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Error estimates for the full discretization of a nonlocal model for type-I superconductors

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Abstract

Two vectorial nonlocal linear problems with applications in superconductors of type-I are studied. The nonlocal term is represented by a (space) convolution with a singular kernel arising in Eringen's model. The well-posedness of the problems is discussed under low regularity assumptions and the error estimates for two time-discrete schemes (based on the backward Euler approximation) are established. In this contribution, the focus lies on a overview of already obtained results and on a full discretization of the considered models.

Key words: integro-partial differential equation, nonlocal superconductors, singular convolution kernel, full discretization, error estimates

MSC 2000: 35L20, 35Q61, 35R09, 65M20, 82D55

1 Introduction

It is well-known that industrial applications require macroscopic models and their mathematical analysis for superconductivity. In their phenomenological theory of superconductivity in 1935, London and London explained that a macroscopic description of type-I superconductors involves a two-fluid model [1, 2]. The current density $J$ is supposed to be the sum of a normal part ($J_n$) and superconducting part ($J_s$). It is assumed that a superconductive material of type-I occupies a bounded domain $\Omega \subset \mathbb{R}^3$ with a Lipschitz continuous boundary $\partial \Omega$. The outward unit normal vector on $\partial \Omega$ is denoted by $\nu$. The primary objective was to derive mathematical models for type-I superconductors. For this reason, the full Maxwell equations ($\delta = 1$) and the quasi-static Maxwell equations ($\delta = 0$)
for linear materials are considered. This means that a linear dependence of the magnetic induction field $B$ and the electric displacement field $D$ on respectively the magnetic field $H$ and the electric field $E$ is assumed,

$$B = \mu H \quad \text{and} \quad D = \epsilon E,$$

where the constant $\mu > 0$ stands for the magnetic permeability and the constant $\epsilon > 0$ denotes the electric permittivity. In agreement with our previous notations, the quasi-static and the full Maxwell equations can be combined as

$$\nabla \times H = J + \delta \partial_t D = J_n + J_s + \delta \epsilon \partial_t E,$$

Ampère's law \hspace{1cm} (1)

$$\nabla \times E = -\partial_t B = -\mu \partial_t H.$$ Faraday's law \hspace{1cm} (2)

Applying the divergence operator to the Faraday law (2) and integrating in time gives

$$\nabla \cdot H(t) = \nabla \cdot H(t = 0).$$

Therefore, assuming $\nabla \cdot H(t = 0) = 0$, it is ensured that the magnetic induction remains divergence free for any time $t$. The normal density current $J_n$ is required to satisfy Ohm's law $J_n = \sigma E$, $\sigma > 0$ being the conductivity of the normal electrons. The nonlocal representation of the superconductive current by Eringen [3] is considered for the superconductive part of the current $J_s$. This representation identifies the state of the superconductor, at time $t$, with the field $H(\cdot, t)$ and is given by the linear functional

$$J_s(x, t) = \int_{\Omega} \sigma_0 \left( \frac{|x - x'|}{(x - x') \times H(x', t)} \right) \, dx' = -(\kappa_0 * H)(x, t),$$

for all $(x, t) \in \Omega \times (0, T)$, where the singular kernel $\sigma_0 : (0, \infty) \rightarrow \mathbb{R}$ is defined by

$$\sigma_0(s) = \begin{cases} \frac{\tilde{C}}{4\pi s^{3/2}} \exp \left( \frac{3}{4}\frac{s}{\xi_0^2} \right) & s < r_0; \\ 0 & s \geq r_0, \end{cases}$$

with $\tilde{C} := \frac{3}{4\pi\xi_0} > 0$. The length $\xi_0$ is called the coherence length of the material and $\Lambda = \frac{n_s}{m_e c^2}$, with $n_s$ the number of superelectrons per unit volume, $m_e$ and $-e$ the mass and the electric charge of an electron respectively. The points which contribute to the integral are separated by distances of order $r_0$ or less, where $r_0$ is defined by

$$\frac{1}{r_0} = \frac{1}{\xi_0} + \frac{1}{l},$$

with $l$ the mean free path of the electrons in the material. Taking the curl of (1) and the time derivative of (2) results into the following parabolic ($\delta = 0$) and hyperbolic ($\delta = 1$) integro-differential equation

$$\delta \epsilon \mu \partial_t \nabla H + \sigma \mu \partial_t \nabla H + \nabla \times \nabla \times H + \nabla \times (\kappa_0 * H) = 0.$$ (3)
2 Mathematical Models

It is assumed without loss of generality that $\epsilon = \mu = \sigma = 1$ in (3). Also a possible source term $f$ is considered in the right-hand side. A distinction is made between $\delta = 0$ and $\delta = 1$. If $\delta = 0$, the following parabolic model ($\delta = 0$) is studied

$$
\begin{align*}
\frac{\partial_t H}{H} + \nabla \times \nabla \times H + \nabla \times (K_0 \ast H) &= f & \text{in } Q_T := \Omega \times (0,T); \\
H \times \nu &= 0 & \text{on } \partial \Omega \times (0,T); \\
H(x,0) &= H_0 & \text{in } \Omega.
\end{align*}
$$

(4)

If $\delta = 1$, the corresponding hyperbolic model is given by

$$
\begin{align*}
\frac{\partial_t H}{H} + \frac{\partial_x H}{H} + \nabla \times \nabla \times H + \nabla \times (K_0 \ast H) &= f & \text{in } Q_T := \Omega \times (0,T); \\
H \times \nu &= 0 & \text{on } \partial \Omega \times (0,T); \\
H(x,0) &= H_0 & \text{in } \Omega; \\
\partial_t H(x,0) &= H'_0 & \text{in } \Omega.
\end{align*}
$$

(5)

The well-posedness of both problems is discussed under low regularity assumptions [4, 5]. The results are summarized in the following section.

3 Results

To address the existence of a solution to both models, the semidiscretization in time is employed. This discretization is based on Rothe's method [6]. The interval $[0,T]$ is divided into $n$ equidistant subintervals $[t_{i-1}, t_i]$ with time step $\tau = \frac{T}{n} < 1$, thus $t_i = i\tau$, $i = 1, \ldots, n$.

The standard notations for the discretized fields are used

$$
h_i = H(t_i) \quad \text{and} \quad \delta h_i = \frac{h_i - h_{i-1}}{\tau}.
$$

Using these notations, the following piecewise linear in time vector field $h_n$ is defined

$$
\begin{align*}
h_n(0) &= H_0 \\
h_n(t_i) &= h_{i-1} + (t - t_{i-1}) \delta h_i & \text{for } t \in (t_{i-1}, t_i), \quad i = 1, \ldots, n.
\end{align*}
$$

The existence and uniqueness of a solution of both problems, as limit of the sequence $\{h_n\}$ if $n \to \infty$, can be shown under appropriate conditions. These can be found in the following theorem.

**Theorem 1**

(i) Let $H_0 \in L^2(\Omega)$ and $f \in L^2((0,T),L^2(\Omega))$. Assume that $\nabla \cdot H_0 = 0 = \nabla \cdot f(t)$ for any time $t \in [0,T]$. Then there exists a vector field $H \in C([0,T],L^2(\Omega)) \cap L^2((0,T),H^1(\Omega))$ with $\partial_t H \in L_2((0,T),H^1(\Omega))$, which is a weak solution of (4). If $H_0 \in H_0(\text{curl},\Omega)$, then $\partial_t H \in L_2((0,T),L^2(\Omega))$;
(ii) Let $H_0 \in H_0(\text{curl}, \Omega), H_0' \in L^2(\Omega)$ and $f \in L^2((0,T), L^2(\Omega))$. Assume that $\nabla \cdot H_0 = \nabla \cdot H_0' = 0 = \nabla \cdot f(t)$ for any time $t \in [0,T]$. Then there exists a vector field $H$ such that $H$ is a weak solution of (5) with $H \in C([0,T], H^{\frac{1}{2}}(\Omega))$, $\partial_t H \in L_2((0,T), H^{\frac{3}{2}}(\Omega)) \cap C([0,T], L^2(\Omega))$ and $\partial_{tt} H \in L^2((0,T), H^{-1}_0(\text{curl}, \Omega))$.

For each model, two time-discrete numerical schemes are developed to approximate the solution. In the first one, the convolution is taken implicitly (from the actual time step). In the second one, the convolution is taken explicitly (from the previous time step). The convergence of this schemes is shown [4, 5]. The following theorem addresses the error estimates for the time discretization. For each model, the same error estimates are obtained for the different schemes.

Theorem 2 Suppose that $f \in \text{Lip}([0,T], L^2(\Omega))$.

(i) Suppose that $\delta = 0$. If $H_0 \in H_0(\text{curl}, \Omega)$ then

$$\max_{t \in [0,T]} \|h_n(t) - H(t)\|^2 + \int_0^T \|\nabla \times [h_n - H]\|^2 \leq C\tau;$$

(ii) Suppose that $\delta = 1$. If $H_0 \in H_0(\text{curl}, \Omega)$ and $H_0' \in L^2(\Omega)$ then

$$\max_{t \in [0,T]} \|h_n(t) - H(t)\|^2 + \max_{t \in [0,T]} \left\| \nabla \times \int_0^t [h_n - H] \right\|^2 \leq C\tau.$$
References


