C_u(S(u) \cup \{e\})$ and $e \in C_v(S(v) \cup \{e\})$ holds.

A partnership problem that has at least one stable partnership will be denoted solvable. Until Section 3.4, we will only consider solvable partnership problems as they are intuitively most appealing to investigate the concept of a path to stability. The example of Figure 3.1 is solvable and has a unique stable partnership, $\{ux, vw, vx\}$.

3.3 Main result.

The central question of this paper is whether a decentralized matching process, as described by Roth and Vande Vate (1990) for the stable marriage problem, always converges to a stable partnership, when such a partnership exists. Consider a partnership M for a given problem $\mathcal{P}$. The starting assumption is that M is individually rational, as each agent can unilaterally decide whether to cut a relationship or not. The decentralized process à la Roth and Vande Vate (1990) then randomly selects a pair of agents with positive probability for each possible pair; as if all agents are in a room together and randomly bump into each other. When a pair is selected, they review their relationship. If their relationship is part of the partnership, nothing happens and each agent involved goes his own way, leaving the relationship intact. If they do not have a relationship in the current partnership, each agent checks his willingness to form that relationship. If one of them objects, again, nothing happens and the relationship is not formed. If both of them are willing – the relationship is a blocking edge – the relationship is formed. In addition, each agent involved in the blocking edge simultaneously cuts relationships that she no longer wants because of the formation of the blocking edge, so as to make the resulting partnership individually rational. The process of forming a blocking edge uv and simultaneously deleting edges that are no longer wanted, is called *satisfying a blocking edge*.[6] In Figure 3.1, vx is a blocking edge for $M = \{uv, vw\}$ and satisfying vx leads to $M' = \{uv, vx\}$, cutting vw in the process. By satisfying blocking edges we obtain an *improving path*, a finite sequence of partnerships where the next partnership in the sequence is obtained by satisfying a blocking edge in the previous partnership (see Jackson and Watts (2002)):

Improving path An *improving path* from an individually rational partnership M to a partnership M' is a finite sequence of partnerships $\{M_t\}_{t=0}^{T}$ such that $M = M_0$ and $M' = M_T$ and for all $t = \{1, 2, \ldots, T\}$, M_t is obtained from M_{t-1} by satisfying a blocking edge.

Now, starting from M, multiple improving paths may exist. For example, for Figure 3.1 and $M = \{uv, vw\}$, besides vx, ux is also a blocking edge and satisfying it would create an alternative improving path. By the assumption that any pair of agents may be selected with positive probability, any improving path is a possible path for the decentralized matching process. As such, if we want to prove that a decentralized matching process converges to some stable partnership with probability one, it suffices to show that, for any M, an improving path exists that ends in a stable partnership. This gives our main theorem.

[6]When dealing with non-simple graphs, the interpretation of satisfying a blocking edge becomes a bit problematic. The problem is that satisfying a blocking edge, say uv_a, may imply the simultaneous deletion of an edge uv_b by agent v, such that it is possible that agent u is worse off from satisfying uv_a. Hence, we can then ask whether uv_a was really a blocking edge, as one of the agents ended up in a worse situation. A possible solution is to assume that u could not foresee agent v's actions. However, u can also see the edge uv_b so he must have known that there was a possibility that satisfying uv_a implies the deletion of uv_b, leaving the question of whether a blocking edge as defined here is an appropriate concept in the case of non-simple graphs.

Theorem 3.3.1 *Consider a partnership problem with linear preferences $\mathcal{P}$ and a partnership M. There always exists an improving path from M to a partnership S such that S is a stable partnership of $\mathcal{P}$.*

Proof See appendix 3.6.1.

An example of an *improving path towards stability* in our example starting from $M = \{uv, vw\}$ is the following:

$$M_0 = M, M_1 = \{\mathbf{ux}, vw\}, M_2 = \{ux, \mathbf{vx}, vw\},$$

where the edges in bold are the blocking edges that were satisfied to reach that partnership. Note that the theorem is an extension of the theorema in Roth and Vande Vate (1990) and Diamantoudi et al. (2004). In the next two subsections, we will give an outline of the proof. First, as the proof builds on Fleiner (2010)'s algorithm that produces a stable partnership if one exists, the algorithm will be presented. Second, we provide an outline of the proof by means of a simple example.

3.3.1 Finding a stable partnership.

Fleiner (2010) extended Irving's algorithm (Irving, 1985) to a two-phase algorithm that produces a stable partnership if one exists.[7] The proof of Theorem 3.3.1 will build on this algorithm as the algorithm can be shown to implicitly draw out a path towards stability. Starting from a partnership problem $\mathcal{P} = \{(V,E),C\}$, the idea is to delete edges of E such that after a number of rounds we end up with a stable partnership of $\mathcal{P}$.

Phase 1. First, consider an edge $uv \in E$ such that $uv \in \overline{C}_v(C_v^-(E) \cup uv)$; we say that uv is *dominated* by the inverted choice set of v for E. Those type of edges, edges dominated by the inverted choice set, can be shown to be never part of a stable partnership of $\mathcal{P}$. The fact that such an edge is dominated by the inverted choice set, the set of incident edges others are always

[7] Fleiner (2010, Section 4.) points out that there is a more efficient one-phase algorithm that is more similar to Irving's algorithm. However, to keep the analysis and the proofs more transparent, we use the two-phase algorithm presented in Fleiner (2010).

willing to form, implies that unformed edges in the inverted choice set will be blocking edges for any partnership that contains a dominated edge.

To see this, consider a partnership M such that $uv \in M$ and

$$uv \in \overline{C}_v(C_v^-(E) \cup uv). \tag{3.1}$$

Can such an M be a stable partnership? A first requirement of stability is that M is individually rational, implying that $uv \in C_v(M(v))$. Second, M does not have a blocking edge. This means that any edge $vw \in T = C_v^-(E) \backslash M(v)$, should be such that $vw \in \overline{C}_v(M(v) \cup vw)$; as vw is in the choice set of w by the definition of the inverted choice set, it must be that it is not in the choice set for v. By MON, it then follows that $T \subseteq \overline{C}_v(M(v) \cup T)$. As INCR states that the choice set of $M(v) \cup T$ should contain at least as many edges as $M(v)$, this then implies that $\overline{C}_v(M(v) \cup T) = T$ and $uv \in C_v(M(v) \cup T)$, contradicting statement (3.1) by MON. Hence, M cannot be a stable partnership. Indeed, deleting edges dominated by the inverted choice set does not eliminate a stable partnership from the set of stable partnerships for a given partnership problem. Phase 1 of the algorithm, therefore, aims to reduce the partnership problem to a partnership problem that has no edges dominated by the inverted choice set, a *reduced* partnership problem:

Reduced partnership problem A partnership problem $\mathcal{P} = \{(V,E),C\}$ is *reduced* if for all $uv \in E$, $uv \in C_v(C_v^-(E) \cup uv)$.

The reduced version of a partnership problem $\mathcal{P} = \{(V,E),C\}$ will be denoted by $\mathcal{P}' = \{(V,E'),C\}$. The example in Figure 3.1 is not a reduced partnership problem, as $uw_b \in \overline{C}_u(C_u^-(E) \cup uw_b)$ with $C_u^-(E) = uv$. The reduced problem of the example is given in Figure 3.2. Formally, Phase 1 entails an iteration of the following step.

Phase 1 While there exists $uv \in \overline{C}_v(C_v^-(E) \cup uv)$ for some v, set $E = E \backslash uv$.

Phase 2. After applying Phase 1 for a partnership problem $\mathcal{P}$, we get a reduced problem $\mathcal{P}'$. For such a problem, we can define a *rotation*. First, define an *edge-pair* as an ordered set of two edges $(a = uv, (a)^r = vw)$ such that $uv \in C_v(E'(v)) \backslash C_u(E'(u))$ and vw is the $E'(v)$-replacement

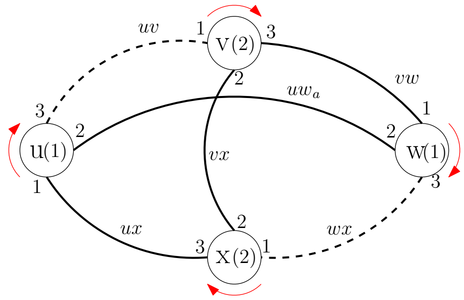

Figure 3.2: A reduced version of the partnership problem given in figure 3.1.

of uv; an edge that is in exactly one choice set and its replacement, form an edge-pair. Second, an edge-pair (uv, vw) *leads* to another edge-pair (wx, xz) if $wx = \overline{C}_w(C_w^-(E') \cup vw)$.[8] Then, a rotation π is a sequence of edge-pairs

$$\pi = (a_1, (a_1)^r, a_2, (a_2)^r, \ldots, a_m, (a_m)^r)$$

such that $(a_i, (a_i)^r)$ leads to $(a_{i+1}, (a_{i+1})^r)$, for any $i = 1, \ldots, m$, *modulo m* – that is, $(a_m, (a_m)^r)$ also leads to $(a_1, (a_1)^r)$.

Fleiner (2010) showed that any reduced partnership problem $\{(V, E'), C\}$ such that E' is not a stable partnership, has at least one rotation. There are two types of rotations. First, the number of different edges in the rotation may be odd – we are dealing with an *odd* rotation. A stable partnership does not exist if and only if a problem has an odd rotation (Fleiner, 2010, Corollary 2.7.).[9] Second, the number of unique edges may be even – we are dealing with an *even* rotation. As we are only considering solvable partnership problems for now, every rotation will be even. Phase 2 then prescribes selecting an (even) rotation and deleting the first edge in every edge-pair of that rotation.

[8] A small note on the definition of *leading*: it is possible for $uv \in C_v(E'(v))$ and vw, the $E'(v)$-replacement of uv, that $\overline{C}_v(C_v^-(E') \cup vw) \neq \emptyset$. However, in the definition of leading, we always look for the edge-pair such that the first element is in $\overline{C}_w(C_w^-(E') \cup vw)$.

[9] This also explains why a stable matching always exists in two-sided matching problems. In that case, an odd rotation is impossible as edges go from one type of agent to another type. If you start from an edge-pair (uv, vw) and end with an edge-pair (xy, yu) such that it leads to (uv, vw), then this sequence will always contain an even number of different edges.

Phase 2 For a reduced problem $\mathcal{P}'$, pick a rotation π and eliminate the first edge in every edge-pair of π.

The unique rotation for the reduced problem in Figure 3.2 is

$$\pi = (uv, vw, wx, xu)$$

Applying Phase 2 would then imply deleting uv and wx. The following argument provides an intuition for why deleting uv and wx is a good idea when searching for a stable partnership. Assume uv is deleted from E'. In that case, we know that, as vw is the $E'(v)$-replacement of uv, $vw \in C_v(E'(v) \backslash uv)$. This, in turn, makes that $wx = \overline{C}_w(C_w^-(E' \backslash uv))$, by which it can never be part of a stable partnership if uv does not exist. Hence, deleting wx makes sense if uv is deleted. Moreover, $xu \in C_x(E'(x) \backslash wx)$ such that $uv = \overline{C}_u(C_u^-(E' \backslash wx))$. Hence, if either of uv or wx is not part of a stable partnership S, then they are both not part of S. In addition, Fleiner (2010, Lemma 3.6.) showed that if there exists a stable partnership, S_1, such that $\{uv, wx\} \subset S_1$, then there exists another stable partnership S_2, such that $\{vw, xu\} \subset S_2$. So to sum up, by deleting uv and wx we are sure to get closer to a stable partnership because, either uv or wx is never part of a stable partnership, and then they are both never part of a stable partnership, or if there exists a stable partnership that contains uv and wx, there exists another stable partnership that is not eliminated by deleting uv and wx.[10]

After having applied Phase 2, we may end up in a partnership problem that is not reduced. Hence, Phase 1 should be re-applied, and if an even rotation exists for the new reduced problem, Phase 2 should also be re-applied. Running Phase 1 and Phase 2 consecutively, by the arguments in the previous paragraphs we are bound to end up in a partnership problem $\{(V, S), C\}$, that has no even rotations, which in the case of solvable partnership problems means that S itself is a stable partnership.

[10]This is where the original term of a rotation, *an all-or-nothing cycle*, due to Irving (1985) stems from; either all first elements of the edge-pairs are in the stable partnership, or none of them.

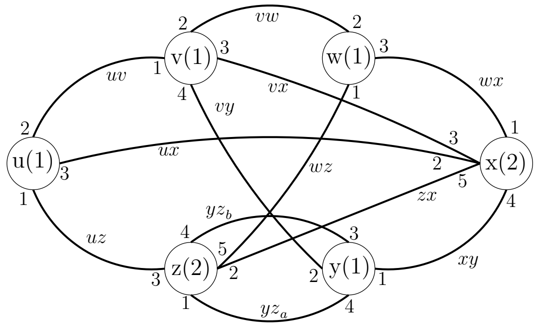

Figure 3.3: A six-agent stable b-matching problem $\mathcal{P}$.

3.3.2 Outline of the proof.

Before giving the outline, we note that from now on, we are only dealing with improving paths starting from individually rational partnerships. It is straightforward to see that there always exists an improving path from a partnership M that is not individually rational towards an individually rational partnership M', simply by letting agents v in some order delete matches in M that are not in their choice set over M. In addition, there can never be an improving path from an individually rational partnership towards a partnership that is not individually rational. Hence, partnerships that are not individually rational are not relevant for the remainder of the proof and are therefore ignored in what follows.

Fleiner (2010) showed that a stable partnership, if it exists, can be found by repeatedly running Phase 1 and Phase 2. These two processes are repeated until the set of even rotations is empty, giving us a stable partnership. The algorithm turns out to be very useful to prove Theorem 3.3.1 as it implicitly maps out an improving path towards a stable partnership. The proof of the theorem has two claims, which suffice to prove that an improving path to stability exists. We will illustrate these claims by means of a six-agent partnership problem, which is essentially a stable b-matching problem, given in figure 3.3.

Claim 1 states that if we apply Phase 1 to a partnership problem, deleting a set of edges Q_1, then for any partnership that contains edges in Q_1, there exists an improving path towards a partnership that does not contain such edges:

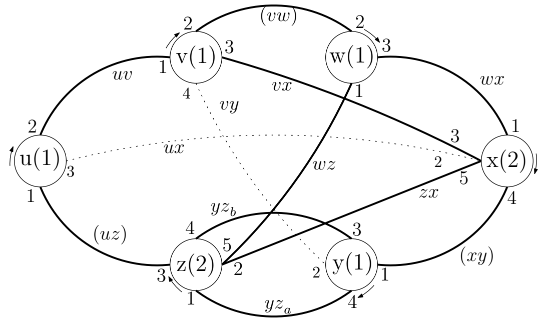

Figure 3.4: Reduced partnership problem $\mathcal{P}'$ of $\mathcal{P}$, the edges deleted when reducing $\mathcal{P}$ are given by a dotted line. The arrows indicate the unique even rotation, with the label of the edge between parentheses if it is the second element in an edge-pair in the rotation.

Claim 1 *In a partnership problem $\mathcal{P} = \{(V,E),C\}$ and its reduced problem $\mathcal{P}' = \{(V,E'),C\}$, there exists an improving path from any individually rational partnership $M \subseteq E$ to a partnership M' such that $M' \subseteq E'$.*

Consider the partnership problem $\mathcal{P}$ in Figure 3.3. Applying phase 1 leads to a reduced partnership problem $\mathcal{P}'$ depicted in Figure 3.4, a problem without ux and vy. Edge ux was deleted since the inverted choice set of agent u for $\mathcal{P}$ contains uv and ux, and ux was dominated by this inverted choice set, i.e. $ux \in \overline{C}_u(C_u^-(E) \cup ux)$. The deletion of ux leads to an intermediate problem $\mathcal{P}''$, where $C_x(E(x)\backslash ux) = \{wx, vx\}$. Hence, while $vy \in C_v(C_v^-(E) \cup vy)$ with $C_v^-(E) = \emptyset$, it is the case that $vy \in \overline{C}_v(C_v^-(E\backslash ux) \cup vy)$ with $C_v^-(E\backslash ux) = vx$. Edge vy is also deleted and we get the reduced problem $\mathcal{P}'$.

Consider an individually rational partnership M_0. If $Q_1 \cup M_0 = \emptyset$, then Claim 1 trivially holds as M_0 itself does not contain deleted edges. Hence, assume that $Q_1 \cap M_0 \neq \emptyset$. To prove claim 1, we show that an improving path exists from M_0 such that edges in $Q_1 \cap M_0$ are deleted in the order in which they are deleted in Phase 1. That is, if e and f are two edges in M_0 that are deleted in Phase 1 with e deleted before f, then there exists an improving path from M_0 such that e is deleted before f on this path. The partnership at the end of such an improving path is such that it does not contain any edge in Q_1. For our running example, consider $M_0 = \{ux, wx, yz_a\}$ and $Q_1 = \{ux, vy\}$. Edge ux can be deleted because no edge deleted before ux in Phase 1 is in M_0 –

in fact ux was deleted first in phase 1 – and $ux \in \overline{C}_u(C_u^-(E) \cup ux)$. This implies, by MON, that

$$ux \in \overline{C}_u(M_0(u) \cup C_u^-(E)) = \overline{C}_u(M_0(u) \cup uv).$$

However, by individual rationality of M_0, $ux \in C_u(M_0(u))$, which implies, by INCR that $uv \in C_u(M_0(u) \cup uv)$. Together with $uv \in C_u^-(E)$, this implies then that uv is a blocking edge and satisfying it will delete ux, giving $M_1 = \{uv, wx, yz_a\}$ with $Q \cap M_1 = \emptyset$. The crucial element of the proof is the fact that we know that $uv \in C_v(M_0(v) \cup uv)$, as no elements that are deleted before ux in Phase 1 are in M_0. Hence, M_0 will be a subset of the edge set that is reached in Phase 1 just before deleting ux, and as uv is in the inverted choice set of u for that edge set, we know that $uv \in C_v(M_0(v) \cup uv)$ by SUB.

Hence, by Claim 1 we know that from any partnership that is not contained in the reduced partnership problem, there exists an improving path towards a partnership that is contained in that problem. Claim 2 is similar but pertains to Phase 2. We apply Phase 2 to a reduced problem with a non-empty set of even rotations, Π^e. Consider any individually rational partnership M contained in this reduced problem. Then, there always exists a rotation $\pi \in \Pi^e$ such that an improving path exists from M to a partnership that does not contain any of the edges that would be deleted if we eliminate π. The partnership that is reached, is contained in the post-Phase 2 partnership problem when eliminating π:

Claim 2 *Consider a reduced partnership problem $\mathcal{P}'$, the set of even rotations Π^e and denote Q_π, the set of deleted edges from eliminating an even rotation $\pi \in \Pi^e$. For every individually rational unstable partnership M such that $M \subseteq E'$, there exists a rotation $\pi \in \Pi^e$ such that an improving path exists from M to $M' \subseteq E' \backslash Q_\pi$.*

The partnership problem in Figure 3.4, $\mathcal{P}'$, has a unique even rotation:

$$\pi = (yz_a, zu, uv, vw, wx, xy). \tag{3.2}$$

Applying Phase 2 and eliminating π would lead to the deletion of $Q_\pi = \{yz_a, uv, wx\}$. Consider the partnership $M_1 = \{uv, wx, yz_a\} = Q_\pi$. By Claim 2 there exists an improving path from M_1 towards a partnership M_2 such that $Q_\pi \cap M_2 = \emptyset$. To see this, note that there exists an edge-pair

(vx, xy) with xy the $E'(x)$-replacement of vx and such that $yz_a = \overline{C}_y(C_y^-(E') \cup xy)$; the edge-pair (vx, xy) leads to (yz_a, zu) which is in the even rotation π. Now $vx \notin M_1$. Whenever we find such a pattern – an edge pair (vx, xy) that leads to an edge-pair (yz_a, zu) such that $vx \notin M_1$ while $yz_a \in M_1$ – an improving path can be constructed towards a partnership M_1' such that both vx and yz_a are not in M_1'. As $vx \notin M_1$ and xy is the $E'(x)$-replacement of vx, $xy \in C_x(M_1(x) \cup xy)$. In addition, LIN implies that $xy \in C_y(M_1(y) \cup xy)$, such that xy is a blocking edge. As $M_1(y) \cup xy = M_1(y) \cup C_y^-(E') \cup xy$, MON implies that $yz_a \in \overline{C}_y(M_1(y) \cup xy)$ and satisfying xy deletes yz_a. This leads to a partnership $M_1' = \{uv, wx, xy\}$ where we again have the same pattern – an edge-pair (yz_a, zu) that leads to (uv, vw), such that $yz_a \notin M_1'$ while $uv \in M_1'$. The previous argument can now be repeated; zu is a blocking edge and satisfying it deletes uv and we reach a partnership $M_1'' = \{uz, wx, xy\}$. Finally, we have (uv, vw, wx, xy) with $uv \notin M_1''$ and $wx \in M_1''$. Satisfying the blocking edge vw, deletes wx and we reach $M_2 = \{uz, vw, xy\}$ with $Q_\pi \cap M_2 = \emptyset$. Hence, the idea behind claim 2 is that, in a sequence of edge-pairs that leads to a rotation, there always exists an edge-pair such that its first element is not part of the partnership. This enables us to delete the first element of the next edge-pair. Repeating this argument, we can *clear out* all edge pairs of the rotation, deleting the first elements of all edge-pairs.

We started from M_0, a partnership that was contained in E but not in E'. Claim 1 states that there is an improving path from M_0 towards M_1 with M_1 being contained in E'. However, M_1 is not contained in the post-Phase 2 set of potential edges for any rotation that is eliminated. Claim 2 then states that there is an improving path from M_1 towards M_2 where M_2 is contained in the post-Phase 2 set of potential edges after deleting some rotation π. The associated reduced partnership problem, $\mathcal{P}''$, is depicted in Figure 3.5. In this case M_2 is already contained in E'' as $yz_b \notin M_2$. Incidentally, M_2 is also contained in the post-Phase 2 set of potential edges if we chose to delete the rotation $\pi' = (vx, xz, zw, wv)$. Applying Phase 2, deleting π', leads to the following problem $\mathcal{P}_S = \{(V, S = \{uz, vw, xz, xy\}), C\}$. It can be verified that S is a stable partnership for $\mathcal{P}$ – the algorithm terminates and we have found a stable partnership. M_2 is contained in S, and by the individual rationality of S, there exists an improving path from M_2 towards S by satisfying the blocking edge xz.

Hence, to sum up, by Claims 1 and 2, every time we apply Phase 1 or Phase 2, we know that

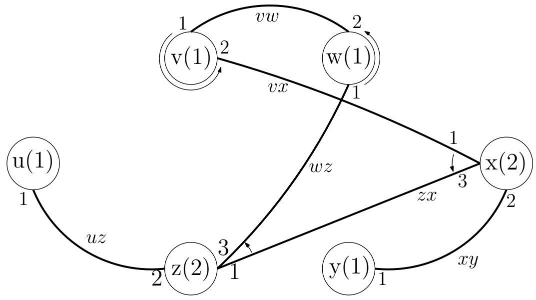

Figure 3.5: Reduced partnership problem $\mathcal{P}''$. The arrows indicate the two rotations, (vx, xz, zw, wv) and (xz, zw, wv, vx).

partnerships that contain a deleted edge will have improving paths towards partnerships that do not contain deleted edges, which eventually points out an improving path towards a partnership M_2 contained in S. As S is individually rational by the definition of a stable partnership, we know that there exists an improving path from M_2 towards S, by satisfying all edges in $S \backslash M_2$. One final remark: Theorem 3.3.1 implies that an improving path exists from an unstable partnership towards *a* stable partnership, rather than towards *any* stable partnership. The reason why convergence towards any stable partnership might not always hold, can be found in Claim 2. Claim 2 does not exclude the possibility that there exists a rotation π in the reduced partnership problem, with associated deleted edge set of Q_π, such that there does not exist an improving path from a partnership M towards a partnership that does not contain edges in Q_π. Hence, from M there might not exist an improving path towards the stable partnership that is reached when deleting this particular rotation π.

3.4 Convergence for non-solvable partnership problems.

So far we have only considered solvable partnership problems, problems that have a stable partnership. Since Gale and Shapley (1962), it is well known that for one-sided matching problems, like the partnership problem, the set of stable matchings may be empty. For non-solvable partnership problems, we extend the results from Iñarra et al. (2008) and Biro and Norman (2012) and show that there exists an improving path towards an instance of a stable half-partnership, that is, a partnership associated with a stable half-partnership, which is itself a generalization of a stable partnership (see Tan (1991) for the roommate problem and Fleiner (2010) for the partnership problem).

A stable half-partnership consists of a set of edges S and a partition of that set of edges $\{S_1, S_2, \dots, S_k\}$ such that any element in the partition contains an odd number of edges.[11] As in the definition of stable partnerships, there are two parts in the definition of half-stability.
First, there is a weaker version of individual rationality, which states that for every element S_i in the partition the following must hold. If $|S_i| = 1$, then assuming $S_i = \{uv\}$, $uv \in C_u(S(u)) \cap C_v(S(v))$; keeping an edge that is a singleton element of the partition must be individually rational in the original sense. If $|S_i| > 1$, then edges in S_i can be ordered, say $(v_m v_1, v_1 v_2, \dots, v_{m-1} v_m)$, such that $v_j v_{j+1}$ is the $S(v_j)$-replacement of $v_{j-1} v_j$. For any such $v_j v_{j+1}$, this implies that $v_j v_{j+1} \in C^-_{v_j}(S) \backslash C_{v_j}(S(v_j))$. In addition, it also follows that $v_j v_{j+1} \in C_{v_j}(C^-_{v_j}(S))$ such that $\overline{C}_{v_j}(C^-_{v_j}(S)) = \emptyset$. Hence, it is no longer the case that for any agent v we have $C_v(S(v)) = S(v)$, as in the definition of individual rationality. However, those edges uv for which $uv \in \overline{C}_v(S(v))$ are in $C^-_v(S)$ and $C_v(C^-_v(S)) = C^-_v(S)$. Second, in the case of half-stability, a *weak blocking edge* is an edge $uv \in E \backslash S$ such that $uv \in C_u(C^-_u(S) \cup uv) \cap C_v(C^-_v(S) \cup uv)$. Hence if there is no weak blocking edge then this means that for every $uv \in E \backslash S$ it must be that uv is not chosen in the option set $C^-_u(S) \cup uv$ or $C^-_v(S) \cup uv$. Formally,

Stable half-partnerships For the problem $\mathcal{P} = \{(V,E), C\}$, a *stable half-partnership*, $(S; S_1, \dots, S_k)$ is such that $S_1, \dots, S_k$ is a partition of $S \subseteq E$ with every S_i containing an odd number of elements and,

- **Weak individual rationality:** For any S_i,
  - if $|S_i| = 1$, then $S_i = uv \in C_u(S(u)) \cap C_v(S(v))$,
  - if $|S_i| = m > 1$, then there exists an ordering $(v_m v_1, v_1 v_2, \dots, v_{m-1} v_m)$ such that any $v_j v_{j+1}$ is the $S(v_j)$-replacement of $v_{j-1} v_j$.

[11] A stable half-partnership can also be defined without the requirement of an odd number of edges. However, a stable half-partnership in which an element in the partition has an even number of edges can always be reduced to a stable half-partnership with a partition that has no even-numbered elements (see Tan (1991)).

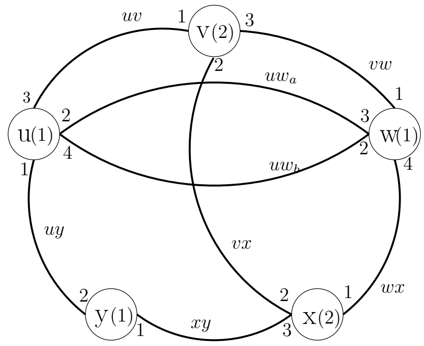

Figure 3.6: A five-agent example. This particular example is a stable b-matching problem: the capacity for each agent is given between parentheses, the preference ordering is indicated by the numbers on the edges close to the vertices.

- **Absence of a weak blocking edge:** There does not exist $e = uv \notin S$ such that both $e \in C_u(C_u^-(S) \cup \{e\})$ and $e \in C_v(C_v^-(S) \cup \{e\})$ holds.

Note that if a stable partnership exists, then that stable partnership is a stable half-partnership as well. For the example in figure 3.1, it can be verified that $\{ux, vw, vx\}$ is equivalent to the stable half-partnership $(\{ux, vw, vx\}; \{ux\}, \{vw\}, \{vx\})$. Figure 3.6 depicts a five-agent stable b-matching problem where a stable partnership does not exist. However, this particular problem has a stable half-partnership,

$$(S = \{uv, wu_a, vw, vx, xy\}; S_1 = \{vx\}, S_2 = \{xy\}, S_3 = \{uv, vw, wu_a\}),$$

depicted in figure 3.7.

In the words of Iñarra et al. (2008), a stable half-partnership can be interpreted as stability over the partition of S. Another way to interpret the stability of a half-partnership is to follow Biro et al. (2008) and assume that agents can assign weights to edges – indicating for example the amount of time or effort spent on a particular relationship. Consider a situation where an edge can have weight 0, $\frac{1}{2}$ or 1. It is as if agents can opt to form *half-time* relationships; for example, if relationship uv means *u and v spend one hour together*, then there is also the possibility of spending half an hour together. Then, to return to our example, if we assume that edges not in S have weight 0, that edges in singleton elements of the partition have weight 1 and that edges

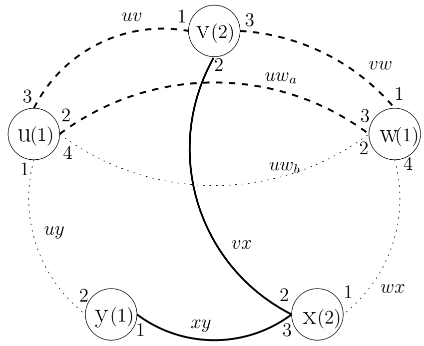

Figure 3.7: Stable half-partnership $(S;S_1,S_2,S_3)$ of the example in figure 3.6. Dotted edges are not in S, S_1 and S_2 are depicted by solid edges, while S_3 is depicted by dashed edges.

in non-singleton elements have weight $\frac{1}{2}$, $(S;S_1,S_2,S_3)$ can be considered stable in the sense of Definition 3.2. First, no agent can improve by reducing the *amount of time* she spends on a relationship (individual rationality). Second, no pair of agents can jointly improve by increasing the *amount of time* spent on a relationship (absence of a blocking edge). For example, agent v may want to increase the time spent on uv and decrease the amount of time spent on vw. However, u will never agree to spend more time on uv, as this will either violate individual rationality, or she will have to decrease time spent on uw_a, which is preferred over uv. Likewise, w will not agree to increase the time spent on uw_a, because she prefers to spend more time on vw, while v does not want to do this because she prefers to spend more time on uv. In addition, if we look at edges with weight 0, it is clear that no pair of agents will jointly agree to increase time spent on those edges. For example, agent w will not want to increase time spent on wx as this would imply decreasing time spent on uw_a or vw, both of which are preferred over wx.

Instances of a stable half-partnership. In the absence of a stable partnership, the set of edges S of a stable half-partnership $(S;S_1,\dots,S_k)$ is not an individually rational partnership. However, we can define a set of individually rational partnerships, denoted *instances*, that are associated with a particular stable half-partnership. For a stable half-partnership $(S;S_1,\dots,S_k)$, an instance I is a subset of S such that it contains all edges that are singleton elements of the partition. If such an edge would not be in I, then by the condition of weak individual rationality, it would always be a blocking edge for I. In addition, for an element S_i such that it is not a singleton, with ordering $(v_m v_1, v_1 v_2, \ldots, v_{m-1} v_m)$, I contains $\frac{m}{2} - 1$ edges of S_i such that no two consecutive edges in the ordering are both in I. For example, if we have the following element $S_i = (v_5 v_1, v_1 v_2, \ldots, v_4 v_5)$, then $I \cap S_i$ can be $\{v_1 v_2, v_3 v_4\}$. Another possibility would be $\{v_5 v_1, v_2 v_3\}$. Hence, for a stable half-partnership there are multiple instances, depending on which edges in the S_i's are formed. The set of all such instances will be called the instance set. Formally,

Instance For a stable half-partnership $(S; S_1, \ldots, S_k)$ of a partnership problem $\mathcal{P}$, an *instance* $I \subseteq S$ is such that, for all S_i:

- if $|S_i| = 1$, then $S_i \subseteq I$,
- if $|S_i| = m > 1$ with the relevant ordering $(v_m v_1, v_1 v_2, \ldots, v_{m-1} v_m)$,
  - if $v_{j-1} v_j \in I$, then $v_j v_{j+1} \notin I$,
  - $|S_i \cap I| = \frac{m}{2} - 1$.

The *instance set*, $I_{(S;S_1,\ldots,S_k)}$, is the set of all instances of $(S; S_1, \ldots, S_k)$.

The stable half-partnership in Figure 3.7 has three instances: $\{xy, vx, uv\}$, $\{xy, vx, vw\}$ and $\{xy, vx, uw_a\}$. Note that every instance contains all edges that are in singleton elements of the partition of S, while they contain exactly one edge of S_3.
In the absence of a stable partnership, we can view the instance set of a stable half-partnership as the closest we can get to stability. Consider an instance I for a stable half-partnership $(S; S_1, \ldots, S_k)$ and let B be the set of blocking edges for I. If $B = \emptyset$, then clearly, as I is individually rational, I is a stable partnership, and the instance set for $(S; S_1, \ldots, S_k)$ is $\{I\}$. Hence, if the set of stable partnerships is empty, $B \neq \emptyset$. Assume $e = uv \in B$ such that $e \notin S$. By the absence of a weak blocking edge in the stable half-partnership, it must be, w.l.o.g., that

$$uv \in \overline{C}_u(C_u^-(S) \cup uv). \tag{3.3}$$

Hence, by LIN, $C_u(I(u) \cup uv) = I(u) \cup uv$. Now, if $|I(u)| = |C_u(S(u))|$, then this is a violation of INCR, hence $|I(u)| < |C_u(S(u))|$. This implies that there must exist an S_i with ordering

$(uv_1, v_1v_2, \ldots, v_{m-1}u)$ such that both uv_1 and $v_{m-1}u$ are not in I. In that case, however, uv_1 is also a blocking edge by SUB, as $uv_1 \in C_{v_1}(S(v_1))$ and $uv_1 \in C_u(S(u) \backslash v_{m-1}u)$. In addition, by (3.3), u will prefer to form uv_1 over uv.This means that whenever there is a blocking edge which leads to a partnership outside the instance set, there is a more preferred blocking edge that leads to a partnership within the instance set. Hence, the instance set can be viewed as almost stable in that, provided the agents can take optimal decisions on which blocking edge to satisfy, the improving path will not lead to partnerships outside the instance set.[12]

Path to the instance set. We can extend the results from Iñarra et al. (2008) and Biro and Norman (2012) and show that for a partnership problem with linear preferences, from any unstable partnership the instance set of a stable half-partnership can be reached by means of an improving path. When the set of stable partnerships is non-empty, this result boils down to Theorem 3.3.1.

Theorem 3.4.1 *Consider a partnership problem with linear preferences $\mathcal{P}$ and a partnership M. There always exists an improving path from M to a partnership I such that I is an instance of a stable half-partnership $(S; S_1, \ldots, S_k)$ of $\mathcal{P}$.*

Proof See appendix 3.6.2.

3.5 Concluding remarks

In this work, we have presented an extension of the result of Diamantoudi et al. (2004) for solvable partnership problems with linear preferences. Theorem 3.3.1 establishes that from any unstable partnership an improving path can be constructed towards a stable partnership. This implies that the decentralized matching process described in Roth and Vande Vate (1990) converges to a stable partnership with probability one. The improving path towards stability was constructed on top of Fleiner (2010)'s algorithm that produces a stable partnership if one exists. This is an intuitively appealing – and maybe not so surprising – result as it uncovers the close relation between an improving path towards stability in a decentralized matching process and the

[12] For a further discussion on this, we refer to the concluding remarks.

centralized algorithm used to find a stable half-partnership. Starting from the same partnership, the decentralized matching process will follow essentially the same route, albeit possibly with detours, as the central planner to get to a stable partnership.

We have focused on partnership problems that are solvable. Intuitively, in a decentralized matching context, when talking about *paths to stability* these are the most interesting problems. For problems that do not have a stable partnership, the concept of a path to stability seems less clear and ill-defined. In the absence of a stable partnership, one can ask what alternative solution concept should be used. We have extended the result from Iñarra et al. (2008) and Biro and Norman (2012) and have shown that there exists an improving path towards an instance set of a stable half-partnership, a set of partnerships associated to one particular stable half-partnership. This result becomes only meaningful when we assume a different decentralized matching process is in order, one where agents, when selected, can approach the other agent involved in their most preferred blocking edge. In this situation, the instance set is never left by an improving path once reached and the concept of a path to (almost) stability becomes meaningful. However, when we resort to the traditional decentralized matching process from Roth and Vande Vate (1990), it is possible that an improving path exists towards a partnership not in the instance set.

An alternative solution concept is the absorbing set. Following the definition of Iñarra et al. (2010), an absorbing set is a set of partnerships such that (i) there exists an improving path between any two partnerships in this set and (ii) there is no improving path from a partnership in the set to a partnership outside the set. Iñarra et al. (2010) and Klaus et al. (2010) showed that a close connection exists between stable half-partnerships and absorbing sets, with the latter being characterized in terms of the former, and it might be interesting to see whether Iñarra et al. (2010)'s result also translates to the partnership problem. Related to this question, Theorem 3.4.1 does not shed light on which particular instance set can be reached from a give unstable partnership – or more particular, which stable partnership can be reached when we are dealing with solvable partnership problems. Further research should try to tackle both questions: (1) Do Iñarra et al. (2010)'s results also hold for the partnership problem and (2) can we say something about which instance set can be reached from a given partnership by means of an improving path?

One final remark pertains to the restriction of linearity imposed on the partnership problems discussed in this paper. Assuming LIN greatly simplified the analysis. Stepping away from the assumption of linearity, the partnership problem allows for a much wider range of possible preference structures. The question whether the result presented here extends towards non-linear partnership problems is important in its own right. However, investigating convergence towards stability for the partnership problem is also interesting for an additional reason. Relaxing either SUB or INCR leads to a matching problem for which examples can be constructed in which our main result does not hold. Hence, if the result should hold for the partnership problem, this would be the most general matching problem in which convergence is guaranteed.

3.6 Appendix

3.6.1 Proof of Theorem 3.3.1.

Proof of Claim 1 Consider the set $Q_M = M \backslash E'$. Pick the edge $uv \in Q_M$ such that uv was deleted in Phase 1 before any other edge of Q_M. Assume that the partnership problem right before deleting uv in Phase 1 is $\{(V,T),C\}$ with $E \subseteq T \subset E'$. There exists an improving path towards a partnership such that uv is not in this partnership: First, uv is deleted in Phase 1, indicating that $uv \in \overline{C}_v(C_v^-(T) \cup uv)$. MON then implies that

$$uv \in \overline{C}_v(M(v) \cup C_v^-(T)). \tag{3.4}$$

By INCR, it follows that there exists $vw \in C_v^-(T) \backslash M(v)$ such that $vw \in C_v(M(v) \cup C_v^-(T))$, as otherwise the number of edges in the choice set of $M(v) \cup C_v^-(T)$ is less than the number of edges in the choice set of $M(v)$. By SUB,

$$vw \in C_v(M(v) \cup vw). \tag{3.5}$$

Now, as uv was deleted in Phase 1 before all other elements in Q_M and because $vw \in T$, $M(w) \cup vw \subseteq T$. By $vw \in C_v^-(T)$ and SUB, this means then that

$$vw \in C_w(M(w) \cup vw). \tag{3.6}$$

Statements (3.5) and (3.6) imply that vw is a blocking edge for M. Satisfying vw leads to a new partnership, say M_1. If $uv \in M_1$, the above argument can be repeated and, again, we know that a blocking pair $vx \in C_v^-(T)$ exists. By (3.4), it is clear that at some point, say after satisfying h blocking edges, uv will be deleted, i.e. $uv \notin M_h$.

Note that the blocking edges on the improving path towards M_h were all in T and were not equal to uv. So the new edges of M_h, in $M_h \backslash M$, are either in E' or they were deleted after uv in Phase 1. Hence, if we again define $Q_{M_h} = M_h \backslash E'$ and look for the first edge in Q_{M_h} that was deleted in phase 1, say yz, then yz was deleted *after* uv was deleted in Phase 1. Apart from that, everything is the same such that the above argument can also be applied here: there exists an improving path towards a partnership $M_{h'}$ such that $yz \notin M_{h'}$.

Repeating this argument, the newly selected edges will be deleted at later rounds of Phase 1. Since the number of deleted edges in Phase 1 is finite, we are bound to end up with a partnership M' where $Q_{M'} = \emptyset$, proving the claim.

Before giving the proof of Claim 2, we present a result from Fleiner (2010) which will be used in the proof. It states that if you have an $E'(v)$-replacement, vw, of some edge uv, then exactly one edge in the inverted choice set will be dominated by the union of the inverted choice set with vw.

Lemma 3.6.1 (Fleiner (2010, Lemma 3.4.)) *For a reduced partnership problem $\mathcal{P}'$, if vw is the $E'(v)$-replacement of uv, then $\overline{C}_w(C_w^-(E') \cup vw)$ contains exactly one edge, say wx. Moreover, $xw \in C_w^-(E')$.*

Proof of Claim 2 We make two assumptions. First, we assume that M is such that

$$\{uv \in E' | uv \in C_u(E'(u)) \cap C_v(E'(v))\} \subset M, \tag{3.7}$$

that is, M contains all edges uv that are in the choice set of both u and v when $E'(u)$, resp. $E'(v)$ is the option set. These edges are always blocking if they are not part of a partnership and are never deleted when satisfying a blocking edge; as such they are not relevant for the proof. Second, if $M \cap Q_\pi = \emptyset$ for some $\pi \in \Pi^e$, the claim holds. Hence, for the remainder of the proof, assume that

$$M \cap Q_\pi \neq \emptyset \tag{3.8}$$

for any $\pi \in \Pi^e$.

Define an edge-pair (uv, vw) as *empty* in M if $uv \notin M$. Note that by the definition of an edge-pair, $uv \in C_v(E'(v)) \backslash C_u(E'(u))$. As E' contains rotations, E' is not individually rational, implying that such an edge-pair will exist. Start from an empty edge-pair, (uv, vw). By Lemma 3.6.1 we know that this edge-pair leads to another edge-pair (wx, xy). The same argument goes for (wx, xy) such that there exists an infinite sequence of edge-pairs leading to other edge-pairs, indicating that we are bound to end up in a loop. That loop is a rotation, say $\pi = (a_1, (a_1)^r, \ldots, a_m, (a_m)^r)$. Hence, we get a sequence of edge-pairs τ that starts with the empty edge-pair (uv, vw) and ends with π:

$$\tau = (a'_1 = uv, (a'_1)^r = vw, \ldots, a'_k, (a'_k)^r, \pi).$$

Denote by $n_\pi(M)$ the number of empty edge-pairs in the rotation π for a partnership M. We will show that for any M, it is possible to construct an improving path towards a partnership M' such that $n_\pi(M) < n_\pi(M')$. As the number of empty edge-pairs in the rotation is bounded from above by m, the number of edge-pairs in the rotation, this implies that there exists an improving path towards a partnership M'' such that $n_\pi(M'') = m$. In this case, $M'' \cap Q_\pi = \emptyset$, proving the claim.

First, assume $n_\pi(M) = 0$; there are no empty edge-pairs in π. Find the empty edge-pair, (uv, vw), that is closest to π in the sequence τ; any edge-pair after (uv, vw) in τ is not empty. Assume (uv, vw) leads to (wx, xy). Our immediate objective is to construct an improving path towards a partnership that does not contain wx, thereby reaching a partnership that has an empty edge-pair closer to π in τ. Now, if $vw \notin M$, then it is a blocking edge for M: As $uv \notin M$ and as vw is the $E'(v)$-replacement of uv, $vw \in C_v(M(v) \cup vw)$. If vw is not a blocking edge, then $vw \in \overline{C}_w(M(w) \cup vw)$. By INCR, this implies that $wx \in C_w(M(w) \cup vw)$, which is a violation of LIN, as $wx \in \overline{C}_w(C_w^-(E') \cup vw)$ while $vw \in C_w(C_w^-(E') \cup vw)$. Hence, vw is a blocking edge for M. If $vw \notin M$, satisfy vw to reach a partnership M'. If $vw \in M$, set $M' = M$. Now, there are two possible situations:

A. <u>$wx \in \overline{C}_w(M(w) \cup vw)$</u>: This implies that by satisfying vw, wx will be deleted, i.e. $wx \notin M'$, reaching our immediate objective.

B. $wx \in C_w(M(w) \cup vw)$: In this case, $wx \in M'$. Consider the following set:

$$B = C_w(M'(w) \cup C_w^-(E')) \backslash M'; \tag{3.9}$$

the edges in the inverted choice set of w that are in the choice set of the union of $M'(w)$ with this inverted choice set. An edge $wy \in B$ will be a blocking edge of M', by SUB and since $wy \in C_w^-(E')$. Moreover, by the same argument, after satisfying any edge in B, all remaining edges in B are still blocking in the new partnership. Note that it might be possible that $uv \in B$, if $u = w$. We satisfy all blocking edges in B, except uv, reaching M''. If $uv \notin B$, then $B \subset M''$. If $vw \notin M''$, then, as $uv \notin M''$, it was deleted by w. Hence, $vw \in \overline{C}_w(M'(w) \cup C_w^-(E'))$, which by LIN implies that $wx \in \overline{C}_w(M'(w) \cup C_w^-(E'))$ and $wx \notin M''$. If $vw \in M''$, then by

$$wx \in \overline{C}_w(C_w^-(E') \cup vw) \tag{3.10}$$

and MON, $wx \notin M''$. Now, assume $uv \in B$, then it might be that $\{vw, wx\} \subset M''$. However, satisfying uv deletes wx, by (3.10) and MON, such that we reach a partnership that does not contain wx.[13]

Hence, there always exists an improving path towards a partnership that has an empty edge-pair closer to π in the sequence τ. Repeating this argument, there exists an improving path towards a partnership that has an empty edge-pair in π, i.e. a partnership M such that $n_\pi(M) > 0$. Next, we will show that there exists an improving path towards M' such that $n_\pi(M') > n_\pi(M)$. By (3.8), $n_\pi(M) < m$ and there exists at least one non-empty edge-pair (wx, xy) that succeeds an empty edge-pair (uv, vw) in π. If situation A applies, then vw blocks wx and satisfying vw leads to a partnership M' such that $n_\pi(M') = n_\pi(M) + 1$. Now, consider situation B applies with, again, the set B as defined in (3.9). Note that B cannot contain edges that are in Q_π. To see this, consider the following edge-pairs in π: (z_1z_2, z_2w) leads to (wz, zz_3). This implies that $\overline{C}_w(C_w^-(E') \cup z_2w) = wz$. However, (3.10) also holds, such that we get a situation where LIN is

[13] It is important to satisfy uv as the last one of the edges in B, as it might otherwise be that by satisfying uv, v deletes vw before wx is deleted.

violated as there exists a partnership, $C_w^-(E') \cup z_2 w$ where wx is in the choice set while wz is not, and a partnership, $C_w^-(E') \cup vw$ where wz is in the partnership while wx is not. Hence, $B \cap Q_\pi = \emptyset$ and in the process of satisfying edges in B, no edge in Q_π is ever satisfied. Hence, we reach a partnership M' such that $n_\pi(M') > n_\pi(M)$.

3.6.2 Proof of Theorem 3.4.1.

The proof consists of two parts. First, we prove that Claims 1 and 2 also apply to the case of non-solvable partnership problems. This shows that, from any partnership M, there exists an improving path towards a partnership M' such that $M \subseteq S$ for some stable half-partnership $(S; S_1, \ldots, S_k)$. Second, we prove that there exists an improving path from M' to an instance of $(S; S_1, \ldots, S_k)$.

As for Claim 1, it is clear that it also applies to the case of non-solvable partnership problems. Claim 2 might not directly apply here because, as you might recall from the proof of Claim 2 for solvable partnership problems, it hinged on the existence of an empty edge-pair that leads towards an even rotation. From the argument in the proof, it is clear that an empty edge-pair still exists for any reduced partnership problem. However, it may be that all empty edge-pairs lead to an odd rotation. Hence, we have to show that in a reduced partnership problem, there always exists an empty edge-pair in M, that does not lead to an odd rotation.

The key to proving the previous statement is to prove the following lemma which states that the incident edge set $E'(v)$ for an agent v that is incident to an edge in an odd rotation, can be partitioned in a set of incident edges that are in the choice set of both agents involved and in a set of exactly two edges that are in that particular odd rotation.

Lemma 3.6.2 *Consider an odd rotation π in $\mathcal{P}'$. If (uv, vw) is an edge-pair in π, then $E'(v) \setminus \{C_v(E'(v)) \cap C_v^-(E')\} = \{uv, vw\}$.*

Proof Consider the following sequences of edge-pairs in the odd rotation π,

$$(az, zu, uv, vw, wx, xy)$$

Then, as the set of first elements of the edge-pairs in π equals the set of second elements in π, there also exists a sequence in π, (zu, uv, vw, wx). As such, vw is both the $E'(v)$-replacement of uv and $vw = \overline{C}_v(C_v^-(E') \cup uv)$. Now, assume that there exists $vb \in E'(v)$ such that $vb \in \overline{C}_v(E'(v))$ and $vb \neq vw$. Then,

$$vb \in \overline{C}_v(E'(v) \backslash uv). \tag{3.11}$$

In addition, since $\mathcal{P}'$ is a reduced partnership, $vb \in C_v(C_v^-(E') \cup vb)$. By INCR and LIN, this implies that $vw \in \overline{C}_v(C_v^-(E') \cup vb)$, violating LIN by (3.11). Hence, no such vb can exist, and $E'(v) \backslash C_v(E'(v)) = vw$. In addition, Fleiner (2010, Lemma 3.2.) states that $|C_v(E'(v))| = |C_v^-(E')|$, which implies that $C_v(E'(v)) \backslash C_v^-(E') = uv$, proving the lemma.

This lemma has two important corollaries: First, there does not exist an agent v such that v is both incident to an edge in an edge-pair not in an odd rotation and to an edge in an edge-pair in an odd rotation. As such, no edge-pair that is not in an odd rotation leads to an edge-pair that is in an odd rotation. Second, for an agent v that is incident to an edge in an edge-pair (uv, vw) in an odd rotation, as $C_v(E'(v)) \cap C_v^-(E') \subset M$, either $uv \in M$ or $vw \in M$ but never both.

Now, by the first corollary, if there does not exist an empty edge-pair in M that is not in an odd rotation, then either individual rationality is violated or we have already reached a stable half-partnership. Hence, we prove that if M is individually rational, then E' is the edge set of a stable half-partnership. Define V' as the set of agents that are not incident to any edge of an odd rotation and note that, for an agent $v \in V'$, $C_v(E'(v)) \subseteq M(v)$, because otherwise there exists an empty edge-pair outside an odd rotation. If $C_v(E'(v)) \subset M(v)$, then individual rationality is violated, hence $C_v(E'(v)) = M(v)$. However, this holds for all agents $v \in V'$, such that $C_v(E'(v)) \cap C_v^-(E') = C_v(E'(v))$ for all v and no edge-pair exists outside an odd rotation. If this is the case, then we have reached a stable half-partnership. Edges uv such that $uv \in C_v(E'(v)) \cap C_u(E'(u))$ are then singleton elements of the partition of E'. The elements in an odd rotation, form a non-singleton element of the partition. Hence, if we assume that we have not reached a stable half-partnership, then there exists an empty edge-pair that is not in an odd rotation.

Now, by claims 1 and 2, from any partnership there is an improving path towards a partnership M

such that the problem $\mathcal{P}$ has a stable half-partnership $(S;S_1,\dots,S_k)$ with $M \subseteq S$ and with $S_i \subseteq M$ if $|S_i| = 1$. Now, consider $\mathcal{S}$, the set of all elements of the partition of S, $(S_1,\dots,S_k)$, such that $S_i \in \mathcal{S}$ if $|S_i| > 1$. We will call such an element *ready* in a partnership M, if for the relevant ordering $(v_m v_1, v_1 v_2, \dots, v_{m-1} v_m)$,

- if $v_{j-1}v_j \in M$, then $v_j v_{j+1} \notin M$,
- $|S_i \cap M| = \frac{m}{2} - 1$.

Since every S_i such that $|S_i| > 1$ will be an odd rotation and since $C_v(S(v)) \cap C_v^-(S) \subset M(v)$ for all $v \in V$, the second corollary of lemma 3.6.2 implies that the definition of a ready element S_i can be simplified to: an element such that $|S_i \cap M| = \frac{m}{2} - 1$. If all elements $S_i \in \mathcal{S}$ are ready then M is an instance of $(S;S_1,\dots,S_k)$ and we are done with the proof. Assume that this is not the case. Pick a non-ready element $S_i \in \mathcal{S}$ such that $|S_i \cap M| = n$. Then, we can show that there exists an improving path towards a partnership M' such that $|S_i \cap M'| > n$.

As S_i is not ready in M, $n < \frac{m}{2} - 1$ with m the number of unique edges. Hence, there exists an edge-pair (uv, vw) such that both uv and vw are not in M. Hence, vw is a blocking edge for M. Assuming (uv, vw) leads to (wx, xy), there are two possibilities. First, if $wx \notin M$, then satisfying vw leads to a partnership $M' = M \cup vw$ and $|S_i \cap M'| > n$. Second, if $wx \in M$, then satisfying vw deletes wx by the second corollary of lemma 3.6.2. Hence, we reach a partnership $M' = M \backslash wx \cup vw$ such that $|S_i \cap M'| = n$. However, if $wx \in M$, then $xy \notin M$ and we can repeat the argument for M'. As $|S_i \cap M'| < \frac{m}{2} - 1$, we are eventually bound to end up in a partnership M'' that was reached by satisfying an edge that did not delete another edge: $|S_i \cap M''| > n$.

On this improving path towards M'', we only satisfied edges in S_i, thereby possibly deleting other edges of S_i. Hence, for all other elements $S_j \in \mathcal{S}$, $|S_j \cap M| = |S_j \cap M''|$. Repeating this argument, there exists an improving path towards a partnership, say M', such that $|S_i \cap M'| = \frac{m}{2} - 1$ and S_i is ready for M'. The set of ready elements has increased and repeating the process, we can construct an improving path such that all elements of $\mathcal{S}$ are ready.

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4

The structure of the partnership problem

4.1 Introduction

Building on the work of Gusfield (1988) and Borbelova and Cechlarova (2010), we study the partnership problem, a very general type of matching problem that encompasses most studied matching problems. We explore the structure of stable matchings for these problems, uncovering their close connection with the concept of a rotation for the partnership problem.

A matching problem consists of a set of agents that wishes to establish some meaningful relationships among each other, based on the preferences that they might have over other agents or subsets of other agents. Within matching theory, a great deal of attention is given to the study of stable matchings, where a stable matching is a matching in which nobody wishes to drop a relationship and in which no pair of agents wishes to establish a relationship.

Since 1962, the publication year of the seminal Gale and Shapley (1962) paper, many different matching problems have been defined and analysed. Types of matching problems are mostly defined by two properties. First, there is the distinction between two-sided and one-sided matching problems. In two-sided matching problems the set of agents is partitioned into two subsets. Agents do not wish to establish a match with other agents of the same subset. In one-sided matching problems, no such restriction is imposed. This distinction, one-sidedness versus two-sidedness, draws the boundary between problems for which a stable matching always exists (two-sided problems) and problems for which this might not be the case (one-sided problems).[1] Second, agents may prefer to establish only one relationship – one-to-one – or they may prefer to establish multiple relationships – many-to-many.[2]

As such, a whole spectrum of problems can be defined. The most restrictive and most popular such problem is the stable marriage problem (Gale and Shapley, 1962) with a group of men and a group of women wishing to find a mate of the other sex. Here, a stable matching will always exist and it is relatively straightforward to find all stable matchings for any given problem. Dropping the property of two-sidedness, we find the stable roommate problem – also by Gale and Shapley (1962) – where any student in a group of n students wishes to find a suitable roommate for $\frac{n}{2}$ available dorm rooms. The roommate problem is already more complex in that a stable matching may not exist and in that the stable marriage algorithm does not always work for the stable roommate problem. Irving (1985) proposed an extended algorithm that decided on the existence of a stable matching and produced one in the case of existence. Irving's algorithm revolved heavily around the notion of so-called even rotations, which can be more or less considered as preference cycles involving an even number of relationships. Later, Gusfield (1988) showed that Irving's algorithm is able to find all stable matchings and presented an efficient enumeration method, showing a close connection between the set of stable matchings and the structure of the set of rotations in the process.

[1]Recently, Ostrovsky (2008) extended the two-sided matching framework to a supply chain context with three subsets – producers, intermediaries and consumers.

[2]For two-sided matching problems, a hybrid class of problems exists where one side prefers to establish multiple relationships while the other side only prefers to form one – many-to-one.

When we no longer assume that agents do not wish to establish multiple relationships, an additional difficulty becomes the modeling of the underlying preferences. In one-to-one matching problems, it suffices to impose a strict preference ordering,[3] as agents can only have preferences over *individual agents*. For many-to-many matching problems, we must model the preference of an agent for one subset of agents over another subset of agents. The most straightforward way of modeling preferences, is to stick to the strict preference ordering assuming that agents have a capacity b and that agents wish to establish a relationship with the b first agents in their ordering. This is the approach taken by Gale and Shapley (1962) for the two-sided case, dubbed the college-admission problem (which is essentially a many-to-one problem) and by Cechlarova and Fleiner (2003) for the one-sided case, later dubbed the stable b-matching problem. For the stable b-matching problem, Borbelova and Cechlarova (2010) — following Gusfield (1988) for the stable roommate problem — further explored the structure of rotations and extended the results on the connection between sets of stable matchings and sets of rotations to the case of stable b-matching problems.

This paper extends these results from Gusfield (1988) and Borbelova and Cechlarova (2010), exploring the structure of stable matchings and uncovering their connection with sets of rotations for *the partnership problem*, a many-to-many one-sided matching problem, first presented in Fleiner (2010), with less restrictive assumptions on the preferences. In the partnership problem, the assumption of linear preferences is dropped. Instead, preferences are modeled through choice functions; functions that map an option set – a set of possible relationships – to a choice set – a set of chosen relationships. These choice functions are assumed to satisfy substitutability and increasingness. Substitutability means that if an agent, when confronted with an option set A of agents, chooses a relationship with agent v, then that agent will also choose that relationship with v when confronted with a subset of A. Increasingness (also known as the law of aggregate demand) states that if k relationships are chosen from a set, then in a superset of that set at least k relationships are chosen; if more options are available, at least as many options will be chosen. Though more complex than the traditional matching problems, Fleiner (2010) showed that Irv-

[3] Not assuming strictness poses a whole different set of problems (see Irving and Manlove (2002)).

ing's algorithm could be extended to find a stable partnership, if one exists.[4] As the partnership problem is a one-sided matching problem, a stable partnership might not exist. However, in this paper, as we are specifically studying the set of stable partnerships, only problems having a stable partnership are discussed.

The aim of this paper is to build further on the work of Fleiner (2010) and show that the results presented in Gusfield (1988) for the stable roommate problem and Borbelova and Cechlarova (2010) for the stable b-matching problem can be extended to the partnership problem. In particular, there are three main results which show that stable partnerships share a common structure on three levels. First, it is shown that, for any given problem, all possible stable partnerships can be found by the algorithm presented in Fleiner (2010). Fleiner (2010)'s algorithm, like Irving (1985)'s algorithm, revolves around eliminating even rotations. Hence, the first result implicitly uncovers a relationship between a stable partnership and the set of rotations that are eliminated on a run of the algorithm producing that particular partnership. The second result shows that there is a one-to-one correspondence between the stable partnership and a set of rotations. That is, if a run of the algorithm eliminates a set of rotations ϕ, then this run can only produce one particular stable partnership and that stable partnership can only be produced by a run of the algorithm that eliminates that set of rotations ϕ. Related to that, it is shown that these sets of rotations are in itself similar in the sense that if a rotation π is in one set but not in another, then this other set has a rotation π' that is very similar to that π. This second set of results then uncovers the structure of a stable partnership on the relationship level – our third result. As an extension of a result from Borbelova and Cechlarova (2010) and due to our second set of results, we can identify relationships that will be part of all stable partnerships, as well as relationships that belong to some stable partnership.

These three sets of results are important in their own right, because they tell us a great deal about stable partnerships, showing that they are structurally the same as stable matchings in a more simple stable roommate context. In addition, extending the original results is not a trivial exercise and by proving these results for the partnership problem, other findings from Gusfield

[4]Fleiner (2010) devised the algorithm to show that, though a stable partnership might not exist, a stable half-partnership will always exist, thus generalizing the findings of Tan (1991) for the stable roommate problem.

(1988) and Borbelova and Cechlarova (2010) can be easily extended as well.

The structure of the paper is as follows. The partnership problem and the extension of Irving's algorithm, here dubbed `SP`, are presented in Section 4.2. The main results are presented in Section 4.3: first it is shown that any stable partnership can be found by a run of `SP` (Theorem 4.3.1). Then, some lemmata are presented which are used to prove Theorems 4.3.5 and 4.3.6, which pertain to the connection between stable partnerships and rotations. The section concludes with the results on relationships in a stable partnership (Theorem 4.3.7). Section 4.4 has some concluding remarks.

4.2 The model.

A partnership problem is a one-sided matching problem in which multiple relationships – even between the same pair of agents – are allowed. In this section, we define the partnership problem and the concept of stability. In Section 4.3, we will present some results on stable partnerships. As their structure is closely related to the concept of rotations, which form the backbone of Fleiner (2010)'s algorithm to find a stable partnership, this section will also explain the algorithm.

4.2.1 The partnership problem.

This paper studies the partnership problem as defined in Fleiner (2010), a one-sided matching problem that allows for multiple relationships for an agent. Let V be a set of agents. Agents in V wish to establish bilateral relationships with other agents in V, given certain preferences. A relationship between an agent u and an agent v will be defined by an edge $e = uv$.[5] The set E contains all *potential edges*, where an edge uv is potential if there exists a partnership where both u and v are willing to form this edge. A partnership M is then a subset of the set of potential edges E, $M \subseteq E$. Further, for a subset X of E and vertex v in V let us denote by $X(v)$ the set of edges that are incident with v, the set of relationships in which v is involved. Finally, note that we are not restricting the model to simple graphs; it may be that edges e_1 and e_2 in E involve the

[5] Note that $uv = \{u,v\} = \{v,u\}$ but for notational simplicity, we write it as uv.

same pair of agents.

The preferences of every agent v are given by a choice function $C_v : 2^{E(v)} \to 2^{E(v)}$ that maps any subset $X(v)$ of edges incident with v – the option set – to a subset of $X(v)$ that v chooses from $X(v)$ – the choice set. The edges in $X(v)$ that are not chosen will be denoted by $\overline{C}_v(X(v)) = X(v) \backslash C_v(X(v))$. The *inverted choice set* for an agent v is the set of edges incident to v that are in the choice set of others for a subset X of E. Formally,

$$C_v^-(X) = \{uv \in X(v) | uv \in C_u(X(u))\}.$$

For an edge $e \in C_v(X(v))$, the $X(v)$-*replacement* of e is the set $C_v(X(v)\backslash e)\backslash C_v(X(v)$; if an option e is in the choice set of an option set $X(v)$ incident to v and is deleted from that option set, then the new elements in the choice set without e will be the $X(v)$-replacements of e; they *replace* e in the choice set. With this notation, every matching problem can be represented by a finite graph (V,E) with $V = \{1,\ldots,|V|\}$ the set of agents (vertices) and E the set of potential matches (edges) and a set of choice functions $C = \big(C_1(.),\ldots,C_{|V|}(.)\big)$. We write $\{(V,E),C\}$.

A partnership problem is a matching problem with two defining characteristics. First, there are no restrictions on the set of potential relationships, i.e. any number of relationships is possible between any pair of agents. Second, the preferences over $E(v)$ for each agent v are modeled by the choice function C_v which is assumed to satisfy *substitutability* (SUB) and *increasingness* (INCR). Substitutability means that when a relationship e is chosen from an option set $X(v)$, then v will still choose option e when some other options are no longer available in $X(v)$. In other words, if e is in the choice set of an option set $X(v)$, then e is in the choice set of any option set that is a subset of $X(v)$.

SUB A choice function, C_v, is *substitutable* if, for $e \neq f$ and $e \in C_v(X(v))$, then $e \in C_v(X(v)\backslash f)$.

Fleiner (2010, Theorem 2.1) finds that a choice function C_v on a finite groundset is substitutable if and only if $\overline{C}_v$ is monotonous; if an option e is not chosen from an option set $Y(v)$, then it will not be chosen either when the option set is expanded:

MON The function of non-chosen options $\overline{C}_v$ is *monotonous* if $Y(v) \subseteq X(v)$ implies that $\overline{C_v}(Y(v)) \subseteq \overline{C_v}(X(v))$.

Another corollary of SUB is that if an edge $uv \in E$ — that is, uv is a potential edge — then $uv \in C_u(uv) \cap C_v(uv)$; if an edge is potential, then it is always chosen by both agents involved, when no other options are available.

Increasingness means that if extra options are added to an option set $Y(v)$, then v chooses at least as many options in the expanded option set as in $Y(v)$; if more options are available, then more options will be chosen. Formally,

INCR A choice function, C_v, is *increasing* if $Y \subseteq X$ implies $|C_v(Y)| \leq |C_v(X)|$.

INCR, together with SUB, guarantees that the $X(v)$-replacement of an edge $e \in C_v(X(v))$ will be at most one element.

A partnership problem can be defined formally as follows:

$\mathcal{P}$ A partnership problem $\mathcal{P}$ is a matching problem $\{(V,E),C\}$ such that any $C_v \in C = (C_1,\ldots,C_{|V|})$ satisfies SUB and INCR.

Stability For a partnership problem $\mathcal{P} = \{(V,E),C\}$, a partnership M is *individually rational* if no agent $v \in V$ has an edge $uv \in M$ such that $uv \in \overline{C}_v(M(v))$; no agent wishes to cut a relationship it has in M. In addition, a *blocking edge uv* of M is such that $uv \notin M$ and $uv \in C_u(M(u) \cup uv) \cap C_v(M(v) \cup uv)$; both u and v wish to form uv in M. A partnership that is individually rational and has no blocking edge is denoted a *stable partnership*. Formally,

Stable partnerships A partnership S is *stable* for the problem $\mathcal{P} = \{(V,E),C\}$ if S satisfies:

- **Individual rationality:** for any agent v we have $C_v(S(v)) = S(v)$, and
- **Absence of a blocking edge:** There does not exist an edge $e = uv \notin S$ such that both $e \in C_u(S(u) \cup \{e\})$ and $e \in C_v(S(v) \cup \{e\})$ holds.

A partnership problem that has at least one stable partnership will be denoted solvable. We will only discuss solvable partnership problems.

4.2.2 Finding a stable partnership.

Fleiner (2010) constructed an algorithm which was proven to produce a stable partnership for a solvable partnership problem. The algorithm is an extension of Irving's algorithm (Irving, 1985), which also forms the basis for Gusfield (1988).[6]

Phase 1 Consider an edge $uv \in E$ such that $uv \in \overline{C}_v(C_v^-(E) \cup uv)$; we say that uv is *dominated* by the inverted choice set of v for E. Edges that are dominated by the inverted choice set for a problem $\{(V,E),C\}$ can never be part of a stable partnership as their presence in a partnership implies the presence of a blocking edge in the inverted choice set for that problem. Phase 1 of the algorithm aims to reduce the partnership problem to a partnership problem that has no edges dominated by the inverted choice set, a *reduced* partnership problem.

Reduced partnership problem A partnership problem $\mathcal{P} = \{(V,E),C\}$ is *reduced* if for all $uv \in E$, $uv \in C_v(C_v^-(E) \cup uv)$.

The reduced version of a partnership problem $\mathcal{P} = \{(V,E),C\}$, reached by applying Phase 1, will be denoted by $\mathcal{P}' = \{(V,E'),C\}$. Formally, Phase 1 entails an iteration of the following step.

Phase 1 While there exists $uv \in \overline{C}_v(C_v^-(E) \cup uv)$ for some v, set $E = E \backslash uv$.

Edges that are deleted in Phase 1 because they are dominated by an inverted choice set of T with $E' \subset T \subseteq E$ at some round of Phase 1, will also be dominated by an inverted choice set of E' (Property 1). In addition, edges dominated by an inverted choice set can never be part of a stable partnership; if a dominated edge uv is in a partnership, then there exists a blocking edge

[6] Two remarks. First, the algorithm consists of two phases. However, as Fleiner (2010, Section 4.) points out, there is a more efficient one-phase algorithm that is even more similar to Irving's algorithm. However, to keep the analysis and the proofs more transparent, we also use the two-phase algorithm presented in Fleiner (2010). Second, Fleiner (2010)'s algorithm is designed to find stable half-partnerships which is a more general concept than a stable partnership. Stable half-partnerships are especially interesting when we are dealing with partnership problems that are not solvable. However, for solvable partnership problems, the set of stable half-partnerships is equal to the set of stable partnerships and the algorithm produces stable partnerships in this case.

in the inverted choice set of u or v. Hence, Phase 1 weeds out those dominated edges as we can be sure that they are never part of a stable partnership. In addition, a result from Fleiner (2010, Lemma 3.1.) shows that applying Phase 1 does not lead to a reduced problem with new stable partnerships, i.e. partnerships that were not stable in the original problem but are stable in the reduced problem. Hence, the set of stable partnerships is left unchanged when going from the original problem to its reduced version (Property 2).

Property 1 *For a partnership problem $\mathcal{P} = \{(V,E),C\}$ and its reduced problem $\mathcal{P}' = \{(V,E'),C\}$, if $uv \in E \backslash E'$ then $uv \in \overline{C}_u(C_u^-(E') \cup uv)$ or $uv \in \overline{C}_v(C_v^-(E') \cup uv)$.*

Proof See appendix 4.5.1.

Property 2 (Lemma 3.1. in Fleiner (2010)) *For a partnership problem $\mathcal{P} = \{(V,E),C\}$ and its reduced problem $\mathcal{P}' = \{(V,E'),C\}$, S is a stable partnership of $\mathcal{P}$ if and only if S is a stable partnership of $\mathcal{P}'$.*

Phase 2. After applying Phase 1 for a partnership problem $\mathcal{P}$, we get a reduced problem $\mathcal{P}'$. For such a problem, we can define a *rotation*. First, define an *edge-pair* as an ordered set of two edges $\{a = uv, (a)^r = vw)$ such that $uv \in C_v(E'(v)) \backslash C_u(E'(u))$ and vw is the $E'(v)$-replacement of uv. That is, an edge that is in exactly one choice set and its replacement, form an edge-pair. Second, an edge-pair (uv, vw) *leads* to another edge-pair (wx, xz) if $wx = \overline{C}_w(C_w^-(E') \cup vw)$.[7] Now, Fleiner (2010, Lemma 3.3.) shows that any $uv \in C_v(E'(v)) \backslash C_u(E'(u))$ will be part of an edge-pair. In addition, Fleiner (2010, Lemma 3.4.) also finds that any edge-pair will lead to exactly one other edge-pair. Hence, there exists an infinite sequence of edge-pairs leading to other edge-pairs. However, as the set of edges in E' is not infinite, this sequence is bound to loop, and this loop will be a rotation. A *rotation* π is a sequence of edge-pairs

$$\pi = (a_1, (a_1)^r, a_2, (a_2)^r, \ldots, a_m, (a_m)^r)$$

[7] A small note on the definition of *leading*: it is possible for $uv \in C_v(E'(v))$ and vw, the $E'(v)$-replacement of uv, that $\overline{C}_v(C_v^-(E') \cup vw) \neq \emptyset$. However, in the definition of leading, we always look for the edge-pair such that the first element is in $\overline{C}_w(C_w^-(E') \cup vw)$.

such that $(a_i, (a_i)^r)$ leads to $(a_{i+1}, (a_{i+1})^r)$, for any $i = 1, \ldots, m$, *modulo m*, that is $(a_m, (a_m)^r)$ leads to $(a_1, (a_1)^r)$. A sequence of edge-pairs that leads to a rotation π but is not part of π is called the *tail* of π. Consider a sequence of edge-pairs

$$(b_1, (b_1)^r, \ldots, b_l, (b_l)^r, a_1, (a_1)^r, \ldots, a_m, (a_m)^r)$$

with $\pi = (a_1, (a_1)^r, \ldots, a_m, (a_m)^r)$ a rotation of $\mathcal{P}'$. Then, $\tau = (b_1, (b_1)^r, \ldots, b_l, (b_l)^r)$ is the tail of π.

Now, if E' is not a stable partnership, $\mathcal{P}'$ has at least one rotation. There are two types of rotations. First, the number of different edges in the rotation may be odd – we are dealing with an *odd* rotation. If a problem has an odd rotation, then a stable partnership does not exist. Second, the number of different edges may be even – we are dealing with an *even* rotation. As we are only considering solvable partnership problems, every rotation will be even (Fleiner, 2010, Corollary 2.7.). Phase 2 then prescribes selecting an (even) rotation out of the set of even rotations $\Pi^e_{\mathcal{P}'}$ for the reduced problem $\mathcal{P}'$ and deleting the first edge in every edge-pair of that rotation – *eliminating a rotation*. Note that for solvable problems $\Pi^e_{\mathcal{P}'} = \Pi_{\mathcal{P}'}$.

Phase 2 For a reduced problem $\mathcal{P}'$, pick a rotation $\pi \in \Pi^e_{\mathcal{P}'}$ and delete the first edge in every edge-pair of π.

Property 3, a combination of two lemmata from Fleiner (2010), gives some insights into why eliminating a rotation brings us closer to a stable partnership. It states that if there exists a stable partnership S such that it contains an edge that is deleted when eliminating π, then (i) S contains all deleted edges of π and (ii) there exists another stable partnership, S', that does not contain any of the deleted edges but contains all of the remaining edges of the rotation π. Hence, either no edge deleted when eliminating π is part of a stable partnership, or they are all part of a particular stable partnership, but then there exists another stable partnership which contains none of those edges; eliminating π brings us closer to a stable partnership.

Property 3 (Adapted from lemmata 3.5. and 3.6. in Fleiner (2010)) *Consider a reduced problem $\mathcal{P}'$ and a rotation $\pi = (a_1, (a_1)^r, \ldots, a_m, (a_m)^r) \in \Pi^e_{\mathcal{P}'}$. Assume S is a stable partnership. If $a_i \in S$ for some $i \leq m$,* *(i)* $a_j \in S$ *for all* $j \leq m$,

(ii) $S' = S \setminus \bigcup_{k \leq m} a_k \cup \bigcup_{k \leq m} (a_k)^r$ *is also a stable partnership.*

After having applied Phase 2, we may end up in a partnership problem that is not reduced. Hence, Phase 1 should be re-applied, and if an even rotation exists for the reduced problem, Phase 2 should also be re-applied. Running Phase 1 and Phase 2 consecutively, we are bound to end up in a partnership problem $\{(V,S),C\}$, that has no even rotations, which in the case of solvable partnership problems means that S itself is a stable partnership. The algorithm, dubbed `SP`, is summarized in figure 4.1.

i. Consider the problem $\mathcal{P} = \{(V,E),C\}$ and set $t = 0$ and $E_t = E$.

ii. Apply Phase 1 for the problem $\mathcal{P}_t = \{(V,E_t),C\}$, obtaining the reduced problem $\mathcal{P}'_t = \{(V,E'_t),C\}$.

iii. Compute the set of even rotations, $\Pi^e_{\mathcal{P}'_t}$.

   1. If $\Pi^e_{\mathcal{P}'_t} = \emptyset$, end `SP` and set $S = E'_t$.
   2. If $\Pi^e_{\mathcal{P}'_t} \neq \emptyset$, apply Phase 2 obtaining $\mathcal{P}_{t+1} = \{(V,E_{t+1}),C\}$. Set $t = t+1$ and go back to step ii.

Figure 4.1: `SP` algorithm

4.3 Main results.

For a given (reduced) partnership problem we can now explore the set of stable partnerships. In particular, we uncover three sets of characteristics of stable partnerships; three ways in which these partnerships are similar. First, all stable partnerships can be found by running `SP` a sufficient number of times. This result, given in Theorem 4.3.1, extends the results from Gusfield (1988, Theorem 2.1.) and Borbelova and Cechlarova (2010, Lemma 7) for the roommate problem and the stable b-matching problem respectively. Second, this observation, that all stable partnerships can be produced by the same algorithm, brings us to the second set of results and a second level of similarity. Each run can be characterized by the set of rotations that are deleted on that run (Lemma 4.3.3). Now, for some rotations there exists a counterpart – a dual rotation – that contains the same edges but at different positions of the rotation. Theorem 4.3.5 then states that if a rotation is not deleted on a run of SP, its dual counterpart will be deleted in that run. Hence, all sets of rotations are similar up to which rotation in a pair of dual rotations is deleted. A related result, Theorem 4.3.6, then shows that there is a one-to-one correspondence between the set of sets of rotations in a run and the set of stable partnerships, hence, the second level of similarity: stable partnerships are similar in that they can be linked to sets of rotations which are in itself similar. Third, this then connects to the third level of similarity and our third set of results, similarity over which edges belong to the stable partnership. There are edges that belong to all stable partnerships and edges that belong to some stable partnerships. Those edges that belong to all stable partnerships are not part of a rotation that has a dual, whereas those edges that belong to some stable partnership, but not to all, are always part of a rotation that has a dual (Theorem 4.3.7).

Additional notation. A run of SP, starting from a reduced partnership problem $\mathcal{P}' = \{(V,E'),C\}$, can be summarized by γ, a sequence of the rotations deleted in Phase 2 – as the selection of which rotation to be deleted determines the deletions in the subsequent Phase 1 of SP. All such runs are given in the set $\Gamma_{\mathcal{P}'}$. After running γ, we obtain the problem $\mathcal{P}'_\gamma = \{(V,E'_\gamma),C\}$ with E'_γ being a stable partnership. A *subrun* of γ of order k, $\gamma[k]$, contains the first k elements of γ. So $\mathcal{P}'_{\gamma[k]}$ is the reduced problem we obtain if we eliminate the first k rotations in γ, applying Phase 1 after each elimination.

Denote by $\phi(\gamma)$, the set of rotations that are eliminated when following the run γ; a rotation π is in $\phi(\gamma)$ if π is in the sequence γ. Then define Φ the set of all rotations that are ever eliminated at some run of SP. Formally,

$$\Phi = \bigcup_{\gamma\in\Gamma_{\mathcal{P}'}} \phi(\gamma).$$

If there exists a rotation $\pi \in \Phi$ such that there is a rotation $\rho \in \Phi$, that contains the same edges while $\rho \neq \pi$, then we say that π and ρ are duals and we write $\rho = \pi^d$ and $\pi = \rho^d$. Formally, the *dual* of a rotation $\pi = (a_1, (a_1)^r, \ldots, a_m, (a_m)^r)$ is a rotation such that $((a_{i-1})^r, a_i)$ is an edge-pair that leads to $((a_i)^r, a_{i+1})$, for every i modulo m. Hence, a dual of a rotation contains the same edges in the same order but the edge-pairs are different. For a set of rotations $\phi \subseteq \Phi$, we say that ϕ *covers* a rotation π, if $\pi \in \phi$ or $\pi^d \in \phi$.

Now, if $\pi \in \Pi_{\mathcal{P}'_{\gamma[k]}}$, we say that running $\gamma[k]$ suffices to *expose* π. Note that it may very well be that π was already exposed for some subrun $\gamma[k']$ with $k' < k$. We will see later on in Lemma 4.3.3 that the order in which rotations are eliminated is not relevant for the reduced problem that is obtained. Hence, the concept of *sufficient to expose* can also be applied to $\phi(\gamma[k])$. A rotation $\pi \in \Pi_{\mathcal{P}'}$ for some reduced problem $\mathcal{P}'$ *removes* another rotation $\pi' \in \Pi_{\mathcal{P}'}$, if deletion of π and subsequent application of Phase 1 leads to a reduced problem $\mathcal{P}''$ such that $\pi' \notin \Pi_{\mathcal{P}''}$.

In Gusfield (1988, Theorem 2.1.) and Borbelova and Cechlarova (2010, Lemma 7) it is shown that, for the roommate problem and the stable b-matching problem, Irving's algorithm can find all possible stable matchings for any given problem. Gusfield (1988) designed an efficient enumeration algorithm to find all stable matchings for a given problem. Theorem 4.3.1 extends this result and shows that the extension of Irving's algorithm, `SP`, also succeeds in finding all possible stable partnerships, provided it is run a sufficient number of times.

Theorem 4.3.1 [8] *For a partnership problem $\mathcal{P}$, if S is a stable partnership of $\mathcal{P}$, then there exists a run of* `SP` *that produces S.*

Proof A run of `SP` will be constructed such that no element in S is ever deleted, implying that the run produces S.

Consider round 0 and the partnership problem $\mathcal{P}_0 = \mathcal{P}$. By Property 2, S is not deleted from the set of stable partnerships by applying Phase 1 in round 0. Assume that for all $v \in V$, $S(v) \subseteq C_v(E'_0(v))$. We can show that in that case $S(v) = C_v(E'_0(v))$ for all $v \in V$: Assume

$$|S(v)| < |C_v(E'_0(v))| \tag{4.1}$$

[8] Adapted from Gusfield (1988, Theorem 2.1.).

and define $T = C_v^-(E_0')\backslash S(v)$, the elements of $C_v^-(E_0')$ not in $S(v)$. Since, by construction of T, for every $uv \in T$, $uv \in C_u(S(u) \cup uv)$, to preserve stability and by MON, it must be that $C_v(S(v) \cup T) = S(v)$. Then, by the inequality in (4.1), $|C_v(S(v) \cup T)| < |C_v(E_0'(v))|$. Now, Fleiner (2010, Lemma 3.2.) states that $|C_v(E_0'(v))| = |C_v^-(E_0')|$ such that

$$|C_v(S(v) \cup T)| < |C_v^-(E_0')| = |C_v(C_v^-(E_0'))|. \tag{4.2}$$

The equality holds since $\mathcal{P}'$ is a reduced problem and there are no edges dominated by the inverted choice set. As

$$C_v^-(E_0') \subseteq S(v) \cup T, \tag{4.3}$$

statement (4.2) implies a violation of INCR. Hence, $|S(v)| \geq |C_v(E_0'(v))|$.[9] Then, if $S(v) \subseteq C_v(E_0'(v))$, it follows that $S(v) = C_v(E_0'(v))$. If $S(v) = C_v(E_0'(v))$, there is no $uv \in S$ such that $uv \in \overline{C}_v(E_0'(v))$, this implies that $S(v) = C_v(E_0'(v)) = C_v^-(E_0')$ for all v. As $\mathcal{P}_0'$ is reduced, there is no uv such that $uv \in \overline{C}_v(C_v^-(E_0') \cup uv)$. Hence, it must be that $C_v(E_0') = E_0'$ and $S = E_0'$. Hence, the lemma holds as SP will directly terminate, producing S.

Now, assume that for some $v \in V$, $S(v) \nsubseteq C_v(E_0'(v))$. As, $|S(v)| = |C_v(E_0'(v))|$, there exists $uv \in C_v(E_0'(v))\backslash S(v)$. As S is stable, $uv \notin C_v^-(E_0')$, so there exists an edge pair $(uv, vw))$ that is either in a rotation $\pi \in \Pi_{\mathcal{P}_0'}$ or in a tail leading to a rotation π. If (uv, vw) is in π with Q_π all edges deleted when eliminating π, then, by Property 3(i), $Q_\pi \cap S = \emptyset$. Hence, eliminating π leads to a new problem $\mathcal{P}_1$ with S a stable partnership of $\mathcal{P}_1$.

Now, assume (uv, vw) is in some tail τ leading to a rotation π. If $Q_\pi \cap S \neq Q_\pi$, then we are in the previous case and there exists $(xy, yz) \in \pi$ such that $xy \notin S$. Hence, assume that $Q_\pi \cap S = Q_\pi$. Consider the sequence

$$(b_1 = uv, (b_1)^r = vw, \ldots, b_l = rx, (b_l)^r = xy, a_i = yz).$$

First, if $yz \in S$, then $xy \notin S$: Assume otherwise, $xy \in S$. Then, by MON, $yz \in \overline{C}_y(C_y^-(E_0') \cup S(y))$, while $yz \in C_y(S(y))$ by individual rationality of S. Hence, there exists a blocking edge in $C_y^-(E_0')$ for S, contradicting the stability of S such that it should be that $xy \notin S$. Second, if $xy \notin S$,

[9] By INCR and $|C_v(E_0'(v))| = |C_v^-(E_0')|$, it holds in fact that $|S(v)| = |C_v(E_0'(v))|$ for all $v \in V$ and all stable partnerships S in $\mathcal{P}'$.

then $rx \in S$: If $rx \notin S$, then $xy \in C_x(S(x) \cup xy)$. In addition, $yz \in \overline{C}_y(C_y^-(E'_0) \cup S(y) \cup xy)$ while $yz \in C_y(S(y))$. Hence, either $xy \in C_y(S(y) \cup xy)$ or there exist a blocking edge in $C_y^-(E'_0)$ for S. Both cases contradict the stability of S, such that rx must be in S.
These two arguments can be repeated such that we have a pattern where all first elements of edge-pairs in τ before (rx, xy) are in S while all second elements are not in S. However, $uv \notin S$, such that we have a contradiction. Hence, if $uv \notin S$, then $Q_\pi \cap S$ must be an empty set, implying that by eliminating π, we did not delete S as a stable partnership. Hence, it is possible, starting from $\mathcal{P}_0$, to execute Phases 1 and 2 such that it results in $\mathcal{P}_1$ for which S is still a stable partnership. This argument can be replicated for all subsequent rounds such that eventually we have a run of `SP` that produces S.

The crucial element in the proof of Theorem 4.3.1 is Property 3 which highlights the relation between stable partnerships and rotations. In words, Property 3 implies that if an edge that is a first element of an edge-pair in a tail-rotation structure is not in a stable partnership, then all the first elements of the edge-pairs in that tail-rotation structure are not in a stable partnership. As such an edge-pair always exists for any stable partnership, assuming a stable partnership is not yet reached by `SP`, this implies that we can always eliminate a rotation without eliminating that stable partnership. Hence, Theorem 4.3.1 implies that the stable partnerships are structurally identical, in the sense that they can all be produced by the same algorithm. The concept that links these stable partnerships is the set of rotations that are eliminated on the different runs of `SP`, which is similar over different runs (Theorem 4.3.5) and which can be mapped onto the set of stable partnerships (Theorem 4.3.6). Before presenting these results, we first present some intermediary results – lemmata – which, besides providing additional intuition into `SP`, are essential to prove the next theorems.

Lemmata on `SP`. The first result shows how choice sets are affected when executing Phase 1 and Phase 2, that is, it deals with the question how $C_v(E_{t+1})$ is different from $C_v(E_t)$. As it turns out, the change is limited to the edges that are part of the rotation that is eliminated in the Phase 2 of that round. If we assume that π was eliminated and that Q_π was the set of deleted edges, then, for v, all edges of $Q_\pi(v)$ that were in the choice set of v drop out of the choice set – quite logically, as they are deleted. In addition, the only new edges in the choice set of v are those second elements of the edge-pairs in π such that they are replacements for the first element of the edge-pair for agent v.

Lemma 4.3.2 *For a reduced problem $\mathcal{P}'_t$, assume that a rotation π is eliminated at round t in Phase 2, with Q_π the set of deleted edges in Phase 2. In addition, define a set $Q_\pi^{r\,v}$, such that $vw \in Q_\pi^{r\,v}$ if there exists an edge-pair $(uv, vw) \in \pi$. Then, for all $v \in V$,*

$$C_v(E'_{t+1}) = C_v(E'_t) \backslash Q_\pi(v) \cup Q_\pi^{r\,v}. \tag{4.4}$$

Proof See appendix 4.5.1.

The second lemma is useful in that it links a set of rotations, associated with a run of SP, to a particular stable partnership; a run γ in which a certain set of rotations $\phi(\gamma)$ is eliminated always leads to the same stable partnership, no matter in which order the rotations in $\phi(\gamma)$ are eliminated. In words, if, starting from the same reduced problem, we have two subruns that eliminate the same set of rotations, then the resulting reduced partnership problems are the same. Hence, if two runs have the same set of rotations, then these runs produce the same stable partnership.

Lemma 4.3.3 *Starting from a reduced problem $\mathcal{P}'$, take two runs of* SP*, γ and δ and subruns $\gamma[k]$ and $\delta[k]$ such that $\phi(\gamma[k]) = \phi(\delta[k])$. Then $\mathcal{P}'_{\gamma[k]} = \mathcal{P}'_{\delta[k]}$.*

Proof See appendix 4.5.1.

While Lemma 4.3.3 will eventually imply that stable partnerships and sets of rotations are linked, the next lemma, Lemma 4.3.4, will be helpful in proving that sets of rotations of different runs cover the same rotations (Theorem 4.3.5) by showing that if we have a subrun $\delta[l]$ that covers all rotations of another subrun $\gamma[k]$ except for a rotation π, then π will be exposed in the problem after running $\delta[l]$. For example if a run (π_1, π_2) exposes a rotation π, then (π_1, π_2^d) or (π_1^d, π_2) or (π_1^d, π_2^d) will also expose π.

Lemma 4.3.4 *Consider a reduced problem $\mathcal{P}'$ and two subruns of order k and l, $\gamma[k]$, and $\delta[l]$ such that any $\rho \in \phi(\gamma[k])$ is covered by $\phi(\delta[l])$. In addition, assume a rotation π that is not covered by $\delta[l]$. If $\pi \in \Pi_{\mathcal{P}'_{\gamma[k]}}$, then $\pi \in \Pi_{\mathcal{P}'_{\delta[l]}}$.*

Proof See appendix 4.5.1 for a formal proof.

Illustration of the proof. The lemma is proven using three sublemmata which are proven in appendix 4.5.1 and can be paraphrased as follows:

- **Lemma 4.5.1:** A rotation exposed in a problem can only be removed by eliminating its dual.
- **Lemma 4.5.2:** If both a rotation π and its dual π^d are exposed, then eliminating either π or π^d does not expose new rotations.
- **Lemma 4.5.3:** If a rotation π is exposed after a subrun δ in which the set of rotations is a subset of the set of rotations of another subrun δ, then π will also be exposed after δ.

Consider a reduced problem $\mathcal{P}'$ and a subrun $\gamma[2] = (\pi_1, \pi_2)$. In addition, there exists another subrun $\delta[3] = (\rho_1, \rho_2, \rho_3)$ such that $\pi_1 = \rho_3^d$ and $\pi_2 = \rho_2^d$, depicted as node (6) in Figure 4.2.

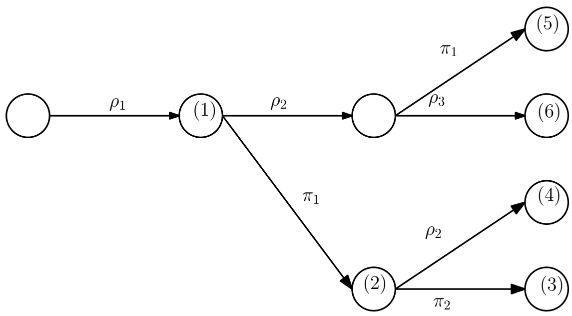

Figure 4.2: Illustration of the proof.

Assume there exist a rotation π such that $\pi \in \Pi_{\mathcal{P}'_{\gamma[2]}}$. The question now is whether $\pi \in \Pi_{\mathcal{P}'_{\delta[3]}}$. Now, there are two pairs of dual rotations in $\phi(\gamma[2]) \cup \phi(\delta[3])$, with $\{\pi_2, \rho_2\}$ being the first to pop up in $\delta[2]$.

(1) Follow the subrun $\delta[3]$ until we have to eliminate ρ_2, i.e. in $\mathcal{P}'_{\delta[1]}$. After this subrun, π_1 is exposed, by Lemma 4.5.1, together with subrun ρ_2.

(2) Eliminating π_1, we get the subrun (ρ_1, π_1). By Lemma 4.5.3, π_2 will be exposed, together with subrun ρ_2, by Lemma 4.5.1.

(3) Eliminating π_2 gives us a subrun (ρ_1, π_1, π_2), for which π is exposed, by Lemma 4.5.3. However, as both ρ_2 and its dual, π_2, are exposed after (ρ_1, π_1) (in node (2)), Lemma 4.5.2 tells us that no new rotation can be exposed after eliminating π_2, such that it has to be that π is also exposed after (ρ_1, π_1) (in node (2)).

(4) Eliminating ρ_2 we get a subrun (ρ_1, π_1, ρ_2) and a problem in which π is exposed by Lemma 4.5.1.

(5) There also exists a subrun (ρ_1, ρ_2, π_1) – by lemma 4.5.1 – and by lemma 4.5.3 π will also be exposed after that subrun.

(6) If we compare the run leading to node (5) with (ρ_1, ρ_2, ρ_3) we see that any rotation in the former is still covered by the latter subrun. In addition, there is only one dual pair left, $\{\pi_1, \rho_3\}$. Sublemma 4.5.2 can be applied again, showing that π is a rotation after the subrun (ρ_1, ρ_2), and by sublemma 4.5.3 also after subrun (ρ_1, ρ_2, ρ_3) as had to be shown.

Connecting sets of rotations to stable partnerships. Theorem 4.3.5 connects sets of rotations of different runs, stating that any run of `SP` covers the same set of rotations. This means that if we have a rotation that is eliminated in some run of `SP`, then this rotation or its dual will be eliminated in all runs of `SP`.

Theorem 4.3.5 [10] *Consider a reduced partnership problem $\mathcal{P}' = \{(V, E'), C\}$. Every $\gamma \in \Gamma_{\mathcal{P}'}$ covers the same set of rotations.*

Proof Consider two runs of `SP`, $\gamma = (\pi_1, \ldots, \pi_K)$ and δ. Note that as γ and δ are full runs, it has to be that

$$\Pi_{\mathcal{P}'_\gamma} = \Pi_{\mathcal{P}'_\delta} = \emptyset.$$

Assume that $\gamma[k]$ is a subrun of γ such that any $\pi_n \in \phi(\gamma[k])$ is covered by $\phi(\delta)$, while π_{k+1} is not covered by $\phi(\delta)$. Then Lemma 4.3.4 states that $\pi_{k+1} \in \Pi_{\mathcal{P}'_{\delta(l)}}$, which violates $\Pi_{\mathcal{P}'_\gamma} = \Pi_{\mathcal{P}'_\delta} = \emptyset$. Hence, no such subrun $\gamma[k]$ will ever exist and all rotations $\pi_n \in \phi(\gamma)$ will be covered by $\phi(\delta[l])$.
By the symmetric argument, all rotations $\rho \in \phi(\delta)$ will be covered by $\phi(\gamma)$, proving that every $\gamma \in \Gamma_{\mathcal{P}'}$ covers the same set of rotations.

[10] Adapted from Gusfield (1988, Theorem 4.1) and Borbelova and Cechlarova (2010, Theorem 2).

Because of Theorem 4.3.5, the rotations in Φ can be partitioned into two sets, Φ^s and Φ^d. The set Φ^s contains singular rotations, that is the rotations that are eliminated at every run of `SP`. The set Φ^d contains dual rotations, rotations that are not always eliminated in a run of `SP`. When a rotation $\pi \in \Phi^d$ is not eliminated in a run γ, then Theorem 4.3.5 implies that π^d is eliminated in a run γ. Hence, every run consists of all singular rotations and exactly one of each pair of dual rotations.

Theorem 4.3.6 connects runs to the stable partnership that is produced, stating that there is a one-to-one correspondence between the sets of rotations in a run and the stable partnerships that are produced. If two runs eliminate the same set of rotations, then they must produce the same stable partnership; and, if two runs of `SP` produce the same stable partnership, then the runs eliminate the same set of rotations.

Hence, to summarize, each stable partnership can be characterized by the unique set of rotations that are eliminated to produce this partnership in `SP`. In addition, differences between stable partnerships are driven by differences in these unique sets of rotations, particularly the different elements of a pair of dual rotations that are eliminated.

Theorem 4.3.6 [11] *Consider a reduced partnership problem $\mathcal{P}'$ and two runs γ and δ of* `SP` *such that executing run γ produces S_γ and executing run δ produces S_δ. Then, $\phi(\gamma) = \phi(\delta)$ if and only if $S_\gamma = S_\delta$.*

Proof

- $\Rightarrow$ By Lemma 4.3.3, $\mathcal{P}'_\gamma = \mathcal{P}'_\delta$, indicating that $S_\gamma = E'_\gamma = E'_\delta = S_\delta$.
- $\Leftarrow$ Assume $S_\gamma = S_\delta$ but $\phi(\gamma) \neq \phi(\delta)$. By Theorem 4.3.5, there must exist $\pi \in \Phi^d$ such that $\pi \in \phi(\gamma)$ while $\pi^d \in \phi(\delta)$. As $S_\gamma = S_\delta$ this implies that no edge in π can be in S_γ, as otherwise this edge will be deleted in the elimination of π^d, and hence will not be in $S_\delta = S_\gamma$. Consider the edge-pair $(uv, vw) \in \pi$. As $\pi \in \phi(\gamma)$, π is eliminated at some round k of the run γ in `SP`. As vw is the $E'_{\gamma[k-1]}(v)$-replacement of uv and as uv is deleted in round k, $vw \in C_u(E'_{\gamma[k]}(v))$ which by SUB implies that

$$vw \in C_v(S_\gamma(v) \cup vw). \tag{4.5}$$

[11] Adapted from Gusfield (1988, Theorem 5.2) and Borbelova and Cechlarova (2010, Theorem 3).

Now, consider the edge-pair $(vw, wx) \in \pi^d$. Assuming π^d is eliminated at some round l of δ, by the definition of an edge-pair, $vw \in C_w(E'_{\delta[l-1]})$. By SUB, it follows that

$$vw \in C_w(S_\delta(w) \cup vw). \tag{4.6}$$

As $S_\gamma = S_\delta$, (4.5) and (4.6) imply that vw is a blocking edge for S_γ which contradicts the stability of S_γ. Hence, $\phi(\gamma) = \phi(\delta)$.

Stable edges. Theorems 4.3.5 and 4.3.6 enable us to extend another result by Gusfield (1988) and Borbelova and Cechlarova (2010), with respect to the type of edges that can be found in a stable partnership. A rotation π is a *predecessor* of another rotation ρ if π has to be eliminated for ρ to become exposed, that is, $\rho \in \Pi_{\mathcal{P}'_{\gamma[k]}}$ only if $\pi \in \gamma[k]$. The predecessor relation can be given by $\prec$, indicating that if $\pi \prec \rho$, then π is a predecessor of ρ. It can be easily shown that for $\pi \in \Phi^s$, if $\pi' \prec \pi$, then $\pi' \in \Phi^s$; a singular rotation can only be preceded by another singular rotation. To see this, note that if $\pi' \in \Phi^d$, then there exists a run γ such that $\pi' \notin \gamma$. However, as π is a singular rotation, π will be part of γ by Theorem 4.3.5 such that it cannot be that π' is a predecessor of π. Hence, $\pi' \in \Phi^s$. By Theorem 4.3.5, this implies that for $\mathcal{P}$ there exists a subrun $\gamma[k]$ such that all singular rotations of the problem are eliminated, while no stable partnership was deleted by this run. Denote $\mathcal{P}^s = \{(V, E^s), C\} = \mathcal{P}'_{\gamma[k]}$, the problem obtained by eliminating every non-dual rotation of $\mathcal{P}'$. Then we get to our final result.
Consider a partnership problem $\mathcal{P} = \{(V, E), C\}$. An edge $e \in E$ is called *stable* if $e \in S$ with S, some stable partnership in $\mathcal{P}$ and e is called *fixed* if $e \in S$ for S, any stable partnership in $\mathcal{P}$. Then, Theorem 4.3.7 states that an edge is a fixed edge if and only if it is in the choice set for both agents for the problem $\mathcal{P}^s$. In addition, e is a non-fixed stable edge if and only if it is an edge in a rotation that has a dual.

Theorem 4.3.7 [12] *For a partnership problem $\mathcal{P}$ and $\mathcal{P}^s$, consider $e \in E^s$. Then:*

(i) $e = uv$ is a fixed edge if and only if $e \in C_v(E^s(v)) \cap C_u(E^s(u))$;

(ii) $e = uv$ is a stable edge that is not fixed if and only if uv is in a rotation $\pi \in \Phi^d$.

Proof (i) $\Rightarrow$: First, note that if $e = uv$ is a fixed edge then uv cannot be in a rotation $\pi \in \Phi^d$: By Theorem 4.3.5, there exists a run γ of SP, starting from $\mathcal{P}^s$ such that $\pi \in \gamma$. Hence, there is no

[12] Adapted from Borbelova and Cechlarova (2010, Lemma 19).

edge-pair $(uv, vw) \in \pi$ for some vw. In addition, there also exists a run δ such that $\pi^d \in \delta$, implying that there is no edge-pair $(xu, uv) \in \pi$ as well for some xu.

Second, pick any stable partnership S and assume that a run γ starting from $\mathcal{P}^s$ produces S. By repeated application of Lemma 4.3.2, we get that for all $w \in V$,

$$C_w(S(w)) \subseteq C_w(E^s(w)) \cup \bigcup_{\pi \in \phi(\gamma)} Q_\pi^{r\,w}. \tag{4.7}$$

Hence, if an edge $f = wx \in \overline{C}_w(E^s(w))$, it can only be in $C_w(S(w))$ when it is in $Q_\pi^{r\,w}$ for some rotation $\pi \in \phi(\gamma)$.

Now, as e is a fixed edge, $e \in C_v(S(v)) \cap C_u(S(u))$ for any stable partnership S. Since e was not in any rotation, it has to be that $e \in C_v(E^s(v)) \cap C_u(E^s(u))$.

$\Leftarrow$: Note that for any stable partnership S, $S \subseteq E^s$. If $e \in C_v(E^s(v)) \cap C_u(E^s(u))$, then SUB implies that $e \in C_v(S(v) \cup e) \cap C_u(S(u) \cup e)$. If $e \notin S$, then e is a blocking edge, contradicting with the stability of S.

(ii) $\Rightarrow$: Assume that $e = uv$ is a stable non-fixed edge and that uv is not in any rotation. As e is a stable edge, $e \in C_v(S(v)) \cap C_u(S(u))$ for some stable partnership S. By (4.7), $e \in C_v(E^s(v)) \cap C_u(E^s(u))$, which by (i) implies that e is a fixed edge. Hence, uv must be in a rotation.

$\Leftarrow$: Assume uv is in a rotation $\pi \in \Phi^d$ such that $(uv, vw) \in \pi$. If uv is not a stable edge then $uv \notin S$ for any stable partnership S. By Property 3(i), no first element xy of an edge-pair $(xy, yz) \in \pi$ is a stable edge. However, as $\pi \in \Phi^d$, $\pi^d \in \Phi^d$, and by Theorem 4.3.5 there exists a run δ such that π^d is eliminated. Hence, δ leads to a stable partnership S such that no edge in π is in S. The proof of Theorem 4.3.6 showed that this contradicts with the stability of S, concluding that uv is a stable edge.

Now, assume $\pi \in \gamma$ for some run γ. Say that π is exposed at $\mathcal{P}^s_{\gamma[k]}$. Then, by the definition of a rotation, $uv \in \overline{C}_u(E^s_{\gamma[k]}(u))$. By MON, this implies that $uv \in \overline{C}_u(E^s(u))$ such that by (i), e is not a fixed edge and the lemma is proven.

4.4 Concluding Remarks

This paper studies the partnership problem and the extended version of Irving's algorithm, SP, that produces a stable partnership if one exists. The main aim of the paper was to analyse the set of stable partnerships and its relation to the concept of rotations on which SP is built. There are three main results. First, any stable partnership can be found by SP (Theorem 4.3.1). In other words, for every stable partnership there exists a run of SP that produces this partnership. This implicitly reveals the similarity between any two stable partnerships, as they seemingly consist of the same type of edges. Second, this similarity is more explicitly established in Theorems 4.3.5 and 4.3.6. Theorems 4.3.5 and 4.3.6 link stable partnerships to sets of rotations. Theorem 4.3.6 states that for any stable partnership there exists a unique set of rotations that produces that stable partnership when eliminated. The similarity then originates in the fact that those sets of rotations are identical in the sense that they all cover the same rotations (Theorem 4.3.5). Third, this implies that there exist rotations that are always eliminated – singular rotations. Edges that are in the choice set for both agents, after these rotations are eliminated, will show up in any stable partnership (Theorem 4.3.7(i); fixed edges). In addition, there are pairs of dual rotations and one rotation of each pair will always be eliminated. Then, a non-fixed edge can only be part of a stable partnership if and only if this edge shows up in one of those rotations (Theorem 4.3.7(ii)).

These results are extensions of results established for the stable roommate problem (Gusfield, 1988)) and the stable b-matching problem (Borbelova and Cechlarova, 2010). This has two implications. First, the main conclusion of this paper should be that, even though the partnership problem is a problem that is much more general than the roommate problem, structurally it still is in essence a roommate problem, in the sense that the structure of the stable partnership is the same as the structure of the stable matching in the roommate problem. Second, other results from Gusfield (1988) and Borbelova and Cechlarova (2010), that build on the theorema presented in this paper, can be extended as well. In particular, we can think of the efficient enumeration of all stable matchings for a given problem (Gusfield, 1988) and the use of Irving (1985)'s algorithm to find stable matchings fulfilling some optimality criteria (Borbelova and Cechlarova, 2010). Appendix 4.5.2 shows that the *dual enumeration method*, the efficient enumeration method introduced in Gusfield (1988) can also be applied to efficiently enumerate all stable partnerships.

4.5 Appendix

4.5.1 Proofs

In this section, Property 1, Lemma 4.3.2, Lemma 4.3.3 and Lemma 4.3.4 will be proven. In the proof of Lemma 4.3.4 we will also present and proof three other (sub)lemmata.

Proof of Property 1. We prove the lemma by induction.
First, consider the last deleted edge uv in Phase 1. For this edge, w.l.o.g.,

$$uv \in \overline{C}_v(C_v^-(E' \cup uv) \cup uv). \tag{1}$$

Now, by SUB,

$$C_v^-(E' \cup uv) \backslash uv \subseteq C_v^-(E'), \tag{2}$$

all elements, except uv, that were in the inverted choice set of v before deleting uv are also in the inverted choice set of v for E'. Statement (1), by MON, then implies that $uv \in \overline{C}_v(C_v^-(E') \cup uv)$.
Now, assume the lemma is valid for the last n deleted edges in phase 1 and consider uv to be the $n+1$-last deleted edge for a problem $\{(V,T),C\}$ with $E' \subset T \subseteq E$. Since uv is deleted, w.l.o.g.,

$$uv \in \overline{C}_v(C_v^-(T) \cup uv). \tag{3}$$

Assume now that, in contradiction with the lemma,

$$uv \in C_v(C_v^-(E') \cup uv). \tag{4}$$

Statement (3) implies, by MON,

$$uv \in \overline{C}_v(C_v^-(T) \cup C_v^-(E') \cup uv). \tag{5}$$

Hence, there should exist an edge

$$vw \in C_v(C_v^-(T) \cup C_v^-(E') \cup uv) \backslash C_v^-(E'), \tag{6}$$

otherwise SUB and INCR would imply a violation of statement (5). In words, there exists an edge vw that is in the choice set for $C_v^-(T) \cup C_v^-(E') \cup uv)$ but is not in $C_v^-(E')$. Note, that by SUB,

$$vw \in C_v(C_v^-(T) \cup C_v^-(E')). \tag{7}$$

Now, if $vw \in E'$, SUB and $E' \subset T$ imply that $vw \in C_v^-(E')$. Thus, $vw \notin E'$, meaning that vw was deleted in some later round T' for which $E' \subset T' \subset T$. As vw was deleted, $vw \in \overline{C}_v(C_v^-(T') \cup vw)$. By induction, $vw \in \overline{C}_v(C_v^-(E') \cup vw)$ which by statement (7) is a violation of SUB. Hence, no such vw can exist, by which statement (4) implies a violation of statement (3), proving the lemma.

Proof of Lemma 4.3.2. First, we prove the following expression:

$$C_v(E_{t+1}(v)) = C_v(E'_t(v)) \backslash Q_\pi(v) \cup Q_\pi^{r\,v}. \tag{1}$$

This amounts to proving two things:

$$C_v(E'_t(v)) \backslash C_v(E_{t+1}(v)) \subseteq Q_\pi(v) \tag{2}$$

$$C_v(E_{t+1}(v)) \backslash C_v(E'_t(v)) = Q_\pi^{r\,v} \tag{3}$$

With respect to (2), consider an edge $uv \in C_v(E'_t(v))$ such that $uv \notin Q_\pi(v)$. In that case, $uv \in E_{t+1}$ and by SUB, $uv \in C_v(E_{t+1}(v))$, such that (2) holds. With respect to (3), note that each $vw \in Q_\pi^{r\,v}$ is an $E'_t(v)$-replacement of some $uv \in Q_\pi(v)$ such that $uv \in C_v(E'_t(v))$. By SUB and INCR, this means that

$$Q_\pi^{r\,v} = C_v(E'_t(v) \backslash Q_\pi(v)) \backslash C_v(E'_t(v)).$$

As $E_{t+1}(v) = E'_t(v) \backslash Q_\pi(v)$, this proves (3).

Next, we prove the following:

$$C_v(E'_{t+1}(v)) = C_v(E_{t+1}(v)), \tag{4}$$

which boils down to the fact that edges deleted in Phase 1 of round $t+1$ do not belong to any choice set over E_{t+1}. We have to show that

$$C_v(C_v^-(E_{t+1})) = C_v^-(E_{t+1}). \tag{5}$$

If this is not the case, then some edge in $C_v^-(E_{t+1})$ is dominated by the inverted choice set, implying by Property 1 that it will be deleted in Phase 1. If (5) holds, then $C_v(C_v^-(T)) = C_v^-(T)$ for all T such that $E_{t+1} \subseteq T \subset E'_{t+1}$ such that eventually no edge in a choice set is deleted in Phase 1 and (4) holds.

By the definition of a reduced problem, we know that for all $v \in V$,

$$C_v(C_v^-(E'_t)) = C_v^-(E'_t). \tag{6}$$

Assume that (5) does not hold and that there exists an edge uv such that

$$uv \in \overline{C}_v(C_v^-(E_{t+1})). \tag{7}$$

Now, denote by F the set $\{\bigcup_{w\neq v} Q_\pi^{r\,w}\}(v)$; the set of edges wv, for some w, in π incident to v such that wv is the $E'(w)$-replacement for some edge incident to w. By (1),

$$C_v^-(E_{t+1}) \subseteq C_v^-(E_t') \cup F, \tag{8}$$

such that by MON and (7),

$$uv \in \overline{C}_v(C_v^-(E_t') \cup F). \tag{9}$$

Now, by the definition of a rotation, for all edges $wv \in F$, there is an edge $vx = \overline{C}_v(C_v^-(E_t') \cup wv)$ and $vx \in Q_\pi(v)$. Denote all such edges vx by F'. By MON and INCR, $\overline{C}_v(C_v^-(E_t') \cup F) = F'$, which by (8) and SUB implies that $uv \in C_v(C_v^-(E_{t+1}))$, contradicting (7), proving (5) and the lemma.

Proof of Lemma 4.3.3. What we have to prove is that $E'_{\gamma[k]} = E'_{\delta[k]}$. Note that, because the set of eliminated rotations is the same, any difference between $E'_{\gamma[k]}$ and $E'_{\delta[k]}$ can only be due to an edge that was deleted at some Phase 1 on, w.l.o.g., the subrun $\gamma[k]$ while this edge was never deleted on the subrun $\delta[k]$. First we show that

$$C_v^-(E'_{\gamma[k]}) = C_v^-(E'_{\delta[k]}). \tag{1}$$

Consider $\mathcal{P}'_{\gamma[k'-1]}$ with $k' \leq k$ and π such that $\gamma[k'] = (\gamma[k'-1], \pi)$, i.e. π is the next rotation to be eliminated after having eliminated the sequence $\gamma[k'-1]$. Then, by Lemma 4.3.2, $C_v(E'_{\gamma[k']}(v)) = C_v(E'_{\gamma[k'-1]}(v)) \setminus Q_\pi(v) \cup Q_\pi^{r\,v}$. This holds for all k' and π such that we obtain

$$C_v(E'_{\gamma[k]}(v)) = C_v(E'(v)) \setminus \{\sum_{\pi\in\gamma[k]} Q_\pi(v)\} \cup \{\sum_{\pi\in\gamma[k]} Q_\pi^{r\,v} \setminus \sum_{\pi\in\gamma[k]} Q_\pi(v)\},^{13} \tag{2}$$

for all $v \in V$. However, as $\gamma[k]$ and $\delta[k]$ contain the same rotations,

$$\sum_{\pi\in\gamma[k]} Q_\pi(v) = \sum_{\pi\in\delta[k]} Q_\pi(v), \tag{3}$$

and

$$\sum_{\pi\in\gamma[k]} Q_\pi^{r\,v} = \sum_{\pi\in\delta[k]} Q_\pi^{r\,v}, \tag{4}$$

such that (1) follows.

Next, we show that if an edge uv is deleted at Phase 1 right after eliminating the k'-th rotation, with $k' \leq k$, on the subrun $\gamma[k]$, then $uv \in \overline{C}_v(C_v^-(E'_{\gamma[k]}) \cup uv)$. By Property 1, we know that $uv \in \overline{C}_v(C_v^-(E'_{\gamma[k']}) \cup uv)$.

[13] $\sum_{\pi\in\gamma[k]} Q_\pi(v)$ is in there two times as it is possible that an edge is in both $\sum_{\pi\in\gamma[k]} Q_\pi(v)$ and $\sum_{\pi\in\gamma[k]} Q_\pi^{r\,v}$

Assume that π' is the rotation that is eliminated after $\gamma[k']$. Denote by F the set $\{\bigcup_{w\neq v} Q^{r\ w}_{\pi'}\}(v)$. Then, by Lemma 4.3.2, $C_v^-(E'_{\gamma[k'+1]}) = C_v^-(E'_{\gamma[k']})\backslash Q_{\pi'}(v)\cup F$. Denote by F' the set of all edges vx such that for $wv \in F$, $vx = \overline{C}_v(C_v^-(E'_{\gamma[k]})\cup wv)$. Then by MON,

$$\overline{C}_v(C_v^-(E'_{\gamma[k']})\cup F\cup uv) = F'\cup uv, \tag{5}$$

such that by INCR $uv \in \overline{C}_v(C_v^-(E'_{\gamma[k'+1]})\cup uv)$. Repeating this argument we get that

$$uv \in \overline{C}_v(C_v^-(E'_{\gamma[k]})\cup uv) = \overline{C}_v(C_v^-(E'_{\delta[k]})\cup uv), \tag{6}$$

where the equality follows from (1). Hence, as $\mathcal{P}'_{\delta[k]}$ is a reduced problem, $uv \notin E'_{\delta[k]}$, proving the lemma.

Proof of Lemma 4.3.4. Lemma 4.3.4 will be proven using three other lemmatas:

Lemma 4.5.1 *For a reduced partnership problem $\mathcal{P}'$, if π and π' are two distinct rotations in $\Pi_{\mathcal{P}'}$, then π removes π' if and only if $\pi' = \pi^d$.*

Consider $\mathcal{P}_{(\pi)}$, the problem after eliminating π and $\mathcal{P}'_{(\pi)}$ the reduced partnership problem after eliminating π and applying Phase 1 from $\mathcal{P}'$.

$\Leftarrow$ If $\pi' = \pi^d$, then it is straightforward that eliminating π removes π', as half of the elements which π' is composed of, are no longer in the subsequent reduced partnership problem.

$\Rightarrow$ We have to show that if $\pi' \neq \pi^d$, then $\pi' \in \Pi_{\mathcal{P}'_{(\pi)}}$. Before it can be proven that $\pi' \in \Pi_{\mathcal{P}'_{(\pi)}}$ we have to show first that all edges in π' are in $E'_{(\pi)}$. This claim has two parts:

(a) No edge in π' is deleted in the elimination of π, $\bigcup_{uv\in\pi'} uv \subseteq E_{(\pi)}$.

(b) No element in π' is deleted in Phase 1 after eliminating π: $\bigcup_{uv\in\pi'} uv \subseteq E'_{(\pi)}$,

Concerning (a) we prove the following statements:

(i) <u>If $\pi' \neq \pi$, then $Q_\pi \cap Q_{\pi'} = \emptyset$.</u> Assume this is not the case: There exists an edge $uv \in E'$ such that $(uv, vw) \in \pi$ and $(uv, vx) \in \pi'$. There is exactly one $E'(v)$-replacement of uv for v such that $vw = vx$. In addition, $\overline{C}_w(C_w^-(E')\cup vw) = wy$ for some x, hence (uv, vw) can only lead to one other edge-pair, say (wy, yz). This argument can be repeated such that eventually we find that $\pi = \pi'$.

(ii) If $\pi' \neq \pi^d$, then $Q_\pi \cap \{\bigcup_{uv\in\pi'} uv \backslash Q_{\pi'}\} = \emptyset$. Assume this is not the case: there exists an edge $uv \in E'$ such that $(uv, vw) \in \pi$ and $(xu, uv) \in \pi'$. Assume (yz, zu) leads to (uv, vw). This implies that

$$uv \in \overline{C}_u(C_u^-(E') \cup zu) \tag{1}$$

$$uv \in C_u(E'(u) \backslash xu). \tag{2}$$

By SUB, (1) and (2) can only hold if

$$xu \in C_u^-(E') \cup zu. \tag{3}$$

However, by the definition of a rotation $xu \notin C_u^-(E')$, such that (3) can only hold if $xu = zu$. Hence, we get the edge-pair $(yx, xu) \in \pi$ and $(xu, uv) \in \pi'$. This argument can then be repeated such that eventually we find that $\pi' = \pi^d$, contradicting the starting assumption.

Statements (i) and (ii) show that if $\pi' \neq \pi^d$, there are no edges that are both in π and in π'. Hence, (a) holds.

Now, we prove (b), $\bigcup_{uv\in\pi'} uv \subseteq E'_{(\pi)}$. Assume on the contrary that $uv \in \pi'$ but $uv \notin E'_{(\pi)}$. Lemma 4.3.2 implies that if uv was deleted in Phase 1, then $uv \notin C_v(E_{(\pi)}(v)) \cup C_u(E_{(\pi)}(u))$, which by MON implies that $uv \notin C_v(E'(v)) \cup C_u(E'(u))$. Hence, uv is such that $(xu, uv) \in \pi'$.

Now, by Property 1, one of two statements must hold:

$$uv \in \overline{C}_v(C_v^-(E'_{(\pi)}) \cup uv) \tag{4}$$

$$uv \in \overline{C}_u(C_u^-(E'_{(\pi)}) \cup uv). \tag{5}$$

Assume first that (4) holds. As $\mathcal{P}'$ is a reduced problem,

$$uv \in C_v(C_v^-(E') \cup uv), \tag{6}$$

and $\overline{C}_v(C_v^-(E') \cup uv) = vw$ for some vw such that (xu, uv) leads to (vw, wy). Define $F = \{\bigcup_{u\neq v} Q_\pi^{r\,u}\}(v)$. Then, by INCR, $vw \in C_v(C_v^-(E') \cup F)$ as the $|F|$ elements in $C_v^-(E') \cap Q_\pi(v)$ will drop out of the choice set. By MON,

$$vw \in \overline{C}_v(C_v^-(E') \cup F \cup uv), \tag{7}$$

such that, by INCR, $uv \in C_v(C_v^-(E') \cup F \cup uv)$. By lemma 4.3.2, $C_v^-(E'_{(\pi)}) \subseteq C_v^-(E') \cup F$. By SUB, this implies that (4) cannot hold.

Next, assume that (5) holds. As (xu,uv) is an edge-pair,

$$uv \in C_u(E'(u)\backslash xu). \tag{8}$$

Hence, by SUB (5) can only hold if $xu \in C_u^-(E'_{(\pi)})$. However, by lemma 4.3.2 this implies that $xu \in F$ which is not possible as there are no edges that are both in π and π'.

Now that it is established that $\bigcup_{uv\in\pi'} uv \subseteq E'_{(\pi)}$, we prove that $\pi' \in \Pi_{\mathcal{P}'_{(\pi)}}$. There are two points to this claim:

(i) Consider an edge pair $(uv,vw) \in \pi'$. First, $uv \in C_v(E'_{(\pi)}(v))$ by SUB and the definition of a rotation. Second, $vw \in \overline{C}_v(E'_{(\pi)}(v))$, by $vw \in \overline{C}_v(E'(v))$ and lemma 4.3.2. In addition, $vw \in C_v(E'_{(\pi)}(v)\backslash uv)$ by SUB. Hence, vw is a $E'_{(pi)}(v)$-replacement of uv.

(ii) Assume now that (uv,vw) leads to $(wx,xy) \in \pi'$. By the same argument as for (7), $wx \in \overline{C}_v(C_v^-(E') \cup F \cup vw)$ with $F = \{\bigcup_{u\neq w} Q_\pi^{r\,u}\}(w)$, which by INCR implies that $wx \in \overline{C}_v(C_v^-(E'_{(\pi)}) \cup vw)$. □

Lemma 4.5.2 *Consider a reduced problem $\mathcal{P}'$ such that $\{\pi,\pi^d\} \subseteq \Pi_{\mathcal{P}'}$. If $\gamma[1] = (\pi)$ is such that $\pi' \in \Pi_{\mathcal{P}'_{(\pi)}}$, then $\pi' \in \Pi_{\mathcal{P}'}$.*

Consider an edge-pair $(uv,(vw)) \in \pi'$ that leads to an edge-pair $(wx,(xy)) \in \pi'$. Proving Lemma 4.5.2 amounts to showing that

(a) $uv \in C_v(E'(v))$,

(b) $vw \in \overline{C}_v(E'(v))$,

(c) $vw \in C_v(E'(v)\backslash uv)$,

(d) $wx = \overline{C}_v(C_v^-(E') \cup vw)$.

(a) By Lemma 4.3.2,

$$C_v(E'_{(\pi)}(v)) = C_v(E'(v))\backslash Q_\pi(v) \cup Q_\pi^{r\,v}. \tag{9}$$

Hence, if $uv \in \overline{C}_v(E'(v))$, then $uv \in Q_\pi^{r\,v}$. However, this implies that $uv \in Q_{\pi^d}(v)$. As $uv \in \overline{C}_u(E'_{(\pi)}(u))$, by MON,

$$uv \in \overline{C}_u(E'(u)). \tag{10}$$

such that $uv \in C_v(E'(v))$, contradicting $uv \in \overline{C}_v(E'(v))$.

(b) As $vw \in \overline{C}_v(E'_{(\pi)}(v))$, $vw \in \overline{C}_v(E'(v))$, by MON.

(c) Note that

$$vw \in C_v(E'_{(\pi)}(v) \backslash uv) \tag{11}$$

and $uv \notin C_v^-(E')$, by (10). If $vw \notin C_v(E'(v) \backslash uv)$, then there exists an edge vz with $vz \neq vw$ such that it is the $E'(v)$-replacement of uv. By SUB, $vz \in C_v(E'_{(\pi)}(v) \backslash uv \cup vz)$, such that there are two possibilities if (11) holds.

(i) $\underline{vz \in C_v(E'_{(\pi)}(v))}$. By Lemma 4.3.2, $vz \in Q_\pi^{r\,v}$. However, then $vz \in Q_{\pi^d}(v)$ such that $vz = \overline{C}_v(C_v^-(E') \cup av)$ for some av. As $av \in Q_\pi(v)$, $av \neq uv$, since $uv \in E'_{(\pi)}$. In addition, $uv \notin C_v^-(E')$ such that by MON, $vz \in \overline{C}_v(E'(v) \backslash uv)$ and vz cannot be the $E'(v)$-replacement of uv.

(ii) $\underline{vz \in Q_\pi(v)}$. As $vz \notin C_v(E'(v))$, this implies that $vz \in C_z(E'(z))$ and

$$vz = \overline{C}_v(C_v^-(E') \cup av) \tag{12}$$

for some av such that $av \in Q_\pi^{r\,a}$. Hence (av, vz) is an edge-pair in π^d, which by SUB implies that $av \in C_v(E'_{(\pi)}(v))$. In addition, as π is eliminated, $av \in C_a(E'_{(\pi)}(a))$. As $av \in C_v(E'_{(\pi)}(v)) \cap C_v^-(E'_{(\pi)}(v))$, $av \neq uv$. However, this implies a contradiction with (12), as $uv \notin C_v^-(E')$, which by MON implies $vz \in \overline{C}_v(E'(v) \backslash uv)$.

Hence, both possibilities are not possible and $vw \in C_v(E'(v) \backslash uv)$.

(d) We know that

$$wx = \overline{C}_w(C_w^-(E'_{(\pi)}) \cup vw) \tag{13}$$

and $vw \notin C_w^-(E')$, by point (b). Define $F = \bigcup_{u \neq w} Q_\pi^{r\,u}$. By Lemma 4.3.2,

$$C_w^-(E') = C_w^-(E'_{(\pi)}) \backslash F \cup Q_{\pi^d}^{r\,w}. \tag{14}$$

By the definition of a rotation and INCR,

$$\overline{C}_w(C_w^-(E'_{(\pi)}) \cup Q_{\pi^d}^{r\,w}) = Q_{\pi^d}^{r\,w}. \tag{15}$$

By MON and (13) and (15) we get

$$\{Q_{\pi^d}^{r\,w}, wx\} \subset \overline{C}_w(C_w^-(E'_{(\pi)}) \cup Q_{\pi^d}^{r\,w} \cup vw). \tag{16}$$

However, each edge in ${Q^r_{\pi^d}}^w$ is also an $E'(w)$-replacement of some element in F such that

$$ {Q^r_{\pi^d}}^w \subseteq C_w(C^-_w(E'_{(\pi)})\backslash F \cup {Q^r_{\pi^d}}^w \cup vw). \quad (17)$$

By INCR, (17) implies that $wx = \overline{C}_w(C^-_w(E') \cup vw)$. □

Lemma 4.5.3 *Consider a reduced problem $\mathcal{P}'$ and two subruns of order k and l, $\gamma[k]$, and $\delta[l]$ such that $\phi(\gamma[k]) \subseteq \phi(\delta[l])$. In addition, consider a rotation π that is not covered by $\phi(\delta[l])$. If $\pi \in \Pi_{\mathcal{P}'_{\gamma[k]}}$, then $\pi \in \Pi_{\mathcal{P}'_{\delta[l]}}$.*

First, consider $k = 0$. Then $\pi \in \Pi_{\mathcal{P}'} = \Pi_{\mathcal{P}'_{\delta[0]}}$ and $\pi \in \Pi_{\mathcal{P}'_{\delta[l]}}$ by lemma 4.5.1. Next, assume that the lemma holds for all subruns, $\delta[l]$ and $\gamma[k']$ such that $k' < k$ and define the following statement,

> $Q(\alpha[a], \beta[b], \rho)$: For two subruns $\alpha[a]$ and $\beta[b]$ such that $\phi(\alpha[a]) \subset \phi(\beta[b])$, if ρ is not covered by $\phi(\beta[b])$ and $\rho \in \Pi_{\mathcal{P}'_{\alpha[a]}}$, then $\rho \in \Pi_{\mathcal{P}'_{\beta[b]}}$.

The *order* of $Q(\alpha[a], \beta[b], \rho)$ is b. A statement $Q(\alpha[a], \beta[b], \rho)$ is called *solved* if we can prove that the statement holds. The goal of the proof is to solve $Q_1 = \{Q(\gamma[k], \delta[l], \pi)\}$.
Consider the sequence $\delta^* = (\rho_1, \rho_2, \ldots, \rho_{l-k})$ that has only the rotations in $\phi(\delta[l])\backslash\phi(\gamma[k])$ but with their order in $\delta[l]$ preserved. Denote by $q(\rho_n)$ the position of ρ_n in $\delta[l]$ for all $n \leq l - k$. To solve Q_1, we need to show that there exists a subrun $\gamma^* = (\gamma[k], \delta^*)$. Then, as $\pi \in \Pi_{\mathcal{P}'_{\gamma[k]}}$, by lemma 4.5.1, $\pi \in \Pi_{\mathcal{P}'_{\gamma^*}}$ and by lemma 4.3.3, $\pi \in \Pi_{\mathcal{P}'_{\delta[l]}}$.
Showing that there exists a subrun γ^* amounts to solving the following set of statements

$$Q_2 = \{Q(\delta[q(\rho_n) - 1], (\gamma[k], \delta^*[n-1]), \rho_n) \text{ for all } \rho_n \in \gamma^*\}. \quad (18)$$

To see this, note that we want to prove that, for all n, $\rho_n \in \Pi_{\mathcal{P}'_{(\gamma[k],\delta^*[n-1])}}$ while we know that $\rho_n \in \Pi_{\mathcal{P}'_{\delta[q(\rho_n)-1]}}$. In addition,

$$\phi(\delta[q(\rho_n) - 1]) \subseteq \phi((\gamma[k], \delta^*[n-1])). \quad (19)$$

Finally, ρ_n is not covered by $(\gamma[k], \delta^*[n-1])$. If ρ_n would be covered by $(\gamma[k], \delta^*[n-1])$ then $\delta[l]$ cannot be a subrun as it would imply that either ρ_n is eliminated twice – which is impossible – or that ρ^d_n is also eliminated – which is impossible by lemma 4.5.1. The statements in Q_2 will have an order that is at most $k + (l - k) - 1 = l - 1$.
If we want to solve a statement in Q_2, say $Q(\delta[q(\rho_n) - 1], (\gamma[k], \delta^*[n-1]), \rho_n)$, then – as the previous argument can be repeated – we need to solve an additional set of statements. The set of all additional

statements (over all statements in Q_2) will be denoted Q_3, where the solvability of Q_2 depends on the solvability of Q_3. The statements in Q_3 will have an order that is at most $l-2$. This process can be repeated such that eventually we are bound to end up with a set of statements Q_i in which the maximal order is lower than k. By induction, all statements in Q_{i+1} can be solved, implying that all statements in Q_i can be solved as well, which eventually implies that Q_1 can be solved, proving the lemma.□

These three lemmata enable us to prove Lemma 4.3.4. Recall that a rotation ρ is covered by $\phi(\delta[l])$ if either $\rho \in \phi(\delta[l])$ or $\rho^d \in \phi(\delta[l])$. Consider $\gamma[k] = (\pi_1, \ldots, \pi_k)$ and $\delta[l] = (\rho_1, \ldots, \rho_k)$. For $\phi(\gamma[k]) \cup \phi(\delta[l])$, denote by n^* the number of dual pairs, i.e. the number of pairs of rotations that are each others duals. If $n^* = 0$, then $\phi(\gamma[k]) \subseteq \phi(\delta[l])$ and the lemma holds by Lemma 4.5.3. Assume that the lemma holds for all $n^* < n$ and set $n^* = n$.

From all n^* dual pairs in $\phi(\gamma[k]) \cup \phi(\delta[l])$ select that pair $\{\pi_p, \rho_q\}$ such that q is as low as possible, i.e. ρ_q is the first rotation in $\delta[l]$ such that it has a dual – π_p – in $\gamma[k]$. Consider $\mathcal{P}'_{\delta[q-1]}$, the problem in $\delta[l]$ just before ρ_q will be eliminated. Rotation π_1 is in $\Pi_{\mathcal{P}'}$ which by Lemma 4.5.1 implies that $\pi_1 \in \Pi_{\mathcal{P}'_{\delta[q-1]}}$. Eliminate π_1 in $\mathcal{P}'_{\delta[q-1]}$ to get a subrun $(\delta[q-1], \pi_1)$. By Lemma 4.5.3,

$$\pi_2 \in \Pi_{\mathcal{P}'_{(\delta[q-1],\pi_1)}}. \tag{20}$$

Repeating this argument and by Lemma 4.5.1, we get that

$$\{\pi_p, \rho_q\} \subset \Pi_{\mathcal{P}'_{(\delta[q-1],\gamma[p-1])}} \tag{21}$$

and

$$\pi_{p+1} \in \Pi_{\mathcal{P}'_{(\delta[q-1],\gamma[p])}}. \tag{22}$$

By Lemma 4.5.2, (21) and (22) imply that

$$\pi_{p+1} \in \Pi_{\mathcal{P}'_{(\delta[q-1],\gamma[p-1])}}. \tag{23}$$

Eliminating π_{p+1} in $\mathcal{P}'_{(\delta[q-1],\gamma[p-1])}$ and by Lemma 4.5.1, we get that

$$\{\pi_p, \rho_q\} \subset \Pi_{\mathcal{P}'_{(\delta[q-1],\gamma[p-1]),\pi_{p+1})}}. \tag{24}$$

As $\pi_{p+2} \in \Pi_{\mathcal{P}'_{(\delta[q-1],\gamma[p+1])}}$, Lemma 4.5.2 implies that $\pi_{p+2} \in \Pi_{\mathcal{P}'_{(\delta[q-1],\gamma[p-1],\pi_{p+1})}}$. This argument can be repeated such that eventually, by Lemma 4.5.3, for $\gamma'[k'] = (\delta[q], \gamma[p-1], \pi_{p+1}, \ldots, \pi_k)$ with $k' = q + (k-1)$ it is shown that

$$\pi \in \Pi_{\mathcal{P}'_{\gamma'[k']}}. \tag{25}$$

Assume $\gamma'[k'] = (\pi'_1, \ldots, \pi'_{k'})$. Now, $\gamma'[k']$ consists of elements of $\gamma[k]$ and elements of $\delta[l]$. Hence, since any π_j in $\gamma[k]$ was covered by $\phi(\delta[l])$, it is also the case that any $\pi'_{j'}$ in $\gamma'[k']$ is covered by $\phi(\delta[l])$. In addition, no new rotations were added, while $\pi_p \notin \gamma'[k']$. Hence the number of dual pairs in $\phi(\gamma'[k']) \cup \phi(\delta[l])$ will be lower than n^*. By induction, (25) then implies that $\pi \in \Pi_{\mathcal{P}'_{\delta[l]}}$.

4.5.2 The dual enumeration method.

Gusfield (1988) presents an efficient method to produce all stable matchings, exploiting the properties of the stable roommate problem. This appendix will show that this enumeration method can be applied to the partnership problem as well.

Exposing a rotation Recall that we defined the *predecessor* relation in section 4.3, where a rotation π was a predecessor of ρ if $\rho \in \Pi_{\mathcal{P}'_{\gamma[k]}}$ only if $\pi \in \gamma[k]$. Define $\Omega(\rho)$ as the set of all predecessors of ρ. Then, if ρ is exposed after a subrun, all rotations in $\Omega(\rho)$ should be eliminated in that subrun. We will also show the reverse. If a subrun eliminates all rotations in $\Omega(\rho)$, then ρ is exposed after the subrun. Hence, the only way to expose a rotation ρ is to eliminate all rotations in $\Omega(\rho)$ and elimination of all rotations in $\Omega(\rho)$ suffices to expose ρ. Formally,

Lemma 4.5.4 *For a rotation ρ and a subrun $\gamma[k]$ such that ρ is not covered by $\gamma[k]$: $\rho \in \Pi_{\mathcal{P}'_{\gamma[k]}}$ if and only if $\Omega(\rho) \subseteq \phi(\gamma[k])$.*

Before giving the proof we first need another result, that states that if a rotation ρ is not exposed for a given problem, then there exists at most one rotation π in that given problem whose elimination can expose ρ:

Lemma 4.5.5 *For a reduced partnership problem $\mathcal{P}'$ and a rotation $\pi \notin \Pi_{\mathcal{P}'}$, if ρ is a rotation in $\Pi_{\mathcal{P}'}$, such that $\pi \in \Pi_{\mathcal{P}'_{(\rho)}}$, then ρ is the unique such rotation in $\Pi_{\mathcal{P}'}$.*

Proof Consider two rotations ρ_1 and ρ_2 such that they are both exposed in $\mathcal{P}'$ and such that π is exposed after eliminating either ρ_1 or ρ_2. We have to prove that $\rho_1 = \rho_2$. For simplicity, call the process of eliminating ρ_1 and subsequently applying Phase 1 *run 1*, and the process of eliminating ρ_2 and applying Phase 1, *run 2*.

Rotation π becomes exposed after either of the following events: First, there may exist two edge pairs in π, (xu, uv) and (vw, wy) such that (xu, uv) does not lead to (vw, wy) in $\mathcal{P}'$ but it does so after run 1 and run

2. Hence,

$$vw \in C_v(C_v^-(E') \cup uv), \tag{26}$$

$$vw \in \overline{C}_v(C_v^-(E'_{(\rho_1)}) \cup uv), \tag{27}$$

$$vw \in \overline{C}_v(C_v^-(E'_{(\rho_2)}) \cup uv). \tag{28}$$

Note that (26) implies, by INCR, that there exists $vz \in \overline{C}_v(C_v^-(E') \cup uv)$ and $vz \neq uv$, by the reducedness of $\mathcal{P}'$. Set $F_{\rho_1} = \{\bigcup_{w \neq v} Q^r_{\rho_1}{}^w\}(v)$. If $vz \notin Q_{\rho_1}$, then by INCR, $vw \in C_v(C_v^-(E') \cup F_{\rho_1} \cup uv)$, which by SUB and Lemma 4.3.2 implies that (27) does not hold. Hence, $vz \in Q_{\rho_1}$. However, the same argument goes for ρ_2 such that vz must be in Q_{ρ_2} contradicting $vz \in Q_{\rho_1}$ unless $\rho_1 = \rho_2$.
Second, (uv, vw) is an edge-pair in $\mathcal{P}'_{(\rho_1)}$ and $\mathcal{P}'_{(\rho_2)}$ but not in $\mathcal{P}'$. Assume $uv \notin C_v(E'(v))$. Then, by Lemma 4.3.2, $uv \in Q^r_{\rho_1}{}^v$ and $uv \in Q^r_{\rho_2}{}^v$, which implies that $\rho_1 = \rho_2$. Hence, $uv \in C_v(E'(v))$. Then there should exist an edge $vy \neq vw$ that is the $E'(v)$-replacement of uv. As vy is no longer an $E'_{(\rho_1)}(v)$- or $E'_{(\rho_2)}(v)$-replacement for uv this means in runs 1 and 2 either vy was deleted or vy is in the choice set after runs 1 and 2. Take run 1 and assume vy was deleted. Edge vy can be deleted in two ways:

(i) vy is deleted in Phase 2; $vy \in Q_{\rho_1}$: This means that uv must be in $C_v^-(E_{(\rho_1)})$, as otherwise vy can never be dominated by the inverted choice set. However, uv is also in $C_v(E'(v))$, by which (uv, vw) cannot be an edge-pair in $\mathcal{P}'_{(\rho_1)}$.

(ii) vy is deleted in Phase 1: The same argument as in (i) implies that vy must be deleted because

$$vy \in \overline{C}_y(C_y^-(E_{(\rho_1)}) \cup vy) \tag{29}$$

as $vy \in C_v(C_v^-(E_{(\rho_1)}) \cup vy)$. Note that, by the reducedness of $\mathcal{P}'$, $vy \in C_y(C_y^-(E') \cup vy)$, indicating, by INCR, that there is an edge in $C_y^-(E')$, say ya, that is in $\overline{C}_y(C_y^-(E') \cup vy)$. If $ya \notin Q_{(\rho_1)}$, then by INCR, (29) cannot hold, hence $ya \in Q_{(\rho_1)}$.

If vy was deleted in run 2 as well, then by (i) and (ii), ya has to be in $Q_{(\rho_2)}$, which violates $ya \in Q_{(\rho_1)}$ unless $\rho_1 = \rho_2$. Hence, if we assume that $\rho_1 \neq \rho_2$, $vy \in C_v(E'_{(\rho_2)})$ and by Lemma 4.3.2, there is an edge-pair in ρ_2, (av, vy) that leads to an edge-pair (yz, zb), indicating that $yz = \overline{C}_y(C_y^-(E') \cup vy)$. As $yz \in Q_{(\rho_2)}$, $yz \notin Q_{(\rho_1)}$, which implies by INCR that $vy \in C_y(C_y^-(E') \cup F_{\rho_1} \cup vy)$ and by SUB (29) cannot hold and $\rho_1 = \rho_2$.
This leaves as a final possibility the case where vy is both in $C_v(E'_{(\rho_1)})$ and $C_v(E'_{(\rho_2)})$. However, by Lemma 4.3.2, this implies that $\rho_1 = \rho_2$.

Proof of Lemma 4.5.4. $\Rightarrow$ If ρ is exposed in a reduced problem $\mathcal{P}'_{\gamma[k]}$, then it follows from the definition of a predecessor that $\Omega(\rho) \subseteq \phi(\gamma[k])$.
$\Leftarrow$ Consider the subrun $\omega[k]$ such that $\Omega(\rho) = \phi(\omega[k])$. If $\Leftarrow$ does not hold, then ρ may not be exposed in $\mathcal{P}'_{\omega[k]}$. However, by the definition of a predecessor, there should exist another subrun $\gamma[k]$ with $\Omega(\rho) \subset \phi(\gamma[k])$ such that ρ is exposed after $\gamma[k]$. As there are elements of $\gamma[k]$ that are not in $\Omega(\rho)$, there should exist another subrun $\delta[l]$ with $\Omega(\rho) \subset \phi(\delta[l])$ such that ρ is exposed after $\delta[l]$ as well and such that $\phi(\delta[l]) \neq \phi(\gamma[k])$. We will show that such a situation is not possible.
We show that it is not possible that we have two distinct subruns $\gamma[k] = (\pi_1, \ldots, \pi_k)$ and $\delta[l] = (\rho_1, \ldots, \rho_l)$ such that ρ is exposed in $\mathcal{P}'_{\gamma[k]}$ and $\mathcal{P}'_{\delta[l]}$ but ρ is not exposed in $\mathcal{P}'_{\gamma[k']}$ and $\mathcal{P}'_{\delta[l']}$ for any $k' < k$ and $l' < l$.
Denote by n^* the number of elements in $\delta[l]$ that are not covered by $\gamma[k]$. First, assume that $\pi_k \neq \rho_l$. By Lemma 4.3.4, we can create a subrun $\gamma^*[k']$ that eliminates all elements of $\gamma[k]$, and then all n^* elements of $\delta[l]$ that are not covered by $\gamma[k]$. Lemma 4.3.4 also tells us that π_k and ρ_l are exposed in $\mathcal{P}'_{\gamma^*[k']}$. In addition, by that same lemma, ρ will be exposed in $\mathcal{P}'_{(\gamma^*[k'],\pi_k)}$ and $\mathcal{P}'_{(\gamma^*[k'],\rho_l)}$. If ρ is not exposed in $\mathcal{P}'_{\gamma^*[k']}$, then Lemma 4.5.5 is violated, hence ρ is exposed in $\mathcal{P}'_{\gamma^*[k']}$.
If, $\pi_k = \rho_l$, then the same argument implies that we can replace the elimination of π_{k-1} and ρ_{l-1} by the elimination of π_k, creating a new subrun $\gamma^*[k']$ and ρ is exposed after this subrun.
Denote $\gamma^*[k']$ by $\delta[l]$. Now, we are faced with a similar situation; a subrun $\gamma[k]$ after which π is exposed and a subrun $\delta[l] = \gamma^*[k']$ after which π is also exposed. However, this time, there are $n * -1$ elements of $\delta[l]$ not covered by $\gamma[k]$. Now, we form another subrun $\gamma^*[k']$ that first eliminates the first $k-1$ elements of $\gamma[k]$ and then all first $l-1$ elements of the new $\delta[l]$ that are not covered by $\gamma[k]$. Again, we can show that ρ is exposed after $\gamma^*[k']$. Repeating this exercise, we create new subruns $\delta[l]$ with a lower n^* and eventually we will have that ρ is exposed after $\gamma[k-1]$ violating the starting assumption.

The method. Call the set of rotations that are eliminated on a particular run of SP a *run set*. In the words of Gusfield (1988), the idea is to simulate SP, forcing it to generate each distinct run set, and associated stable partnership, exactly once. Consider a directed tree B where a node x represents a reduced problem $\mathcal{P}'_x$ – the root node is $\mathcal{P}'$. A leaf is a node that has no arcs leaving. When a node x is not a leaf of the tree, there are at most two arcs leaving x. The first arc – the *left arc* – will be labeled by a rotation π that is exposed in $\mathcal{P}'_x$. The left arc leaving x arrives at a node y that represents the reduced problem after eliminating π and applying Phase 1. If this rotation is singular, there is no second arc leaving. If this rotation has a dual, π^d, then there will be a second arc – the *right arc* – that will be labeled by $\Omega(\pi^d)$. This right arc arrives at a node z that represents the reduced problem after eliminating all rotations in $\Omega(\pi^d)$ and π^d and applying Phase 1. Note that the path from the root of B along all left arcs to the leaf, represents one particular run of `SP`.

Each leaf of B represents a reduced problem that has no rotations. The idea is now that any leaf of B represents a stable partnership and that no stable partnership is represented by more than one leaf. Hence, by applying the dual enumeration method, we enumerate all stable partnerships while avoiding producing the same stable partnership twice. Formally,

Theorem 4.5.6 (Theorem 6.2. in Gusfield (1988).) *For the directed tree B, if $\mathcal{P}'_x$ is the reduced partnership problem at a leaf x, then E'_x is a stable partnership of $\mathcal{P}$ and there does not exist another leaf x' such that $E'_{x'} = E'_x$.*

We will not give the proof of theorem 4.5.6 as it is in large part the same as the proof of Gusfield (1988, Theorem 6.2.), because it builds on the same results that we extended in this paper (theorems 4.3.1, 4.3.5 and 4.3.6). However, there is one property, used in the proof of Gusfield (1988), that is yet to be proven in a partnership context:

Lemma 4.5.7 *Let π be such that $\pi \in \Phi^d$. Then, if there exists a subrun $\gamma[k]$ such that $\pi \in \Pi_{\mathcal{P}'_{\gamma[k]}}$, there exists a subrun $\gamma^*[k+k'] = (\gamma[k], \gamma'[k'])$ such that $\pi^d \in \Pi_{\mathcal{P}'_{\gamma^*[k+k']}}$.*

Proof By Theorem 4.3.5, there exists a subrun $\delta[l]$ such that $\pi^d \in \Pi_{\mathcal{P}'_{\delta[l]}}$. Note that $\pi \notin \delta[l]$. Consider the first rotation, ρ_1, in the subrun $\delta[l]$, that is not covered by $\gamma[k]$. As all previous rotations in $\delta[l]$ are covered by $\gamma[k]$, by Lemma 4.3.4, ρ_1 is exposed in $\mathcal{P}'_{\gamma[k]}$. Eliminating ρ_1 leads to the reduced problem $\mathcal{P}'_{(\gamma[k],\rho_1)}$. Now, there is a new first rotation, say ρ_2, in the subrun $\delta[l]$ that is not covered by $(\gamma[k], \rho_1)$. Again, by Lemma 4.3.4, ρ_2 is exposed in $\mathcal{P}'_{(\gamma[k],\rho_1)}$ such that it can be eliminated. Hence, repeating this argument, we can create a subrun $\gamma^*[k+k']$ such that $\delta[l]$ is covered by this subrun and such that $\pi \notin \gamma^*[k+k']$. Lemma 4.3.4 can be applied and $\pi^d \in \Pi_{\mathcal{P}'_{\gamma^*[k+k']}}$.

Complexity Set $n = |V|$ and $m = |E|$ for a partnership problem $\mathcal{P}$. If we assume that there are k stable partnerships, we want to know how long it takes to find those k stable partnerships, assuming we use an oracle that gives the function $C_v(.)$ for all v. Fleiner (2010, Theorem 4.1.) showed that a modified version of SP runs in $O(n+m)$ time, that is, the algorithm uses $O(n+m)$ calls of the oracle.[14] The dual enumeration method contains a number of steps:

(1) Finding all rotations and their types: Running the modified version of SP once, we get a set of rotations that covers all rotations eliminated in any run of SP – by Theorem 4.3.5. Hence, obtaining the set of rotations takes $O(n+m)$ time. To determine the type of a rotation π – singular or dual – we go back to where this rotation was eliminated in the particular run and continue SP without eliminating π. This boils down to another run of SP and requires $O(n+m)$ time. There are maximally m rotations, such that this whole process uses $O(m(n+m))$ calls.

(2) Construct the predecessor relations: Following the analysis of Gusfield (1988, Section 6.2.3.), this takes $O(m)$ calls.

(3) The dual enumeration method: By Gusfield (1988, Section 6.2.4.), producing one stable partnership takes $O(m)$. Hence, this process runs in $O(k.m)$ time.

This implies that the whole process takes at most $O(m(n+m)+km)$ time.

[14] As mentioned before, in footnote 6, we presented a two-phase algorithm here to make the analysis more straightforward. The modified, more efficient, version starts with a first application of phase 1. Then phase 2 and subsequent phase 1 can be combined, as the edges eliminated in phase 2 indicate which edges might be dominated by the inverted choice set in the post-phase 2 partnership problem.

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