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Moufang sets: polarities, mixed Moufang sets and inclusions

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1 Introduction

The main objects in this thesis are Moufang sets. Moufang sets were introduced by Jacques Tits in [33] as an axiomatization of the simple algebraic groups of relative rank one, and they are, in fact, the buildings corresponding to these algebraic groups, together with some of the group structure (which comes from the root groups of the algebraic group). In this way, Moufang sets are a powerful tool to study these algebraic groups. On the other hand, the notion of a Moufang set is more general, and includes many more examples that do not directly arise by this procedure. In fact, it is still a wide open question whether every Moufang set is, in some sense, of algebraic origin.

We start the first chapter of this thesis with the general definition of a Moufang building and introduce in this context Moufang sets and Moufang polygons. Next, we state some properties of Moufang sets and give a few important examples. A notion closely related to Moufang buildings is the notion of a BN-pair for buildings of rank at least two and split, saturated BN-pairs of rank one for Moufang sets. Moufang buildings of rank at least two have been completely classified, see for instance [34] for the classification of all Moufang buildings of rank two and [32] for a classification of all buildings of rank higher than two. Moufang sets are then defined as buildings of rank one for which the root groups satisfy certain Moufang-like conditions. For the rank one case, there does not yet exist a classification at all.

An important topic in the theory of Moufang sets is their link with linear algebraic groups. Every linear algebraic group of relative rank one gives rise to a Moufang set. Quite often, the structure of this Moufang set is a lot easier to deal with than the actual linear algebraic group. We hope to get more information about the linear algebraic groups using the theory and properties of these Moufang sets. Therefore, an important part of this thesis is about the determination of the Moufang sets corresponding to certain types of linear algebraic groups of relative rank one. We give a definition of linear algebraic groups and try to introduce the basic concepts needed to be able to work with them.

The third chapter of this thesis discusses polarities on octonion planes. Very recently, all polarities on Moufang planes have been classified by N. Knarr and M. Stroppel in [19] and [18] for arbitrary characteristics. We describe a general procedure to construct Moufang sets from polarities on these planes. More specifically, we show how the set of fixed flags under this involution has in a natural way the structure of a Moufang set. Next, we use the classification of polarities to obtain a classification of the corresponding Moufang sets. Eventually, we find four different types of Moufang sets; three of them are already known, but one type had not yet been described in the literature. We prove that this new class of Moufang sets corresponds to a class of linear algebraic groups of relative rank one. More precisely, we obtain that the centralizer in the little projective group of the octonion plane of a polarity corresponding to this new class is exactly a group of type ${}^2\mathsf{E}_{6,1}^{29}$. In fact, we found an elegant description for Moufang sets corresponding to linear algebraic groups of type ${}^2\mathsf{E}_{6,1}^{29}$.

In the fourth chapter, we construct a new class of Moufang sets arising from mixed buildings of type F_4. These are buildings in characteristic two defined over two fields K and L such that $L^2 \leq K \leq L$. We have a closer look at the groups corresponding to these buildings; these groups (often denoted by $\mathsf{F}_4(K, L)$) are mixed Chevalley groups of type F_4. We now can construct Moufang sets out of these mixed groups by defining an involution σ as follows. Every mixed Chevalley group has a natural BN-pair, so our goal is to construct σ in such a way that we find a subgroup of $\mathsf{F}_4(K, L)$ fixed by σ which has a split, saturated BN-pair of rank one. Since there is a one-to-one relationship between Moufang sets and split, saturated BN-pairs of rank one, we then obtain from such a BN-pair a nice description for the corresponding Moufang sets. We call this new type of Moufang set "Moufang sets of type mixed F_4". We remark that the methods used in this chapter can be applied to construct and describe several 'algebraic' Moufang sets; i.e. Moufang sets corresponding to linear algebraic groups of relative rank one.

Finally, in the last chapter we study Moufang subsets of some classes of algebraic Moufang sets in characteristic different from two. In [10], T. De Medts studied the problem of classifying all subpolygons of a given Moufang polygon.

As all Moufang polygons have been classified, this classification can be used to check each type of Moufang polygon separately. For Moufang subsets of algebraic Moufang sets on the contrary, we cannot use such a classification because it is not known whether or not all Moufang sets are of algebraic origin. We show that in characteristic different from two a Moufang subset of a non-abelian Moufang set obtained from a composition division algebra again corresponds to a composition division algebra or is isomorphic to $\mathbb{M}(J)$, with $\mathbb{M}(J)$ a Moufang set corresponding to some quadratic Jordan division algebra J.

In the second section of this chapter, we investigate Moufang subsets of hermitian Moufang sets $\mathbb{M}(U, \tau)$. Here, we do not obtain a full classification, but manage to show that a large class of Moufang subsets is again hermitian or a Moufang set corresponding to a quadratic Jordan division algebra J.

2
Preliminaries

We start this first chapter by recalling some basic definitions and theorems of algebra and geometry. We introduce composition algebras, quadratic Jordan algebras, Moufang buildings, Moufang sets and linear algebraic groups and briefly explain their connection.

2.1 Algebras

2.1.1 Composition algebras

Let V be a vector space over a field k and let N be a quadratic form on V. Then N is called non-degenerate if the corresponding bilinear form f is non-degenerate, i.e. $V^{\perp} = 0$.

A *composition algebra* C over a field k is a (not necessary associative) algebra over k with identity element e such that there exists a multiplicative non-degenerate quadratic form N on C. A composition subalgebra C' is then a subalgebra of C such that the restriction of N to C' remains non-degenerate. The *standard involution* σ on C is defined as

$$\sigma : C \to C; x \mapsto \langle x, e \rangle e - x,$$

with $\langle , \rangle$ the inner product corresponding to N. It follows easily from this definition that $\mathrm{N}(x) = \sigma(x)x = x\sigma(x)$. The fact that such a multiplicative norm exists imposes big restrictions on the algebra C as the following theorem shows:

Theorem 2.1.1. *The only possible dimensions of a composition algebra C over k are 1, 2, 4 or 8. A composition algebra of dimension two is a separable quadratic extension of k. Composition algebras of dimension 4 over k are called quaternion algebras, while composition algebras of dimension 8 are called octonion algebras.*

Proof. See [27, Theorem 1.6.2] for a proof. □

One can show that quaternions (or quaternion algebras) are associative but not commutative, while octonions (or octonion algebras) are not commutative and not associative. *Composition division algebras*, i.e. with N anisotropic, often appear in the theory of linear algebraic groups and Moufang sets as we will see later on.

Octonion algebras are non-associative composition algebras. Every composition algebra C however does satisfy the following *Moufang identities* (see for instance [27, Proposition 1.4.1]) :

$$\begin{aligned}(ax)(ya) &= a((xy)a) \\ a(x(ay)) &= (a(xa))y \\ x(a(ya)) &= ((xa)y)a\end{aligned} \tag{2.1.1}$$

for all $a, x, y \in C$.

2.1.2 Quadratic Jordan algebras

We define the concept of a quadratic Jordan algebra as in [21]. Let k be an arbitrary field, J a vector space over k and let 1 be a distinguished element in $J^* := J \setminus \{0\}$. Suppose there exists a linear map $U_x : J \to J$ for every $x \in J$, such that $U : J \to \mathrm{End}_k(J);\ x \mapsto U_x$ is quadratic, i.e. $U_{tx} = t^2 U_x$ for every $t \in k$ and

$$\phi : J \times J \to \mathrm{End}_k(J);\ (x, y) \mapsto U_{x,y} := U_{x+y} - U_x - U_y$$

is bilinear. Define $V_{x,y}z$ as $U_{x,z}y$, then the triple $(J, U, 1)$ is a *quadratic Jordan algebra* if the following identities hold strictly (i.e. they should continue to hold over scalar extensions of J):

(QJ1) $U_1 = \mathrm{id}_J$
(QJ2) $V_{x,y}U_x = U_x V_{y,x}$

(QJ3) $U_{U_x y} = U_x U_y U_x$.

An element x of J is invertible if U_x is invertible and the inverse is given by $x^{-1} := U_x^{-1}(x)$. In this thesis, we will only be interested in quadratic Jordan division algebras; these are quadratic Jordan algebras such that every element of $J^* := J \setminus \{0\}$ is invertible.

2.2 Geometries

2.2.1 Simplicial complexes

Complex

A *simplicial complex* (S, X) is a set S together with a set X of subsets of S, such that the union of all elements of X forms the set S and such that every subset of an element of X belongs to X. We call the elements of X *simplices*, the maximal elements *chambers* and the objects obtained by removing one element of a chamber a *panel*. Two chambers are called *adjacent* if they have a panel in common.

Chamber complex

A *chamber complex* is a simplicial complex (S, X) such that all chambers are finite and have the same cardinality. Furthermore, every two chambers can be joined by a sequence of chambers such that every two subsequent chambers are adjacent. The minimal length of such a sequence is called the *distance* between two maximal simplices. Two elements of X are called *incident* if they have at least one upper bound in common.

A *chamber subcomplex* (S', X') of a chamber complex (S, X) is a chamber complex such that $S' \subseteq S$ and $X' := \{A \in X : A \subseteq S'\}$. A chamber complex is called *thick* if every panel belongs to at least three chambers and *thin* if every panel is in exactly two chambers.

Coxeter complex

A Coxeter matrix M is defined as a symmetric n by n matrix $M = (m_{ij})$ whose entries m_{ij} are elements of $\mathbb{N} \cup \{\infty\}$ such that $m_{ii} = 1$ for all i and $m_{ij} \geq 2$ for all $i \neq j$.

The *Coxeter group* W corresponding to M is then defined as

$$W := \langle w_1, \ldots w_n \mid (w_i w_j)^{m_{ij}} = 1,\ \forall i, j \in \{1, \ldots, n\}\rangle.$$

Out of W, we can construct a chamber complex (W, S), which we call a *Coxeter complex*. Suppose J is a subset of $\{1, \ldots, n\}$, then the corresponding generators in W generate a subgroup W_J. We define the elements of the complex as the left cosets of Coxeter subgroups W_J with $|J| = n - 1$ and the simplices as the left cosets of all subgroups W_J, $J \subseteq \{1, \ldots, n\}$. An element then belongs to a simplex if the corresponding maximal coset contains the coset corresponding to the simplex. We define the *rank* of the Coxeter complex (W, S) as n.

A *spherical Coxeter complex* (W, S) is a complex for which W is finite. Suppose next that two chambers C and D of (W, S) contain the same panel, then we define the *root* corresponding to C (and D) as the set of all chambers for which the distance to C is strictly smaller than the distance to D.

2.2.2 Buildings

Let Δ be a chamber complex and let $\mathcal{A}$ be a set of subcomplexes of Δ. The pair $(\Delta, \mathcal{A})$ is called a *building*, of which the elements of $\mathcal{A}$ are called *apartments*, if the following conditions hold:

(B1) Δ is thick.
(B2) The elements of $\mathcal{A}$ are thin chamber complexes.
(B3) Any two elements of Δ belong to an apartment.
(B4) If two apartments Σ and Γ contain two elements $A, B \in \Delta$, there exists an isomorphism from Σ onto Γ which leaves invariant A, B and all their simplices.

One can show this definition implies that all apartments of Δ are isomorphic and are in fact Coxeter complexes. If the corresponding Weyl group is finite, we call the building *spherical*. The *rank* of Δ is then defined as the rank of the Coxeter complex.

An important subclass of the buildings is the class of *Moufang buildings*, we only consider spherical Moufang buildings. The roots of these buildings play an important role in their definition. We first define the interior α° of a root α as the set of all panels contained in two chambers of α. Next, we associate to every root α a subgroup U_α of the automorphism group of Δ. More specifically, we define U_α as the group of all automorphisms which act trivially on each chamber containing a panel of α°.

A building satisfies the Moufang condition or is Moufang if every root group U_α acts transitively on the set of apartments containing α. For the sequel, we will mostly be interested in Moufang buildings of rank two, which are the Moufang polygons and in Moufang sets, which are slightly more general objects than Moufang buildings of rank one.

Remark 2.2.1. We briefly mention that one can associate to every building an incidence geometry. Simplices of the building then correspond to flags in the geometry, so vertices (i.e. simplices consisting of only one element) are exactly points, lines, planes and so on of the geometry. On the other hand, chambers correspond to maximal flags. Also, two vertices are incident if and only if the corresponding points, lines, planes and so on are incident.

2.3 Moufang sets

2.3.1 Definition

A *Moufang set* $\mathbb{M} = \big(X, (U_x)_{x\in X}\big)$ is a set X together with a collection of groups $U_x \leq \mathrm{Sym}(X)$, such that for each $x \in X$:

(1) U_x fixes x and acts sharply transitively on $X \setminus \{x\}$;
(2) $U_x^\varphi = U_{x\varphi}$ for all $\varphi \in G := \langle U_z \mid z \in X\rangle$.

The group G is called the *little projective group* of the Moufang set.

2.3.2 An explicit construction of Moufang sets

We will now explain how any Moufang set can be reconstructed from a single root group together with one additional permutation [14].

Let $(U, +)$ be a group, with identity 0, and where the operation + is *not necessarily commutative*. Let $X = U \cup \{\infty\}$, where ∞ is a new symbol. For each $a \in U$, we define a map $\alpha_a \in \mathrm{Sym}(X)$ by setting

$$\alpha_a\colon \begin{cases} \infty \mapsto \infty \\ x \mapsto x + a \quad \text{ for all } x \in U. \end{cases} \tag{2.3.1}$$

Let

$$U_\infty := \{\alpha_a \mid a \in U\}\,.$$

Now let τ be a permutation of $U^* := U \setminus \{0\}$. We extend τ to an element of $\mathrm{Sym}(X)$ (which we also denote by τ) by setting $0^\tau = \infty$ and $\infty^\tau = 0$. Next we set

$$U_0 := U_\infty^\tau \text{ and } U_a := U_0^{\alpha_a} \tag{2.3.2}$$

for all $a \in U$. Let

$$\mathbb{M}(U,\tau) := \big(X, (U_x)_{x\in X}\big) \tag{2.3.3}$$

and let

$$G := \langle U_\infty, U_0\rangle = \langle U_x \mid x \in X\rangle \,.$$

Then $\mathbb{M}(U,\tau)$ is not always a Moufang set, but every Moufang set can be obtained in this way. The next lemma shows us how to do this.

Lemma 2.3.1. *Let $\mathbb{M} = (X, (U_x)_{x\in X})$ be a Moufang set. Choose two elements $0, \infty \in X$ and define U as $X \setminus \{\infty\}$.*
For every $a \in U$, define $\alpha_a \in U_\infty$ as the unique element such that $\alpha_a(0) = a$. Let $a+b := \alpha_b(a)$ for every $a, b \in U$ and $\tau \in \mathrm{Sym}(X)$ be a permutation interchanging 0 and ∞ such that $U_\infty^\tau = U_0$. Then $\mathbb{M} = \mathbb{M}(U,\tau)$.

Proof. This is obvious from the above construction of $\mathbb{M}(U,\tau)$. □

Note that, for a given Moufang set, the map τ is certainly not unique: different choices for τ can give rise to the same Moufang set.

2.3.3 Hua maps and μ-maps

For every element $a \in U^*$, we define the Hua map h_a of $\mathbb{M}(U,\tau)$ as

$$h_a : X \to X : x \mapsto \tau(\tau^{-1}[\tau x + a] - \tau^{-1}[a]) - \tau(-\tau^{-1}a)$$

with $X := U \cup \{\infty\}$. For every $a \in U^*$, the map h_a fixes 0 and ∞. These maps can be used in the following way to determine whether or not the construction given in the previous subsection actually gives rise to a Moufang set:

Theorem 2.3.2. *The set $\mathbb{M}(U,\tau)$ obtained via subsection 2.3.2 is a Moufang set if and only if for every $a \in U^*$ the Hua map h_a restricted to U is additive.*

Proof. For a proof of this theorem, we refer to [14, Theorem 3.2]. □

Let $\mathbb{M}(U,\tau)$ be a Moufang set, then one can show (see for example [12, Proposition 4.1.1]) that if $a \in U^*$, there is a unique permutation $\mu_a \in U_0^*\alpha_a U_0^*$ interchanging 0 and ∞. For every $a \in U^*$, this map has the following form:

$$\mu_a : X \to X;\; x \mapsto \alpha^\tau_{-(\tau^{-1}a)}(\alpha^\tau_{\tau^{-1}(-a)}[x] + a).$$

Unlike the Hua maps, the maps μ_a (often called μ-maps) are independent of the choice of τ. Also, every μ-map μ_a has the nice property that $\mathbb{M}(U,\tau) = \mathbb{M}(U,\mu_a)$. We finish by mentioning the nice relationship between Hua maps and μ-maps; for every $a \in U^*$ we have that

$$\mu_a = h_a \tau^{-1}.$$

A proof of this equality can be found in [12, Proposition 4.3.1].

2.3.4 Some classes of Moufang sets

Moufang sets from quadratic Jordan division algebras

Let $(J, U, 1)$ be a quadratic Jordan division algebra. Now let $(U, +) = (J, +)$ be the additive group of J and define the following permutation on U^*:

$$\tau \colon U^* \to U^* \colon x \mapsto -x^{-1}.$$

Then by [14], $\mathbb{M}(U,\tau)$ is a Moufang set which we will denote by $\mathbb{M}(J)$. The Hua maps of this Moufang set are given by

$$h_a : X \to X;\ x \mapsto a - (a^{-1} - (a - x^{-1})^{-1})^{-1}$$

and one can show (see for instance [14, Theorem 4.1]) they coincide with the maps U_a for every $a \in U^*$.

An important subclass of these Moufang sets are the *projective Moufang sets*, which we now describe.

Let $(D, +, \cdot)$ be an alternative division ring, i.e. a not necessarily associative ring (with 1) such that for each $a \in D^{*}$[1] there exists some element $a^{-1} \in D^*$ for which $a \cdot a^{-1}b = b = ba^{-1} \cdot a$ for every $b \in D$ and such that the following 'alternative laws' hold for every $x, y \in D$:

$$x(xy) = (xx)y$$
$$(yx)x = y(xx).$$

By the Bruck–Kleinfeld theorem, every alternative division ring is either associative (hence a skew field), or it is an octonion division algebra.

By defining $U_a(b) := aba$, the algebra $(D, +, \cdot)$ becomes a quadratic Jordan division algebra.

[1] In general, if $(D, +, .)$ is a ring then we denote by D^* the set of all elements of D without the identity element of the addition $+$.

Now let $(U,+) = (D,+)$ be the additive group of D, and define the permutation on U^* as:

$$\tau\colon U^* \to U^*\colon x \mapsto -x^{-1}.$$

Then $\mathbb{M}(U,\tau)$ is a Moufang set, Moufang sets obtained in this way are called *projective Moufang sets* and are often denoted by $\mathbb{M}(D)$. In this case, the Hua maps simplify to

$$h_a : X \to X;\ x \mapsto axa.$$

Hermitian Moufang sets

Let K be a field or a skew field equipped with an involution σ. Let V be a right vector space over K and set

$$K_\sigma^- = \{a - a^\sigma \mid a \in K\}.$$

A map $h : V \times V \to K$ is called *hermitian* with respect to σ if $h(at, bs) = t^\sigma h(a,b)s$ and if moreover $h(a,b)^\sigma = h(b,a)$ for all $a, b \in V$ and all $t, s \in K$.

A map q from V to K is a hermitian pseudo-quadratic form on V with respect to σ if there is a form h on V which is hermitian with respect to σ such that q and h satisfy

(i) $q(a+b) \equiv q(a) + q(b) + h(a,b) \mod K_\sigma^-$,
(ii) $q(at) \equiv t^\sigma q(a)t \mod K_\sigma^-$

for all $a, b \in V$ and $t \in K$. We say that q is anisotropic if

(iii) $q(a) \equiv 0 \mod K_\sigma^-$ only if $a = 0$.

Let $(T,\cdot)$ denote the group with underlying set

$$\{(a,t) \in V \times K \mid q(a) - t \in K_\sigma^-\}$$

with $(a,t)\cdot(b,u) := (a+b, t+u+h(b,a))$. Define a permutation τ on T^* by setting

$$\tau(a,t) = (at^{-1}, t^{-1}). \tag{2.3.4}$$

Then $\mathbb{M}(T,\tau)$ is a Moufang set. Moufang sets obtained in this way are called *hermitian Moufang sets*. For a more detailed description, we refer to [34, (11.15), (11.16) and (16.18)].

Moufang sets of type F_4

Let $\mathcal{O}$ be an octonion division algebra over a commutative field k. Let $x \mapsto \overline{x}$ be the standard involution, N the multiplicative norm with $\mathrm{N}(x) = x \cdot \overline{x}$ and T the trace map $\mathrm{T}(x) = x + \overline{x}$ on $\mathcal{O}$. We define a set U by

$$U := \{(a, b) \in \mathcal{O} \times \mathcal{O} \mid \mathrm{N}(a) + \mathrm{T}(b) = 0\}$$

and the following (non-abelian) group operation on U:

$$(a, b) + (c, d) := (a + c, b + d - \overline{c} \cdot a)$$

for all $(a, b),\ (c, d) \in U$. Define a permutation τ on U^* by setting

$$\tau(a, b) = (-a \cdot b^{-1}, b^{-1})$$

for all $(a, b) \in U^*$, then $\mathbb{M}(U, \tau)$ is a Moufang set. Moufang sets obtained in this way are called *Moufang sets of type* F_4 since they are Moufang sets arising from linear algebraic groups of relative rank one and of absolute type F_4. We refer to [13] for more details.

2.4 Moufang polygons

2.4.1 Generalized polygons

A *generalized n-gon* ($n \geq 3$) is a geometry $\Gamma = (\mathcal{P}, \mathcal{L}, I)$ such that:

(GP1) Γ does not contain ordinary k-gons, for every $k \in \{2, \ldots n-1\}$.
(GP2) Every $x, y \in \mathcal{P} \cup \mathcal{L}$ are contained in an ordinary n-gon of Γ.

Every ordinary n-gon is called an *apartment*. The subgraph spanned by the vertices of an n-path in Γ is called a *root* of Γ.

2.4.2 The Moufang property

Let $\alpha = (x_0, \ldots, x_n)$ be a root of Γ and let γ denote the subpath $(x_1, \ldots, x_{n-1})$ of α. We define the *root group* U_α as the group of all γ-elations. Here, a γ-*elation* is an automorphism g of Γ such that g fixes all elements of Γ incident with at least one element of γ.

We say Γ satisfies the Moufang condition or Γ is a *Moufang polygon* if for every root α the root group U_α acts regularly on the set of apartments through α.

Fix a flag $x = (x_n, x_{n+1})$ and an apartment $\Sigma = (x_0 = x_{2n}, x_1, \ldots, x_{2n-1})$ of Γ containing x. We define the group U_+ corresponding to (Σ, x) as the group $U_{\alpha_1} \ldots U_{\alpha_n}$ with α_i the root $(x_i, \ldots, x_{i+n})$. So in fact, U_+ is the group generated by those root groups corresponding to the roots of Σ containing the flag x.

One can show that the Moufang polygons described here are in fact the same objects as the Moufang buildings of rank two.

2.5 BN-pairs

We give definitions for BN-pairs of arbitrary rank. These objects are of great importance because of their natural link with buildings.

2.5.1 BN-pairs

A BN-pair is a pair of subgroups (B, N) of a group G such that

(BN1) $G = \langle B, N \rangle$.

(BN2) $B \cap N$ is a normal subgroup of N.

(BN3) $W = N/B \cap N$ is generated by elements w_i such that $w_i^2 = 1$, $i \in I$. For each $i \in I$, we choose a preimage $n_i \in N$ of w_i.

(BN4) For each $i \in I$ and each $n \in N$, we have

$$Bn_iB \cdot BnB \subseteq Bn_inB \cup Bnb.$$

(BN5) For each $i \in I$, we have $n_iBn_i \neq B$.

The *rank* of a BN-pair is defined as the cardinality of I. A subgroup P of G is called a parabolic subgroup if P contains B or some conjugate of B. One can show that the parabolic subgroups containing B are of the form BN_JB with N_J the group of elements sent to W_J under the natural morphism from N to W. Here, W_J is the group generated by the elements $\{w_j \mid j \in J\}$ with $J \subseteq I$. Also, the nice property $G = BNB$ holds.

Out of every BN-pair, we can construct a building $\Delta(G, B)$. We indicate how to do this, for more details we refer to [2, Section 6.2.6]. The simplices of the building $\Delta(G, B)$ are defined as the parabolic subgroups P of G, ordered by the inverse inclusion relation. The standard apartment is defined as

$$\Sigma = \{wPw^{-1} \mid w \in W, P \geq B\},$$

and the set of apartments is $\mathcal{A} := \{\Sigma^g \mid g \in G\}$. The building $(\Delta(G, B), \mathcal{A})$ is called the building corresponding to the BN-pair (B, N). If we look at the incidence geometry corresponding to this building, we notice that the parabolic subgroups P of G are exactly the stabilizers of their corresponding flags.

2.5.2 Saturated, split BN-pairs of rank one

We introduce the notion of a saturated split BN-pair of rank one because of the correspondence with Moufang sets. We notice that the first three conditions of the definition correspond with those of a general BN-pair of rank one, while the last two conditions are supplementary.

A *BN-pair of rank one* in a group G is a system (B, N) of two subgroups B and N of G such that

(BN1) $G = \langle B, N\rangle$
(BN2) $H := B \cap N \trianglelefteq N$
(BN3) There is an element $\omega \in N \setminus H$ with $\omega^2 \in H$ such that $N = \langle H, w\rangle$, $G = B \cup B\omega B$ and $\omega B\omega \neq B$.

We call such a pair split[2] and saturated if additionally

(BN4) There exists a normal subgroup U of B such that $B = U \rtimes H$,
(BN5) $H = B \cap B^{\omega}$.

Such a BN-pair gives rise to a Moufang set:

Lemma 2.5.1. *Let G be a group with a saturated split BN-pair of rank one and let $X := \{U^g \mid g \in G\}$ be the set of conjugates of U in G. Then $(X, (x)_{x\in X})$ is a Moufang set.*

Proof. For a proof, see [12, Proposition 2.1.3.]. □

Also, the converse holds:

Lemma 2.5.2. *Let $\mathbb{M} = (X, (U_x)_{x\in X})$ be an arbitrary Moufang set, with little projective group G. Let $0, \infty$ be two arbitrary elements of X, let $B := G_\infty$, $N := G_{\{0,\infty\}}$, $H := G_{0,\infty}$ and $U := U_\infty$ and let ω be an element of G interchanging 0 and ∞. Then B, N, ω and U satisfy all the axioms of a saturated, split BN-pair of rank one.*

Proof. For a proof, we refer to [12, Proposition 2.1.5.] □

2.6 Linear algebraic groups

There are several ways to introduce linear algebraic groups; for instance one can define linear algebraic groups as some affine algebraic sets or use a functorial way

[2] For BN-pairs of rank at least two it is customary to assume that a split BN-pair is nilpotent as well.

to introduce them. We choose to restrict ourselves to the, in our opinion, most intuitive notion of linear algebraic groups, namely that of closed subgroups of n by n invertible matrices over some algebraically closed field K. This definition has as a disadvantage that it is not an intrinsic one, but will be more than general enough for our needs.

The following subsections are mainly based on a review paper [3] on linear algebraic groups of Armand Borel, who certainly may be considered as one of the experts in the research of linear algebraic groups. Also, deeper results were obtained from [4], a joint article of A. Borel and J. Tits. We end this section by explaining the connection between Moufang sets and linear algebraic groups of relative rank one.

2.6.1 Definition

Let k be an arbitrary field and K an algebraic closure of k. Let $K[X_1, \ldots, X_m]$ be a polynomial ring over K, then K^m can be equipped with the Zariski topology. The closed sets of this topology are of the form $V(I) := \{x \in K^m \mid f(x) = 0,\ \forall f \in I\}$, with I an ideal in the polynomial ring $K[X_1, \ldots, X_m]$. Because $M_n(K) \cong K^{n^2}$, it also has the Zariski topology. For the sequel, we will only be interested in the open subgroup $\mathsf{GL}(n, K)$ of $M_n(K)$ consisting of all n by n invertible matrices over K.

Definition 2.6.1. We define a *linear algebraic group* $\mathbf{G}$ as a closed subgroup of $\mathsf{GL}(n, K)$. Concretely, this means that there exists a set of polynomials $\{f_i \mid i \in J\}$ in $K[X_{11}, \ldots, X_{nn}]$ such that

$$\mathbf{G} := \{x \in \mathsf{GL}(n, K) \mid f_i(x) = 0, \forall i \in J\},$$

where x is interpreted as an element of K^{n^2}.

By mapping every element $g = (g_{ij}) \in \mathbf{G}$ to $(g_{11}, g_{12}, \ldots, g_{nn}, det(g)^{-1})$, the set $\mathbf{G}$ can be identified with the closed subgroup

$$\{(x, y) \in K^{n^2} \times K \mid f_i(x) = 0\ \forall i \in J,\ f((x, y)) := det(x) \cdot y = 1\}$$

of K^{n^2+1}. Here, we interpret the polynomials f_i as elements of $K[X_{11}, \ldots, X_{nn}, Y]$. The *coordinate ring* $K[\mathbf{G}]$ is then the quotient ring $K[X_{11}, \ldots, X_{nn}, Y]/I$ with I the ideal $\langle \{f_i \mid i \in J\} \cup f \rangle$ of $K[X_{11}, \ldots, X_{nn}, Y]$.

We say that the algebraic group is *defined over* a subfield k or equivalently that $\mathbf{G}$ is a k-group if I is generated by elements of $k[X_{11}, \ldots, X_{nn}, Y]$. Let $I_k := I \cap k[X_{11}, \ldots, X_{nn}, Y]$, then the *coordinate ring* $k(\mathbf{G})$ *of* $\mathbf{G}$ *over* k is

$k[X_{11}, \ldots, X_{nn}, Y]/I_k$. Finally, we can introduce the concept of k-rational points $\mathbf{G}(k)$ of $\mathbf{G}$, which is the set $\mathbf{G} \cap k^{n^2+1}$ or

$$\mathbf{G}(k) = \{x \in \mathsf{GL}(n,k) \mid f(x) = 0, \ \forall f \in I_k\}.$$

Later on, when we refer to linear algebraic groups over some arbitrary field k, we actually mean the group of k-rational points of some linear algebraic group $\mathbf{G}$ over an algebraic closure K of k.

2.6.2 Some important maps and subgroups of G

Let $\mathbf{G}$ and $\mathbf{G}'$ be two algebraic groups over K, then a map $\phi : \mathbf{G} \to \mathbf{G}'$ is a morphism of algebraic groups if ϕ is a group morphism and

$$\phi^* : K[G'] \to K[G]; f \mapsto f \circ \phi$$

is a ring morphism. If $\mathbf{G}$ and $\mathbf{G}'$ are defined over k, then ϕ is a *k-morphism* if $\phi^*(k[\mathbf{G}']) \subseteq k[\mathbf{G}]$.

A *character* χ of $\mathbf{G}$ is a morphism from $\mathbf{G}$ to $\mathsf{GL}(1, K)$. We denote the set of all characters of $\mathbf{G}$ by $X(\mathbf{G})$. The multiplication in $\mathsf{GL}(1, K)$ induces an abelian group structure on $X(\mathbf{G})$. Moreover, one can show that $X(\mathbf{G})$ is finitely generated.

Let $\mathbf{H}$ be a closed subgroup of $\mathbf{G}$, then it is a *k-subgroup* of $\mathbf{G}$ if $\mathbf{H}$ is defined over k. Next, we introduce some important types of linear algebraic (sub)groups.

Definition 2.6.2. (i) An algebraic group $\mathbf{G}$ is a *torus* if it is isomorphic to d copies of $\mathsf{GL}(1, K)$.

(ii) We call an algebraic group *solvable* or *nilpotent* if it is solvable or nilpotent as an abstract group. An algebraic group $\mathbf{G}$ is called *unipotent* if every element of $\mathbf{G}$ is unipotent.

(iii) The *radical* $R(\mathbf{G})$ of $\mathbf{G}$ is the largest connected normal solvable subgroup of $\mathbf{G}$, while the *unipotent radica*l $R_u(\mathbf{G})$ is the largest connected unipotent normal solvable subgroup of $\mathbf{G}$. We say $\mathbf{G}$ is *semisimple* if $R(\mathbf{G}) = \{e\}$ and *reductive* if $R_u(\mathbf{G}) = \{e\}$.

We state the following theorem without proof:

Theorem 2.6.3. *Let $\mathbf{G}$ be a connected linear algebraic group.*

1. *All maximal tori of $\mathbf{G}$ are conjugate. The (absolute)* rank *of an algebraic group is the common dimension of the maximal tori and is often denoted by $rk(\mathbf{G})$.*

2. *All maximal connected solvable subgroups of* **G** *are conjugate. These subgroups are called Borel subgroups.*

The closed subgroups of **G** containing a Borel subgroup are called *parabolic subgroups.*

From now on, we assume that **G** is connected, reductive and defined over k. In this case, one can show that **G** contains a maximal torus, which is defined over k. Let **T** be a torus defined over k, then we say **T** is a k-split torus if all characters of **T** are defined over k.

The following theorem is a nice result about the maximal k-split tori:

Theorem 2.6.4. *The maximal k-split tori of* **G** *are conjugate over k; this means they are conjugate by an element of* $\mathbf{G}(k)$*. Let* **S** *be such a torus, then the dimension of* **S** *is called the k-*rank *of* **G** *and is denoted by* $rk_k(\mathbf{G})$*.*

2.6.3 The absolute and relative root system of G

Besides the (absolute and relative) rank, another important invariant of linear algebraic groups is their root system. The root system can be determined using the Lie algebra of **G**. More information about the link between tangent spaces, Lie algebras and linear algebraic groups can be found in [1, Chapter 3].

Suppose A is a K-algebra and B an A-module. Then a K-linear map $\delta : A \to B$ is called a *K-derivation* if $\delta(a_1a_2) = a_1\delta(a_2) + a_2\delta(a_1)$ for all $a_1, a_2 \in A$. The set of all these K-derivations is denoted by $\mathrm{Der}_K(A, B)$.

Definition 2.6.5. We define the Lie algebra $\mathcal{L}(\mathbf{G})$ of **G** as

$$\mathcal{L}(\mathbf{G}) := \{\delta \in \mathrm{Der}_K(K[\mathbf{G}], K[\mathbf{G}]) \mid \delta \circ \lambda_g = \lambda_g \circ \delta,\ \forall g \in \mathbf{G}\}$$

with $\lambda_g : K[\mathbf{G}] \to K[\mathbf{G}]$ such that $\lambda_g(f)(x) = f(g^{-1}x)$ for all $g, x \in \mathbf{G}$ and $f \in K[\mathbf{G}]$.

With Lie bracket $[\delta_1, \delta_2] := \delta_1 \circ \delta_2 - \delta_2 \circ \delta_1$ for all $\delta_1, \delta_2 \in \mathcal{L}(\mathbf{G})$, $\mathcal{L}(\mathbf{G})$ becomes a Lie algebra. Next, every affine variety, so every linear algebraic group **G** has a *tangent space* $\mathcal{T}(\mathbf{G})_x$ in x, for all $x \in \mathbf{G}$. One can show that for every $x \in \mathbf{G}$ this space is isomorphic to $\mathrm{Der}_K(K[\mathbf{G}])_x := \mathrm{Der}_K(K[\mathbf{G}], K)$. The field K is made into a $K[\mathbf{G}]$-module by defining $k_1 * f := k_1 f(x)$ for all $k_1 \in K$ and all $f \in K[\mathbf{G}]$. Moreover, $\mathcal{T}(\mathbf{G})_e$ (with e the neutral element of **G**) and $\mathcal{L}(\mathbf{G})$ are isomorphic as vector spaces over K. Let $\phi : \mathbf{G} \to \mathbf{G}'$ be a morphism of linear algebraic groups, then the differential map of ϕ in $x \in \mathbf{G}$ is

$$(d\phi)_x : \mathcal{T}(\mathbf{G})_x \to \mathcal{T}(\mathbf{G}')_{\phi(x)}; d \mapsto d \circ \phi^*$$

with $\phi^* : K[\mathbf{G}'] \to K[\mathbf{G}];\ f \mapsto f \circ \phi$.

Let $\mathbf{G}$ be a reductive group and $\mathbf{T}$ a maximal torus, then $\mathbf{G}$ acts on itself by the inner automorphisms $\mathrm{Int}_g : \mathbf{G} \to \mathbf{G}; x \mapsto gxg^{-1}$ for every $g \in \mathbf{G}$. This map induces a K-morphism $\mathrm{Ad} : \mathbf{G} \to \hom(\mathcal{L}(\mathbf{G}), \mathcal{L}(\mathbf{G}));\ g \mapsto (d\mathrm{Int}_g)_e$.

We proceed by mentioning the following lemma, for a proof we refer to [1, Lemma 6.1.6]:

Lemma 2.6.6. *A linear algebraic group* $\mathbf{G}$ *over* K *is diagonalizable if and only if for each morphism* $\psi : \mathbf{G} \to \mathsf{GL}(n, K)$ *holds that*

$$K^n = \bigoplus_{\chi \in X(\mathbf{G})} V_\chi$$

with $V_\chi = \{v \in K^n \mid \psi(g)(v) = \chi(g)v,\ \forall g \in \mathbf{G}\}$.

As $\mathbf{T}$ is diagonalizable by definition, we can apply the previous lemma to $\mathrm{Ad}|_{\mathbf{T}}$ and find that

$$\mathcal{L}(\mathbf{G}) = \bigoplus_{\chi \in X(\mathrm{T})} \mathfrak{g}_\chi$$

with $\mathfrak{g}_\chi = \{x \in \mathcal{L}(\mathbf{G}) \mid \mathrm{Ad}(t)(x) = \chi(t)x,\ \forall t \in \mathbf{T}\}$.

We define the *root system* of $\mathbf{G}$ with respect to the maximal torus $\mathbf{T}$ as

$$\Phi(\mathbf{G}, \mathbf{T}) := \{0 \neq \alpha \in X(\mathbf{T}) \mid \mathfrak{g}_\alpha \neq 0\}.$$

The *Weyl group* $W(\mathbf{G}, \mathbf{T})$ is the finite quotient $N_{\mathbf{G}}(\mathbf{T})/C_{\mathbf{G}}(\mathbf{T})$, with $N_{\mathbf{G}}(\mathbf{T}) := \{g \in \mathbf{G} \mid \mathbf{T}^g = \mathbf{T}\}$ and $C_{\mathbf{G}}(\mathbf{T}) := \{g \in \mathbf{G} \mid gt = tg,\ \forall\ t \in \mathbf{T}\}$.

In a completely similar fashion, we define the *relative root system* as $\Phi(\mathbf{G}, \mathbf{S})$ with $\mathbf{S}$ a maximal k-split torus. The *Weyl group* ${}_kW(\mathbf{G}, \mathbf{S})$ *relative to* k is then defined as $N_{\mathbf{G}}(\mathbf{S})/C_{\mathbf{G}}(\mathbf{S})$.

One can show that both root systems are root systems in the traditional sense, i.e. abstract root systems.

Let V be a finite dimensional vector space over $\mathbb{R}$ and Φ a finite subset of V, then Φ is an *abstract root system* if

(R1) Φ spans V, $0 \notin \Phi$.
(R2) If $\alpha \in \Phi$, then the map

$$s_\alpha : V \to V; v \mapsto v - 2\frac{\langle v, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha,$$

with $\langle , \rangle$ being the standard inner product on V, satisfies $s_\alpha(\Phi) = \Phi$.

(R3) If $\alpha, \beta \in \Phi$, then $s_\alpha(\beta) - \beta$ is an integer multiple of α.

A root system is called *reduced* if for every $\alpha \in \Phi$, $\lambda\alpha \in \Phi$ implies that $\lambda = \pm 1$. A root system is *irreducible* if it cannot be written as the union of two other root systems. The reduced irreducible root systems are classified and are of type A_n, B_n, C_n, D_n, F_4, G_2, E_6, E_7 or E_8. For a description of these types of root systems, we refer to [8, Section 3.6]. The *Weyl group* $W(\Phi)$ of Φ is defined as the subgroup of $\mathsf{GL}(V)$ generated by s_α, $\alpha \in \Phi$.

Finally, we introduce the notion of a *Weyl-chamber*. We call a hyperplane singular if it is orthogonal to some $\alpha \in \Phi$. A Weyl-chamber is then a connected component of the complement of the union of the singular hyperplanes. One can show that the Weyl group acts simply transitively on the set of Weyl-chambers. We say that a root α is positive with respect to a certain Weyl-chamber C if $\langle \alpha, v\rangle > 0$ for every $v \in C$. A positive root is called simple if it is not the sum of two positive roots. The set of all simple roots is often denoted by Π.

Let $V := \Phi(\mathbf{G},\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{R}$, then one can show that V together with $\Phi(\mathbf{G},\mathbf{T})$ satisfies all the above axioms. The *type* of the linear algebraic group is defined as the type of the corresponding root system. We remark that the abstract Weyl group $W(\Phi(\mathbf{G},\mathbf{T}))$ is isomorphic to the Weyl group $W(\mathbf{G},\mathbf{T})$.

Similarly, the relative root system is an abstract root system and the corresponding abstract Weyl group is isomorphic to ${}_kW(\mathbf{G},\mathbf{S})$.

We end this subsection by introducing some important subgroups of $\mathbf{G}$ (see for instance [15, Section 7]).

Definition 2.6.7. Let α be a root of $\mathbf{G}$ with respect to a maximal torus $\mathbf{T}$, then we define the *root group* U_α of $\mathbf{G}$ as the unique closed connected unipotent subgroup of $\mathbf{G}$ fixed under conjugation with elements of $\mathbf{T}$ and with corresponding Lie algebra $\mathfrak{g}_\alpha$.

Moreover, one can show that there exists an isomorphism

$$u_\alpha : (K,+) \to U_\alpha$$

with $(K,+)$ the additive group of the field $(K,+,.)$ and such that

$$tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x)$$

for all $t \in \mathbf{T}, x \in K$.

Definition 2.6.8. Similarly, one can associate a *k-root group* $U_{(\beta)}$ to every root β of the relative root system (with respect to a maximal k-split torus $\mathbf{S}$). For every $\beta \in {}_k\Phi$, set

$$\mathfrak{g}_{(\beta)} := \begin{cases} \mathfrak{g}_\beta & \text{if } 2\beta \notin {}_k\Phi \\ \mathfrak{g}_\beta \oplus \mathfrak{g}_{2\beta} & \text{if } 2\beta \in {}_k\Phi. \end{cases}$$

The k-root group $U_{(\beta)}$ is then defined as the unique closed connected unipotent k-subgroup normalized by $C_{\mathbf{G}}(\mathbf{S})$ and with Lie algebra $\mathfrak{g}_{(\beta)}$.

2.6.4 Properties of tori and parabolic subgroups

Let $\mathbf{G}$ be a connected reductive linear algebraic group defined over k. We state without proof an important theorem which explains the relation between k-split tori and parabolic k-subgroups (for a proof, see [4, 4.13 and 4.18]):

Theorem 2.6.9. *(i) The minimal parabolic k-groups $\mathbf{P}$ of $\mathbf{G}$ are conjugate over k.*

(ii) Let $\mathbf{P}$ be a minimal k-parabolic subgroup, then there exists a k-split torus $\mathbf{S}$ such that $\mathbf{P} = C_{\mathbf{G}}(\mathbf{S}) \ltimes R_u(\mathbf{P})$. The group $C_{\mathbf{G}}(\mathbf{S})$ is called a Levi subgroup of $\mathbf{P}$. Every two minimal k-parabolic subgroups $\mathbf{P}$ and $\mathbf{P}'$ contain a maximal k-split torus $\mathbf{T}$ such that $C_{\mathbf{G}}(\mathbf{T}) \subseteq \mathbf{P} \cap \mathbf{P}'$.

(iii) The minimal k-parabolic subgroups containing $C_{\mathbf{G}}(\mathbf{S})$, with $\mathbf{S}$ a maximal k-split torus, are in one-to-one correspondence with the Weyl chambers. From this, it follows that the Weyl group ${}_kW(\mathbf{G})$ acts simply transitively on the set of minimal parabolic k-groups containing $C_{\mathbf{G}}(\mathbf{S})$.

Let $\mathbf{S}$ be a maximal k-split torus and C a Weyl-chamber. Denote the set of simple roots with respect to C by ${}_k\Pi$ and let ${}_k\Theta$ be a subset of ${}_k\Pi$. One can show that $\mathbf{S}_\Theta$, the identity component of $\cap_{\alpha\in\Theta} \ker \alpha$, is a k-split torus. The *standard parabolic k-subgroup* defined by ${}_k\Theta$ is then the group generated by $C_{\mathbf{G}}(\mathbf{S}_\Theta)$ and $U_{{}_k\Theta} := \langle \{U_\alpha \mid \alpha \in {}_k\Phi^+ \setminus \langle {}_k\Theta \rangle_{\mathbb{Z}}\}\rangle$. One can show that every parabolic k-subgroup $\mathbf{P}$ is conjugate over k to exactly one standard parabolic k-subgroup. Furthermore, it can be shown that $R_u(\mathbf{P})$, with $\mathbf{P}$ the standard parabolic subgroup defined by ${}_k\Theta$, is exactly the group $U_{{}_k\Theta}$.

We finish this section by introducing the concept of opposite parabolic subgroups. The properties of such opposite groups will play an important role when we define the Moufang set corresponding to a linear algebraic group of relative rank one later on.

Definition 2.6.10. Two parabolic subgroups $\mathbf{P}$ and $\mathbf{P}'$ are called *opposite* if $\mathbf{P} \cap \mathbf{P}'$ is a Levi subgroup of both $\mathbf{P}$ and $\mathbf{P}'$, i.e. $\mathbf{P} = (\mathbf{P} \cap \mathbf{P}') \ltimes R_u(\mathbf{P})$ and $\mathbf{P}' = (\mathbf{P} \cap \mathbf{P}') \ltimes R_u(\mathbf{P}')$ and both $R_u(\mathbf{P}) \cap \mathbf{P}'$, $R_u(\mathbf{P}') \cap \mathbf{P}$ are trivial.

Opposite parabolic subgroups have some nice properties as the following theorem illustrates:

Theorem 2.6.11. *(i) Every parabolic k-subgroup $\mathbf{P}$ is opposite to at least one parabolic k-subgroup $\mathbf{Q}$.*

(ii) Let $\mathbf{P}_1$ and $\mathbf{P}_2$ be two parabolic k-subgroups opposite to the same parabolic k-subgroup $\mathbf{P}$, then there exists a unique element $u \in R_u(\mathbf{P})(k)$ such that $\mathbf{P}_1^u = \mathbf{P}_2$.

2.6.5 Moufang buildings and linear algebraic groups

We show a standard way to construct Moufang buildings out of linear algebraic groups, more details can be found in [32, Chapter 5].

Let $\mathbf{G}$ be a connected reductive linear algebraic group defined over a field k. Let Δ denote the set of all parabolic k-subgroups of $\mathbf{G}$, then this set ordered by the opposite of the inclusion relation forms a chamber complex. For every maximal k-split torus $\mathbf{S}$, define $A_{\mathbf{S}}$ as the set of all parabolic k-subgroups containing $\mathbf{S}$.

Then $(\Delta, \mathcal{A})$ with $\mathcal{A} := \{A_{\mathbf{S}} \mid \mathbf{S}$ is a maximal k-split torus$\}$ is a Moufang building, and $\mathbf{G}(k)$ acts on this building by conjugation. The Weyl group of the building is exactly the relative Weyl group $N_{\mathbf{G}}(\mathbf{S})/C_{\mathbf{G}}(\mathbf{S})$. In particular, every two parabolic k-subgroups contain a common maximal k-split torus, so every two elements are contained in common apartment. We can say something more about the length of the chains of k-parabolic subgroups if $\mathbf{G}$ is semisimple. In this case, the k-rank d of $\mathbf{G}$ equals the rank of the relative root system ${}_k\Phi$, which is the dimension of the corresponding vector space over $\mathbb{R}$ or also the number of simple roots $\{\alpha_1, \dots, \alpha_d\}$. Concretely, we obtain that for a semisimple linear algebraic group of k-rank d, every maximal k-split torus $\mathbf{S}$ is contained in a chain of d parabolic k-subgroups.

2.6.6 Moufang sets and linear algebraic groups of relative rank one

One of the main motivations (but certainly not the only one) for studying Moufang sets, is that they provide a tool to understand linear algebraic groups of relative rank one. We briefly explain the connection.

Suppose $\mathbf{G}$ is a semisimple algebraic k-group of relative k-rank one. Let X be the set of all parabolic k-subgroups of $\mathbf{G}$. Using the remark made at the end of the previous section these are all minimal parabolic k-subgroups, so all elements of X are conjugate under $\mathbf{G}(k)$.

For each $x \in X$, we let U_x be the k-rational points of the root subgroup corresponding to the k-parabolic subgroup x (which coincides, using the fact that ${}_k\Phi$ is generated by one simple root, with the k-rational points of the k-unipotent radical of x).

$\mathbf{G}$ has k-rank one, so its relative root system ${}_k\Phi$ is one-dimensional. Consequently, the relative Weyl group ${}_kW(\mathbf{G})$ has order two. Therefore, using Theorem 2.6.9, every maximal k-split torus is contained in exactly two minimal parabolic k-subgroups and the intersection of every two parabolic k-subgroups contains such a maximal k-split torus.

We show that every two minimal parabolic k-groups are opposite, Theorem 2.6.11 then implies that an arbitrary root group $R_u(\mathbf{P})(k)$ acts regularly (by conjugation) on the set of minimal parabolic k-subgroups without $\mathbf{P}$. Indeed, let $\mathbf{P}$ and $\mathbf{Q}$ be two arbitrary minimal parabolic k-subgroups, then there exists a maximal k-split torus $\mathbf{T}$ such that $C_{\mathbf{G}}(\mathbf{T}) \subseteq \mathbf{P} \cap \mathbf{Q}$. Now $\mathbf{P}$ is opposite to some minimal parabolic k-group $\mathbf{R}$, so $\mathbf{P} \cap \mathbf{R} = C_{\mathbf{G}}(\mathbf{S})$ with $\mathbf{S}$ a maximal k-split torus. Then, using Theorem 2.6.4, there exists an element $g \in \mathbf{G}(k)$ such that $\mathbf{S}^g = \mathbf{T}$, so $C_{\mathbf{G}}(\mathbf{T}) = \mathbf{P}^g \cap \mathbf{R}^g$. This implies, as $\mathbf{P}$ and $\mathbf{Q}$ are the only two minimal parabolic k-groups containing $C_{\mathbf{G}}(\mathbf{T})$, that $\mathbf{P} = \mathbf{P}^g$ and $\mathbf{Q} = \mathbf{R}^g$ or $\mathbf{P} = \mathbf{R}^g$ and $\mathbf{Q} = \mathbf{P}^g$. In both cases, we find that $\mathbf{P}$ and $\mathbf{Q}$ are opposite.

This proves that $\big(X, (U_x)_{x\in X}\big)$ is a Moufang set, which we will denote by $\mathbb{M}(\mathbf{G}, k)$.

If we define $\mathbf{G}^+(k)$ to be the subgroup of $\mathbf{G}(k)$ generated by all the root subgroups, then $\mathbf{G}^+(k)$ modulo its center acts faithfully on X.

2.6.7 Example

We try to clarify the concepts introduced earlier in this section with a (class) of examples. As we are mainly interested in Moufang sets, we give a construction of a class of linear algebraic groups of relative rank one.

Let k be an arbitrary field and K its algebraic closure. Assume V_k is a $(2n+1)$-dimensional vector space over k with basis $\mathcal{B} := \{e_1, \dots, e_{2n+1}\}$ and denote by $V := V_k \otimes_k K$. Furthermore, let

$$q_k : V_k \to k : \ (x_1, x_2, x_3, \dots, x_{2n+1}) \mapsto x_1x_2 + (q_0)_k(x_3, \dots, x_{2n+1})$$

be the representation of a non-degenerate quadratic form q_k on V_k with respect to $\mathcal{B}$ and such that $(q_0)_k$ is anisotropic. We notice that $q := q_k \otimes_k K$ is a quadratic form on V over K.

Next, we consider the linear algebraic group

$$\mathsf{SO}(q) := \{g \in \mathsf{GL}(V) \mid \det(g) = 1 \text{ and } q(g(v)) = q(v),\ \forall v \in V\}.$$

As all polynomials needed to define $\mathsf{SO}(q)$ only have coefficients in k, the group $\mathsf{SO}(q)$ is defined over k. We claim that $\mathsf{SO}(q)$ is a linear algebraic group of absolute rank n and of k-rank one. Equivalently, the building corresponding to this linear algebraic group is a Moufang set. In what follows, we try to reconstruct the building stones of these Moufang sets.

We verify that the common dimension of the maximal tori equals n and that the common dimension of the maximal k-split tori equals one. We remark that every quadratic form over an algebraically closed field has maximal Witt index, so in the case of a $(2n+1)$-dimensional form this is n. Equivalently, the form q is isometric to the split quadratic form

$$Q : V \to K; (x_1, \ldots, x_{2n+1}) \mapsto x_1x_2 + x_3x_4 + \ldots x_{2n-1}x_{2n} + x_{2n+1}^2.$$

This implies there exists a basis $\mathcal{C}$ in V such that q with respect to this basis is exactly Q. With respect to $\mathcal{C}$, we see that the maps ϕ with

$$\phi : V \to V;\ (x_1, \ldots, x_{2n+1}) \mapsto (\lambda_1 x_1, \lambda_1^{-1}x_2, \lambda_2 x_3, \lambda_2^{-1}x_4, \ldots, \lambda_n x_{2n-1}, \lambda_n^{-1}x_{2n}, x_{2n+1})$$

with $\lambda_i \in K^*$ for all $i \in \{1 \ldots, n\}$ have as corresponding matrix representation

$$\operatorname{diag}(\lambda_1, \lambda_1^{-1}, \lambda_2, \lambda_2^{-1}, \ldots \lambda_n, \lambda_n^{-1}, 1).$$

The collection of all these maps indeed is a maximal torus $\mathbf{T}$ of dimension n.

To determine a maximal k-split torus of $\mathsf{SO}(q)$, we use the following lemma:

Lemma 2.6.12. *Let q be a non-degenerate quadratic form on a k-vector space V_k with coefficients in k. Let $\mathbf{G} = \mathsf{O}(q \otimes_k K)$ be the orthogonal group of $q \otimes_k K$, which is a linear algebraic group over the algebraic closure K of k. Then $\mathbf{G}$ has a non-trivial k-split torus if and only if q is isotropic over k.*

Proof. See [3, Section 6.4] for a proof of this statement. □

As $(q_0)_k$ is anisotropic over k, the previous lemma shows that $\mathsf{SO}((q_0)_k \otimes_k K)$ has no non-trival k-split torus. So similarly as before, we find that the elements φ with

$$\varphi : V \to V; (x_1, x_2, x_3, \ldots, x_{2n+1}) \mapsto (\lambda x_1, \lambda^{-1}x_2, x_3, \ldots, x_{2n+1})$$

for all $\lambda \in K^*$ form a k-split torus $\mathbf{S}$. The matrix representation of such an element φ is $\operatorname{diag}(\lambda, \lambda^{-1}, 1, \ldots, 1)$, we see that indeed this torus is isomorphic to K^* or one-dimensional.

The minimal parabolic k-subgroups are exactly the k-groups stabilizing the maximal totally isotropic flags. Here, the maximal totally isotropic flags are one-dimensional isotropic subspaces of V. So for instance the subgroup $\mathbf{P}$ of $\mathbf{G}$ stabilizing the flag $\langle e_1 \rangle$ is a minimal parabolic k-subgroup. From quadratic form theory, we know that two parabolic k-subgroups are conjugate over k if and only if the corresponding flags have the same type. As there is only one type of flag, every two k-parabolic subgroups are indeed conjugate by an element of $\mathbf{G}(k)$.

3
Polarities on Moufang planes

In this chapter we study Moufang sets arising from polarities of Moufang planes $\mathbb{P}^2(\mathcal{O})$, with $\mathcal{O}$ an octonion division algebra. (A *polarity* is a duality of order two, i.e. an incidence-preserving but not type-preserving automorphism of $\mathbb{P}^2(\mathcal{O})$ whose square is the identity.) Both F.D. Veldkamp [36], and more recently N. Knarr and M. Stroppel [18, 19] studied these polarities.

Veldkamp gave a classification of all polarities with absolute points in characteristic different from two. He showed that there are only three different types of non-elliptic polarities (i.e. polarities with at least one absolute flag):

(1) the standard polarity;
(2) a polarity that only exists if the center E of $\mathcal{O}$ is a separable quadratic extension of some smaller field k; and
(3) a polarity arising from an automorphism of order 2 fixing a sub-quaternion algebra (which relies explicitly on the fact that the characteristic is different from two).

On the other hand, N. Knarr and M. Stroppel give a full classification of all polarities of octonion planes in all characteristics. It turns out that only the first two types described by Veldkamp exist as well in characterstic two, but there is also an additional polarity that only exists when the characteristic is equal to 2. (The paper [19] also deals with polarities having no absolute points or exactly one absolute point, but these cases do not give rise to Moufang sets.) At the end of [19], two open questions appear; one of them is to determine the centralizer in $\mathrm{Aut}(\mathcal{P}_2(\mathcal{O}))$ for each of the polarities they describe. This question is closely related to determining the Moufang sets corresponding to these polarities, as the little projective group of each of these Moufang sets turns out to be an important subgroup of the centralizer in $\mathrm{Aut}(\mathcal{P}_2(\mathcal{O}))$ of the corresponding polarities.

Our goal is to describe all the Moufang sets that arise from these polarities, thereby partially answering this question. More specifically, we obtain these Moufang sets by looking at the natural action of the centralizer of the polarity on the little projective group of the octonion plane. We find that the two types of polarities that do exist in all characteristics give rise to Moufang sets of type F_4 —arising from the standard polarity, as described by T. De Medts and H. Van Maldeghem [13]— and (new) Moufang sets of type ${}^2\mathsf{E}_6$, corresponding to forms of algebraic groups of type ${}^2\mathsf{E}_{6,1}^{29}$. The polarity only existing in characteristic different from two corresponds to a class of hermitian Moufang sets, while the polarity only existing in characteristic two induces projective Moufang sets over a 5-dimensional subspace of $\mathcal{O}$.

We first give two different descriptions of octonion planes. In the next section we describe the (already known) types of Moufang sets that correspond to these polarities. Next, we give a general procedure how to construct these Moufang sets from polarities of the Moufang plane. Finally, we discuss for each of these polarities what their corresponding Moufang set is and construct an isomorphism between already existing types of Moufang sets. We find that one of the polarities results in a new type of Moufang set, and we show that these Moufang sets are of algebraic origin. More specifically, we obtain that the centralizer of the polarity is an algebraic group of type ${}^2\mathsf{E}_{6,1}^{29}$.

The results in this chapter are joint work with Tom De Medts and are accepted for publication in Manuscripta Mathematica [5].

3.1 Octonion planes

We start by explaining the basic objects that we will use, namely projective planes coordinatized by an octonion division algebra. We will refer to such planes as *octonion planes* or as *Moufang planes*. (The latter terminology is perhaps somewhat ambiguous, since there are of course many other projective planes with the Moufang property, but it is customary to refer to the non-desarguesian projective planes with the Moufang property as *Moufang planes*.)

We describe octonion planes $\mathbb{P}^2(\mathcal{O})$, where $\mathcal{O}$ is an octonion division algebra with center k, in two different ways as a point-line incidence geometry $(\mathcal{P}, \mathcal{L}, *)$.

The first way is the most natural way to describe octonion planes. The point set $\mathcal{P}$ consists of three different types of points. Points of the first type are elements of the form (a, b) with $a, b \in \mathcal{O}$, points of the second type are (c) with $c \in \mathcal{O}$ and the last type is only one point which we denote by (∞).

Similarly, there are three types of lines. The first type consists of the elements $[m, k]$ with $m, k \in \mathcal{O}$, lines of the second type are elements $[l]$ with $l \in \mathcal{O}$, the third is the line $[\infty]$.

The incidence relation $*$ between points and lines is as follows:

$$\begin{aligned}
(a,b) * [m,k] \quad &\Longleftrightarrow \quad ma + b = k,\\
(a,b) * [l] \quad &\Longleftrightarrow \quad a = l,\\
(c) * [m,k] \quad &\Longleftrightarrow \quad c = m,\\
(c) * [\infty] \quad &\text{for all } c \in \mathcal{O},\\
(\infty) * [l] \quad &\text{for all } l \in \mathcal{O},\\
(\infty) * [\infty]. &
\end{aligned}$$

For the second description of the octonion plane, which we denote by $\hat{\mathbb{P}}^2(\mathcal{O})$ to avoid confusion, we define $\mathcal{O}_3$ as the vector space of 3×3 matrices with entries in $\mathcal{O}$. The set $\mathcal{H}(\mathcal{O}_3)$ is then defined as the subspace of $\mathcal{O}_3$ consisting of elements x of the form

$$x = \begin{pmatrix} \alpha_1 & -a_3 & \overline{a_2} \\ \overline{a_3} & \alpha_2 & a_1 \\ a_2 & -\overline{a_1} & \alpha_3 \end{pmatrix}$$

with $\alpha_i \in k$, $a_i \in \mathcal{O}$ for all $i \in \{1, 2, 3\}$ and $\bar{\cdot}$ the standard involution on $\mathcal{O}$. We will often abbreviate the matrix element x as $(\alpha_1, \alpha_2, \alpha_3; a_1, a_2, a_3)$. The space

$\mathcal{H}(\mathcal{O}_3)$ is a *cubic norm structure*, with norm $\mathrm{N}\colon \mathcal{H}(\mathcal{O}_3) \to k$ given by

$$\begin{aligned}&\mathrm{N}(\alpha_1,\alpha_2,\alpha_3;a_1,a_2,a_3)\\ &\qquad = \alpha_1\alpha_2\alpha_3 + \alpha_1\,\mathrm{N}(a_1) - \alpha_2\,\mathrm{N}(a_2) + \alpha_3\,\mathrm{N}(a_3) - \mathrm{T}(a_1a_2a_3),\end{aligned}$$

trace $\mathrm{T}\colon \mathcal{H}(\mathcal{O}_3)\times\mathcal{H}(\mathcal{O}_3) \to k$ given by

$$\begin{aligned}&\mathrm{T}\big((\alpha_1,\alpha_2,\alpha_3;a_1,a_2,a_3),(\beta_1,\beta_2,\beta_3;b_1,b_2,b_3)\big)\\ &\qquad = \textstyle\sum_{i=1}^{3}\alpha_i\beta_i - \mathrm{T}(a_1\overline{b_1}) + \mathrm{T}(a_2\overline{b_2}) - \mathrm{T}(a_3\overline{b_3}),\end{aligned}$$

and with adjoint map $\sharp\colon \mathcal{H}(\mathcal{O}_3) \to \mathcal{H}(\mathcal{O}_3)$ given by

$$\begin{aligned}&(\alpha_1,\alpha_2,\alpha_3;a_1,a_2,a_3)^\sharp\\ &\qquad = \big(\alpha_2\alpha_3 + \mathrm{N}(a_1), \alpha_1\alpha_3 - \mathrm{N}(a_2), \alpha_1\alpha_2 + \mathrm{N}(a_3);\\ &\qquad\qquad - \overline{a_2a_3} - \alpha_1a_1, \overline{a_3a_1} - \alpha_2a_2, -\overline{a_1a_2} - \alpha_3a_3\big),\end{aligned}$$

where on the right-hand side N and T are the norm and trace map of $\mathcal{O}$. Together with the quadratic maps $U_x\colon \mathcal{H}(\mathcal{O}_3) \to \mathcal{H}(\mathcal{O}_3)$ (for each x in $\mathcal{H}(\mathcal{O}_3)$) given by

$$U_x(y) = \mathrm{T}(x,y)x - x^\sharp \times y,$$

the space $\mathcal{H}(\mathcal{O}_3)$ becomes a *quadratic Jordan algebra* over k, which we will denote by $J(\mathcal{H}(\mathcal{O}_3))$.

We will now describe the projective plane $\hat{\mathbb{P}}^2(\mathcal{O})$. The set of points $\hat{\mathcal{P}}$ consists of elements (x) with $x \in \mathcal{H}(\mathcal{O}_3)$, $x \neq 0$ and $x^\sharp = 0$. Two elements (x) and (x') represent the same point if and only if $kx = kx'$. The line set $\hat{\mathcal{L}}$ consists similarly of elements $[x]$, with $x \in \mathcal{H}(\mathcal{O}_3)$ different from 0 and satisfying $x^\sharp = 0$. Two elements $[x]$ and $[x']$ denote the same line if and only if $kx = kx'$.

We define an incidence relation $\hat{*}$ by

$$(x)\ \hat{*}\ [y] \Longleftrightarrow T(x,y) = 0.$$

Next, we define a map ϕ between the point and line sets of $\mathbb{P}^2(\mathcal{O})$ and $\hat{\mathbb{P}}^2(\mathcal{O})$:

$$\begin{aligned}(a,b) &\mapsto (N(b), -N(a), 1; a, \overline{b}, b\overline{a})\\ (c) &\mapsto (N(c), -1, 0; 0, 0, -c)\\ (\infty) &\mapsto (1,0,0;0,0,0)\\ [m,k] &\mapsto [-1, N(m), -N(k); -\overline{m}k, \overline{k}, m]\\ [l] &\mapsto [0,1,-N(l); -l, 0, 0]\\ [\infty] &\mapsto [0,0,1;0,0,0]\end{aligned}$$

One can show that ϕ is an isomorphism of projective planes. For a more detailed description, we refer to [13, Section 3].

3.2 The polarities of $\mathbb{P}^2(\mathcal{O})$ and $\hat{\mathbb{P}}^2(\mathcal{O})$

In this paragraph, we investigate all polarities of $\hat{\mathbb{P}}^2(\mathcal{O})$ (and thus of $\mathbb{P}^2(\mathcal{O})$ as well) having at least three absolute points. Our goal is to describe each type of polarity together with the associated Moufang set. We were inspired by an article of Veldkamp [36], which considers polarities with absolute points together with their associated groups (over fields of characteristic different from 2). We will extend these results to include fields of characteristic 2, and we will translate some of his result into the more modern language of Moufang sets. We first give a general approach for constructing Moufang sets from polarities of Moufang planes. We will use a similar method as in [13, Section 5].

3.2.1 Recovering Moufang sets from polarities of the Moufang plane

Definition 3.2.1. Let Δ be a projective plane (or more generally, a generalized polygon) with point set $\mathcal{P}$ and line set $\mathcal{L}$. A map $\Psi\colon \mathcal{P}\cup\mathcal{L} \to \mathcal{P}\cup\mathcal{L}$ is called a *polarity* of Δ if:

- $\Psi(\mathcal{P}) = \mathcal{L}$ and $\Psi(\mathcal{L}) = \mathcal{P}$;
- Ψ preserves incidence, i.e. $p * L \iff \Psi(L) * \Psi(p)$ for all $p \in \mathcal{P}, L \in \mathcal{L}$;
- $\Psi^2 = 1$.

An element $x \in \mathcal{P}\cup\mathcal{L}$ is called *absolute* if $x * \Psi(x)$. Similarly, a flag (p, L) with $p * L$ is called an *absolute flag* if $L = \Psi(p)$ (and consequently also $p = \Psi(L)$).

Now suppose Ψ is a polarity of $\mathbb{P}^2(\mathcal{O})$ having at least three absolute points and G is the *little projective group* of $\mathbb{P}^2(\mathcal{O})$, i.e. the subgroup of $\mathrm{Aut}(\mathbb{P}^2(\mathcal{O}))$ generated by the root groups (or equivalently, generated by all elations).

First, we determine the set X of absolute points and the subgroup $C_G(\Psi)$ of G, which is the group of all elements of G that commute with Ψ. For every element $\sigma \in C_G(\Psi)$ and every $x \in X$, the image $\sigma(x)$ is again an absolute point. In this way, we obtain a natural action of $C_G(\Psi)$ on the set of all absolute points (or equivalently, of all absolute flags) of the polarity Ψ. As the following result shows, this gives rise to a Moufang set.

This result seems to be well known, and is used in [20] for instance, but we could not find a proof in the literature. It turns out that the details are somewhat more intricate than one would expect.

Proposition 3.2.2. *Let Δ be a Moufang n-gon, and let G be a subgroup of* $\mathrm{Aut}(\Delta)$ *containing all root groups. Let Ψ be a polarity of Δ, and let X be the set* *of absolute flags of* Ψ*; assume that* $|X| \geq 3$*. Let* $C = C_G(\Psi)$*; then* C *acts on* X*. Let* K *be the kernel of this action, and let* $\overline{C} = C/K$*. Then:*

(i) *For each* $x \in X$*, let* U_x *be the intersection of* $\overline{C}$ *with the unipotent radical* $U_+ = U_1 \cdots U_n$ *of* Δ *with respect to the pair* (Σ, x)*, where* Σ *is an (arbitrary) apartment of* Δ *containing* x*. Then* $\big(X, (U_x)_{x \in X}\big)$ *is a Moufang set; its little projective group is a normal subgroup of* $\overline{C}$*.*

(ii) *If either* n *is even, or* $n = 3$ *and each non-absolute line through a given absolute point of* Δ *contains a second absolute point (so not all absolute points are contained in a single line), then* $K = 1$*, and hence the little projective group of* X *is a normal subgroup of* $C = C_G(\Psi)$ *itself.*

Proof. Notice that the absolute flags of Δ with respect to Ψ are two by two opposite, i.e. they lie at maximal distance.

(i) Choose two arbitrary elements x and y of X, and let Σ be an apartment containing the flags x and y. Label the roots of Σ in such a way that the root groups $U_1, \dots, U_n$ are precisely those fixing the flag x. Then for each $i \in \{1, \dots, n\}$, the conjugate U_i^Ψ is precisely U_{n+1-i}, so in particular U_+ is normalized by Ψ.
We now claim that $V_+ := U_+ \cap \overline{C}$ acts sharply transitively on $X \setminus \{x\}$. To show that it acts transitively, let $z \in X \setminus \{x\}$ be arbitrary. By [34, (4.11) and (5.3)], there exists an element $u \in U_+$ mapping z to y. It follows that $[u, \Psi] = u^{-1}u^\Psi$ is an element of U_+ fixing y, but then this element fixes Σ pointwise. This can only be true if $[u, \Psi] = 1$, and hence $u \in V_+$. This shows that V_+ acts transitively on $X \setminus \{x\}$. Since no non-trivial element of U_+ fixes y, we conclude that V_+ acts sharply transitively, as claimed.
Similarly, the group $V_- := U_- \cap \overline{C}$ obtained by interchanging the roles of x and y, acts sharply transitively on $X \setminus \{y\}$. Moreover, note that U_+ is normalized by every element of G fixing the flag x, and hence V_+ is a normal subgroup of $\mathrm{Stab}_C(x)$. This shows that $\langle V_+, V_- \rangle$ is the little projective group of a Moufang set with underlying set X. Finally, observe that every element of $\overline{C}$ conjugates root groups to root groups, and hence $\langle V_+, V_- \rangle$ is a normal subgroup of $\overline{C}$.

(ii) If either n is even, or $n = 3$ and each non-absolute line through a given absolute point of Δ contains a second absolute point, we will show that every element of G fixing all elements of X fixes all elements of Δ, implying that the kernel K of the action is trivial.
Assume first that $n = 3$, and that (p, p^Ψ) is an absolute flag with the property that every line through p contains a second absolute point (and hence every point on the line p^Ψ is contained in a second absolute line). If

$g \in G$ fixes all absolute points and all absolute lines, then it has to fix all points on the line p^Ψ and all lines through the point p, and at least one additional point q not on the line p^Ψ. This can only be true if $g = 1$.
Assume next that $n = 2m$ is even. Then the absolute flags of Ψ form an ovoid-spread pairing. Indeed, observe that every non-absolute flag (p, p^Ψ) induces an absolute flag in the middle of the unique minimal path γ connecting p and p^Ψ (as γ is fixed under Ψ). Let (q, q^Ψ) be an absolute flag, we show (as for $n = 3$) that if $g \in G$ fixes all absolute points and all absolute lines then g also fixes all points on q^Ψ (and consequently all lines through q). Let t be arbitrary point on q^Ψ distinct from q and let r be an arbitrary element at distance $m-1$ from t and distance m from q^Ψ. By the definition of an ovoid, there is a absolute point s at distance at most m from r. As q^Ψ is a line lying at distance m from r, we conclude that the distance $d(s, r)$ between s and r is at most $m - 1$ and s is unique with that property. We notice that $s \neq q$ since otherwise both q and t would be at distance $m - 1$ from r, a contradiction. Since every two absolute points are opposite, we find that $d(q, s) = n$ and therefore $d(s, t) = n-2$ and $d(s, q^\Psi) = n-1$. This implies t lies on the unique minimal path γ' from s to q^Ψ. Since s is fixed by g, the path γ' is fixed as well. Together with $g(q^\Psi) = q^\Psi$, we obtain that $g(t) = t$. Hence g fixes all points of q^Ψ and dually, g fixes all lines through q. From [34, (5.2)], it now follows immediately that the subgroup G_{q,q^Ψ} of $\mathrm{Aut}(\Delta)$ fixing all lines through q and all points on q^Ψ is contained in the group U_+ corresponding to $((q, q^\Psi), Y)$ with Y an apartment containing (q, q^Ψ) and some other absolute flag (u, u^Ψ). Using [35, Lemma 5.2.6], we find that U_+ acts regularly on the set of flags of Δ opposite to (q, q^Ψ). As $g \in G_{q,q^\Psi} \leq U_+$ fixes the flag (u, u^Ψ), we find that $g = 1$. □

Remark 3.2.3. The additional condition when $n = 3$ in part (ii) of the previous Proposition is necessary, as is illustrated by the polarities of type IV, for which all absolute points lie on a single line of $\mathbb{P}^2(\mathcal{O})$ (see Section 3.3.5); in this case, the action of C on X is not faithful. This type of polarity turns out to be the only one where K differs from zero.

We will now make this result explicit in our situation where Δ is the projective plane $\mathbb{P}^2(\mathcal{O})$, following the approach of [13, Section 5].

We choose an arbitrary element of X, and denote it by ∞; the corresponding flag of $\mathbb{P}^2(\mathcal{O})$ is denoted by $((\infty), [\infty])$. Next, we choose an apartment Σ, for instance the apartment through $(0), (0, 0)$ and (∞), through the flag $((\infty), [\infty])$. The corresponding unipotent radical U_+ is the product $U_1U_2U_3$ of the three root groups through the pair $(\Sigma, ((\infty), [\infty]))$, i.e. the root groups determined by the roots of Σ containing the flag $((\infty),[\infty])$. The first root group U_1 is the group of collineations fixing all the points on $[0]$ and the lines through (∞), the second group U_2 fixes all lines through (∞) and all points on $[\infty]$ while U_3 fixes all points on $[\infty]$ and all lines through (0). We find that these root groups have the following action on $\mathbb{P}^2(\mathcal{O})$:

$$U_1 := \{x_1(M) \mid M \in \mathcal{O}\} \text{ where } x_1(M)\colon \begin{cases} (a,b) \mapsto (a, b - Ma), \\ [m,k] \mapsto [m+M, k], \end{cases}$$

$$U_2 := \{x_2(B) \mid B \in \mathcal{O}\} \quad \text{where } x_2(B)\colon \begin{cases} (a,b) \mapsto (a, b + B), \\ [m,k] \mapsto [m, k+B], \end{cases}$$

$$U_3 := \{x_3(A) \mid A \in \mathcal{O}\} \quad \text{where } x_3(A)\colon \begin{cases} (a,b) \mapsto (a+A, b), \\ [m,k] \mapsto [m, k+mA]. \end{cases}$$

Therefore, an arbitrary element of U_+ is of the form $x(A,B,M) := x_1(M) \circ x_2(B) \circ x_3(A)$ with

$$x(A,B,M)\colon \begin{cases} (a,b) \mapsto (a+A, b+B-Ma), \\ [m,k] \mapsto [m+M, k+B+mA+MA]. \end{cases}$$

The root group U_∞ can now be obtained as the subgroup of U_+ consisting of those elements that map X to itself.

Finally, we have to find a permutation τ of U_∞^*. Therefore, it suffices to find a collineation σ of $\mathbb{P}^2(\mathcal{O})$ that commutes with Ψ and interchanges the points $(0,0)$ and (∞). Let τ be the restriction of σ to X; then U_∞^τ will be a root groop and therefore will coincide with the root group $U_{(0,0)}$ of the Moufang set. This implies that $\mathbb{M}(U_\infty, \tau)$ is the Moufang set obtained from Ψ as in Proposition 3.2.2 above.

3.2.2 Description of the different types of polarities

We describe all different types of polarities of the Moufang plane with at least three absolute points. In characteristic different from two this was already done by Veldkamp in the late sixties [36]. He used the description of the octonion plane we defined as $\hat{\mathbb{P}}^2(\mathcal{O})$ and showed that in this case there only exist three types of polarities. On the other hand, N. Knarr and M. Stroppel described all the polarities in characteristic two on $\mathbb{P}^2(\mathcal{O})$. It turns out that also in this case, there are three types of polarities; the first two coincide with two of the polarities found by Veldkamp, but the third type is a polarity that only exists in the characteristic two case.

Remark 3.2.4. All the polarities we present can be seen as the composition of some standard polarity and a collineation on the Moufang plane; see [18, Theorem 3.4]. Furthermore, we may assume (by conjugating the polarity if necessary) that this collineation is induced by an automorphism of the octonion division algebra $\mathcal{O}$; see [18, Theorem 3.6]. Such a collineation is easy to write down explicitly, both in $\mathbb{P}^2(\mathcal{O})$ and in $\hat{\mathbb{P}}^2(\mathcal{O})$: for $\mathbb{P}^2(\mathcal{O})$, it suffices to apply the automorphism on its coordinates; for $\hat{\mathbb{P}}^2(\mathcal{O})$, the collineation is given by applying the automorphism on each of the entries of the matrix for each point and each line.

By Remark 3.2.4, it suffices to go over the different types of automorphisms on $\mathcal{O}$ in order to describe all possible polarities with at least three absolute points.

Polarities of type I – the standard polarity

We define a natural polarity π on $\hat{\mathbb{P}}^2(\mathcal{O})$:

$$\pi\colon \hat{\mathbb{P}}^2(\mathcal{O}) \to \hat{\mathbb{P}}^2(\mathcal{O})\colon (x) \mapsto [x].$$

It is easy to see that this indeed forms a polarity; we refer to [13, Theorem 4.5] for a proof that this polarity has enough absolute points. For obvious reasons, we call this polarity the *standard polarity.*

We use the isomorphism ϕ between $\hat{\mathbb{P}}^2(\mathcal{O})$ and $\mathbb{P}^2(\mathcal{O})$ to transform the above polarity into the following polarity of $\mathbb{P}^2(\mathcal{O})$:

$$\begin{aligned}
(a,b) &\leftrightarrow [-\overline{ab^{-1}}, -\overline{b^{-1}}]\\
(a,0) &\leftrightarrow [\overline{a^{-1}}]\\
(0,0) &\leftrightarrow [\infty]\\
(c) &\leftrightarrow [\overline{c^{-1}}]\\
(0) &\leftrightarrow [0]\\
(\infty) &\leftrightarrow [0,0]
\end{aligned}$$

An easy transformation (see [13, Section 4.5] for more details) reduces these polarities to the following more elegant form:

$$\begin{aligned}
(a,b) &\leftrightarrow [\overline{a}, -\overline{b}]\\
(c) &\leftrightarrow [\overline{c}]\\
(\infty) &\leftrightarrow [\infty].
\end{aligned}$$

Taking into account the remarks we just made about the form of a general polarity, we find that each polarity of $\mathbb{P}^2(\mathcal{O})$ is conjugate to a polarity of the following form, for some $\eta \in \mathrm{Aut}(\mathcal{O})$:

$$\begin{aligned} (a,b) &\leftrightarrow [\eta(\overline{a}), -\eta(\overline{b})] \\ (c) &\leftrightarrow [\eta(\overline{c})] \\ (\infty) &\leftrightarrow [\infty], \end{aligned} \tag{3.2.1}$$

as in [18, Section 3].

Polarities of type II

This type of polarity only exists if the center E of the octonion division algebra $\mathcal{O}$ is a separable quadratic extension of a subfield k and $\mathcal{O}$ is obtained by extending scalars from an octonion division algebra over k. So let $\mathcal{O}_k$ be an octonion division algebra over k, and assume that E/k is a separable quadratic extension such that $\mathcal{O} = \mathcal{O}_k \otimes_k E$ remains a division algebra. Let γ be the non-trivial element of $\mathrm{Gal}(E/k)$; then γ is an involution on E, which extends to a non-linear automorphism η of $\mathcal{O}$ by applying the involution to each coefficient with respect to a basis of $\mathcal{O}_k$. This automorphism gives rise to a polarity described in equation (3.2.1), and we will refer to this class of polarities as the *polarities of type II*.

Polarities of type III

The third type of polarity only exists if the characteristic of the center k of $\mathcal{O}$ is different from two. (More precisely, when $\mathrm{char}(k) = 2$, it coincides with the standard polarity.) Let D be an arbitrary quaternion subalgebra of $\mathcal{O}$. Then $\mathcal{O}$ decomposes as the direct sum of D and $D^\perp$, and the map

$$\eta\colon \mathcal{O} \to \mathcal{O}\colon d + d' \mapsto d - d',$$

for all $d \in D$ and $d' \in D^\perp$, is an automorphism of $\mathcal{O}$.

Again, such an automorphism induces a polarity by equation (3.2.1). We will refer to this class of polarities as the *polarities of type III*.

Polarities of type IV

In contrast with the polarities of type III, we will now describe a type of polarity (or automorphism) that only exists when the characteristic of the field is two. The reason for this is the following: in characteristic two, all octonion division algebras possess a totally singular subalgebra D of dimension 4. If such an algebra exists, one can show it is in fact a subfield of the octonion division algebra; see [16, Theorem 4.11].

We will show explicitly how such an algebra is naturally contained in an octonion division algebra in characteristic two. For this, we rely on the fact that each octonion algebra in characteristic two has a so-called symplectic basis. This is a basis of the form e, a, b, ab, c, ac, bc, $(ab)c$ with $N(a)N(b)N(c) \neq 0$, such that

$$\langle e, a\rangle = 1, \quad \langle b, ab\rangle = N(b), \quad \langle c, ac\rangle = N(c), \quad \langle bc, (ab)c\rangle = N(b)N(c),$$

and all other inner products between distinct basis vectors are zero; see [27, Section 1.6]. Now let D be the subspace of $\mathcal{O}$ spanned by the vectors e, b, c and bc; then D is a totally singular subalgebra, which is a 4-dimensional subfield of the octonion algebra.

It is easy to see that for each element $z \in \mathcal{O} \setminus D$, there is a decomposition $\mathcal{O} = D \oplus Dz$. Furthermore, we can choose this element z in such a way that $\mathrm{T}(z) = 1$, or equivalently, such that $\overline{z} = z + e$. The map

$$\eta\colon\ \mathcal{O} \to \mathcal{O} \colon d + d'z \mapsto d + d'\overline{z}$$

for all $d, d' \in D$, is an automorphism of order two on $\mathcal{O}$. The *polarities of type IV* are now those induced by such an automorphism.

3.3 Description of the Moufang sets

In the previous section we described all different types of polarities with at least three absolute points that can occur in a Moufang plane. Each of these polarities now induces a certain type of Moufang set. In this subsection we give a detailed description of all these Moufang sets. In most of the cases we can identify the Moufang sets we find with some known algebraic Moufang sets. Polarities of type II on the other hand result in Moufang sets that have not yet been described in literature. We will show that this type of Moufang set is also of algebraic nature; more specifically, these Moufang sets arise from algebraic groups of type ${}^2\mathsf{E}_{6,1}^{29}$.

3.3.1 General method

For each of the four types of polarities described in section 3.2.2, we compute the Moufang set corresponding to such a polarity Ψ with the methods discussed in section 3.2.1. Let $\eta \in \mathrm{Aut}(\mathcal{O})$ be the automorphism of the octonion algebra $\mathcal{O}$

chosen in Section 3.2 corresponding to the type of Ψ. Then the set of absolute points of Ψ is

$$X = \{(a,b) \in \mathcal{O} \times \mathcal{O} \mid \eta(\overline{a}) \cdot a + \eta(\overline{b}) + b = 0\} \cup \{\infty\}.$$

Notice that the flag $(0,0) * [0,0]$ is fixed under the action of Ψ, so for an arbitrary element $x(A,B,M)$ of U_∞, the flag

$$x(A,B,M)(0,0) * x(A,B,M)[0,0] = (A,B) * [M, B + MA]$$

has to be fixed under Ψ. Since

$$\Psi((A,B) * [M, B + MA]) = [\eta(\overline{A}), \eta(\overline{B})] * \big(\eta(\overline{M}), \eta(\overline{B + MA})\big),$$

this implies $A = \eta(\overline{M})$ and $\eta(\overline{B}) = B + MA$. Because A and B alone determine the element $x(A,B,M)$ completely, we will denote this element by $x(A,B)$.

The composition of two arbitrary elements $x(A,B)$ and $x(C,D)$ is the same action on X as the element $x\big(A + C, B + D - \eta(\overline{C})A\big)$. This implies that the group $(U,+)$ of the Moufang set we are describing is

$$U = \big\{(a,b) \in \mathcal{O} \times \mathcal{O} \mid \eta(\overline{a}) \cdot a + \eta(\overline{b}) + b = 0\big\}, \tag{3.3.1}$$

with the group operation $+$ on U given by

$$(a,b) + (c,d) = \big(a + c, b + d - \eta(\overline{c}) \cdot a\big)$$

for all $(a,b),(c,d) \in U$.

We determine an appropriate permutation τ on U^*. Therefore we need a collineation on $\mathbb{P}^2(\mathcal{O})$ commuting with Ψ. Inspired by [13], we find that the following collineation σ has the desired properties:

$$\sigma\colon \begin{cases} (a,b) \mapsto (-ab^{-1}, b^{-1}) \\ [m,k] \mapsto [k^{-1}m, k^{-1}]. \end{cases} \tag{3.3.2}$$

The permutation τ can be defined as the restriction of σ to X, and is thus defined by

$$\tau(a,b) = (-ab^{-1}, b^{-1}) \tag{3.3.3}$$

for all $(a,b) \in U^*$.

3.3.2 Polarities of type I – Moufang sets of type F_4

The polarities of type I are exactly those described in [13]. There, it is shown that the Moufang sets corresponding to the standard polarity are precisely the Moufang sets of type F_4 or thus Moufang sets arising from an algebraic group of type $F_{4,1}^{21}$.

3.3.3 Polarities of type II – Moufang sets of type ${}^2\mathsf{E}_6$

Assume now that Ψ is a polarity of type II. We will prove that in this case, the Moufang set arises indeed from a linear algebraic group of type ${}^2\mathsf{E}_{6,1}^{29}$ over k. Therefore we rely on a paper by S. Garibaldi [17] dealing with the algebraic structure of the linear algebraic group in terms of elements of an Albert algebra.

Recall from section 3.2.2 that $\mathcal{O} = \mathcal{O}_k \otimes_k E$, where $\mathcal{O}_k$ is an octonion division algebra over k. Let $J := J(\mathcal{H}((\mathcal{O}_k)_3))$ be a quadratic Jordan algebra over k with corresponding norm and trace maps N and T, and define the group $M_1(J)$ as the group of isometries on J. Every element g of $M_1(J)$ has a natural action on $\hat{\mathbb{P}}^2(\mathcal{O}_k)$; more precisely, g induces a collineation $\rho(g)$ on the Moufang plane by defining

$$\rho(g)\colon \begin{cases} \mathcal{P} \to \mathcal{P}\colon (x) \mapsto (g(x)) \\ \mathcal{L} \to \mathcal{L}\colon [y] \mapsto [g^\dagger(y)], \end{cases}$$

where $g^\dagger$ is the unique element of $M_1(J)$ such that $\mathrm{T}(g(x), g^\dagger(y)) = \mathrm{T}(x, y)$ for all $x, y \in J$. Moreover, the map ρ is a k-isomorphism between $M_1(J)$ and the little projective group G generated by all elations of the Moufang plane.

Furthermore, we define a map τ as

$$\tau\colon J \to J\colon (\epsilon_1, \epsilon_2, \epsilon_3;\ c_1, c_2, c_3) \mapsto (\epsilon_1, \epsilon_3, \epsilon_2;\ \overline{c_1}, \overline{c_3}, \overline{c_2})$$

for all $a = (\epsilon_1, \epsilon_2, \epsilon_3;\ c_1, c_2, c_3) \in J$.

We now construct the twisted algebraic group ${}^2M_1(J)$ corresponding to $M_1(J)$. It follows from [17, Proposition 2.2] that, if k a field of characteristic not 2 or 3, this is a linear algebraic group of type ${}^2\mathsf{E}_{6,1}^{29}$ over k. The argument given in the proof of [13, Theorem 4.1] (which is based on results by J. Faulkner [16]) shows that this description remains valid over fields of arbitrary characteristic.

The group ${}^2M_1(J)$ is an algebraic group such that ${}^2M_1(J) \otimes_k E \cong M_1(J) \otimes_k E \cong M_1(J \otimes E)$, this last group being the group of all similarities of $J \otimes_k E$ that preserve the extended norm $\mathrm{N} \otimes_k E$.

The non-trivial element η of the Galois group $\mathrm{Gal}(E/k)$ acts differently on the groups $M_1(J)$ and ${}^2M_1(J)$. Indeed, the non-twisted action on an element of $M_1(J)(E) := M_1(J) \otimes_k E$ which we denote by ι, is as follows:

$$\iota\colon M_1(J)(E) \to M_1(J)(E)\colon g \otimes \ell \mapsto g \otimes \eta(\ell)$$

for all $g \in M_1(J), \ell \in E$. On the other hand, the twisted Galois action on $M_1(J)(E)$ is given by

$$\iota^*\colon M_1(J)(E) \to M_1(J)(E)\colon g \otimes \ell \mapsto \tau \circ [\iota(g \otimes \ell)]^\dagger \circ \tau^{-1},$$

where τ and $\dagger$ are the natural extensions to $M_1(J)(E)$ of the equally named maps on $M_1(J)$; see [17, Section 2]. We will also write $\iota * g$ for $\iota^*(g)$.

We conclude that the corresponding groups over k are given by

$$M_1(J) = \{g \in M_1(J)(E) \mid \iota(g) = g\} \quad \text{and}$$
$${}^2M_1(J) = \{g \in M_1(J)(E) \mid \iota * g = g\}.$$

For a more general and detailed description on the process of twisting of linear algebraic groups, we refer for instance to [24, Kapitel I, §3] or [25, Section 5.3].

Next, we define the following map ψ on $\hat{\mathbb{P}}^2(\mathcal{O})$, the Moufang plane over $\mathcal{O} := \mathcal{O}_k \otimes_k E$:

$$\psi\colon \mathcal{P} \to \mathcal{L}\colon (x) \mapsto [(\tau \circ \tilde{\eta})(x)],$$

with $\tilde{\eta} := \mathrm{id}_J \otimes \eta$. It is easy to check this map induces a polarity of the Moufang plane $\hat{\mathbb{P}}^2(\mathcal{O})$.

Observe that the isomorphism ρ from $M_1(J)$ to G extends naturally to an isomorphism from $M_1(J)(E)$ to $G(E)$. The following theorem shows that under this extended isomorphism, the elements of the twisted group ${}^2M_1(J)$ are mapped to elements of $G(E)$ commuting with the polarity ψ.

Theorem 3.3.1. *The map ρ induces a k-isomorphism between the group ${}^2M_1(J)$ of elements $g \in M_1(J)(E)$ such that $\iota * g = g$, and the group $C_{G(E)}(\psi)$ of elements of $G(E)$ that commute with the polarity ψ.*

Proof. Since we have already a k-isomorphism between $M_1(J)(E)$ and $G(E)$, it suffices to prove that $\psi \circ \rho(g) = \rho(g) \circ \psi$ if and only if $\iota * g = g$.

Write $E = k(\delta)$; an arbitrary element $g \in M_1(J)(E)$ can then be written in the form

$$g = g_1 \otimes 1 + g_2 \otimes \delta,$$

with $g_1, g_2 \in M_1(J)$.

Let $a = x_1 \otimes m_1 + \cdots + x_s \otimes m_s \in J_E$ with $x_i \in J$, $m_i \in E$ be an element with $a^\sharp = 0$; then

$$\psi\big(\rho(g)\{(a)\}\big) = \psi\{(g(a))\} = [\tau\tilde{\eta}g(a)]$$

while

$$\rho(g)\big(\psi\{(a)\}\big) = \rho(g)([\tau\tilde{\eta}(a)]) = [g^\dagger\tau\tilde{\eta}(a)].$$

We conclude that $\psi \circ \rho(g) = \rho(g) \circ \psi$ if and only if $\tau\tilde{\eta}g = g^\dagger\tau\tilde{\eta}$.

Next, we investigate when $\iota * g = g$ or equivalently when $(\iota * g)^\dagger = g^\dagger$. First, we show that $(\iota * g)^\dagger = \tau[\iota(g)]\tau$. We compute

$$\begin{aligned}
\mathrm{T}(\tau x, \tau y) &= \mathrm{T}(x, y) \\
&= \mathrm{T}\big((\iota * g)x, (\iota * g)^\dagger y\big) \\
&= \mathrm{T}\big(\tau[\iota(g)]^\dagger \tau x, (\iota * g)^\dagger y\big) \\
&= \mathrm{T}\big(\iota(g)^\dagger \tau x, \tau(\iota * g)^\dagger y\big) \\
&= \mathrm{T}\big(\tau x, [\iota(g)]^{-1}\tau(\iota * g)^\dagger y\big),
\end{aligned}$$

for all $x, y \in J_E$. Since the trace T is non-degenerate, we can conclude that $\tau = [\iota(g)]^{-1}\tau(\iota * g)^\dagger$.

The problem is reduced to proving that $\tau[\iota(g)]\tau = g^\dagger$ if and only if $\tau\tilde{\eta}g = g^\dagger\tau\tilde{\eta}$, or equivalently, that $\iota(g)\tilde{\eta} = \tilde{\eta}g$. Notice that

$$\iota(g) = g_1 \otimes 1 + g_2 \otimes \eta(\alpha),$$

and hence

$$\begin{aligned}
\iota(g)\tilde{\eta}(x_i \otimes m_i) &= \iota(g)\big(x_i \otimes \eta(m_i)\big) \\
&= g_1(x_i) \otimes \eta(m_i) + g_2(x_i) \otimes \eta(\alpha)\eta(m_i) \\
&= \tilde{\eta}\big(g_1(x_i) \otimes m_i + g_2(x_i) \otimes (\alpha m_i)\big) \\
&= \tilde{\eta}\big(g(x_i \otimes m_i)\big);
\end{aligned}$$

it follows that $\iota(g)\tilde{\eta}(a) = \tilde{\eta}g(a)$ for all $a \in J_E$. □

In order to be able to invoke Proposition 3.2.2, we have to check the condition in part (ii) of that proposition.

Lemma 3.3.2. *Let $\mathbb{P}^2(\mathcal{O})$ and Ψ be as above. Every non-absolute line $[a]$ through the point (∞) contains an absolute point of the form (a, b).*

Proof. We have to check that for every $a \in \mathcal{O}$, the equation

$$\eta(\overline{a}) \cdot a + \eta(\overline{b}) + b = 0$$

has a solution $b \in \mathcal{O}$. Since $\eta(\overline{a}) \cdot a$ is fixed by the map $x \mapsto \eta(\overline{x})$, this follows immediately from [19, Proposition 3.2]. □

It now follows from Proposition 3.2.2 that the Moufang set corresponding to Ψ has a little projective group which is a normal subgroup of the centralizer

$C_G(\Psi)$ of the polarity. If we can show that this polarity Ψ gives rise to the same Moufang set (up to isomorphism) as the Moufang set we obtained from the polarity ψ, then it will indeed follow from Theorem 3.3.1 that the Moufang set corresponding to Ψ arises from a twisted algebraic group of type ${}^2\mathsf{E}_{6,1}^{29}$.

We calculate what this polarity ψ looks like on $\mathbb{P}^2(\mathcal{O})$, and we find

$$\begin{aligned} (a,b) &\mapsto [-\eta(\bar{b})^{-1}, -\eta(\overline{ab^{-1}})] \\ (a,0) &\mapsto [-\eta(\overline{a})] \\ (0,0) &\mapsto [0] \\ (c) &\mapsto [0, -\eta(\bar{c})^{-1}] \\ (0) &\mapsto [\infty] \\ (\infty) &\mapsto [0,0] \end{aligned}$$

for all $a \in \mathcal{O}$ and all $b, c \in \mathcal{O}\setminus\{0\}$. All that is left now is to construct an incidence preserving coordinate transformation mapping Ψ to ψ. After some calculations, we find that the following transformation $T\colon \mathbb{P}^2(\mathcal{O}) \to \mathbb{P}^2(\mathcal{O})$ does the job:

$$\begin{aligned} (a,b) &\mapsto (b,-a) & [m,k] &\mapsto [-m^{-1}, -m^{-1}k] \\ (c) &\mapsto (-c^{-1}) & [0,b] &\mapsto [b] \\ (0) &\mapsto (\infty) & [a] &\mapsto [0,-a] \\ (\infty) &\mapsto (0) & [\infty] &\mapsto [\infty] \end{aligned}$$

for all $a, b, k \in \mathcal{O}$ and all $c, m \in \mathcal{O} \setminus \{0\}$. The existence of such a transformation proves that both Moufang sets are indeed isomorphic. We obtain the following result:

Theorem 3.3.3. *The Moufang set $\mathbb{M}(U,\tau)$ obtained from a polarity of type II, given by equations* (3.3.1) *and* (3.3.3)*, is the Moufang building associated to a twisted linear algebraic group of type ${}^2\mathsf{E}_{6,1}^{29}$. Conversely, every Moufang set corresponding to such an algebraic group of type ${}^2\mathsf{E}_{6,1}^{29}$ can be obtained from a polarity of type II and is therefore of the form $\mathbb{M}(U,\tau)$, with U and τ as in equations* (3.3.1) *and* (3.3.3)*, and with η as in section* 3.2.2.

3.3.4 Polarities of type III – Moufang sets of hermitian type

We reconstruct the Moufang structure on X arising from of a polarity Ψ of type III, and we will prove that the Moufang set we obtain is indeed a hermitian Moufang set (of type C_4), by constructing an explicit isomorphism.

First, we determine the structure of the Moufang set obtained by the polarity of the octonion plane. We begin in the same fashion as in section 3.3.1; in particular, the group U is given by equation (3.3.1). However, in order to simplify our proof of the isomorphism with the hermitian Moufang set, we will choose a slightly different τ, as follows. Instead of the collineation σ given by equation (3.3.2), we choose the following collineation σ (which still commutes with the polarity Ψ):

$$\sigma\colon \begin{cases} (a,b) \mapsto \big(-\eta(ab^{-1}), \eta(b)^{-1}\big) \\ [m,k] \mapsto \big[\eta(k^{-1}m), \eta(k)^{-1}\big]. \end{cases}$$

(Recall that $\eta \in \mathrm{Aut}(\mathcal{O})$ is as in section 3.2.2.) The corresponding permutation τ on U^* is thus defined by

$$\tau(a,b) = \big(-\eta(ab^{-1}), \eta(b)^{-1}\big) \tag{3.3.4}$$

for all $(a,b) \in U^*$.

Theorem 3.3.4. *Let k be a field with* $\mathrm{char}(k) \neq 2$*, and let $\mathcal{O}$ be an octonion division algebra over k. Consider a decomposition $\mathcal{O} = D \oplus cD$ with D a quaternion subalgebra of $\mathcal{O}$ and some $c \in D^\perp$ with $\beta = N(c) \neq 0$. Let*

$$\eta\colon \mathcal{O} \to \mathcal{O}\colon a_1 + ca_2 \mapsto a_1 - ca_2,$$

and let

$$h\colon \mathcal{O} \times \mathcal{O} \to D\colon (a_1 + ca_2, b_1 + cb_2) \mapsto \overline{a_1}b_1 + \beta\overline{a_2}b_2$$

for all $a_1, a_2, b_1, b_2 \in D$. Then:

- (i) *h is a hermitian form on $\mathcal{O}$ (considered as a 2-dimensional right vector space over D);*
- (ii) *the Moufang set corresponding to h (as defined in section* 2.3.4*) is isomorphic to the Moufang set $\mathbb{M}(U,\tau)$ arising from the polarity Ψ corresponding to η, with U given by equation* (3.3.1) *and τ given by equation* (3.3.4)*;*
- (iii) *$\mathbb{M}(U,\tau)$ arises from a linear algebraic group of type C_4.*

Proof. (i) This is obvious from the definitions. Note that the corresponding hermitian pseudoquadratic form q is equal to $\frac{1}{2}N_{\mathcal{O}}$:

$$q(a_1 + ca_2) = \tfrac{1}{2}\big(\mathrm{N}(a_1) + \beta\,\mathrm{N}(a_2)\big).$$

(ii) We will denote an element $a_1 + ca_2 \in \mathcal{O}$ as (a_1, a_2), where $a_1, a_2 \in D$. Observe that the multiplication in $\mathcal{O}$ with respect to this decomposition is given by

$$(a_1,a_2)(b_1,b_2) = \big(a_1b_1 - \beta b_2\overline{a_2},\, b_1a_2 + \overline{a_1}b_2\big).$$

Now let $a = (a_1, a_2)$ and $b = (b_1, b_2)$; then the condition

$$\eta(\overline{a}) \cdot a + \eta(\overline{b}) + b = 0$$

occuring in equation (3.3.1) can be rewritten as the system of the following two equations:

$$\overline{a_1} \cdot a_1 - \beta \overline{a_2} \cdot a_2 + \overline{b_1} + b_1 = 0 \quad \text{and}$$
$$a_1 a_2 + b_2 = 0.$$

Now consider the group $T = \{(a, t) \in \mathcal{O} \times D \mid q(a) - t \in D_\sigma^-\}$ as defined in section 2.3.4, with group operation $(a, t) \cdot (b, u) := (a + b, t + u + h(b, a))$. Since $\text{char}(k) \neq 2$, the space D_σ^- is precisely the subspace of trace zero elements of D, and hence

$$T = \big\{(a_1, a_2, t) \in D \times D \times D \mid \mathrm{N}(a_1) + \beta\,\mathrm{N}(a_2) = \mathrm{T}(t)\big\},$$

with

$$(a_1, a_2, t) \cdot (b_1, b_2, u) = \big(a_1 + b_1, a_2 + b_2, t + u + \overline{b_1} a_1 + \beta \overline{b_2} a_2\big).$$

It turns out that the map

$$\chi \colon T \to U \colon (a_1, a_2, t) \mapsto \big((a_1, \overline{a_2}), (-t + \beta\,\mathrm{N}(a_2), -a_1 \overline{a_2})\big)$$

is a group isomorphism.
It remains to check that the map τ given by equation (3.3.4) corresponds to the map τ given by equation (2.3.4) under the isomorphism χ. This follows from another short calculation, keeping in mind that

$$\mathrm{N}\big(-t + \beta\,\mathrm{N}(a_2)\big) + \beta\,\mathrm{N}\big(-a_1 \overline{a_2}\big) = N(t)$$

for all $(a_1, a_2, t) \in T$. We leave the details to the reader.

(iii) By (ii), this now follows, for instance, from [31, p. 56]. □

3.3.5 Polarities of type IV

We finally assume that Ψ is a polarity of type IV; in particular $\text{char}(k) = 2$. Again, U and τ are given by equations (3.3.1) and (3.3.3), respectively. In this case, however, the group U takes a very simple form. Indeed, it follows from [19, Theorem 7.3] that

$$U = \big\{(0, y) \mid y \in \mathcal{O},\ \eta(\overline{y}) = y\big\},$$

and hence U becomes an abelian group, with

$$(0, y_1) + (0, y_2) = (0, y_1 + y_2)$$

for all $(0, y_1), (0, y_2) \in U$. The map τ is now simply given by

$$\tau(0, y) = (0, y^{-1})$$

for all $(0, y) \in U$.

Suppose D is the 4-dimensional subfield of $\mathcal{O}$ fixed by η. Then [19, Theorem 3.1] tells us that the set of elements fixed under $x \mapsto \eta(\overline{x})$ is the set $\mathrm{Fix}(\eta) \oplus kz = D \oplus kz$, with z as in section 3.2.2. Observe that the map τ does indeed preserve the subset U^*.

We conclude that the Moufang set associated to Ψ is (isomorphic to) a Moufang subset of the projective Moufang set $\mathbb{M}(\mathcal{O})$ over the octonion division algebra $\mathcal{O}$, as defined in section 2.3.4; the root groups are 5-dimensional subspaces of the 8-dimensional vector space $\mathcal{O}$ over k.

4

Moufang sets of mixed type F_4

In this chapter, we are studying Moufang sets arising from so-called *mixed groups* of type F_4. These groups exist only over fields of characteristic 2, and they are defined over a *pair* of fields (k, ℓ) such that $\ell^2 \leq k \leq \ell$. There has been an increasing interest in a systematic study of these inseparable situations over non-perfect fields, most notably by the recent work on *pseudo-reductive groups* by B. Conrad, O. Gabber and G. Prasad [9].

In the previous chapter, we described Moufang sets corresponding to polarities of type I. In [13] it is shown that these are exactly the Moufang sets corresponding to algebraic groups of type F_4. The techniques used to prove this however, rely heavily on the fact that the algebraic groups of type F_4 arise as the automorphism groups of certain 27-dimensional algebraic structures known as Albert algebras, and such a description is not available for the mixed groups of type F_4. We therefore take a completely different approach, using the description of the corresponding Chevalley groups, and replacing the geometric ingredients of the algebraic approach (namely polarities of the octonion plane) by group theoretic ingredients (namely involutions of the Chevalley groups). More specifically, we show how to construct a split saturated BN-pair of rank one from a well-chosen involution. Such a BN-pair is essentially equivalent to a Moufang set.

This chapter is organized as follows. In the first section, we recall the necessary basics about Chevalley groups. Section 4.2 deals with the basic theory of mixed groups as introduced by J. Tits. In section 4.3, we specifically look at mixed Chevalley groups of type F_4, and in section 4.4, we study involutions of these mixed groups such that the centralizer of the involution is a split BN-pair of rank one. These BN-pairs give rise to the Moufang sets we are interested in, and in section 4.5 we proceed to explicitly describe these Moufang sets. This culminates in our main result (Theorem 4.5.2). In the last section 4.6, we point out that in the algebraic case, we recover the known description of algebraic Moufang sets of type F_4 as in section 2.3.4.

The results in this chapter are joint work with Tom De Medts and are submitted for publication [6].

4.1 Chevalley groups

We briefly recall some basics about Chevalley groups that we will need in the sequel. Our main reference is [8].

4.1.1 The Cartan decomposition of a complex simple Lie algebra

Let $\mathcal{L}$ be a Lie algebra over $\mathbb{C}$, with Lie bracket $[\cdot,\cdot]$. A Cartan subalgebra $\mathcal{H}$ is a subalgebra of $\mathcal{L}$ which is nilpotent, and such that $\mathcal{H}$ is not contained as an ideal in any larger subalgebra of $\mathcal{L}$, i.e. if $x \in \mathcal{L}$ is such that $[x,h] \in \mathcal{H}$ for all $h \in \mathcal{H}$, then $x \in \mathcal{H}$.

Now if $\mathcal{L}$ is simple over $\mathbb{C}$, then $\mathcal{L}$ can be decomposed into a direct sum of $\mathcal{H}$ with a number of one-dimensional $\mathcal{H}$-invariant subspaces:

$$\mathcal{L} = \mathcal{H} \oplus \mathcal{L}_{r_1} \oplus \cdots \oplus \mathcal{L}_{r_m}.$$

The one-dimensional subspaces $\mathcal{L}_{r_i}$ are called the root spaces of $\mathcal{L}$ (w.r.t. $\mathcal{H}$).

In each one-dimensional subspace $\mathcal{L}_r$, we choose a non-zero element e_r. Then for each $h \in \mathcal{H}$, we have

$$[h, e_r] = r(h)e_r \tag{4.1.1}$$

for some $r(h) \in \mathbb{C}$. This defines a linear map

$$r\colon \mathcal{H} \to \mathbb{C}; h \mapsto r(h).$$

It can be shown that each element f of the dual space $\mathcal{H}^*$ of $\mathcal{H}$ is of the form

$$f : \mathcal{H} \to \mathbb{C}; h \mapsto (x, h)$$

for some unique $x \in \mathcal{H}$, where $(\cdot,\cdot)$ is the Killing form on $\mathcal{H}$. In this way r corresponds to a unique element in $\mathcal{H}$, which we also denote by r. We can repeat this procedure for each root space and denote by Ψ the subset of $\mathcal{H}$ we obtain in this way.

One can show that Ψ forms a root system in $\mathcal{H}_{\mathbb{R}}$, where $\mathcal{H}_{\mathbb{R}}$ is the set of linear combinations of elements of Ψ with real coefficients. As the Killing form induces an isomorphism between $\mathcal{H}$ and its dual space $\mathcal{H}^*$, there is a corresponding root system Φ in $\mathcal{H}^*$. The elements of $\mathcal{H}^*$ are called the *roots* of $\mathcal{L}$ (w.r.t. $\mathcal{H}$).

We can rewrite the Cartan decomposition of $\mathcal{L}$ as

$$\mathcal{L} = \mathcal{H} \oplus \bigoplus_{r\in\Phi} \mathcal{L}_r,$$

in such a way that for any pair of roots $r, s \in \Phi$, we have

$$\begin{aligned}
[\mathcal{L}_r, \mathcal{L}_s] &= \mathcal{L}_{r+s} && \text{if } r+s \in \Phi,\\
[\mathcal{L}_r, \mathcal{L}_s] &= 0 && \text{if } r+s \notin \Phi, r+s \neq 0,\\
[\mathcal{L}_r, \mathcal{L}_{-r}] &= \mathbb{C}r, &&\\
[\mathcal{H}, \mathcal{L}_r] &= \mathcal{L}_r. &&
\end{aligned}$$

These relations can be made more precise. Indeed, if r is any root, then the element $h_r \in \mathcal{H}$ corresponding to $(2r)/\langle r, r\rangle$ under the isomorphism is called the *coroot* of r. Now let Π be a set of fundamental roots for Φ; if we choose $e_{-r} \in \mathcal{L}_{-r}$ such that $[e_r, e_{-r}] = h_r$ for each $r \in \Phi$ then

$$\{h_r \mid r \in \Pi\} \cup \{e_r \mid r \in \Phi\} \tag{4.1.2}$$

forms a basis for $\mathcal{L}$, called a *Chevalley basis*, satisfying

$$\begin{aligned}
[h_r, h_s] &= 0, &&\\
[h_r, e_s] &= A_{rs} e_s, &&\\
[e_r, e_{-r}] &= h_r, &&\\
[e_r, e_s] &= 0 && \text{if } r+s \notin \Phi, r+s \neq 0,\\
[e_r, e_s] &= N_{rs} e_{r+s} && \text{if } r+s \in \Phi.
\end{aligned}$$

The constants A_{rs} are easily determined by the root system, as is the absolute value of the constants N_{rs}; determining the *sign* of the N_{rs} is much more delicate, however. Since we will be working over fields of characteristic 2, we need not worry about these signs.

4.1.2 Chevalley groups

Let $\mathcal{L}$ be a simple Lie algebra over $\mathbb{C}$ with Chevalley basis as in (4.1.2). Now let $\mathcal{L}_{\mathbb{Z}}$ be the subset of $\mathcal{L}$ of all *integral* linear combinations of the basis elements; then $\mathcal{L}_{\mathbb{Z}}$ becomes a Lie algebra over $\mathbb{Z}$.

Now let k be any field. Then we can form the tensor product of the additive group of k with the additive group of $\mathcal{L}_\mathbb{Z}$ and define

$$\mathcal{L}_k = k \otimes \mathcal{L}_\mathbb{Z},$$

which is in a natural way a Lie algebra over k.

We now introduce certain automorphisms of $\mathcal{L}_k$. For every root $r \in \Phi$ and every element $t \in k$, we define an automorphism $u_r(t)$ as follows:

$$\begin{aligned} u_r(t) \cdot e_r &= e_r, \\ u_r(t) \cdot e_{-r} &= e_{-r} + th_r - t^2 e_r, \\ u_r(t) \cdot e_s &= \sum_{i=0}^{q} M_{r,s,i} t^i e_{ir+s} && \text{if } r, s \text{ are linearly independent}, \\ u_r(t) \cdot h_s &= h_s - A_{sr} t e_r && \text{for } s \in \Pi, \end{aligned}$$

where in the rule for $u_r(t) \cdot e_s$, q is the largest integer such that $qr + s \in \Phi$, and where the constants $M_{r,s,i}$ are defined in terms of the structure constants N_{rz}.

The *(adjoint) Chevalley group* of type $\mathcal{L}$ over k is now defined as the group

$$\mathcal{L}(k) := \langle u_r(t) \mid r \in \Phi, t \in k \rangle.$$

It turns out that this group is independent of the choice of the Chevalley basis; its isomorphism type depends only on $\mathcal{L}$ and k. Note that $u_r(s)u_r(t) = u_r(s+t)$ for all $r \in \Phi$ and all $s, t \in k$. The subgroups $U_r = \{u_r(t) \mid t \in k\}$ are called the *root subgroups* of $G = \mathcal{L}(k)$.

Remark 4.1.1. (i) If $k = \mathbb{C}$, then $u_r(t) = \exp(t \operatorname{ad} e_r)$, and in fact, this is where the definition of the automorphisms $u_r(t)$ in the general case comes from.

(ii) We have been following Chevalley's original approach to construct Chevalley groups. This construction has later been generalized to include not only adjoint groups but also more general connected semisimple split linear algebraic groups. The corresponding building, and consequently also the Moufang set that we will construct, does not detect this distinction (its little projective group always corresponds to the adjoint representation) so it is no loss of generality to restrict to the adjoint case.

(iii) The relation between (not necessarily adjoint) Chevalley groups and linear algebraic groups is as follows. Let k be an arbitrary field, and let K be its algebraic closure. Then a Chevalley group $\mathcal{L}(K)$ is a connected semisimple linear algebraic group $\mathbf{G}$ over K of type $\mathcal{L}$, defined and split over k (and in fact over the prime subfield of k). The Chevalley group $\mathcal{L}(k)$ is the commutator subgroup of the group $\mathbf{G}(k)$ of k-rational points of $\mathbf{G}$.

An important feature of Chevalley groups and of linear algebraic groups is that the root groups satisfy certain *commutator relations*. We will discuss those relations (and the extension of this concept to mixed groups) in section 4.2 below.

4.1.3 Weyl groups and subgroups of Chevalley groups

We introduce some notation for certain important subgroups of Chevalley groups that we will need in the future.

Let Φ be the root system associated to an arbitrary Chevalley group $\mathcal{L}(k)$, let Π be a fundamental root system of Φ, Φ^+ be the set of positive roots and Φ^- be the set of negative roots of Φ. One can associate to every root $r \in \Phi$ a reflection w_r; the group generated by all these reflections is called the Weyl group W of $\mathcal{L}(k)$. More information on Weyl groups can be found in [8, Chapter 2].

Furthermore, every Chevalley group G has an associated BN-pair. Indeed, define $n_r(t) := u_r(t)u_{-r}(t)u_r(t)$, $n_r := n_r(1)$ and $h_r(t) := n_r(t)n_r(-1)$ for all $r \in \Phi$ and all $t \in k$. Let

$$\begin{aligned} N &:= \langle n_r(t) \mid r \in \Phi, t \in k\rangle, \\ H &:= \langle h_r(t) \mid r \in \Phi, t \in k\rangle, \\ U &:= \langle u_r(t) \mid r \in \Phi^+, t \in k\rangle \text{ and} \\ B &:= UH. \end{aligned}$$

Then one can show that (B, N) forms a BN-pair for $\mathcal{L}(k)$ with $B \cap N = H$ and $N/H = W$ with W the Weyl group of $\mathcal{L}(k)$.

Finally, we introduce some standard notation. Let J be a subset of Π, then we define Φ_J as $\Phi \cap \langle J\rangle$ and W_J as the Weyl group generated by all w_α with $\alpha \in J$. We denote by w_0 the longest element in W and similarly w_0^J is the longest element in W_J. We then define

$$\begin{aligned} U_J &:= \langle U_r \mid r \in \Phi^+ \setminus \Phi_J\rangle, \\ L_J &:= \langle H, U_r \mid r \in \Phi_J\rangle, \\ P_J &:= U_J L_J. \end{aligned}$$

4.2 Mixed groups

In this section, we recall some basic facts about mixed groups. Our main reference is [32, Section 10.3].

Let G be an adjoint split simple algebraic group of type X defined over a field k of characteristic p, where either $X = \mathsf{B}_n, \mathsf{C}_n, \mathsf{F}_4$ and $p = 2$, or $X = \mathsf{G}_2$ and $p = 3$. Assume moreover that ℓ is a field such that $\ell^p \le k \le \ell$.

Let T be a maximal k-split torus, let $N = N_G(T)$ be the normalizer of T in G, let B be a Borel subgroup of G containing T, and let Φ be a root system of type X corresponding to the maximal torus T. Since X is not simply laced, Φ consists of long and short roots, and we write $\Phi = \Phi_\ell \cup \Phi_s$, where Φ_ℓ and Φ_s denote the sets of long and short roots, respectively. In the algebraic group G, all k-root groups (respectively ℓ-root groups) are isomorphic to the additive group of the field k (respectively ℓ). For each $a \in \Phi$, we choose an isomorphism u_a from k (respectively ℓ) to $U_a(k)$ (respectively $U_a(\ell)$), so $U_a(k) = u_a(k)$ and $U_a(\ell) = u_a(\ell)$. We also define Φ^+ to be the set of positive roots of Φ, i.e. the roots $a \in \Phi$ such that $U_a \subseteq B$; correspondingly, we write $\Phi_\ell^+ := \Phi_\ell \cap \Phi^+$ and $\Phi_s^+ := \Phi_s \cap \Phi^+$.

Now let

$$\begin{aligned} T(k,\ell) &:= \left\{ t \in T \;\middle|\; \begin{array}{ll} a(t) \in k & \text{for all } a \in \Phi_l \text{ and} \\ a(t) \in \ell & \text{for all } a \in \Phi_s \end{array} \right\}, \\ N(k,\ell) &:= N(k)\, T(k,\ell)\,, \\ B(k,\ell) &:= \langle T(k,\ell) \cup \{U_a(k) \mid a \in \Phi_l^+\} \cup \{U_a(\ell) \mid a \in \Phi_s^+\}\rangle\,, \end{aligned}$$

and finally

$$G(k,\ell) := \langle T(k,\ell) \cup \{U_a(k) \mid a \in \Phi_l\} \cup \{U_a(\ell) \mid a \in \Phi_s\}\rangle\,.$$

The group $G(k,\ell)$ is the *mixed group* of type X corresponding to the pair of fields (k,ℓ), and it is also denoted by $X(k,\ell)$, particularly when X is specified. One can show that the pair $(B(k,\ell), N(k,\ell))$ forms a BN-pair of $G(k,\ell)$.

Example 4.2.1 ([32, p. 204])**.** *Let (k,ℓ) be a pair of fields of characteristic 2 with $\ell^2 \le k \le \ell$, and let q be the "mixed quadratic form" from $k^{2n} \times \ell$ to k given by*

$$q(x_0, x_1, \ldots, x_{2n-2}, x_{2n-1}, x_{2n}) = x_0^2 + x_1x_2 + \cdots + x_{2n-1}x_{2n}\,.$$

Then the mixed group $\mathsf{B}_n(k,\ell)$ is isomorphic to the group $\mathsf{PGO}(q)$, i.e. the quotient of the group of all invertible similitudes of q by the subgroup $k^\times$. This group is also isomorphic to the mixed group $\mathsf{C}_n(\ell^2, k)$.

When we are considering the corresponding building, i.e. the "mixed quadric" consisting of the isotropic vectors of q, it will often be convenient to drop the last coordinate x_{2n}, since it is uniquely determined from the other coordinates by the equation $q(x_0, \ldots, x_{2n-1}, x_{2n}) = 0$. Thus, the mixed quadric will then consist of *points in* $\mathsf{PG}(2n-1,k)$ *with (projective) coordinates* $(X_0,\ldots,X_{2n-1})$ *satisfying the condition*

$$X_1X_2+\cdots+X_{2n-1}X_{2n}\in\ell^2\,, \tag{4.2.1}$$

and the higher-dimensional objects of the building are now simply the subspaces of the underlying projective space lying on this mixed quadric.

For split algebraic groups, it is well known that the root groups satisfy certain commutator relations depending on the root system. More precisely, it is possible to renormalize the parametrizations u_a in such a way that there are constants $c_{r,a,s,b}\in\{\pm1,\pm2,\pm3\}$, called the *structure constants*, such that

$$[u_a(x),u_b(y)]=\prod_{\substack{r,s\in\mathbb{Z}_{>0}\\ ra+sb\in\Phi}}u_{ra+sb}\left(c_{r,a,s,b}\cdot x^ry^s\right) \tag{4.2.2}$$

for all $a,b\in\Phi$ and all $x,y\in k$; see, for example, [26, Propositions 9.2.5 and 9.5.3], or [8, Theorem 5.2.2] for the analogous statement for Chevalley groups.

This goes through for mixed groups without any change: we get the same commutator relations (4.2.2), but this time for all $a,b\in\Phi$ and all $x,y\in k$ or ℓ depending on whether the corresponding roots a and b are long or short, respectively. Observe that the condition $\ell^p\le k\le\ell$ is exactly the condition which is needed for these commutator relations to make sense, i.e. the elements x^ry^s belong to k whenever the root $ra+sb$ is a long root.

In the case $p=\mathrm{char}(k)=2$, which will be the only case we will be dealing with in this chapter, the constants $c_{r,a,s,b}$ are all equal to 0 or 1, so equation (4.2.2) simplifies further. In the case $p=2$ and $X=\mathsf{F}_4$, which is the case that we are interested in in this chapter, we can summarize the commutator relations as follows; see, for instance, [23, (2.2)–(2.5)]:

$$\begin{aligned}
&[u_a(x),u_b(y)]=1 && \text{if } a,b\in\Phi \text{ but } a+b\notin\Phi\,,\\
&[u_a(x),u_b(y)]=u_{a+b}(xy) && \text{if } a,b\in\Phi_s \text{ and } a+b\in\Phi_s\,,\\
&[u_a(x),u_b(y)]=1 && \text{if } a,b\in\Phi_s \text{ and } a+b\in\Phi_\ell\,,\\
&[u_a(x),u_b(y)]=u_{a+b}(xy) && \text{if } a,b\in\Phi_\ell \text{ and } a+b\in\Phi_\ell\,,\\
&[u_a(x),u_b(y)]=u_{a+b}(xy)\,u_{2a+b}(x^2y)\\
&\qquad\qquad \text{if } a\in\Phi_s, b\in\Phi_\ell \text{ and } a+b\in\Phi_s, 2a+b\in\Phi_\ell\,,
\end{aligned} \tag{4.2.3}$$

with a,b linearly independent and for all $x,y\in k$ or ℓ depending on whether the corresponding roots a and b are long or short, respectively. Note that this list is exhaustive: if a and b are long roots with $a+b\in\Phi$, then $a+b\in\Phi_\ell$; and if a is a short root and b a long root with $a+b\in\Phi$, then $2a+b\in\Phi$ as well and $a+b$ is short and $2a+b$ is long. See [23, (1.2) and (1.3)].

4.3 Mixed Chevalley groups of type F_4

Let k and ℓ be fields of characteristic 2 such that $\ell^2 \leq k \leq \ell$. Assume that $\delta \in k$ is such that the polynomial $x^2 + x + \delta$ is irreducible over k. Let γ be a solution of $x^2 + x = \delta$, and let $K = k(\gamma)$ and $L = \ell(\gamma)$. Then $L^2 \leq K = \langle k, L^2 \rangle \leq L$, and K and L are separable quadratic extensions of k and ℓ, respectively. We denote the standard involution on both L and K corresponding to γ by $x \mapsto \overline{x}$.

Let Φ be a root system of type F_4 with fundamental system $\Pi := \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. We can represent the fundamental roots with respect to an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $\mathbb{R}^4$ as $\alpha_1 = \frac{1}{2}(-e_1 - e_2 - e_3 + e_4)$, $\alpha_2 = e_3$, $\alpha_3 = e_2 - e_3$, $\alpha_4 = e_1 - e_2$ and the full system of roots is given by

$$\Phi = \begin{cases} \pm e_i \pm e_j \text{ for } 1 \leq i < j \leq 4, \\ \pm e_i \text{ for } 1 \leq i \leq 4, \\ \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4). \end{cases}$$

We define the mixed Chevalley group $\mathsf{F}_4(K, L)$ of type F_4 as the mixed group that can be obtained from the ordinary Chevalley group $\mathsf{F}_4(L)$ of type F_4. For this, we remark (using the definitions introduced in section 4.1.3) that H is a maximal K-split torus, $N = N_{\mathsf{F}_4(L)}(H)$ is the normalizer of H in $\mathsf{F}_4(L)$ and B is a Borel subgroup of $\mathsf{F}_4(L)$. Then

$$\mathsf{F}_4(K, L) = \Big\langle \{u_r(s) \mid r \in \Phi_\ell, s \in K\} \cup \{u_r(t) \mid r \in \Phi_s, t \in L\} \cup T(K, L) \Big\rangle$$

is the mixed group of type F_4 corresponding to the pair of fields (K, L) of $\mathsf{F}_4(L)$. Using the same procedure, we can construct mixed Chevalley groups of type B_n, C_n and G_2. In general, we denote a mixed Chevalley group by $X(K, L)$.

The next lemma shows that we can omit the subgroup $T(K, L)$ in the generating set for $X(K, L)$ if $X(K, L)$ is of type F_4 or B_n with n odd.

Lemma 4.3.1. *Let $X(K, L)$ be a mixed Chevalley group of type F_4 or B_n with n odd. Let*

$$T(K, L) = \big\{ h \in T(L) \mid a(h) \in K \textit{ for all } a \in \Phi_l \textit{ and } a(h) \in L \textit{ for all } a \in \Phi_s \big\}$$

as before. Then

$$T(K, L) = \Big\langle \{h_r(t) \mid r \in \Phi_l,\ t \in K^*\} \cup \{h_r(t) \mid r \in \Phi_s,\ t \in L^*\} \Big\rangle.$$

In particular,

$$X(K,L) = \Big\langle \{u_r(s) \mid r \in \Phi_\ell, s \in K\} \cup \{u_r(t) \mid r \in \Phi_s, t \in L\} \Big\rangle.$$

Proof. Let $H = \langle \{h_r(\lambda) \mid r \in \Phi, \lambda \in L^*\}\rangle$. We claim that

$$H = \langle h_{\alpha_i}(\lambda_i) \mid \alpha_i \in \Pi \text{ and } \lambda_i \in L^* \rangle, \tag{4.3.1}$$

where $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ is the set of fundamental roots of Φ.

For Chevalley groups, each element $h_r(\lambda)$ (with $r \in \Phi$ and $\lambda \in L^*$) corresponds to an L-character

$$\chi_{r,\lambda} \colon \mathbb{Z}\Phi \to L^* \colon a \mapsto \chi_{r,\lambda}(a) := \lambda^{2\langle r,a\rangle/\langle r,r\rangle}$$

(see [8, Chapter 7]). The product of generators in H then corresponds to the product of the corresponding characters.

Two characters are equal if and only if the corresponding elements of H are the same. Therefore, it is enough to show that every character χ corresponding to an element of H can be written as a product of elements $\chi_{\alpha_i,\lambda_i}$ with $\alpha_i \in \Pi$, $\lambda_i \in L^*$.

Now let $\{q_1, \ldots, q_l\}$ be the basis of $\mathcal{H}^*_{\mathbb{R}}$ dual to the basis $\{h_{\alpha_1}, \ldots, h_{\alpha_l}\}$ of fundamental coroots of $\mathcal{H}_{\mathbb{R}}$. These elements are called the fundamental weights of $\mathcal{L}$. They have the nice property that

$$2 \cdot \frac{\langle \alpha_i, q_j \rangle}{\langle \alpha_i, \alpha_i \rangle} = \delta_{ij}$$

for all $i, j \in \{1, \ldots, l\}$. Let Q be the set of integral linear combinations of the fundamental weights; notice that $\mathbb{Z}\Phi$ is a finite index subgroup of Q. For each $h \in H$, [8, Theorem 7.1.1] then implies that the L-character χ corresponding to h extends to an L-character χ of Q.

Now let $h \in H$ be arbitrary, and let χ be the corresponding L-character of Q. For each i, we let $\lambda_i = \chi(q_i)$; then $\chi = \prod_i \chi_{\alpha_i,\lambda_i}$ since $\chi_{\alpha_i,\lambda_i}(q_j) = \lambda_i^{\delta_{ij}}$. It follows that $h = h(\chi) = \prod h_{\alpha_i}(\lambda_i)$, proving our claim (4.3.1).

Next, we notice that

$$S := \{h_r(t) \mid r \in \Phi_s,\ t \in L^*\} \cup \{h_r(t) \mid r \in \Phi_l,\ t \in K^*\} \subseteq T(K,L).$$

We prove that any element of $T(K,L)$ can be written as a product of the elements in S. Let $h \in T(K,L)$, then $h = \prod_{i \mid \alpha_i \in \Pi} h_{\alpha_i}(\lambda_i)$. Since all the elements

$h_{\alpha_i}(\lambda_i)$ with $\alpha_i \in \Phi_s$ are already in S by definition and as H is abelian, it suffices to consider only elements $h = \prod_{j|\alpha_j \in \Pi_l} h_{\alpha_j}(\lambda_j)$, where Π_l is the set of long fundamental roots. Furthermore, we may assume that all λ_j are in $L \setminus K$, since otherwise $h_{\alpha_j}(\lambda_j)$ is already contained in S as well.

So, it only remains to show that elements of the form $h = \prod_i h_{\alpha_i}(\lambda_i)$ with $\lambda_i \in L$ and $\alpha_i \in \Pi_l$ can only be contained in $T(K, L)$ if each $\lambda_i \in K$. To prove this, we have a closer look at the root systems of type B_n and F_4. With respect to an orthonormal basis $\{e_1, \ldots, e_n\}$, the root system of type B_n can be chosen in such a way that the fundamental roots are $e_1 - e_2, \ldots, e_{n-1} - e_n, e_n$. So for both B_n and F_4, the set of long fundamental roots is of the form

$$\{\alpha_1, \ldots, \alpha_{n-1}\} := \{e_1 - e_2, \ldots, e_{n-1} - e_n\}$$

(where $n = 3$ for F_4). Notice that $n \geq 3$ since n is odd. Now let $h = \prod_{i=1}^{n-1} h_{\alpha_i}(\lambda_i)$ with $\lambda_i \in L$. We observe that

$$\chi(e_1 + e_2) = \prod_{i=1}^{n-1} \chi_{\alpha_i,\lambda_i}(e_1 + e_2) = \lambda_2 \in K;$$

$$\chi(e_{n-1} + e_n) = \prod_{i=1}^{n-1} \chi_{\alpha_i,\lambda_i}(e_{n-1} + e_n) = \lambda_{n-2}^{-1} \in K;$$

and when $n \geq 4$, we have for each $2 \leq k \leq n-2$

$$\chi(e_k + e_{k+1}) = \prod_{i=1}^{n-1} \chi_{\alpha_i,\lambda_i}(e_k + e_{k+1}) = \lambda_{k-1}^{-1}\lambda_{k+1} \in K. \tag{4.3.2}$$

Since $\lambda_2 \in K$, we can repeatedly apply (4.3.2) to obtain $\lambda_4, \lambda_6, \ldots, \lambda_{n-1} \in K$. On the other hand, since $\lambda_{n-2} \in K$, we can repeatedly apply (4.3.2) to obtain $\lambda_{n-2}, \lambda_{n-4}, \ldots, \lambda_1 \in K$. We conclude that all λ_i are contained in K. □

Remark 4.3.2. For Chevalley groups of mixed type B_n (n even), C_n and G_2 the argument in the proof of Lemma 4.3.1 fails. We do not know whether the statement of Lemma 4.3.1 continues to hold in these cases.

The previous lemma will allow us to transfer known facts about BN-pairs of (ordinary) Chevalley groups to BN-pairs of mixed Chevalley groups. Indeed, when $X(K, L)$ is a mixed group, the subgroups

$$N(K, L) := N(K)\, T(K, L) \quad \text{and}$$
$$B(K, L) := \big\langle T(K, L) \cup \{U_a(K) \mid a \in \Phi_l^+\} \cup \{U_a(L) \mid a \in \Phi_s^+\}\big\rangle$$

form a BN-pair for $X(K,L)$. Using Lemma 4.3.1, we actually get

$$B(K,L) = B(L) \cap X(K,L) \quad \text{and} \quad N(K,L) = N(L) \cap X(K,L)$$

if $X(K,L)$ is of the appropriate type, where $(B(L), N(L))$ is the natural BN-pair of $X(L)$. This implies (using the general properties of a BN-pair) that

$$X(K,L) = B(K,L)\ N(K,L)\ B(K,L)$$

and that all parabolic subgroups containing $B(K,L)$ are of the form

$$P_J(K,L) := B(K,L)\ N_J(K,L)\ B(K,L) = P_J \cap X(K,L).$$

Notice that $N(K,L)/T(K,L)$ is also isomorphic to the Weyl group of $X(L)$. So $N_J(K,L)$ is the preimage of W_J under the canonical epimorphism from $N(K,L)$ to W.

We end this section with a unique decomposition lemma for mixed Chevalley groups.

Lemma 4.3.3. *Let $X(K,L)$ be a mixed Chevalley group of type F_4 or of type B_n with n odd. If $g \in X(K,L)$ is such that*

$$P_J(K,L) g P_J(K,L) = P_J(K,L) n P_J(K,L) \tag{4.3.3}$$

with $n \in N(K,L)$ such that $nT(K,L) = w \in \mathrm{Stab}(\Phi_J)$, then g has a unique decomposition $g = ulnu'$ with $u \in U_J(K,L) := U_J \cap X(K,L)$, $l \in L_J(K,L) := L_J \cap X(K,L)$ and $u' \in U^-_{w,J}$, where

$$U^-_{w,J} := \langle X_r \mid r \in \Phi^+ \setminus \Phi_J, w(r) \in \Phi^- \rangle \cap X(K,L).$$

Proof. From the equality (4.3.3), we find that every element of the form $p_1 g p_2$ (p_1, p_2 arbitrary in $P_J \cap X(K,L)$) equals $p_3 n p_4$ for some $p_3, p_4 \in P_J \cap X(K,L)$. Therefore, we get that $g = p_1^{-1} p_3 n p_4 p_2^{-1}$. As ordinary Chevalley groups have a Levi-decomposition $P_J = L_J \cdot U_J$, we find that

$$P_J(K,L) = L_J(K,L) \cdot U_J(K,L).$$

Assume $p_4 p_2^{-1} = l'u'$ for some $l' \in L_J(K,L)$ and $u' \in U_J(K,L)$, then (as $nH \in \mathrm{Stab}(\Phi_J)$), we can switch l' to the left of n. We find that indeed $g = pnu' = ulnu'$ for some $u \in U_J(K,L)$ and $l \in L_J(K,L)$.

Suppose that $g = u_1 l_1 n u_1' = u_2 l_2 n u_2'$, then $n u_1' {u_2'}^{-1} n^{-1} = u_1 {l_1}^{-1} u_2 l_2 \in U_J^- \cap P_J = 1$, so uniqueness follows. □

4.4 Construction of a split saturated BN-pair of rank one

In this section we construct a split saturated BN-pair out of an involution on $\mathsf{F}_4(K, L)$. For Chevalley groups there exists a general procedure to construct a BN-pair from an involution σ satisfying certain conditions, as carried out in [29] and [30].

Using a similar procedure, we show that we can construct a split saturated BN-pair of rank one from a suitable involution on $\mathsf{F}_4(K, L)$. From a geometric point of view, we actually have constructed, starting from a mixed building of type F_4, a new type of Moufang set; these Moufang sets will be called *Moufang sets of mixed type* F_4.

4.4.1 Construction of an involution on $\mathsf{F}_4(K, L)$

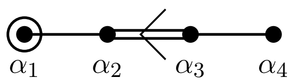
We follow the ideas from [29], but in order to deal with the situation of mixed Chevalley groups, we impose slightly adjusted conditions on the involution σ on $\mathsf{F}_4(K, L)$. More precisely, we fix a set $J \subsetneq \Pi$, and we choose σ in such a way that

(1) σ permutes root groups and $N(K, L)$ is invariant under σ.
(2) If P is a parabolic subgroup of $L_J(K, L) = L_J(L) \cap \mathsf{F}_4(K, L)$, invariant under σ, then $P = L_J(K, L)$.
(3) $\langle \{U_r(K) \mid r \in \Phi_l^- \setminus \Phi_J\} \cup \{U_r(L) \mid r \in \Phi_s^- \setminus \Phi_J\} \rangle \cap \mathrm{Fix}(\sigma) \neq 1$.

In order to take care of the first condition, we consider an involution σ of $\mathsf{F}_4(K, L)$ with the following action on the generators of the mixed group (where we denote the corresponding action on the root system also by σ):

$$\sigma : \mathsf{F}_4(K, L) \to \mathsf{F}_4(K, L) \colon u_r(t) \mapsto u_{\sigma(r)}(c_r\, \bar{t}).$$

In analogy with the situation in the algebraic case, we will choose the action of σ on the root system so that the corresponding Tits index is as follows:



If we now look at the F_4-building corresponding to $\mathsf{F}_4(K, L)$, our goal is to construct the involution σ on $\mathsf{F}_4(K, L)$ in such a way that the corresponding fixbuilding has only points of the first type (these are the points corresponding to α_1). Therefore, we choose J to be the subset $\{\alpha_2, \alpha_3, \alpha_4\}$ of Π. In particular, the action of σ on the root system Φ is given by the longest element ${w_0}^J$ in the Weyl group W_J generated by w_{α_2}, w_{α_3} and w_{α_4}. This implies (since σ is an involution) that the action of σ on Π is given by

$$\begin{cases} \alpha_1 \mapsto \frac{1}{2}(e_1+e_2+e_3+e_4) = \alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4 \\ \alpha_2 \mapsto -\alpha_2 \\ \alpha_3 \mapsto -\alpha_3 \\ \alpha_4 \mapsto -\alpha_4. \end{cases}$$

Our next step will be to determine the coefficients c_r so that σ does indeed give rise to a Moufang set. We will first focus on the second condition for σ; since this condition only concerns the subgroup $L_J(K,L)$ of $\mathsf{F}_4(K,L)$, we will achieve this by looking at the subgroup $\mathsf{B}_3(K,L) \leq \mathsf{F}_4(K,L)$ (this is the subgroup of $\mathsf{F}_4(K,L)$ generated by the root groups $U_{\alpha_2}(L)$, $U_{\alpha_3}(K)$ and $U_{\alpha_4}(K)$). Once we will have constructed an involution σ such that the second condition is satisfied, we will see it is not very hard to check that also the third condition for σ holds.

In the non-mixed case, we know that the action of σ on the B_3-building has to be chosen in such a way that the group fixed under σ is isomorphic to the projective orthogonal group of an anistropic quadratic form of dimension 7 with trivial Hasse invariant; see [24, Section 3.4] for more details. One can show that every such a quadratic form can be obtained as the restriction to the trace zero part of an 8-dimensional norm form of an octonion division algebra (i.e. of a 3-fold Pfister form). In a completely similar way, we obtain that the fixed point set of the involution σ on the mixed B_3-subbuilding has to be isomorphic to $\mathsf{PGO}(q)$ with q the trace zero part of the 'mixed' norm form of an octonion division algebra. In the next subsection, we determine explicitly what this action should be and deduce in this way the coefficients c_r.

4.4.2 The action on the B_3-subbuilding

As we have seen in Example 4.2.1, we can identify $\mathsf{B}_3(K,L)$ with the group $\mathsf{PGO}(\mathcal{Q})$ where $\mathcal{Q}$ is the mixed quadratic form

$$\mathcal{Q} : L \oplus K^6 \to K; (x_0, x_1, x_{-1}, x_2, x_{-2}, x_3, x_{-3}) \mapsto x_0^2 + x_1x_{-1} + x_2x_{-2} + x_3x_{-3}$$

with respect to a well chosen hyperbolic basis $\mathcal{C}$; this group consists of all (K,L)-linear maps φ (modulo scalars) such that $\mathcal{Q}(\varphi(v)) = \mathcal{Q}(v)$ for all $v \in L \oplus K^6$. There is a bijective correspondence between (K,L)-linear maps φ and the invertible 7 by 7 matrices A such that the first row[1] consists of elements in L, while the

[1] We will always use *left* multiplication by matrices on column spaces.

others consist of elements in K and all the elements in the first column, except the first one, are zero.

First, we will determine the isomorphism between $\mathsf{B}_3(K, L)$ and $\mathsf{PGO}(\mathcal{Q})$ explicitly, because this will allow us to describe the action of σ on the B_3-subbuilding entirely in terms of matrices.

Construction of an isomorphism between $\mathsf{B}_3(K, L)$ and $\mathsf{PGO}(J)$

We use the correspondence between $\mathsf{B}_3(L)$ and $\mathsf{PGO}(\tilde{\mathcal{Q}})$ (with $\tilde{\mathcal{Q}}$ the unique extension of $\mathcal{Q}$ to a quadratic form on L^7) mentioned in [8, Section 11.3] to determine a matrix representation for the elements of $\mathsf{B}_3(K, L)$. Therefore we return to the original definition of $\mathsf{B}_3(L)$ as the group generated by some automorphisms on $L \otimes \mathcal{L}_{\mathbb{Z}}$. We know $\mathcal{L} := \mathcal{L}_{\mathbb{C}}$ has a Cartan decomposition

$$\mathcal{L} = \mathcal{H} \oplus \bigoplus_{r \in \Phi_{\mathsf{B}_3}} \mathcal{L}_r = \mathcal{H} \oplus \bigoplus \mathbb{C} e_r,$$

where e_r runs through the list ([8, p. 180])

$$\begin{array}{ccc} E_{i,j} - E_{-j,-i} & E_{i,-j} - E_{j,-i} & 2E_{i,0} - E_{0,-i} \\ -E_{-i,-j} + E_{j,i} & -E_{-i,j} + E_{-j,i} & -2E_{-i,0} + E_{0,i} \end{array}$$

for $0 < i < j \leq 3$. The matrices $E_{i,j}$ are the 7 by 7 matrices with a 1 on the (i, j)-th position, with rows and columns indexed by $\{0, 1, -1, 2, -2, 3, -3\}$.

Next, we want to find an explicit correspondence between the roots of Φ_{B_3} and the root spaces $\mathbb{C} e_r$ of $\mathcal{L}$. Therefore, it suffices to find an identification between the fundamental roots of $\mathcal{H}^*$ and those of Φ_{B_3}. According to [8, p. 180], the elements of $\mathcal{H}$ are of the form $\mathrm{diag}(0, \lambda_1, \lambda_2, \lambda_3, -\lambda_1, -\lambda_2, -\lambda_3)$, $\lambda_i \in \mathbb{C}$. We find a fundamental system $\{\tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\alpha}_4\}$ for $\mathcal{H}^*$ with

$$\begin{aligned} \tilde{\alpha}_2 &\colon \mathcal{H} \to \mathbb{C}; \mathrm{diag}(0, \lambda_1, \lambda_2, \lambda_3, -\lambda_1, -\lambda_2, -\lambda_3) \mapsto \lambda_3, \\ \tilde{\alpha}_3 &\colon \mathcal{H} \to \mathbb{C}; \mathrm{diag}(0, \lambda_1, \lambda_2, \lambda_3, -\lambda_1, -\lambda_2, -\lambda_3) \mapsto \lambda_2 - \lambda_3, \\ \tilde{\alpha}_4 &\colon \mathcal{H} \to \mathbb{C}; \mathrm{diag}(0, \lambda_1, \lambda_2, \lambda_3, -\lambda_1, -\lambda_2, -\lambda_3) \mapsto \lambda_1 - \lambda_2. \end{aligned}$$

With the use of equation (4.1.1), we obtain that the elements $\tilde{\alpha}_2$, $\tilde{\alpha}_3$, $\tilde{\alpha}_4$ of $\mathcal{H}^*$ correspond to the elements $2E_{3,0} - E_{0,-3}$, $E_{2,3} - E_{-3,-2}$ and $E_{1,2} - E_{-2,-1}$ of the Chevalley basis, respectively. The obvious identification one can make between the roots of $\mathcal{H}^*$ and those of Φ_{B_3} is

$$\begin{aligned} \tilde{\alpha}_2 &\leftrightarrow \alpha_2 = e_3 \\ \tilde{\alpha}_3 &\leftrightarrow \alpha_3 = e_2 - e_3 \\ \tilde{\alpha}_4 &\leftrightarrow \alpha_4 = e_1 - e_2. \end{aligned}$$

We can now identify the elements $u_r(t)$ with matrices. This can be done using the epimorphism

$$G \to \mathcal{L}(L) : \exp(te_r) \mapsto u_r(t),$$

with $\exp(te_r)$ being the matrices described on [8, p.183] and G being the group generated by all these matrices. The kernel of this map turns out to be the center of G.

In this way we can identify the following elements for all $i, j \in \{1, \dots, 3\}$:

$$\begin{array}{rcl} u_{e_i-e_j}(\lambda) & \leftrightarrow & \mathrm{I} + \lambda(E_{i,j} + E_{-j,-i}) \\ u_{e_i+e_j}(\lambda) & \leftrightarrow & \mathrm{I} + \lambda(E_{i,-j} + E_{j,-i}) \\ u_{-e_i-e_j}(\lambda) & \leftrightarrow & \mathrm{I} + \lambda(E_{-i,j} + E_{-j,i}) \\ u_{e_i}(\lambda) & \leftrightarrow & \mathrm{I} + \lambda^2 E_{i,-i} \\ u_{-e_i}(\lambda) & \leftrightarrow & \mathrm{I} + \lambda^2 E_{-i,i}. \end{array}$$

for all $\lambda \in L$, where I is the 7 by 7 identity matrix.

The action of σ on $\mathsf{PGO}(J)$

As mentioned in the previous subsection, we wish to construct σ in such a way that the fixed points form a group isomorphic to $\mathsf{PGO}(q)$ with q, the trace zero part of a mixed norm form of an octonion division algebra. Such a quadratic form is defined over the fields k and ℓ and is of the form

$$\begin{aligned} \mathrm{N} := \mathrm{N}_L \perp \alpha\,\mathrm{N}_K \perp \beta\,\mathrm{N}_K \perp \alpha\beta\,\mathrm{N}_K : & \\ L \oplus K \oplus K \oplus K \to \ell; & \\ (y_1, y_2, y_3, y_4) \mapsto y_1\overline{y_1} + \alpha y_2\overline{y_2} + \beta y_3\overline{y_3} + \alpha\beta y_4\overline{y_4}\,, & \end{aligned}$$

where α, β are constants in $k^\times$.

Remark 4.4.1. Denote by N_ℓ the extension of N to the octonion algebra $\mathcal{O}_\ell = L \oplus L \oplus L \oplus L$. Although the norm on $\mathcal{O}_\ell$ is uniquely determined, there is no canonical way to define the product of two octonions (in terms of the decomposition $\mathcal{O}_\ell = L^4$), although all of the octonion algebras are isomorphic. The most common way to define such a multiplication uses the fact that every composition algebra of dimension $d > 1$ can be obtained from a $(d/2)$-dimensional subalgebra by the so-called Cayley–Dickson doubling process; see, for example, [27, Proposition 1.5.3]. For our purposes, it will be more convenient to use the following description.

Let $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$ be two arbitrary elements of $\mathcal{O}_\ell$, then we define the product $x \cdot y$ to be equal to

$$\big(x_1y_1 + \alpha\overline{x_2}y_2 + \beta\overline{x_3}y_3 + \alpha\beta\overline{x_4}y_4,\ x_2y_1 + \overline{x_1}y_2 + \beta x_4\overline{y_3} + \beta\overline{x_3}y_4,\\ x_3y_1 + \overline{x_1}y_3 + \alpha x_4\overline{y_2} + \alpha\overline{x_2}y_4,\ x_3y_2 + x_2y_3 + x_4\overline{y_1} + x_1y_4\big);$$

this makes $\mathcal{O}_\ell = L^4$ into an octonion algebra with norm N_ℓ.

The restriction q of N to the subspace of trace zero elements is then of the following form:

$$q : \ell \oplus K \oplus K \oplus K \to k; (y_1, y_2, y_3, y_4) \mapsto y_1^2 + \alpha y_2\overline{y_2} + \beta y_3\overline{y_3} + \alpha\beta y_4\overline{y_4}.$$

We will now show how to construct such an involution σ.

Viewing the mixed quadratic form q as a form over (K, L) in the obvious way (we denote this extended form by $Q := q \otimes_{k,\ell} (K, L)$), the quadratic forms Q and $\mathcal{Q}$ are isometric. Indeed, looking at the matrix representations $[Q]_\mathcal{B}$ and $[\mathcal{Q}]_\mathcal{C}$ of the corresponding bilinear forms with respect to the standard bases $\mathcal{B}$ and $\mathcal{C}$ we see that

$$[Q]_\mathcal{B} = \begin{pmatrix} 1 &&&&&&\\ & \alpha & \alpha &&&&\\ & 0 & \alpha\delta &&&&\\ &&& \beta & \beta &&\\ &&& 0 & \beta\delta &&\\ &&&&& \alpha\beta & \alpha\beta\\ &&&&& 0 & \alpha\beta\delta \end{pmatrix} \text{ and } [\mathcal{Q}]_\mathcal{C} = \begin{pmatrix} 1 &&&&&&\\ & 0 & 1 &&&&\\ & 0 & 0 &&&&\\ &&& 0 & 1 &&\\ &&& 0 & 0 &&\\ &&&&& 0 & 1\\ &&&&& 0 & 0 \end{pmatrix}.$$

A change of the base of $\mathcal{B}$ using the transition matrix

$$S := \begin{pmatrix} 1 &&&&&&\\ & 1 & \gamma &&&&\\ & \alpha & \alpha\overline{\gamma} &&&&\\ &&& 1 & \gamma &&\\ &&& \beta & \beta\overline{\gamma} &&\\ &&&&& 1 & \gamma\\ &&&&& \alpha\beta & \alpha\beta\overline{\gamma} \end{pmatrix}$$

will do the job.

It follows that $\mathsf{PGO}(Q)^{S^{-1}} = \mathsf{PGO}(\mathcal{Q})$, and since $\mathsf{PGO}(q)$ is equal to $\{A \in$

$\mathsf{PGO}(Q) \mid A = \overline{A}\}$ we obtain

$$\begin{aligned}\mathsf{PGO}(q)^{S^{-1}} &= \{SAS^{-1} \mid A \in \mathsf{PGO}(Q) \text{ and } A = \overline{A}\}\\ &= \{B \in \mathsf{PGO}(\mathcal{Q}) \mid S^{-1}BS = \overline{S^{-1}BS}\}\\ &= \{B \in \mathsf{PGO}(\mathcal{Q}) \mid B = (S\overline{S}^{-1})\overline{B}(\overline{S}S^{-1})\}\\ &= \{B \in \mathsf{PGO}(\mathcal{Q}) \mid B = M^{-1}\overline{B}M\}\end{aligned}$$

with

$$M = \overline{S}S^{-1} = \begin{pmatrix} 1 & & & & & & \\ & 0 & \alpha^{-1} & & & & \\ & \alpha & 0 & & & & \\ & & & 0 & \beta^{-1} & & \\ & & & \beta & 0 & & \\ & & & & & 0 & \alpha^{-1}\beta^{-1} \\ & & & & & \alpha\beta & 0 \end{pmatrix}.$$

We conclude that we can describe the restriction of the involution σ (using the isomorphism between $\mathsf{PGO}(\mathcal{Q})$ and $\mathsf{B}_3(K,L)$) as

$$\sigma_{|_{\mathsf{B}_3(K,L)}} : \mathsf{PGO}(\mathcal{Q}) \to \mathsf{PGO}(\mathcal{Q}); x \mapsto M^{-1}\overline{x}M. \tag{4.4.1}$$

We will from now on identify $\mathsf{PGO}(\mathcal{Q})$ and $\mathsf{B}_3(K,L)$ without explicitly mentioning the isomorphism.

Calculation of the coefficients c_r

Using the identification between $\mathsf{PGO}(\mathcal{Q})$ and $\mathsf{B}_3(K,L)$, we can now write the involution σ as

$$\sigma(u_r(t)) = M^{-1}\overline{u_r(t)}M = u_{\sigma(r)}(c_r\overline{t}).$$

for all $r \in \Phi_{\mathsf{B}_3}$. Using (4.4.1) we find for the generators α_2, α_3 and α_4 of Φ_{B_3} that $c_{\alpha_2} = \alpha\beta$, $c_{\alpha_3} = \alpha^{-1}$ and $c_{\alpha_4} = \alpha\beta^{-1}$.

It remains to determine the coefficient c_{α_1} because then all c_r follow using the Chevalley commutator relations. Since the anisotropic subbuilding is of the right form, the only thing we still have to express is that σ should be an involution. This is fulfilled if $\overline{c_{\alpha_1}}c_{\sigma(\alpha_1)} = 1$.

We would like to deduce all coefficients c_r with $r \in \Phi^+$ arbitrary and consequently $c_{\sigma(\alpha_1)}$. By applying σ on the non-trivial relations from (4.2.3), we see that

$$\begin{aligned} c_r c_s &= c_{r+s} && \text{when } r,s\in\Phi_s \text{ and } r+s\in\Phi_s\,,\\ c_r c_s &= c_{r+s} && \text{when } r,s\in\Phi_\ell \text{ and } r+s\in\Phi_\ell\,,\\ c_r c_s &= c_{r+s} \text{ and } c_r^2 c_s = c_{2r+s} && \text{when } r\in\Phi_s, s\in\Phi_\ell \text{ and } r+s\in\Phi_s\,. \end{aligned}$$

We remark that the only root $r = \frac{1}{2}(-e_1 - e_2 - e_3 + e_4)$ obtained as a positive linear combination of the short fundamental roots e_3 and $\frac{1}{2}(-e_1 - e_2 - e_3 + e_4)$ is again short. Every other positive root therefore is a positive linear combination containing at least one long fundamental root, so again we can apply the above multiplication rule to find their coefficients. This means that if $r = \sum_i \lambda_i \alpha_i \in \Phi^+$, then $c_r = \prod c_{\alpha_i}^{\lambda_i}$ holds. We therefore find that $\overline{c_{\alpha_1}} c_{\sigma(\alpha_1)} = 1$ for $c_{\alpha_1} = \alpha^{-1}\beta^{-1}$.

The other coefficients (belonging to negative roots) can be found using the relation $\overline{c_r} c_{\sigma(r)} = 1$ for every $r \in \Phi$. As before, this relation follows from the fact that σ is an involution.

4.4.3 Description of the split saturated BN-pair

We show in this section that $G^1 = \langle U^1, V^1 \rangle$ with

$$\begin{aligned} U^1 &:= U_J \cap \mathrm{Fix}(\sigma)\\ V^1 &:= U_J^- \cap \mathrm{Fix}(\sigma) \end{aligned}$$

has a split saturated BN-pair of rank one.

We verify that σ satisfies condition (2) on page 66:

Lemma 4.4.2. *No parabolic subgroups of $L_J(K, L)$ are fixed.*

Proof. We prove that no parabolic subgroups of $\mathsf{B}_3(K, L)$ are fixed. This is enough since if $L_J(K, L)$ has a parabolic subgroup P fixed by σ, then $P \cap \mathsf{B}_3(K, L)$ is a fixed parabolic subgroup of $\mathsf{B}_3(K, L)$.

As the group $\mathsf{B}_3(K, L)$ has a BN-pair $(\mathsf{B}_3(K, L) \cap B(K, L), \mathsf{B}_3(K, L) \cap N(K, L))$, we can have a closer look at the building corresponding to $\mathsf{B}_3(K, L)$. By Section 2.5.1, the parabolic subgroups of $\mathsf{B}_3(K, L)$ ordered by the opposite of the inclusion relation form the simplicial complex of the building. In particular, the chambers (i.e. maximal flags) correspond to conjugates under $\mathsf{B}_3(K, L)$ of $\mathsf{B}_3(K, L) \cap B(K, L)$. Also, the group $\mathsf{B}_3(K, L) \cap B(K, L)$ is exactly the stabilizer in $\mathsf{B}_3(K, L)$ of the chamber corresponding to $\mathsf{B}_3(K, L) \cap B(K, L)$. Furthermore, the parabolic subgroups of $\mathsf{B}_3(K, L)$ are exactly the stabilizers in $\mathsf{B}_3(K, L)$ of their corresponding flags.

Since $\mathsf{PGO}(\mathcal{Q})$ is isomorphic to $\mathsf{B}_3(K,L)$, we know from the theory of buildings that the building we obtain is the mixed quadric corresponding to $\mathcal{Q}$. More specifically, the quadric corresponding to $\mathcal{Q}$ has points, lines and planes (since $\mathcal{Q}$ has Witt index 3). The flags of this quadric are then exactly the flags of the building of $\mathsf{PGO}(\mathcal{Q})$. So conjugates of $\mathsf{B}_3(K,L) \cap B(K,L)$ correspond to triples (p, L, π) with p a point of L and L a line on the plane π, while maximal parabolic subgroups correspond to points, lines or planes of the mixed quadric. From now on, we will assume that (p, L, π) is the maximal flag corresponding to the standard minimal parabolic subgroup $\mathsf{B}_3(K,L) \cap B(K,L)$.

Let $\mathcal{C} = (x, x_1, y_1, x_2, y_2, x_3, y_3)$ be a hyperbolic basis for the mixed quadratic form $\mathcal{Q}$, i.e. a basis for the K-vector space $L \oplus K^6$ such that

$$\mathcal{Q}(x) = 1, \langle x, x\rangle = \langle x, x_i\rangle = \langle x, y_i\rangle = \langle x_i, x_j\rangle = \langle y_i, y_j\rangle = 0, \langle x_i, y_j\rangle = \delta_{ij},$$

where $\langle \cdot, \cdot\rangle$ is the bilinear form corresponding to $\mathcal{Q}$.

We observe that $(\langle x_1\rangle, \langle x_1, x_2\rangle, \langle x_1, x_2, x_3\rangle)$ forms a chamber in the building of $\mathsf{PGO}(\mathcal{Q})$. We claim that this chamber is precisely the chamber (p, L, π) corresponding to the standard minimal parabolic $\mathsf{B}_3(K,L) \cap B(K,L)$ under the isomorphism between $\mathsf{B}_3(K,L)$ and the matrix group corresponding to $\mathsf{PGO}(\mathcal{Q})$ constructed in section 4.4.2. To prove our claim, we have to show that all generators of $\mathsf{B}_3(K,L) \cap B(K,L)$ fix the subspaces $\langle x_1\rangle, \langle x_1, x_2\rangle$ and $\langle x_1, x_2, x_3\rangle$. Consider the generators of the form $x_{e_1-e_2}(t)$ with $t \in K$. These elements correspond to matrices $A = \mathrm{I} + t(e_{1,2} + e_{-2,-1})$, so we need to check that

$$\begin{aligned} &A(0, \lambda_1, 0, 0, 0, 0, 0)^t \in \langle x_1\rangle \\ &A(0, \lambda_1, 0, \lambda_2, 0, 0, 0)^t \in \langle x_1, x_2\rangle \\ &A(0, \lambda_1, 0, \lambda_2, 0, \lambda_3, 0)^t \in \langle x_1, x_2, x_3\rangle \end{aligned}$$

for all $\lambda_1, \lambda_2, \lambda_3 \in K$, which is easily verified. The other generators can be treated similarly, and this proves our claim.

Now suppose that a parabolic subgroup S of $\mathsf{B}_3(K,L)$ is fixed by σ; our goal is to derive a contradiction. As σ is type-preserving, we know that if a flag is fixed, then certainly a point, line or plane must be fixed. As all parabolic subgroups are conjugate, there is some $g \in \mathsf{B}_3(K,L)$ such that $S = P^g$, where P is a standard parabolic subgroup, i.e. P contains $(\mathsf{B}_3(K,L) \cap B(K,L))$. So S corresponds to a flag contained in the chamber $(g(p), g(L), g(\pi))$, and hence one of the subspaces $\langle g(x_1)\rangle$, $\langle g(x_1), g(x_2)\rangle$ or $\langle g(x_1), g(x_2), g(x_3)\rangle$ has to be fixed under σ.

We claim that the involution σ maps $g(p)$, $g(L)$ and $g(\pi)$ to $\langle \sigma(g)(y_1)\rangle$, $\langle \sigma(g)(y_1), \sigma(g)(y_2)\rangle$ and $\langle \sigma(g)(y_1), \sigma(g)(y_2), \sigma(g)(y_3)\rangle$, respectively. From this we deduce that in each of the 3 cases (fixed point, line or plane), $\sigma(g)(y_1) \in \langle g(x_1), g(x_2), g(x_3)\rangle$. In particular, $\langle g(x_1)\rangle \perp \langle \sigma(g)(y_1)\rangle$.

In order to prove our claim, we need to show that $\sigma(B^g)$, where B^g is the stabilizer of the flag

$$\big(g(\langle x_1\rangle),\ g(\langle x_1, x_2\rangle),\ g(\langle x_1, x_2, x_3\rangle)\big),$$

fixes the flag

$$\big(\langle \sigma(g)(y_1)\rangle,\ \langle \sigma(g)(y_1), \sigma(g)(y_2)\rangle,\ \langle \sigma(g)(y_1), \sigma(g)(y_2), \sigma(g)(y_3)\rangle\big).$$

This is equivalent to showing that $\sigma(B)$ is the stabilizer of the flag

$$\big(\langle y_1\rangle, \langle y_1, y_2\rangle, \langle y_1, y_2, y_3\rangle\big).$$

This last statement can again easily be checked on each of the generators of B, and this proves our claim.

Suppose next that $g(x_1) = (z, z_1, a_1, z_2, a_2, z_3, a_3)$, then $z^2 + \sum_i z_i a_i = 0$ since $\mathcal{Q}(g(x_1)) = \mathcal{Q}(x_1) = 0$. Notice that $g(x_1)$ is the second column of the matrix corresponding to g and that $\sigma(g)(y_1) = M^{-1}\overline{g}M(y_1)$ is the third column of the matrix $M^{-1}\overline{g}M$; this implies, using the explicit description of the matrix M, that

$$\sigma(g)(y_1) = \alpha^{-1}\big(\overline{z}, \alpha^{-1}\overline{a_1}, \alpha\overline{z_1}, \beta^{-1}\overline{a_2}, \beta\overline{z_2}, \alpha^{-1}\beta^{-1}\overline{a_3}, \alpha\beta\overline{z_3}\big).$$

Since $\langle g(x_1)\rangle \perp \langle \sigma(g)(y_1)\rangle$, we get

$$\alpha z_1\overline{z_1} + \alpha^{-1}a_1\overline{a_1} + \beta z_2\overline{z_2} + \beta^{-1}a_2\overline{a_2} + \alpha\beta z_3\overline{z_3} + \alpha^{-1}\beta^{-1}a_3\overline{a_3} = 0.$$

This is equivalent with

$$(z+\overline{z})^2 + \alpha(\overline{z_1} + \alpha^{-1}a_1)(z_1 + \alpha^{-1}\overline{a_1}) + \beta(\overline{z_2} + \beta^{-1}a_2)(z_2 + \beta^{-1}\overline{a_2}) \\ + \alpha\beta(\overline{z_3} + \alpha^{-1}\beta^{-1}a_3)(z_3 + \alpha^{-1}\beta^{-1}\overline{a_3}) = 0.$$

Since q is anistropic, this implies $a_1 = \alpha\overline{z_1}$, $a_2 = \beta\overline{z_2}$ and $a_3 = \alpha\beta\overline{z_3}$. Finally, by expressing again that $\mathcal{Q}(g(y_1)) = 0$, we obtain that

$$z^2 + \alpha z_1\overline{z_1} + \beta z_2\overline{z_2} + \alpha\beta z_3\overline{z_3} = 0,$$

and hence $z = 0$ and $z_i = 0$ for all i, a contradiction. We conclude that no parabolic subgroup of $\mathsf{B}_3(K, L)$ is fixed. □

To proceed, we assemble a few lemmas about mixed BN-pairs. We write W for the Weyl group $N(K, L)/T(K, L)$, which is isomorphic to the Weyl group corresponding to a root system of type F_4.

Lemma 4.4.3. *Let $g \in \mathsf{F}_4(K,L)$ such that $\sigma(g) \in P_J(K,L)gP_J(K,L)$. If $P_J(K,L)gP_J(K,L) = P_J(K,L)nP_J(K,L)$ for some $n \in N(K,L)$ corresponding to the shortest element w in W_JwW_J. Then $w \in C_W(\sigma)$.*

Proof. See [29, Lemma 2.4]. Although the proof is not stated for mixed Chevalley groups, it can be copied almost ad verbatim, by replacing P_J and H by $P_J(K,L)$ and $T(K,L)$, respectively. □

The next lemma is a mixed version of [29, Lemma 2.5]. We notice that in the proof of this lemma we need the assumption (2) made in Section 4.4.1, page 66, which we proved in Lemma 4.4.2 above.

Lemma 4.4.4. *Let $g \in \mathsf{F}_4(K,L)$ with ${}^gP_J(K,L) = gP_J(K,L)g^{-1}$ invariant under σ. If $P_J(K,L)gP_J(K,L) = P_J(K,L)nP_J(K,L)$, with $n \in N(K,L)$ such that the corresponding element w of W is the shortest element in W_JwW_J, then $w \in C_W(\sigma) \cap \mathrm{Stab}(\Phi_J)$.*

Proof. Let $g = pnp'$ with $n \in N(K,L)$ and $p, p' \in P_J(K,L)$. Let $I := J \cap w(J)$; then

$$P_I(K,L) = U_J(K,L)(P_J(K,L) \cap {}^nP_J(K,L)).$$

Hence ${}^pP_I(K,L) = U_J(K,L)(P_J(K,L) \cap {}^gP_J(K,L))$ is σ-invariant. Furthermore, if $p = lu$ with $l \in L_J(K,L)$ and $u \in U_J(K,L)$, then

$${}^l(L_J \cap P_I(K,L)) = L_J(K,L) \cap {}^pP_I(K,L)$$

is a parabolic subgroup of $L_J(K,L)$. By Lemma 4.4.2, $L_J(K,L) \cap {}^pP_I(K,L) = L_J(K,L)$. We conclude that $P_J(K,L) \subseteq P_I(K,L)$, so $J = I$ and therefore $w(J) = J$. □

Lemma 4.4.5. *Let $1 \neq w \in \Phi$ with $w \in C_W(\sigma)$ and $w(\Phi^+) = \Phi^+$, then $w_0^J w = w_0$.*

Proof. See [29, Lemma 2.6]. □

In the next paragraph, we will prove that $B^1 := P_J(K,L) \cap G^1$, together with a suitable N^1 (which we will construct on the way) forms a split saturated BN-pair for G^1. We let $H^1 := L_J(K,L) \cap G^1$.

We use the proof of [29, Lemma 2.7] to construct an element $\tilde{n} \in (n_0 L_J) \cap G^1$ with n_0 an arbitrary element of $N(K,L)$ such that $n_0T(K,L) = w_0$, the longest element in $W = N(K,L)/T(K,L)$.

Lemma 4.4.6. *Let $n_0 \in N(K,L)$ be such that $n_0 T(K,L) = w_0$, then $n_{e_4} \in n_0 L_J(K,L) \cap G^1$.*

Proof. We notice that $u_{-e_4}(1) \in G^1 \cap U_{\Phi^- \setminus \Phi_J^-}$. Furthermore, $P_J u_{-e_4}(1) P_J = P_J n_{e_4} P_J$ with w_{e_4} the shortest element in $W_J w_{e_4} W_J$. Lemma 4.4.4 shows that $w_{e_4} \in C_W(\sigma) \cap \mathrm{Stab}(\Phi_J)$. This together with Lemma 4.4.5 allows us to conclude that $w_0 = w_0^J w_{e_4}$, with w_0^J being the longest element in Φ_J.

Since $W \cong N(K,L)/T(K,L)$, this yields that $n_{e_4} = n_0 n_0^J h$ for some $h \in T(K,L)$, so $n_{e_4} \in n_0 L_J(K,L)$ and is fixed by σ, which proves the lemma. □

Next, we define

$$N^1 := \langle n_0,\ L_J(K,L)\rangle \cap G^1.$$

This group is certainly non-trivial since $n_{e_4} \in N^1$.

Similarly as with ordinary Chevalley groups, one can associate to the root sytem Φ and corresponding vector space V a new root system $\tilde{\Phi}$ and vector space $\tilde{V}$ using the action of σ on V. Indeed, define $\tilde{V}$ as $C_V(\sigma) \cap J^\perp$, then for every $v \in V$, $\tilde{v}$ denotes the orthogonal projection of v in $\tilde{V}$. One can prove that $\tilde{\Phi} := \{\tilde{r} \mid r \in \Phi \setminus \Phi_J\}$ forms a root system of $\tilde{V}$. We denote the Weyl group corresponding to $\tilde{\Phi}$ by $\tilde{W}$.

Lemma 4.4.7. *Let $n^1 \in N^1$, then there exists an element $n \in N(K,L)$ such that $n^1 L_J(K,L) = n L_J(K,L)$ where $nT(K,L)$ corresponds to the shortest element w in wW_J. Then $w \in C_W(\sigma) \cap \mathrm{Stab}(\Phi_J)$ and $w|_{\tilde{V}} \in \tilde{W}$. The map*

$$\phi : N^1/H^1 \to \tilde{W}; n^1 H^1 \mapsto w|_{\tilde{V}}$$

is an isomorphism.

Proof. Again, the proof of [29, Lemma 2.9] holds almost verbatim. □

We compute the root system $\tilde{\Phi}$ and find that all roots of $\Phi \setminus \Phi_J$ are mapped under the projection map to a root system of type BC_1, which implies that the corresponding Weyl group $\tilde{W}$ has order 2. From this, we can immediately conclude that $N^1 = \langle n_{e_4}\rangle H^1$.

Before we can proceed with the actual proof of the existence of a split saturated BN-pair, we formulate a last important theorem, again inspired by [29].

Theorem 4.4.8. *Every $g \in G^1 \setminus B^1$ can be written as*

$$g = u l n_{e_4} u'$$

with $u \in U_J(K,L) \cap \mathrm{Fix}(\sigma)$, $u' \in U^-_{w_{e_4},J} \cap \mathrm{Fix}(\sigma)$ and $l \in L_J(K,L) \cap \mathrm{Fix}(\sigma)$.

Proof. Using Lemmas 4.4.4 and 4.4.7, the proof can be taken over from [29, Theorem 2.10]. □

We now have enough information to prove the existence of a split saturated BN-pair.

Theorem 4.4.9. *G^1 together with (B^1, N^1) forms a saturated, split BN-pair of rank one.*

Proof. We check that all five conditions are satisfied.

(i) We show that $G^1 = \langle B^1, N^1\rangle$. This follows immediately from Theorem 4.4.8.

(ii) We first prove that $H^1 = B^1 \cap N^1$. It is easy to see that $H^1 \subseteq B^1 \cap N^1$. If on the other hand $n_{e_4}h \in B^1$ for some $h \in H^1$, this would imply that $n_{e_4} \in P_J(K, L)$, a contradiction. From this it follows immediately that $H^1 \trianglelefteq N^1$ since $|\tilde{W}| = 2$ and therefore $[N^1 : H^1] = 2$.

(iii) The element $\omega := n_{e_4} \in N^1 \setminus H^1$ with $n_{e_4}^2 = e$ such that $N^1 = \langle H^1, n_{e_4}\rangle$. We also have $G^1 = B^1 \cup B^1 n_{e_4} B^1$ since $N^1 = n_{e_4}H^1 \cup H^1$ and $n_{e_4}B^1 n_{e_4} \neq B^1$ because $x_{-e_4}(1) \notin B^1 \subseteq P_J$.

(iv) The group $U^1 \trianglelefteq B^1$ since $U_J(K, L) \cap \mathrm{Fix}(\sigma) \trianglelefteq P_J(K, L) \cap G^1$. As $B^1 \subseteq P_J(K, L)$ with $P_J(K, L) = U_J(K, L) \rtimes L_J(K, L)$, we find that every $b \in B^1$ can be written uniquely as $b = ul$ for some $u \in U_J(K, L)$ and $l \in L_J(K, L)$. Now σ fixes $U_J(K, L)$ and $L_J(K, L)$ which means that u, l have to fixed by σ as well, so we get that $u \in U^1$ and $l \in H^1$. This shows that $B^1 = U^1 \rtimes H^1$.

(v) Because $H^1 \trianglelefteq N^1$, we obtain that $n_{e_4}H^1 n_{e_4} = H^1$, so $H^1 \subseteq B^1 \cap n_{e_4}B^1 n_{e_4}$. Remains to check the direction $B^1 \cap (n_{e_4}B^1 n_{e_4}) \subseteq H^1$. We have that $x \in B^1 \cap (n_{e_4}B^1 n_{e_4}) \subseteq B^1$ so the only thing left to check is that x belongs to N^1. Since x is in the intersection of the above groups, we can write $x = b_1 = n_{e_4}b_2 n_{e_4} = n_{e_4}uhn_{e_4} = n_{e_4}un_{e_4}h'$ for certain $b_1, b_2 \in B^1$, $u \in U^1$ and $h, h' \in H^1$. This implies $b_1 h'^{-1} \in V^1 \cap B^1 = \{e\}$ or $b_1 = h' \in H^1$. □

4.5 The Moufang set of mixed type F_4

From Lemma 2.5.1 we find that the set $X := \{(U^1)^g \mid g \in G^1\}$ together with the set of subgroups $\{(U^1)^g \mid g \in G^1\}$ acting on X by conjugation, forms a Moufang set $\mathbb{M} = (X, (V_x)_{x\in X})$.

In general, every Moufang set of the form $\mathbb{M} = (X, (U_x)_{x\in X})$ can be written as $\mathbb{M}(U, \tau)$ for some group U and some permutation τ on X using Lemma 2.3.1. We use this lemma to find a representation of our Moufang set in terms of a group $(U, +)$ and a permutation τ.

We choose $\infty := U^1$ and $0 := (U^1)^{n_{e_4}} = V^1$, and we define

$$U := \{(U^1)^g \mid g \in G^1\} \setminus \{U^1\}.$$

Notice that every $g \in G^1 \setminus B^1$ can be written in a unique way as $g = bn_{e_4}u$ with $b \in B^1$ and $u \in U^1$; equivalently, for any two elements $g = bn_{e_4}u$ and $g' = b'n_{e_4}u'$ with $b, b' \in B^1$, $u, u' \in U^1$, we have $(U^1)^g = (U^1)^{g'}$ if and only if $u = u'$. Therefore, the map

$$\zeta\colon U^1 \to U\colon u \mapsto (U^1)^{n_{e_4}u}$$

is a bijection. In particular, this makes U into a group which is isomorphic to U^1. Finally, we can set $\tau := n_{e_4}$, which acts on the set U^* of non-trivial elements of U by conjugation.

We conclude that the corresponding Moufang set $\mathbb{M}$ is given by $\mathbb{M} = \mathbb{M}(U, n_{e_4})$. In section 4.5.1, we will determine the group U; in section 4.5.2, we will determine the action of τ on U^*.

4.5.1 Description of the group U

We determine what an arbitrary element of U^1 looks like, and we will describe the group structure of $U \cong U^1$.

Since $U^1 = U_J \cap \mathrm{Fix}(\sigma)$, we find

$$U^1 := U_{r_1}U_{r_2}\cdots U_{r_{15}} \cap \mathrm{Fix}_{\mathsf{F}_4(K,L)}(\sigma),$$

with

$$\begin{array}{ll}
r_1 = e_4 & \\
r_2 = e_1 + e_4 & r_3 = -e_1 + e_4 \\
r_4 = e_2 + e_4 & r_5 = -e_2 + e_4 \\
r_6 = e_3 + e_4 & r_7 = -e_3 + e_4 \\
r_8 = \frac{1}{2}(e_1 + e_2 - e_3 + e_4) & r_9 = \frac{1}{2}(-e_1 - e_2 + e_3 + e_4) \\
r_{10} = \frac{1}{2}(e_1 - e_2 + e_3 + e_4) & r_{11} = \frac{1}{2}(-e_1 + e_2 - e_3 + e_4) \\
r_{12} = \frac{1}{2}(-e_1 + e_2 + e_3 + e_4) & r_{13} = \frac{1}{2}(e_1 - e_2 + e_3 + e_4) \\
r_{14} = \frac{1}{2}(-e_1 - e_2 - e_3 + e_4) & r_{15} = \frac{1}{2}(e_1 + e_2 + e_3 + e_4),
\end{array}$$

$U_r = \{u_r(t) \mid t \in L\}$ if $r \in \Phi_s$ and $U_r = \{u_r(t) \mid t \in K\}$ if $r \in \Phi_l$. This implies that an arbitrary element x of U_{α_1} is of the form

$$u_{r_1}(t_1)u_{r_2}(t'_2)u_{r_3}(t'_3)\cdots u_{r_7}(t'_7)u_{r_8}(t_8)u_{r_9}(t_9)\cdots u_{r_{15}}(t_{15})$$

with $t_i \in L$ and $t'_j \in K$ for all i, j and satisfies the relation $x = \sigma(x)$.

After rearranging some factors, using the commutator relations in (4.2.3), we find that

$$\sigma(x) = u_{r_1}(c_1\overline{t_1} + \overline{t_8t_9} + \overline{t_{10}t_{11}} + \overline{t_{12}t_{13}} + \overline{t_{14}t_{15}})u_{r_2}(c_3\overline{t'_3}) \cdots u_{r_{15}}(c_{14}\overline{t_{14}})$$

We now determine the values for each c_{r_i}, using the already known values for $c_{\alpha_1}, \ldots, c_{\alpha_4}$ from paragraph 4.4.2 and the product formula. We get

$$\begin{aligned}
&c_{r_1} = 1 \\
&c_{r_2} = \alpha && c_{r_3} = \alpha^{-1} \\
&c_{r_4} = \beta && c_{r_5} = \beta^{-1} \\
&c_{r_6} = \alpha\beta && c_{r_7} = \alpha^{-1}\beta^{-1} \\
&c_{r_8} = 1 && c_{r_9} = 1 \\
&c_{r_{10}} = \alpha && c_{r_{11}} = \alpha^{-1} \\
&c_{r_{12}} = \beta && c_{r_{13}} = \beta^{-1} \\
&c_{r_{14}} = \alpha^{-1}\beta^{-1} && c_{r_{15}} = \alpha\beta.
\end{aligned}$$

and so the relation $x = \sigma(x)$ implies the following relations:

$$t_1 + \overline{t_1} + \overline{t_8} \cdot \overline{t_9} + \overline{t_{10}} \cdot \overline{t_{11}} + \overline{t_{12}} \cdot \overline{t_{13}} + \overline{t_{14}} \cdot \overline{t_{15}} = 0,$$
$$t'_3 = \alpha\overline{t'_2},\ t'_5 = \beta\overline{t'_4},\ t'_7 = \alpha\beta\overline{t'_6},$$
$$t_9 = \overline{t_8},\ t_{11} = \alpha\overline{t_{10}},\ t_{13} = \beta\overline{t_{12}},\ t_{14} = \alpha\beta\overline{t_{15}}.$$

Replacing t'_3, t'_5, t'_7, t_9, t_{11}, t_{13} and t_{14} shows that an arbitrary element x of U is therefore of the form

$$u_{r_1}(t_1)u_{r_2}(t'_2)u_{r_3}(\alpha\overline{t'_2}) \cdots u_{r_{14}}(\alpha\beta\overline{t_{15}})u_{r_{15}}(t_{15})$$

with $t_1 + \overline{t_1} + t_8\overline{t_8} + \alpha t_{10}\overline{t_{10}} + \beta t_{12}\overline{t_{12}} + \alpha\beta t_{15}\overline{t_{15}} = 0$.

Now let $x, y \in U^1$ be arbitrary, and write

$$x = u_{r_1}(t_1)u_{r_2}(t'_2)u_{r_3}(t'_3) \cdots u_{r_7}(t'_7)u_{r_8}(t_8)u_{r_9}(t_9) \cdots u_{r_{15}}(t_{15}),$$
$$y = u_{r_1}(s_1)u_{r_2}(s'_2)u_{r_3}(s'_3) \cdots u_{r_7}(s'_7)u_{r_8}(s_8)u_{r_9}(s_9) \cdots u_{r_{15}}(s_{15});$$

then the product $x \cdot y$ is equal to

$$u_{r_1}(t_1 + s_1 + \overline{t_8}s_8 + \alpha\overline{t_{10}}s_{10} + \beta\overline{t_{12}}s_{12} + \alpha\beta\overline{t_{14}}s_{14})$$
$$\cdot u_{r_2}(t'_2 + s'_2) \cdot u_{r_3}(t'_3 + s'_3) \cdots u_{r_{15}}(t_{15} + s_{15}).$$

To simplify the notation, we will identify an arbitrary element

$$x = u_{r_1}(t_1)u_{r_2}(t'_2)u_{r_3}(t'_3) \cdots u_{r_7}(t'_7)u_{r_8}(t_8)u_{r_9}(t_9) \cdots u_{r_{15}}(t_{15})$$

with the element

$$((t_8, t_{10}, t_{12}, t_{15}), (t_1, t'_2, t'_4, t'_6)) \in (L \oplus L \oplus L \oplus L) \oplus (L \oplus K \oplus K \oplus K).$$

We conclude that $(U, +)$ is isomorphic with

$$U := \big\{((x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)) \in (L^4) \oplus (L \oplus K^3) \mid \\ x_1\overline{x_1} + \alpha x_2\overline{x_2} + \beta x_3\overline{x_3} + \alpha\beta x_4\overline{x_4} + \mathrm{T}(y_1) = 0\big\},$$

where the group addition $+$ is given by the formula

$$\big((x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)\big) + \big((a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4)\big) \\ = \big((x_1 + a_1, x_2 + a_2, x_3 + a_3, x_4 + a_4), \\ (y_1 + b_1 + \overline{x_1}a_1 + \alpha\overline{x_2}a_2 + \beta\overline{x_3}a_3 + \alpha\beta\overline{x_4}a_4, y_2 + b_2, y_3 + b_3,\ y_4 + b_4)\big).$$

4.5.2 Description of the action of τ on U^*

We present a way to calculate the image of $\tau = n_{e_4}$ on an arbitrary non-trivial element $u \in U^1$. Using the isomorphism with U, we need to determine which element of U^1 corresponds to the element $(U^1)^{n_{e_4}un_{e_4}}$ of U. As mentioned in the previous section, this comes down to rewriting an arbitrary element of the form $g = n_{e_4}un_{e_4}$ with $u \in U^1$ as $g = bn_{e_4}u'$ with $b \in B^1$ and $u' \in U^1$. We have shown in the previous section that this can be done in a unique way and we therefore have $\tau(u) = u'$.

Bringing such an arbitrary element $n_{e_4}un_{e_4}$ into the right form comes down to quite long, but systematic calculations, as we will explain below. We implemented our algorithm in the computer algebra software package `Sage` [28]. We refer to [7] for the detailed implementation and the output of the program.

We briefly describe the methods we use to rewrite $n_{e_4}un_{e_4}$ as $bn_{e_4}u'$ for some $b \in B^1$ and $u' \in U^1$. Suppose $u = x_{r_1}(t_1)x_{r_2}(t_2)\cdots x_{r_{14}}(t_{14})x_{r_{15}}(t_{15})$, then using the relations

$$n_s x_r(t) n_s = x_{w_s(r)}(t) \quad \text{and} \quad n_r(t) = h_r(t)n_r$$

for all $r, s \in \Phi$ and all $t \in L$, we find that

$$n_{e_4}x_{r_1}(t_1)x_{r_2}(t_2)x_{r_3}(t_3)\cdots x_{r_{14}}(t_{14})x_{r_{15}}(t_{15})n_{e_4} \\ = x_{-r_1}(t_1)x_{-r_3}(t_2)x_{-r_2}(t_2)\cdots x_{-r_{15}}(t_{14})x_{-r_{14}}(t_{15})$$

or that this element has the same action on U^1 as the element

$$x = n_{e_4}(t_1)x_{-r_1}({t_1}^{-1})x_{-r_3}(t_2)x_{-r_2}(t_2)\cdots x_{-r_{15}}(t_{14})x_{-r_{14}}(t_{15}),$$

provided $t_1 \neq 0$.

Remark 4.5.1. If $t_1 = 0$, then the norm condition implies that $t_8, \dots, t_{15}$ are also equal to zero. Again, we have to distinguish between $t_2 \neq 0$ and $t_2 = 0$ to proceed, and a similar distinction has to be made for t_4 and t_6, but in each case, the process is very similar (but gets easier as more and more elements become zero), and we omit the details.

We now describe our algorithm to rewrite x in the required form, i.e. in the form

$$x = bn_{e_4}u'$$

with $b \in B^1$ and $u' \in U^1$. Our strategy is to try to swap all "bad root elements", i.e. all root elements $x_r(t)$ not belonging to U^1, and all "Hua elements" $h_r(t)$, in x, from the right side of n_{e_4} to the left, in such a manner that the elements that we get at the left of n_{e_4} all belong to P_J. At the end, we will then indeed have rewritten x as $bn_{e_4}u'$. As n_{e_4} and u' are in G^1 at the end of the algorithm, so is b, and therefore $b \in B^1$ as required.

Step 1. We always start with the leftmost element on the right side of n_{e_4}. This element will be of one of the following types:

(1) a Hua element $h_r(t)$,
(2) a root element $x_r(t)$, with r containing no term in e_4,
(3) a root element $x_r(t)$, with r containing a negative term in e_4,
(4) a root element $x_r(t)$, with r containing a positive term in e_4.

We point out that elements of the the first three types can be swapped to the left side of n_{e_4} without any problem. Only if we encounter an element $x_r(t) = x_{r_i}(t)$ of the fourth type, we have a look at the element next to $x_{r_i}(t)$.

Step 2. Depending on the form of this second element, we can distinguish a few cases.

(a) This second element is a Hua element $h_s(t')$. In this case, we can use the conjugation relation

$$x_r(t)h_s(t') = h_s(t')x_r\big(t\lambda^{-2\langle r,s\rangle/\langle s,s\rangle}\big)$$

to reverse the order of $x_r(t)$ and $h_s(t')$ in the product.

(b) This second element is of the form $x_r(t')$. In this case we simply combine both elements to the single root element $x_r(t + t')$.

(c) This second element is of the form $x_s(t')$ with either
  (i) $s = r_j$, with $j > i$, or
  (ii) $s = r_j$, with $j < i$, or

(iii) s contains no positive term in e_4.

In case (i), there is nothing to do, and we proceed to the next element in the product. In cases (ii) and (iii), we distinguish between the case $s = -r$ (i.e. s and r are opposite roots) and $s \neq -r$. If $s \neq -r$, we use the commutator relations to switch the roots $x_r(t)$ and $x_s(t')$, and we add the possible new element(s) to the right of $x_s(t')x_r(t)$. If on the other hand $s = -r$, we use the equality

$$x_r(t)x_{-r}(t') = x_{-r}\Big(\frac{t'}{tt'+1}\Big)\, h_r(tt'+1)\, x_r\Big(\frac{t}{tt'+1}\Big)$$

to proceed. In both cases, we then return to step 1.

By repeating these steps, we end up with an element in U^1 on the right side of n_{e_4}, as we wanted.

Applying the algorithm on an arbitrary element of U^1 we get, using our `Sage` program [7], that the corresponding map $\tau\colon U^1 \to U^1$ (which maps u to u' as explained in the beginning of this section) is explicitly given by

$$\begin{aligned}&\tau\colon U^1 \to U^1 :\\ &\qquad (a,b) \mapsto \big(a\cdot\big(b+f(a)\big)^{-1}, \big(b+f(a)\big)^{-1} + f\big(a\cdot(b+f(a))^{-1}\big)\big),\end{aligned}$$

where

$$\begin{aligned}&f\colon L\oplus L\oplus L\oplus L \to L\oplus L\oplus L\oplus L :\\ &(a_1,a_2,a_3,a_4) \mapsto (a_1\overline{a_1} + \alpha a_2\overline{a_2} + \beta a_3\overline{a_3} + \alpha\beta a_4\overline{a_4},\\ &\qquad\qquad a_1a_2 + \beta\overline{a_3}a_4,\ a_1a_3 + \alpha\overline{a_2}a_4,\ a_2a_3 + \overline{a_1}a_4),\end{aligned}$$

and where the multiplication of the elements in L^4 is the octonion multiplication described in Remark 4.4.1.

4.5.3 Conclusion

We now summarize our results.

Theorem 4.5.2. *Let (k,ℓ) be a pair of fields of characteristic 2 such that $\ell^2 \leq k \leq \ell$. Let $\mathcal{O}$ be an octonion division algebra over k, with norm N, and let $\mathcal{O}_\ell = \mathcal{O}\otimes_k \ell$, with norm N_ℓ. Let K and L be separable quadratic field extensions of k and ℓ, respectively, such that $L^2 \leq K \leq L \leq \mathcal{O}_\ell$, and identify $\mathcal{O}_\ell$ with L^4. Under this identification, we define a subspace $\mathcal{O}_{\mathrm{mixed}} := L\oplus K^3$ of $\mathcal{O}_\ell$. Assume that the restriction of N_ℓ to $\mathcal{O}_{\mathrm{mixed}}$ is anisotropic.*

There exist constants $\alpha, \beta \in k^\times$ such that the norm N *is given by*

$$\mathrm{N}\colon \mathcal{O}_\ell = L^4 \to K\colon (a_1, a_2, a_3, a_4) \mapsto a_1\overline{a_1} + \alpha a_2\overline{a_2} + \beta a_3\overline{a_3} + \alpha\beta a_4\overline{a_4}.$$

Let

$$f\colon \mathcal{O}_\ell \to \mathcal{O}_\ell\colon (a_1, a_2, a_3, a_4) \mapsto \big(\mathrm{N}(a_1, a_2, a_3, a_4), \\ a_1a_2 + \beta\overline{a_3}a_4,\ a_1a_3 + \alpha\overline{a_2}a_4,\ a_2a_3 + \overline{a_1}a_4\big),$$

and let

$$g\colon \mathcal{O}_\ell \times \mathcal{O}_\ell \to L\colon \big((a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4)\big) \mapsto \\ \overline{a_1}b_1 + \alpha\overline{a_2}b_2 + \beta\overline{a_3}b_3 + \alpha\beta\overline{a_4}b_4.$$

Define

$$U := \big\{(a, b) \in \mathcal{O}_\ell \oplus \mathcal{O}_{\text{mixed}} \mid \mathrm{N}(a) + \mathrm{T}(b) = 0\big\},$$

and make U into a group by setting

$$(a, b) + (c, d) := (a + c, b + d + g(a, c))$$

for all $(a, b), (c, d) \in U$. Define a permutation τ on U^* by

$$\tau(a, b) := \Big(a \cdot \big(b + f(a)\big)^{-1}, \big(b + f(a)\big)^{-1} + f\big(a \cdot (b + f(a))^{-1}\big)\Big)$$

for all $(a, b) \in U$. Then $\mathbb{M}(U, \tau)$ is a Moufang set corresponding to a mixed group of type F_4.

4.6 Algebraic Moufang sets of type F_4 in characteristic 2

We conclude with having a closer look at what happens to the above results when $K = L$, which is in fact the algebraic case in characteristic 2. More precisely, we show that every Moufang set $\mathbb{M}(U, \tau)$ we obtained in section 4.5 in this case is isomorphic to an algebraic Moufang set of type F_4 (as we expect).

To see this, we apply the transformation φ on U with

$$\varphi\colon U \to U\colon (a, b) \mapsto (a, b + f(a))$$

Then $\varphi((a, b) + (c, d)) = \varphi(a, b) \tilde{+} \varphi(c, d)$ with

$$(x_1, y_1) \tilde{+} (x_2, y_2) = (x_1 + x_2, y_1 + y_2 + \overline{x_2} \cdot x_1)$$

for all $(x_1, y_1),\ (x_2, y_2) \in U$. Furthermore, $\varphi(\tau((x, y)) = \tilde{\tau}(x, y)$ for all $(x, y) \in U$ with

$$\tilde{\tau}\colon U^* \to U^*\colon (x, y) \mapsto (x \cdot y^{-1}, y^{-1}).$$

We find that $\mathbb{M}(U, \tau)$ is isomorphic to $\mathbb{M}(U, \tilde{\tau})$, which is indeed a Moufang set of type F_4.

Remark 4.6.1. It is an interesting open question whether or not all Moufang sets of mixed type F_4 can be embedded in some algebraic Moufang set of type F_4. This problem can be reformulated to the question whether the extension of the mixed norm form N on $\mathcal{O}_{\text{mixed}}$ to the larger space $\mathcal{O}_\ell$ remains anisotropic.

5

Moufang subsets of algebraic Moufang sets

In this chapter, we take a closer look at Moufang subsets of certain classes of algebraic Moufang sets, i.e. Moufang sets corresponding to a class of linear algebraic groups of relative rank one in characteristic different from two. We try to show that all Moufang subsets of algebraic Moufang sets are again of algebraic origin.

Let $\mathbb{M}(U, \tau)$ be an algebraic Moufang set and let $(T, +)$ be a subgroup of $(U, +)$, then T determines a *Moufang subset* if T is closed under the action of μ_x for all $x \in T$. Using the definition of the μ-maps one can easily see this last condition is equivalent with requiring that T is closed under the action of μ_x for some $x \in T$. For that purpose, we can denote the corresponding Moufang subset by $\mathbb{M}(T, \mu_x)$. In order to be able to determine the structure of $\mathbb{M}(T, \mu_x)$, we first need to determine the μ-maps or equivalently the Hua maps corresponding to τ.

We exclude the characteristic two case because in characteristic two special phenomena can occur as the following basic example illustrates. Let $\mathbb{M}(k) = \mathbb{M}(k, \tau)$ with $\tau(x) = -x^{-1}$ be a projective Moufang set over a some field $(k, +, .)$. Assume next that identity element e of $(k, .)$ is contained in some Moufang subset $\mathbb{M}(V, \mu_e)$ of $\mathbb{M}(k)$. Using the Hua maps, it is immediately clear that in characteristic different from two V is closed under the multiplication of k and is therefore V is a field. In characteristic two on the other hand, the only thing one can prove is that V is a subset of k closed under addition and inverting.

It appears that, contrary to what we might expect, more difficult Moufang sets (i.e. corresponding to exceptional groups of relative rank one) are easier to handle, thanks to their specific form. We therefore start with Moufang sets of type F_4 and ${}^2\mathsf{E}_6$ and some related classical Moufang sets. Because their description is quite similar, we try to handle all these types of Moufang sets at once.

In a second part of this chapter, we study the class of hermitian Moufang sets. Here, we encounter a problem that frequently appears when studying Moufang subsets. More specifically, let $h : V \times V \to k$ be a skew-hermitian form, then it is often not clear whether the subset of V corresponding to the Moufang subset we are studying, inherits the vector space structure of V. Due to this problem, we only proved some partial results for Moufang subsets of hermitian Moufang sets.

5.1 Non-abelian Moufang sets corresponding to composition algebras

We investigate Moufang subsets of non-abelian Moufang sets obtained from composition division algebras. We start by giving a description of each of these Moufang sets, in such a way that we can treat all of them at once. We also compute their Hua maps explicitly.

5.1.1 Description of the Moufang sets

Let E/k be an extension of k which is either trivial or a separable quadratic extension such that $\tilde{\cdot}$ is the non-trivial element of $\mathrm{Gal}(E/k)$. Furthermore, let A_k be either a separable quadratic extension field of k, a quaternion division algebra over k or an octonion division algebra over k such that $A = A_k \otimes_k E$ remains division. Then, in the second case, $\tilde{\cdot}$ is an involution on E, which extends to a non-linear automorphism $\tilde{\cdot}$ of A by applying the involution to each coefficient with respect to a basis of A_k. Denote by $\bar{\cdot}$ the standard involution of A/E and let $\mathrm{N}(a) := \overline{a}a$ and $\mathrm{T}(a) = a + \overline{a}$ be the norm and trace map of A/E respectively. Let

$$U := \{(a, b) \in A \times A \mid \eta(\overline{a})a + \eta(\overline{b}) + b = 0\}$$

with η either equal to the identity map on A if $E = k$ and equal to $\tilde{\cdot}$ if E is a quadratic separable extension of k. Then we can make U into a group by defining an addition on U as follows:

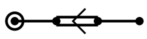
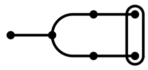
$$(a, b) + (c, d) := (a + b, c + d - \eta(\overline{c})a)$$

for all $(a,b),(c,d) \in U$. This is a group with neutral element $(0,0)$ and with the inverse given by $-(a,b) = (-a, \eta(\overline{b}))$. Now we define a permutation τ on U^* by setting

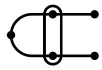
$$\tau(a,b) = (-ab^{-1}, b^{-1}) \tag{5.1.1}$$

for all $(a,b) \in U^*$. Then for every choice of η and A, we get that $\mathbb{M}(U,\tau)$ is a Moufang set corresponding to a linear algebraic group of rank one. As A is each time a composition division algebra, we call the collection of these Moufang sets *"non-abelian Moufang sets corresponding to composition algebras"*.

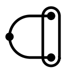
Remark 5.1.1. We remark that if A is an octonion division algebra over E, then the above description exactly corresponds (depending on whether η is the identity or equal to $\tilde{\ }$) to the description of Moufang sets of type F_4 and ${}^2\mathsf{E}_6$ respectively. The diagrams corresponding to these groups are:

F_4 :  and ${}^2\mathsf{E}_6$: 

If on the other hand, A is a quaternion division algebra over E, then one can show that (again depending on the choice of η) $\mathbb{M}(U,\tau)$ is isomorphic to a Moufang set of hermitian type with corresponding linear algebraic group of type C_3 or ${}^2\mathsf{A}_5$ respectively.[1] The diagrams corresponding to these Moufang sets are:

C_3 :  and ${}^2\mathsf{A}_5$: 

Finally, if A is a quadratic separable extension of E, we find both times a hermitian Moufang set corresponding to a linear algebraic group of type ${}^2\mathsf{A}_3$ and relative rank one. The diagram corresponding to this linear algebraic group looks as follows:

${}^2\mathsf{A}_3$: 

Description of the Hua maps

We now explicitly compute the action of the Hua maps $h_{(a,b)}$ on an arbitrary element $(c,d) \in U^*$. During the calculations, we ignore the fact that for some

[1] We show in the appendix of this thesis that if A is a quaternion division algebra over E, then the corresponding Moufang sets are isomorphic to hermitian Moufang sets of type C_3 or ${}^2\mathsf{A}_5$.

very specific choice of elements (c,d) we might divide by zero. The computations for such elements can be done in a completely similar fashion. Also, we frequently use the Moufang identities for composition algebras (see Formula 2.1.1).

Theorem 5.1.2. *Let $\mathbb{M}(U,\tau)$ be a non-abelian Moufang set corresponding to a composition algebra, then*

$$h_{(a,b)}(c,d) = \big(ab^{-1}[\eta(\bar{b})a^{-1}\cdot c\eta(\bar{b})], ba^{-1}[ad\cdot\eta(\bar{b})]\big)$$

for all $(a,b),(c,d)\in U^*$ with $a\neq 0$. If $a=0$,

$$h_{(0,b)}(c,d) = (cb, -bdb).$$

Proof. Recall that the Hua maps $h_{(a,b)}$ are defined as

$$h_{(a,b)}(c,d) := \tau(\tau^{-1}[\tau(c,d)+(a,b)] - \tau^{-1}[(a,b)]) - \tau(-\tau^{-1}(a,b))$$

for all $(a,b),(c,d)\in U^*$. As in our setting $\tau^2=1$, we have that $\tau^{-1}=\tau$. Then

$$\begin{aligned}
&-\tau^{-1}(a,b) = (ab^{-1}, \eta(\overline{b}^{-1}))\\
&-\tau(-\tau^{-1}(a,b)) = (ab^{-1}\cdot\eta(\bar{b}), b).
\end{aligned}$$

Furthermore,

$$\tau(c,d)+(a,b) = (-cd^{-1}+a, d^{-1}+b+\eta(\bar{a})\cdot cd^{-1})$$

and hence

$$\tau^{-1}[\tau(c,d)+(a,b)] = ((cd^{-1}-a)(b+d^{-1}+\eta(\bar{a})\cdot cd^{-1})^{-1}, (b+d^{-1}+\eta(\bar{a})\cdot cd^{-1})^{-1}).$$

Now let

$$\begin{aligned}
A :=& (cd^{-1}-a)(b+d^{-1}+\eta(\bar{a})\cdot cd^{-1})^{-1} + ab^{-1}\\
B :=& (b+d^{-1}+\eta(\bar{a})\cdot cd^{-1})^{-1} + \eta(\overline{b}^{-1})\\
&- \eta(\overline{ab^{-1}})\cdot(cd^{-1}-a)(b+d^{-1}+\eta(\bar{a})\cdot cd^{-1})^{-1}
\end{aligned}$$

then

$$h_{(a,b)}(c,d) = (-AB^{-1}+ab^{-1}\eta(\bar{b}), B^{-1}+b+\eta(\overline{ab^{-1}\cdot\eta(\bar{b})})\cdot AB^{-1}).$$

We can rewrite B as

$$\begin{aligned}
B =& [(cd^{-1}-a)^{-1} + \eta(\bar{b})^{-1}\cdot(b+d^{-1}+\eta(\bar{a})\cdot cd^{-1})(cd^{-1}-a)^{-1} - \eta(\overline{ab^{-1}})]\\
&\cdot[(cd^{-1}-a)(b+d^{-1}+\eta(\bar{a})\cdot cd^{-1})^{-1}].
\end{aligned}$$

Using the fact that $\eta(\overline{a})a + b + \eta(\overline{b}) = 0$, the first factor of B becomes

$$(cd^{-1} - a)^{-1} + \eta(\overline{b})^{-1}[(-\eta(\overline{b}) + d^{-1} + \eta(\overline{a}) \cdot (cd^{-1} - a))(cd^{-1} - a)^{-1}] - \eta(\overline{ab^{-1}})$$
$$= \eta(\overline{b})^{-1} \cdot d^{-1}(cd^{-1} - a)^{-1}$$

so

$$B = [\eta(\overline{b})^{-1} \cdot d^{-1}(cd^{-1} - a)^{-1}][(cd^{-1} - a)(b + d^{-1} + \eta(\overline{a}) \cdot cd^{-1})^{-1}].$$

Then

$$\begin{aligned} B^{-1} &= [(b + d^{-1} + \eta(\overline{a}) \cdot cd^{-1})(cd^{-1} - a)^{-1}] \cdot [\eta(\overline{b})^{-1} \cdot d^{-1}(cd^{-1} - a)^{-1}]^{-1} \\ &= [(-\eta(\overline{b}) + d^{-1} + \eta(\overline{a}) \cdot (cd^{-1} - a))(cd^{-1} - a)^{-1}] \cdot [\eta(\overline{b})^{-1} \cdot d^{-1}(cd^{-1} - a)^{-1}]^{-1} \\ &= [(-\eta(\overline{b}) + d^{-1}) \cdot (cd^{-1} - a)^{-1} + \eta(\overline{a})] \cdot [\eta(\overline{b})^{-1} \cdot d^{-1}(cd^{-1} - a)^{-1}]^{-1} \\ &= \eta(\overline{b}) - \eta(\overline{b})d\eta(\overline{b}) + \eta(\overline{a}) \cdot [(cd^{-1} - a)d \cdot \eta(\overline{b})] \\ &= \eta(\overline{b}) - \eta(\overline{b})d\eta(\overline{b}) + \eta(\overline{a}) \cdot [(c - ad) \cdot \eta(\overline{b})] \end{aligned}$$

and

$$\begin{aligned} AB^{-1} &= (c - ad)\eta(\overline{b}) + ab^{-1} \cdot \big[\eta(\overline{b}) - \eta(\overline{b})d\eta(\overline{b}) + \eta(\overline{a}) \cdot [(c - ad) \cdot \eta(\overline{b})]\big] \\ &= c\eta(\overline{b}) - ad \cdot \eta(\overline{b}) + ab^{-1} \cdot \eta(\overline{b}) - (ab^{-1}) \cdot [\eta(\overline{b})d\eta(\overline{b}) - \eta(\overline{a}) \cdot (c - ad)\eta(\overline{b})] \\ &= c\eta(\overline{b}) + ab^{-1} \cdot \eta(\overline{b}) + ab^{-1} \cdot [\eta(\overline{a}) \cdot c\eta(\overline{b})]. \end{aligned}$$

The last equation uses the identity

$$ba^{-1} \cdot (ad \cdot \eta(\overline{b})) = -\eta(\overline{b})d\eta(\overline{b}) - \eta(\overline{a}) \cdot (ad \cdot \eta(\overline{b})).$$

It follows that

$$\begin{aligned} B^{-1} + b + \eta(\overline{ab^{-1} \cdot \eta(\overline{b})}) \cdot AB^{-1} &= -\eta(\overline{b})d\eta(\overline{b}) + \eta(\overline{a}) \cdot [(c - ad)\eta(\overline{b})] \\ &\quad - \eta(\overline{ab^{-1} \cdot \eta(\overline{b})}) \cdot [c\eta(\overline{b}) + ab^{-1} \cdot (\eta(\overline{a}) \cdot c\eta(\overline{b}))] \\ &= -\eta(\overline{b})d\eta(\overline{b}) + \eta(\overline{a}) \cdot [(c - ad)\eta(\overline{b})] \\ &\quad - \eta(\overline{ab^{-1} \cdot \eta(\overline{b})}) \cdot [(\eta(\overline{a})^{-1} + ab^{-1})(\eta(\overline{a}) \cdot c\eta(\overline{b}))] \\ &= -\eta(\overline{b})d\eta(\overline{b}) - \eta(\overline{a})[ad \cdot \eta(\overline{b})] \end{aligned}$$

using among other things that $[\eta(\overline{ab^{-1} \cdot \eta(\overline{b})})][ab^{-1} \cdot \eta(\overline{b})] = -\eta(\overline{b}) - b$. So we finally get that

$$\begin{aligned} h_{(a,b)}(c,d) &= \big(-c\eta(\overline{b}) - ab^{-1} \cdot [\eta(\overline{a}) \cdot c\eta(\overline{b})], -\eta(\overline{b})d\eta(\overline{b}) - \eta(\overline{a})[ad \cdot \eta(\overline{b})]\big) \\ &= \big(ab^{-1}[\eta(\overline{b})a^{-1} \cdot c\eta(\overline{b})], -\eta(\overline{b})d\eta(\overline{b}) - \eta(\overline{a})[ad \cdot \eta(\overline{b})]\big) \\ &= \big(ab^{-1}[\eta(\overline{b})a^{-1} \cdot c\eta(\overline{b})], ba^{-1}[ad \cdot \eta(\overline{b})]\big) \end{aligned}$$

If $a = 0$, the above equation is not well-defined. In this case $\eta(\overline{b}) = -b$ and we find that

$$h_{(0,b)}(c, d) = (cb, -bdb).$$

□

5.1.2 Description of Moufang subsets of $\mathbb{M}(U, \tau)$

We determine the structure of all Moufang subsets of $\mathbb{M}(U, \tau)$. We show that in characteristic different from two all Moufang subsets $\mathbb{M}(T, \mu_{(a,b)}|_T)$ with $(a, b) \in T$ of $\mathbb{M}(U, \tau)$ are of algebraic origin.

To determine the structure of T, we start with a few remarks about the set U. Define $X := \{a \mid (a, b) \in U\}$ and $Y := \{b \mid (a, b) \in U\}$, then we notice that X coincides with the set A since $(a, -\eta(\overline{a})a/2) \in U$ and consequently $Y \subseteq X$. Therefore, our first goal is to get a grip on what the corresponding sets $V := \{a \mid (a, b) \in T\}$ and $W := \{b \mid (a, b) \in T\}$ of T are.

From now on, we assume that both V and $S := \{b \mid (0, b) \in T\}$ are nontrivial. We treat the cases where V or S are trivial separately afterwards. As S is nontrivial, we may assume that T is closed under some element $\mu_{(0,d_0)}$ as well.

We notice that the element $(e, -e/2)$ belongs to U. We show that we may assume, without loss of generality, that $(e, -e/2)$ belongs to $\mathbb{M}(T, \tau|_T)$. We start with the following lemma:

Lemma 5.1.3. *(i) Let $(0, d) \in T$, then $(0, d/2) \in T$.*

(ii) If $(a, b) \in T$ with $a \neq 0$, then $(a, -\eta(\overline{a})a/2) \in T$ if and only if $(0, \eta(\overline{b}) - b) \in T$.

(iii) Let $(c, d) \in T$ be arbitrary with $c \neq 0$, then $(c, -\eta(\overline{c})c/2) \in T$.

Proof. (i) If $(0, d) \in T$, then also $\mu_{(0,d_0)}(2 \cdot \mu_{(0,d_0)}(0, d)) = (0, d/2) \in T$.

(ii) If $(a, b) \in T$, then $-(a, b) = (-a, \eta(\overline{b})) \in T$. Consequently, $(a, -\eta(\overline{a})a/2) \in T$ if and only if

$$\begin{aligned}(a, -\eta(\overline{a})a/2) + (-a, \eta(\overline{b})) &= (0, \eta(\overline{b}) + \eta(\overline{a})a/2) \\ &= (0, [\eta(\overline{b}) - b]/2)) \in T.\end{aligned}$$

Using (i), this is equivalent with $(0, \eta(\overline{b}) - b) \in T$.

(iii) As $(c, d) \in T$ with $c \neq 0$, also $\mu_{(0,d_0)}\mu_{(0,d_0)}(c, d) = (-c, d) \in T$. Therefore

$$(c, d) + (-c, d) = (0, 2d + \eta(\overline{c})c) = (0, d - \eta(\overline{d})) \in T,$$

so using (ii) we are done.

□

From the previous lemma we deduce that $\mathbb{M}(T, \mu_{(a,b)}|_T) = \mathbb{M}(T, \mu_{(a,-\eta(\overline{a})a/2)}|_T)$. Now we we have enough information to state the following theorem:

Theorem 5.1.4. *Let $\mathbb{M}(T, \mu_{(a,-\eta(\overline{a})a/2)}|_T)$ be a Moufang subset of $\mathbb{M}(U, \tau)$ containing the element $(a, -\eta(\overline{a})a/2)$ with $a \neq 0$. Then there exists a non-abelian Moufang set $\mathbb{M}(U^\sharp, \tau^\sharp)$ associated to a composition algebra with identity element e and an embedding Ψ of $\mathbb{M}(T, \mu_{(a,-\eta(\overline{a})a/2)}|_T)$ as a Moufang subset of $\mathbb{M}(U^\sharp, \tau^\sharp)$ such that $\Psi(T)$ is closed under $\mu^\sharp_{(e,-e/2)}$ and contains the element $(e, -e/2)$.*

Proof. We split up the proof of this theorem in several parts.

(i) In a first step of the proof, we construct a composition algebra $(A, +, \sharp)$ out of $(A, +, .)$ by defining a new multiplication $\sharp$ on A such that a is the identity element with respect to this multiplication.[2]
Let x, y be two arbitrary elements of A, then we define the $\sharp$-multiplication by setting

$$x \sharp y := \eta(\overline{a})^{-1}[(\eta(\overline{a})x \cdot a^{-1})y].^3$$

It is easily verified that $(A, +, \sharp)$ has identity element a and for every $b \in A^*$, the inverse element is $x^{\sharp-1} := ax^{-1}a$. Also, one can define a norm map

$$\mathrm{N}^\sharp : A \to k;\ x \mapsto N(x)/N(a)$$

on $(A, +, \sharp)$ with corresponding standard involution

$$\overline{\cdot}^\sharp : A \to A;\ x \mapsto N(a)^{-1}a\overline{x}a.$$

This implies that $(A, +, \sharp)$ together with $\mathrm{N}^\sharp$ is a composition algebra over k.

(ii) Secondly, we construct a Moufang set corresponding to $(A, +, \sharp)$ and $\mathrm{N}^\sharp$. For this, we notice that

$$\eta^\sharp : A \to A;\ x \mapsto \eta(x) \cdot \eta(a)^{-1}a$$

is an automorphism of order two of $(A, +, \sharp)$ and that

$$\eta^\sharp(\overline{x}^\sharp) = \eta(\overline{a})^{-1} \cdot \eta(\overline{x})a.$$

[2]More information about isotopes of alternative algebras can be found in [22].
[3]At first sight, our choice of multiplication does not seem to be the most obvious one. A motivation why to take exactly this multiplication, can be found in the appendix of this thesis.

Next, we define

$$U^\sharp := \{(x,y) \in A \times A \mid \eta^\sharp(\overline{x}^\sharp) \sharp x + y + \eta^\sharp(\overline{y}^\sharp) = 0\}$$

and

$$(x_1,y_1) \oplus (x_2,y_2) := (x_1 + x_2, y_1 + y_2 - \eta^\sharp(\overline{x_2}^\sharp) \sharp x_1)$$

for all $(x_1,y_1),(x_2,y_2) \in U^\sharp$. Finally, we set

$$\tau^\sharp : U^{\sharp *} \to U^{\sharp *};\ (x,y) \mapsto (-x \sharp y^{\sharp -1}, y^{\sharp -1})$$

Then $\mathbb{M}(U^\sharp, \tau^\sharp)$ is a Moufang set of the same shape as $\mathbb{M}(U,\tau)$.

(iii) In a last step, we construct an explicit isomorphism Ψ between $\mathbb{M}(U,\tau)$ and $\mathbb{M}(U^\sharp,\tau^\sharp)$ and show that $\mathbb{M}(T, \mu_{(a,-\eta(\overline{a})a/2)})$ is isomorphic to $\mathbb{M}(\Psi(T), \mu^\sharp_{(a,-a/2)})$ with $(a, a/2) \in \Psi(T)$.
By applying several times the Moufang identities for composition algebras, we can rewrite $U^\sharp$ in terms of $\bar{\cdot}$, η and the original multiplication. We obtain that

$$\begin{aligned} U^\sharp &:= \{(x,y) \in A \times A \mid \eta^\sharp(\overline{x}^\sharp) \sharp x + y + \eta^\sharp(\overline{y}^\sharp) = 0\} \\ &= \{(x,y) \in A \times A \mid \eta(\overline{a})^{-1} \cdot \eta(\overline{x})x + y + \eta(\overline{a})^{-1} \cdot \eta(\overline{y})a = 0\} \\ &= \{(x,y) \in A \times A \mid \eta(\overline{x})x + \eta(\overline{a})y + \eta(\overline{y})a = 0\}. \end{aligned}$$

An easy calculation shows that

$$\Psi : (U,+) \to (U^\sharp, \oplus);\ (c,d) \mapsto (c, \eta(\overline{a})^{-1}d)$$

is an isomorphism of groups.
We end by noticing that $\Psi((a, -\eta(\overline{a})a/2)) = (a, -a/2)$ and that Ψ transforms $\mu_{(a,-\eta(\overline{a})a/2)}$ to the map $\mu^\sharp_{(a,-a/2)}$ with a being the identity element of $(A,\sharp)$.

□

Structure of V

We notice that using the previous theorem, we may assume that the Moufang subset we are considering contains the element $(e, -e/2)$ and is closed under $\mu_{(e,-e/2)}$. This implies that $e \in V$, we start by proving that T is closed under τ as well.

Lemma 5.1.5. *Every Moufang subset* $\mathbb{M}(T, \mu_{(e,-e/2)})$ *of* $\mathbb{M}(U, \tau)$ *containing* $(e, -e/2)$ *is closed under* τ.

Proof. We show that for every element $(c, d) \in T^*$, $\tau(c, d) = (-cd^{-1}, d^{-1}) \in T$. As T is closed under $\mu_{(e,-e/2)}$, we find that

$$\mu_{(e,-e/2)}(c, d) = (-cd^{-1}/2, d^{-1}/4) \in T$$

for every $(c, d) \in T^*$. We obtain that

$$\begin{aligned} 2 \cdot (-cd^{-1}/2, d^{-1}/4) &= (-cd^{-1}, 3d^{-1}/4 + \eta(\overline{d})^{-1}/4) \\ &= (-cd^{-1}, d^{-1} + \eta(\overline{d})^{-1}/4 - d^{-1}/4) \in T \end{aligned}$$

or using Lemma 5.1.3(iii) that $(-cd^{-1}, d^{-1}) \in T$ for every $(c, d) \in T^*$. □

We show that V is closed under inverting. Let (a_1, b_1), $(a_2, b_2) \in T$, then the commutator $(0, \eta(\overline{a_2})a_1 - \eta(\overline{a_1})a_2) \in T$. In particular, we find that $(0, a - \eta(\overline{a})) \in T$ for all $a \in V$. If $(0, b) \in T$, then also $h_{(0,b)}(e, -e/2) = (b, b^2/2)$, consequently $b \in V$. We get that

$$\eta(\overline{a_2})a_1 - \eta(\overline{a_1})a_2 \in V \tag{5.1.2}$$

for every $a_1, a_2 \in V$. Moreover, as both $a - \eta(\overline{a})$ and a belong to V, $\eta(\overline{a}) \in V$.

Using Lemma 5.1.3, we find for every $c \in V^*$ that $(c, -\eta(\overline{c})c/2) \in T$. Therefore $\tau(c, -\eta(\overline{c})c/2) \in T$, so $\eta(\overline{c})^{-1} \in V$. As $\eta(\overline{a})$ belongs to V for every $a \in V$, we obtain that $a^{-1} \in V$ for every $a \neq 0$.

Next, we show that V is closed under multiplication with other elements of V. Let a be an arbitrary element of V then

$$-2h_{(a,-\eta(\overline{a})a/2)}(e) = \eta(\overline{a})a \in V.$$

Let $a_1, a_2 \in V$, then the above result applied to a_1, $\eta(\overline{a_2})$ and $a_1 + \eta(\overline{a_2})$ shows that $\eta(\overline{a_1})\eta(\overline{a_2}) + a_2 a_1 \in V$. So combining this with (5.1.2) applied to $\eta(\overline{a_1})$ and a_2, we obtain that

$$\begin{aligned} a_1 a_2 + \eta(\overline{a_2})\eta(\overline{a_1}) \in V, \\ a_1 a_2 - \eta(\overline{a_2})\eta(\overline{a_1}) \in V. \end{aligned}$$

We find that $a_1 a_2 \in V$, so V is closed under multiplication.

We now discuss separately what happens when V is a field. In this case, the map η composed with $\bar{\cdot}$ is an involution ξ on V, so either every element of V

is fixed by ξ or V is a separable quadratic extension over $\tilde{V} := \{x \in V \mid \xi(x) = x\}$. In the first situation, we find that $\mathbb{M}(V, \tau_V)$ is isomorphic to the projective Moufang set $\mathbb{M}(V)$ over V while in the second case, we obtain a non-abelian Moufang set corresponding to a separable quadratic extension V over $\tilde{V}$.

We claim that in all other cases the standard involution $\bar{\cdot}$ preserves V. From this, it follows immediately that for every $a \in V$, $\overline{a}a$ and $\mathrm{T}(a)$ are also contained in V. The set V has the structure of an algebra over the subfield of E corresponding to $E \cap V$. Using our claim V is also equipped with a norm and trace map, namely the restriction of N and T to V. This means that in fact V is a composition algebra over some subfield l of E. We end this section with proving our claim.

Lemma 5.1.6. *If V is not a field, then the algebra V is closed under the standard involution.*

Proof. Let a be an arbitrary element of $V \setminus Z(V)$, then there exists at least one element $b \in V$, such that $ab - ba \neq 0$. We find that

$$\begin{aligned} \mathrm{T}(a)(ab - ba) &= \mathrm{T}(a)ab - ba\,\mathrm{T}(a) \\ &= a^2b - ba^2. \end{aligned}$$

Now $ab - ba$ and $a^2b - ba^2$ are contained in V, so $\mathrm{T}(a) \in V$. We conclude that $\overline{a} \in V$.

Suppose next that $x \in Z(V)$ arbitrary, then $x - a$ and a both belong to $V \setminus Z(V)$. Using the previous argument, we obtain that $\mathrm{T}(x - a)$ and $\mathrm{T}(a)$ belong to V, consequently $\mathrm{T}(x) \in Z(V)$. We find for every element v of V that $\overline{v} \in V$. □

Remark 5.1.7. We are very grateful to the MathOverflow user Aakumadula, who provided us in [11] with the main idea for the proof of the previous lemma.

Structure of W and T

Let $(a, b) \in T$, then as $b + \eta(\overline{b}) = -\eta(\overline{a})a$ and $b - \eta(\overline{b})$ belong to V, so does b. We find that $W \subseteq V$. Next, we show that every element $(a, b) \in U$ with $a, b \in V$ belongs to T. As $a \in V$, the element $(-a, -\eta(\overline{a})a/2)$ belongs to T. We find that (a, b) belongs to T if and only if

$$(a, b) + (-a, -\eta(\overline{a})a/2) = (0, b + \eta(\overline{a})a/2) = (0, b/2 - \eta(\overline{b})/2) \in V.$$

As $b \in V$, we find that $(0, b - \eta(\overline{b})) \in T$. We conclude that

$$T := \{(a, b) \in V \times V \mid \eta(\overline{a})a + \eta(\overline{b}) + b = 0\}.$$

Conclusion

From the classification of composition algebras, see for example [27, Theorem 1.6.2], we find that (in general) V is an octonion division algebra, a quaternion division algebra, a quadratic field extension over ℓ or ℓ itself, with ℓ a subfield of A. This implies that Moufang subsets $\mathbb{M}(T, \mu_{(a,b)}|_T)$ of $\mathbb{M}(U, \tau)$, with V and S nontrivial, are again non-abelian Moufang sets corresponding to some composition algebra or are a projective Moufang set $\mathbb{M}(\ell)$ with ℓ a subfield of A.

Special cases

We discuss separately the two special cases that can occur.

(i) The first special case is the one where S is trivial. We show that in this case the Moufang set is a projective Moufang set over some subfield ℓ of A. We already found we may assume without loss of generality that $(e, -e/2) \in T$. We notice that for every element $(a, b) \in T$ the commutator $[(e, -e/2), (a, b)]$ belongs to T. This implies that $(0, a - \eta(\overline{a})) \in T$, so $a = \eta(\overline{a})$ for every $a \in V$. Computing the commutator

$$[(a, b), (c, d)] = (0, \eta(\overline{a})c - \eta(\overline{c})a) = (0, ac - ca) = (0, 0)$$

for every $(a, b), (c, d) \in T$, we see that $ac = ca$ for every $a, c \in V$. If we apply this to (a, b), $(-ab^{-1}, b^{-1}) \in T$, we find that $ab^{-1} = b^{-1}a$. Combining this with $\eta(\overline{ab^{-1}}) = ab^{-1}$ and $\eta(\overline{a}) = a$, we find that $b = \eta(\overline{b})$. We obtain that every element of T is of the form $(a, -\eta(\overline{a})a/2) = (a, -a^2/2)$. Using τ, we can conclude from this that $a^{-1} \in V$ for every $a \in V^*$.
Completely similar as in the general case, we find that a_1^2, a_2^2 and $(a_1 + a_2)^2$ are contained in V for all $a_1, a_2 \in V$. We obtain that $a_1 a_2 \in V$ for every $a_1, a_2 \in V$, so V is a field. We end with remarking that if we replace τ by

$$\mu_{(-e,-e/2)} : T \to T;\ (a, -a^2/2) \mapsto (-a^{-1}, -a^{-2}/2)$$

the Moufang set we obtain is the projective Moufang set $\mathbb{M}(V)$, with V a subfield of A.

(ii) The second special case is the one where $V = \{0\}$. We show that $\mathbb{M}(T, \mu_{(0,x)}|_T)$ with $(0, x) \in T$ is a Moufang set isomorphic to a Moufang set coming from a quadratic Jordan division algebra. Fix an arbitrary element $(0, x)$ of T and let $(A, +, *)$ be the isotope of $(A, +, .)$ such that

$$y * z := (yx)(x^{-2}z),$$

then x is the identity element of $(A, +, *)$ and $y^{*-1} = xy^{-1}x$. The map

$$\Gamma : T \to A;\ (0, y) \mapsto y$$

is a group isomorphism from T to $\Gamma(T)$. Furthermore Γ transforms

$$\mu_{(0,x)}|_T : T \to T;\ y \mapsto -xy^{-1}x$$

into the map

$$\tau^* : \Gamma(T) \to \Gamma(T);\ y \mapsto -y^{*-1}.$$

We observe that $\mathbb{M}(\Gamma(T), \tau^*)$ is a submoufang set of $\mathbb{M}(A, \tau^*)$ containing the identity element of $(A, +, *)$. We find that $y^{*2} \in \Gamma(T)$ for every $y \in \Gamma(T)$, so $y * z + z * y \in \Gamma(T)$ for every $y, z \in \Gamma(T)$. Take $z \in Ex$, then the set $\Gamma(T)$ becomes a vector space over the field $\Gamma(T) \cap Ex$. Also, the set $\Gamma(T)$ is closed under inversion and addition, so using the Hua identity, $\Gamma(T)$ is closed as well under the maps

$$U_y : \Gamma(T) \to \Gamma(T);\ z \mapsto y * z * y$$

for every $y \in \Gamma(T)$. It follows that $(\Gamma(T), U, x)$ is a quadratic Jordan algebra with corresponding Moufang set $\mathbb{M}(\Gamma(T))$ equal to $\mathbb{M}(\Gamma(T), \tau^*)$. We end this section with the following summarizing theorem:

Theorem 5.1.8. *Moufang subsets of non-abelian Moufang sets corresponding to a composition algebra are again of the same type or are a Moufang set corresponding to some quadratic Jordan division algebra.*

5.2 Hermitian Moufang sets

We investigate Moufang subsets $\mathbb{M}(U', \mu_{(a,t)}|_{U'})$ with $(a,t) \in U'$ of hermitian Moufang sets in characteristic different from two. Here, we did not obtain a full classification yet. We present some (partial) results.

5.2.1 Description of the Moufang sets

One can also define hermitian Moufang sets using skew-hermitian forms, for simplicity reasons, we this time take this approach. In characteristic different from two, this new description is obtained by replacing K_σ^- in Definition 2.3.4 by $K_0 := \{x \in K \mid x = x^\sigma\}$, the hermitian form by a skew-hermitian form and with

$$\tau(a,t) = (-a \cdot t^{-1}, -t^{-1}).$$

The triple (K, K_0, σ) is called an involutory set.

So let (K, K_0, σ) be an involutory set and let $q : L_0 \to K$ be a pseudo-quadratic form on K with skew-hermitian form h. Then

$$U := \{(a,t) \in L_0 \times K \mid q(a) - t \in K_0\}$$

with $(a,t) + (b,v) = (a+b, t+v+h(b,a))$ for all $(a,t), (b,v) \in U$, together with a permutation

$$\tau(a,t) = (-a \cdot t^{-1}, -t^{-1})$$

is a hermitian Moufang set. One can show that the Hua maps corresponding to this Moufang set are

$$h_{(a,t)}(b,v) = (b \cdot t^\sigma - a \cdot t^{-1} h(a,b) t^\sigma, t v t^\sigma)$$

for all $(a,t), (b,v) \in U$. We notice that the Hua maps can also be viewed as maps acting on L_0, we use the same notation for this action.

Using [34, 11.28], we may assume that $q(a) = h(a,a)/2$ and consequently that $q(a \cdot r) = r^\sigma q(a) r$ for all $a \in L_0,\ r \in K$.

5.2.2 General properties of Moufang subsets

We start with studying the properties of general Moufang subsets of hermitian Moufang sets. We define $V_0 := \{a \in L_0 \mid \exists t \in K :\ (a,t) \in U'\}$ and D_0 as the subset of K_0 corresponding to $Z(U) \cap U'$.

As in the case of the non-abelian Moufang sets corresponding to composition algebras, Moufang subsets containing an 'identity element' are easier to work with. The following theorem essentially proves we may assume without loss of generality that $(0,e)$ is contained in every Moufang subset with $D_0 \neq \{0\}$.

Theorem 5.2.1. *Let $\mathbb{M}(U', \mu_{(0,d_0)}|_{U'})$ be a Moufang subset of $\mathbb{M}(U,\tau)$ containing the element $(0,d_0) \neq (0,0)$. Then there exists a hermitian Moufang set $\mathbb{M}(U^\sharp, \tau^\sharp)$ with e being the identity element of the corresponding skew-field and an embedding Ψ of $\mathbb{M}(U', \mu_{(0,d_0)}|_{U'})$ as a Moufang subset of $\mathbb{M}(U^\sharp, \tau^\sharp)$ such that $\Psi(U')$ is closed under $\mu^\sharp_{(0,e)}$ and contains the element $(0,e)$.*

Proof. Let $(K,+,.)$ be the skew-field corresponding to $\mathbb{M}(U,\tau)$, then as before we can define an isotope $(K,+,\sharp)$ of $(K,+,.)$ by setting

$$x \sharp y := x d_0^{-1} y$$

for all $x, y \in K$. We notice that the involution σ corresponding to $\mathbb{M}(U,\tau)$ is also an involution on $(K,+,\sharp)$ and therefore the set $((K,+,\sharp), K_0, \sigma)$ is an involutory set. Also, the maps h and q corresponding to $\mathbb{M}(U,\tau)$ are as well hermitian and pseudo-quadratic with respect to $((K,+,\sharp), K_0, \sigma)$. For that purpose we can define the group $(U^\sharp, +)$ as the group $(U,+)$ with corresponding involutory set $((K,+,\sharp), K_0, \sigma)$. The permutation $\tau^\sharp$ on $U^\sharp$ is then defined as

$$\tau^\sharp : U^{\sharp *} \to U^{\sharp *};\ (a,t) \mapsto (-a \sharp t^{\sharp -1}, -t^{\sharp -1})$$

with $t^{\sharp -1} = d_0 t^{-1} d_0$.

It is now easy to notice that $\mathbb{M}(U,\tau)$ is isomorphic to the Moufang set $\mathbb{M}(U^\sharp, \tau^\sharp)$ with Ψ the identity map on U. We end by remarking that Ψ sends $\mu_{(0,d_0)}$ to $\mu^\sharp_{(0,d_0)} = \tau^\sharp$ with d_0 the identity element of $(K,+,\sharp)$. □

From now on, we exclude the case where U' is completely contained in the center $Z(U)$ of U and the Moufang subsets for which $Z(U') \cap U = \{(0,0)\}$. We treat these special cases separately afterwards. This implies we may assume without loss of generality that $(0,e) \in U'$ and that the Moufang subset we are considering is of the form $\mathbb{M}(U', \tau|_{U'})$.

In the next lemma, we summarize a few facts about the elements contained in D_0, V_0 and U' that will turn out to be useful for the following sections:

Lemma 5.2.2. *Let $(a,t), (b,v), (c,s)$ be three elements of U' and $d_0 \in D_0$, then:*

(i) $h(a,b) - h(b,a) \in D_0$
(ii) $tt^\sigma \in D_0$, *so* $q(a)^2 \in D_0$

(iii) $d_0/2 \in D_0$
(iv) $a \cdot d_0 \in V_0$
(v) $(a, q(a)) \in U'$
(vi) $a \cdot q(a)^{-1} \in V_0$, $a \cdot q(a) \in V_0$
(vii) $q(a)h(a,b) + h(b,a)q(a) \in D_0$
(viii) $q(a)h(b,a) + h(a,b)q(a) \in D_0$
(ix) $a \cdot q(b) + b \cdot h(a,b) \in V_0$
(x) $a \cdot h(b,c) + b \cdot h(a,c) + c \cdot h(a,b) \in V_0$
(xi) $q(a)q(b) + q(b)q(a) + h(a,b)h(b,a) \in D_0$
(xii) $q(a)d_0q(a) \in D_0$

Proof. (i) Compute the commutator $[(a,t),(b,v)]$.
(ii) This follows from $h_{(a,t)}(0,e) = (0, tt^\sigma) \in U'$.
(iii) As $(0, d_0) \in U'$, also $\tau[2 \cdot \tau((0, d_0))] = (0, d_0/2) \in U'$.
(iv) This follows from $h_{(0,d_0)}(a,t) = (a \cdot d_0, d_0td_0) \in U'$.
(v) As U' is closed under the action of τ, we obtain that $\tau^2(a,t) = (-a,t) \in U'$. Consequently, $(a,t) + (-a,t) = (0, 2t - h(a,a)) = (0, 2(t - q(a))) \in U'$. It now follows easily that $(a,t) + (0, q(a) - t) = (a, q(a)) \in U'$.
(vi) Trivially from $\tau((a, q(a))) = (-a \cdot q(a)^{-1}, -q(a)^{-1}) \in U'$ together with $q(a)^2 \in D_0$.
(vii) Apply (i) to the elements $a \cdot q(a)$ and b of V_0.
(viii) Use (vii) together with $a \cdot q(a)(h(a,b) - h(b,a)) \in V_0$.
(ix) Compute $h_{(a,q(a))}(b,v) = (-b \cdot q(a) + a \cdot q(a)^{-1}h(a,b)q(a), -q(a)vq(a))$, then together with (viii) the assertion follows.
(x) Apply (ix) to a and $b + c$.
(xi) Express that $(q(a) + q(b) + h(a,b))(q(a) + q(b) + h(a,b))^\sigma \in D_0$.
(xii) This follows from $h_{(a,q(a))}(0, d_0) = (0, -q(a)d_0q(a)) \in U'$.

□

The following lemma gives us some more information about the actual form of the elements of U' in terms of D_0:

Lemma 5.2.3. *Let $(a,t), (a,s) \in U'$, then $t - s \in D_0$ and all elements of U' are of the form $(a, q(a) + r)$ with $r \in D_0$.*

Proof. The first statement follows immediately by noticing that

$$(a,t) - (a,s) = (a, q(a) + r_1) - (a, q(a) + r_2) = (0, r_1 - r_2) \in U' \cap Z(U)$$

for some $r_1, r_2 \in K_0$. We find that $t - s = r_1 - r_2 \in D_0$. Let $(a,t) \in U'$, then also $(a, q(a)) \in U'$. The second assertion then follows by applying the first one to t and $q(a)$. □

5.2.3 The vector space structure on V_0

In this subsection, we show that if the set V_0 is closed under scalar multiplication with certain elements of K that $\mathbb{M}(U', \tau_{U'})$ is a hermitian Moufang set. The following definition specifies which elements of K we mean:

Definition 5.2.4. We say that the subset V_0 of L_0 *preserves the vector space structure of* L_0 if for every $(a, t) \in U'$, $b \cdot t \in V_0$ for every $b \in V_0$.

If V_0 does preserve the vector space structure of L_0, we can make V_0 into a right module over the associative ring D with:

$$D := \langle \{t \mid \exists a \in L_0 : (a, t) \in U'\} \rangle_{\text{ring}} \subseteq K. \tag{5.2.1}$$

We show that D is a skew field or equivalently that every element $0 \neq d \in D$ has an inverse in D.

Lemma 5.2.5. *If $d \in D^*$, then $d^{-1} \in D^*$.*

Proof. Let $d \in D^*$ then $d = \sum_i \prod_j d_{ij}$, with d_{ij} such that there exists an element $a_{ij} \in V_0$ for which $(a_{ij}, d_{ij}) \in U'$. We show that $dd^\sigma \in D_0$ and therefore also $d^{-\sigma} d^{-1} \in D_0$. Using the Hua maps corresponding to $h_{(a_{ij}, d_{ij})}$, it is easy to see that for a fixed i,

$$(\prod_j d_{ij})(\prod_j d_{ij})^\sigma \in D_0,$$

so we already found that the assertion is true for all elements $d_i := \prod_j d_{ij}$. As d is the sum of such elements, it easy to notice (using the same argument as before) that $d_l d_m^\sigma + d_m d_l^\sigma \in D_0$ for all l, m.

Since $d^\sigma \in D$, it follows that $d^{-1} = d^\sigma \cdot d^{-\sigma} d^{-1}$ belongs to D.

□

We show that all conditions for $\mathbb{M}(U', \tau_{U'})$ to be a hermitian Moufang set are fulfilled.

Theorem 5.2.6. *Let $\mathbb{M}(U', \tau|_{U'})$ be a Moufang subset of a hermitian Moufang set $\mathbb{M}(U, \tau)$. Suppose that $Z(U) \cap U' \neq \{(0, 0)\}$ and that $V_0 := \{a \in L_0 \mid \exists t \in K : (a, t) \in U'\} \neq \{0\}$ preserves the vector space structure of the vector space L_0 corresponding to U. Then $\mathbb{M}(U', \tau|_{U'})$ is a hermitian Moufang set.*

Proof. (i) In a first step, we show that (D, D_0, σ), with D as defined in 5.2.1, is an involutory set. Therefore, we need to check that $D_0 = \{d \in D \mid d^\sigma = d\}$.

It is obvious that $D_0 \subseteq \{d \in D \mid d^\sigma = d\}$, so it remains to check that every element of D fixed by σ belongs to D_0.
We remark that in characteristic different from two, the set $\{d \in D \mid d^\sigma = d\}$ coincides with the set $\{d + d^\sigma \mid d \in D\}$ because $a = a/2 + a^\sigma/2$ for every a fixed by σ. It therefore suffices to prove that for every element $d \in D$, $d + d^\sigma \in D_0$. Using the vector space structure of V_0, we notice that for every $(a,t) \in U'$, $(a \cdot h(a,a)^{-1}d, q(a \cdot h(a,a)^{-1}d))$ belongs to U'. As the commutator

$$[(a \cdot h(a,a)^{-1}d, q(a \cdot h(a,a)^{-1}d)), (a, q(a))] \in D_0,$$

we find that

$$\begin{aligned} h(a \cdot h(a,a)^{-1}d, a) - h(a, a \cdot h(a,a)^{-1}d) &= d^\sigma \cdot h(a,a)^{-\sigma} \cdot h(a,a) \\ &\quad - h(a,a) \cdot h(a,a)^{-1} \cdot d \\ &= -(d^\sigma + d) \in D_0, \end{aligned}$$

what we needed to prove.

(ii) We prove that $\mathbb{M}(U', \tau|_{U'})$ is a hermitian Moufang set. We start by noticing that $U' := \{(a,t) \in V_0 \times D \mid q(a) - t \in D_0\}$. Here, V_0 is a right vector space over D.
The only thing we still need to verify is that the restriction $q|_{V_0}$ of q is a pseudo-quadratic form on V_0 with respect to D and σ and that $h|_{V_0 \times V_0} : V_0 \times V_0 \to D$ is a skew-hermitian form. This comes down to verifying that $q(a+b) - q(a) - q(b) = (h(a,b) + h(b,a))/2 \equiv h(a,b) \mod D_0$ for all $a, b \in V_0$, which is obviously fulfilled.

$\square$

We finish this section with proving that for a large class of hermitian Moufang sets $\mathbb{M}(U,\tau)$, every Moufang subset of $\mathbb{M}(U,\tau)$ (with $D_0 \neq \{0\}$ and $V_0 \neq \{0\}$) is again hermitian:

Theorem 5.2.7. *If K is a field, then $\mathbb{M}(U', \tau_{U'})$ is a hermitian Moufang set.*

Proof. Using the previous theorem, it suffices to show that V_0 preserves the vector space structure of L_0.

Lemma 5.2.2 shows that $q(b)h(a,b) + h(b,a)q(b) \in D_0$ for every $a, b \in V_0$. This implies that

$$b \cdot q(b)^{-1}[q(b)h(a,b) + h(b,a)q(b)] \in V_0.$$

Using the fact that K is a field, we find that

$$b \cdot [h(a,b) + h(b,a)] \in V_0.$$

Together with $b \cdot [h(a,b) - h(b,a)] \in V_0$, this results in

$$b \cdot h(b,a) \in V_0,$$

so again using Lemma 5.2.2, we obtain that $a \cdot q(b) \in V_0$ for all $a, b \in V_0$ or that indeed V_0 preserves the vector space structure of L_0. □

5.2.4 Conditions for V_0 to preserve the desired vector space structure

From now on, using Theorem 5.2.7, we may assume that K is not a field.

Let a, b be two arbitrary elements of V_0. We investigate under which conditions $a \cdot q(b)$ and $b \cdot q(a)$ belong to V_0. We start with noticing that it is sufficient to investigate when $a \cdot q(b) \in V_0$:

Lemma 5.2.8. *Let $a, b \in V_0$ arbitrary, then $a \cdot q(b) \in V_0$ if and only if $b \cdot q(a) \in V_0$.*

Proof. Assume that $b \cdot q(a) \in V_0$, we prove that $a \cdot q(b) \in V_0$. The other direction then follows by switching the roles of a and b.

Using Lemma 5.2.2, we find that $a \cdot h(b,a) \in V_0$ and consequently

$$h_{(-a \cdot q(a)^{-1}, -q(a)^{-1})}(a \cdot h(b,a)) = -a \cdot h(b,a) q(a)^{-1} \in V_0.$$

Let $L_0 = \langle a \rangle \oplus a^{\perp}$, then $b = a\lambda + a'$ for some $\lambda \in K$ and $a' \in a^{\perp}$, so we reduced the problem to proving that $a \cdot [q(a') + q(a \cdot \lambda)] \in V_0$. Because $a \cdot h(b,a) q(a)^{-1} \in V_0$, we already found that $a\lambda^{\sigma} \in V_0$. As

$$-\frac{1}{2}[h(a \cdot q(a)^{-1}, b) - h(b, a \cdot q(a)^{-1})] = \lambda + \lambda^{\sigma} \in D_0,$$

we obtain that $a \cdot \lambda \in V_0$ and therefore $a' \in V_0$. Now, $h(a, a') = 0$, so $a \cdot q(a') \in V_0$. Therefore, it remains to check that $a \cdot q(a\lambda) \in V_0$. This follows easily using the fact that $a \cdot \lambda^{\sigma} \in V_0$ and $a \in V_0$. Indeed, together with Lemma 5.2.2, we find that

$$a \cdot \lambda^{\sigma} q(a) \lambda + a \cdot \lambda^{\sigma} h(a, a \cdot \lambda^{\sigma}) \in V_0,$$

so we can finish the proof by remarking that $a \cdot \lambda^{\sigma} q(a)(\lambda + \lambda^{\sigma}) \in V_0$ as $a \cdot h(b,a) \in V_0$ and $\lambda + \lambda^{\sigma} \in D_0$. □

Next, we define $\overline{D_0}$ as the ring generated by all elements of D_0 and prove the following nice property of D_0:

Lemma 5.2.9. *If $\overline{D_0}$ is not abelian, then $\mathbb{M}(U', \tau_{U'})$ is hermitian.*

Proof. Using Lemma 5.2.2, we know that for every $a, b, c \in V_0$

$$a \cdot h(b, c) + b \cdot h(a, c) + c \cdot h(a, b) \in V_0.$$

If we replace c by $a \cdot d$ with $d \in \overline{D_0}$, we obtain that

$$[a \cdot h(b, a) + 2b \cdot q(a)]d + a \cdot dh(a, b) \in V_0$$

for every $d \in \overline{D_0}$. Furthermore, using Lemma 5.2.2, we obtain that $[a \cdot h(b, a) + b \cdot q(a)]d \in V_0$ and $a \cdot [h(b, a \cdot d^\sigma) - h(a \cdot d^\sigma, b)] \in V_0$. From this, it follows that

$$b \cdot q(a)d + a \cdot h(b, a)d^\sigma \in V_0$$

and therefore

$$b \cdot q(a)[d - d^\sigma] \in V_0$$

for every $d \in \overline{D_0}$. Suppose next that $\overline{D_0}$ is not abelian, then there exist two elements $d_0, d_1 \in D_0$ such that $d_0 d_1 \neq d_1 d_0$. Define $d := d_0 d_1$, then $d - d^\sigma \neq 0$, so $b \cdot q(a) \in V_0$ for all $a, b \in V_0$. This means that V_0 preserves the vector space structure of L_0, so indeed $\mathbb{M}(U', \tau_{U'})$ is hermitian. □

So from now on, we only consider Moufang subsets for which $\overline{D_0}$ is abelian, since otherwise we already know they are of hermitian type. We remark that if $\overline{D_0}$ is abelian, then D_0 is a field. Indeed, we only need to show that D_0 is closed under multiplication. For every $d_0, d_1 \in D_0$, we find that

$$h(a, a \cdot q(a)^{-1} d_0 d_1) - h(a \cdot q(a)^{-1} d_0 d_1, a) \in D_0,$$

or $d_0 d_1 \in D_0$. We can even prove more about the elements of D_0 as the following statement shows:

Lemma 5.2.10. *Let a be an arbitrary element of V_0, then $q(a)$ commutes with all elements of D_0.*

Proof. Let d_0 be arbitrary in D_0, then from Lemma 5.2.2 we obtain that $q(a)d_0q(a) \in D_0$. We notice that the following equation holds:

$$\begin{aligned} q(a)d_0q(a) &= q(a)\{d_0q(a) - q(a)d_0\} + q(a)^2 d_0 \\ &= \frac{q(a)}{2}\{h(a \cdot d_0, a) - h(a, a \cdot d_0)\} + q(a)^2 d_0. \end{aligned}$$

Now, the left hand side of the above equation and the second term of the right hand side both belong to D_0, which implies that

$$q(a)\{h(a \cdot d_0, a) - h(a, a \cdot d_0)\} \in D_0.$$

Consequently $h(a, a \cdot d_0) = h(a \cdot d_0, a)$ since otherwise the element $q(a)$ which is not fixed by σ would belong to D_0, a contradiction. As d_0 is arbitrary in D_0, this proves the lemma. □

Let $D \subseteq K$ be the skew field as in Definition 5.2.1, then

$$V := \{\sum_i a_i \cdot d_i \mid a_i \in V_0,\ d_i \in D\}$$

is a vector space over D and

$$\overline{U'} := \{(v, q(v) + r) \mid v \in V, r \in \mathrm{Fix}_\sigma(D)\}$$

is a subgroup of U containing U'. As V preserves the vector space structure of L_0, the corresponding Moufang subset $\mathbb{M}(\overline{U'}, \tau|_{\overline{U'}})$ is a hermitian Moufang set containing $\mathbb{M}(U', \tau|_{U'})$.

5.2.5 Special cases

Finally, we discuss what happens if $V_0 = \{0\}$ or $D_0 = \{0\}$.

(i) The first exceptional case is the one where V_0 is trivial. Let $(0, x) \in T$, then as before x is the identity element with respect to the (x, x^{-2})-isotope $(K, +, *) := K^{(x,x^{-2})}$ of the skew field K. For that purpose, we find that $J := \{y \mid (0, y) \in T\}$ is a subset of the skew field $(K, +, *)$ containing the identity element. Also, J is closed under inversion and addition, i.e. $\mathbb{M}(U', \tau|_{U'})$ is isomorphic to a Moufang set $\mathbb{M}(J)$ arising from a quadratic Jordan division algebra J with $U_y(z) := y * z * y$ for all $y \in J$.

(ii) The second exceptional case is the one where $D_0 = \{0\}$. In this case, the Moufang set can only contain elements of the form $(a, q(a))$, so $(U', +)$ is isomorphic to $(V_0, +)$. This implies that all such Moufang subsets are abelian and isomorphic to some Moufang set arising from a quadratic Jordan division algebra J.

5.2.6 Conclusion

The only Moufang subsets $\mathbb{M}(U', \tau|_{U'})$ of $\mathbb{M}(U, \tau)$ we did not classify yet are those for which D_0 is abelian and K is not a field. Moreover, since $\mathbb{M}(U', \tau|_{U'})$ is also a Moufang subset of $\mathbb{M}(\overline{U'}, \tau|_{\overline{U'}})$, we may also assume that D is not a field.

The Moufang subsets we did classify turn out to be Moufang sets of hermitian type or Moufang sets arising from a quadratic Jordan division algebra J.

A

Appendices

A.1 Explicit isomorphism between $\mathbb{M}(Q,\tau)$ and hermitian Moufang sets of type C_3 and ${}^2\mathsf{A}_5$

We prove that the non-abelian Moufang sets described in Section 5.1.1 with the composition algebra being a quaternion division algebra Q are isomorphic to hermitian Moufang sets of type C_3 and ${}^2\mathsf{A}_5$ depending on the choice of η.

Let

$$U := \{(a,b) \in Q \times Q \mid \eta(\overline{a})a + \eta(\overline{b}) + b = 0\}$$

with η either equal to the identity map or $\tilde{\cdot}$ on Q. The addition on U is given by

$$(a,b) + (c,d) := (a+b, c+d - \eta(\overline{c})a)$$

for all $(a,b),(c,d) \in U$ and the permutation τ on U^* is defined as

$$\tau(a,b) = (-ab^{-1}, b^{-1})$$

for all $(a,b) \in U^*$.

Next, let

$$T := \{(x,t) \in Q \times Q \mid q(x) - t \in K_\iota^-\}$$

with $q(x) := \eta(\overline{x})x/2$ and $K_\iota^- := \{x - \eta(\overline{x}) \mid x \in Q\}$. We can make T into a group by defining the following addition on T:

$$(x,t) + (y,v) := (x + y, t + v + h(y,x))$$

for all $(x,t),(y,v) \in T$ and with $h(y,x) := \eta(\overline{y})x$.

Finally, set $\kappa((x,t)) := (x \cdot t^{-1}, t^{-1})$ for all $(x,t) \in T^*$, then $\mathbb{M}(T,\kappa)$ is a hermitian Moufang set. Moreover, the linear algebraic group corresponding to $\mathbb{M}(T,\kappa)$ is of type C_3 or ${}^2\mathsf{A}_5$ (depending on η) as can be found in [31].

It therefore remains to construct an isomorphism Ψ between $\mathbb{M}(U,\tau)$ and $\mathbb{M}(T,\kappa)$. Define

$$\Psi : U \to T;\ (a,b) \mapsto (-a,-b),$$

then one can easily check that Ψ is indeed an isomorphism between both Moufang sets.

A.2 Argument why the $\sharp$-multiplication in Theorem 5.1.4 works

We briefly show how we came up with the $\sharp$-multiplication of Theorem 5.1.4.

The original problem of the corresponding section of this thesis was to determine all Moufang subsets of a fixed Moufang set $\mathbb{M}(U,\tau)$ over some composition algebra A. Eventually, we tried to show that all nontrivial Moufang subsets $\mathbb{M}(T,\tau|_T)$, i.e. those where (using the notation of Section 5.1) $V \neq \{0\}$ and $S \neq \{0\}$, were again corresponding to some composition algebra. Under the assumption that $(e,-e/2) \in T$, this followed quite naturally, so the problem remained for solving the question in the case that $(e,-e/2) \notin T$.

Our guess was that, even though $(e,-e/2) \notin T$, those Moufang subsets were still isomorphic to some Moufang set $\mathbb{M}(U',\kappa)$ corresponding to a composition algebra $(A',+,\sharp)$. So

$$U' := \{[x,y] \in A' \times A' \mid \eta^\sharp(\overline{x}^\sharp) \sharp x + y + \eta^\sharp(\overline{y}^\sharp) = 0\},$$

with $\eta^\sharp$ an automorphism of order two on A' with respect to $\sharp$. The addition on U' is defined as

$$[x_1,y_1] + [x_2,y_2] = [x_1 + x_2, y_1 + y_2 - \eta^\sharp(\overline{x_2}^\sharp) \sharp x_1]$$

for all $[x_1,y_1],[x_2,y_2] \in U'$ and $\kappa([x,y]) = [-x \sharp y^{\sharp-1}, y^{\sharp-1}]$ for all $[x,y] \in U'$.

Moreover we assumed that the isomorphism Ψ can be chosen in such a way that the first component of T is fixed by Ψ. Expressing that $\mathbb{M}(U', \kappa) = \mathbb{M}(T, \tau|_T)$ eventually showed that if Ψ exists, the multiplication on A' should be as follows;

$$\sharp : A' \times A' \to A';\ (c_1, c_2) \mapsto \eta(\overline{a})^{-1}[(\eta(\overline{a})c_1 \cdot a^{-1})c_2]$$

with a the identity element of A', so $\Psi((a, -\eta(\overline{a})a/2)) = [a, -a/2]$ for some $a \in V$.

B

Nederlandstalige samenvatting

In deze Nederlandstalige samenvatting bespreken we bondig de resultaten uit deze thesis. We vermelden de belangrijkste begrippen, definities en stellingen. Voor bewijzen en niet vermelde definities verwijzen we naar de corresponderende hoofdstukken in het Engelstalig gedeelte van deze thesis.

B.1 Inleiding

De basisobjecten in deze thesis zijn Moufang verzamelingen. Deze objecten vormen een deelklasse van de klasse van gebouwen, meer specifiek zijn het gebouwen van rang één waarvoor de wortelgroepen voldoen aan bepaalde transitiviteitseigenschappen. Ze werden oorspronkelijk ingevoerd door J. Tits omwille van hun link met lineaire algebraïsche groepen van rang 1. Moufang verzamelingen zijn echter veel algemenere objecten, een grote open vraag is bijvoorbeeld of alle Moufang verzamelingen van algebraïsche origine zijn.

Definitie B.1.1. Een *Moufang verzameling* $\mathbb{M} = \big(X, (U_x)_{x \in X}\big)$ is een verzameling X samen met een verzameling groepen $U_x \leq \mathrm{Sym}(X)$, zodanig dat voor alle $x \in X$ geldt:

(1) U_x fixeert x en werkt scherp transitief op $X \setminus \{x\}$;
(2) $U_x^\varphi = U_{x\varphi}$ voor alle $\varphi \in G := \langle U_z \mid z \in X \rangle$.

De groep G wordt de *kleine projectieve groep* van de Moufang verzameling genoemd.

B.1.1 Een expliciete constructie van Moufang verzamelingen

Elke Moufang verzameling kan geconstrueerd worden uit één enkele groep samen met een extra permutatie op de elementen van die groep. Onderstaande constructie toont hoe we dit kunnen doen.

Stel $(U, +)$ een groep met eenheidselement 0 en (niet noodzakelijk commutatieve) bewerking + en definieer X als $U \cup \{\infty\}$, met ∞ een nieuw symbool. Voor elke $a \in U$, definiëren we ook een afbeelding $\alpha_a \in \mathrm{Sym}(X)$ als volgt:

$$\alpha_a \colon \begin{cases} \infty \mapsto \infty \\ x \mapsto x + a & \text{voor alle } x \in U. \end{cases} \tag{B.1.1}$$

Stel vervolgens

$$U_\infty := \{\alpha_a \mid a \in U\}\,.$$

Stel τ een permutatie van $U^* := U \setminus \{0\}$. Dan kunnen we τ uitbreiden tot een element van $\mathrm{Sym}(X)$ (die we ook met τ noteren) door $0^\tau = \infty$ en $\infty^\tau = 0$ te stellen.

Vervolgens stellen we

$$U_0 := U_\infty^\tau \text{ en } U_a := U_0^{\alpha_a} \tag{B.1.2}$$

voor alle $a \in U$. Stel

$$\mathbb{M}(U, \tau) := \big(X, (U_x)_{x \in X}\big) \tag{B.1.3}$$

en

$$G := \langle U_\infty, U_0 \rangle = \langle U_x \mid x \in X \rangle\,.$$

Dan is $\mathbb{M}(U, \tau)$ niet altijd een Moufang verzameling, maar elke Moufang verzameling kan op deze manier bekomen worden.

B.1.2 Voorbeeld

We beschrijven de meest eenvoudige klasse van Moufang verzamelingen; dit is de klasse van *projectieve Moufang verzamelingen* $\mathsf{PSL}_2(D)$. We verwijzen naar [12, Sectie 5] voor een meer gedetailleerde uitleg.

Stel $(D, +, \cdot)$ een *alternatieve delingsring*, i.e. een niet noodzakelijk associatieve ring (met 1) zodanig dat voor elke $a \in D^*$ een element $a^{-1} \in D^*$ bestaat waarvoor $a \cdot a^{-1}b = b = ba^{-1} \cdot a$ voor elke $b \in D$ en zodanig dat volgende 'alternatieve wetten' gelden voor alle $x, y \in D$:

$$x(xy) = (xx)y$$
$$(yx)x = y(xx).$$

De Bruck–Kleinfeld stelling vertelt ons dan dat elke alternatieve delingsring ofwel associatief is ofwel een octonendelingsalgebra vormt.

Stel vervolgens $(U, +) := (D, +)$ en definieer volgende permutatie op U^*:

$$\tau\colon U^* \to U^* : x \mapsto -x^{-1}.$$

Dan vormt $\mathbb{M}(U, \tau)$ een Moufang verzameling die we noteren met $\mathbb{M}(D)$ en die de *projectieve Moufang verzameling over D* genoemd wordt.

B.2 Polariteiten op Moufang vlakken

In het eerste grote onderdeel van deze thesis bespreken we een methode om uit een polariteit met minstens drie absolute punten op een Moufang vlak een Moufang verzameling te construeren. In twee recente artikels [19, 18] van R. Knarr en M. Stroppel classificeren ze alle mogelijke types polariteiten op een Moufang vlak. Wij maken gebruik van hun classificatie om een classificatie te vinden van de corresponderende Moufang verzamelingen.

B.2.1 Moufang vlakken

We geven een definitie van een Moufang vlak $\mathbb{P}^2(\mathcal{O})$. Deze projectieve vlakken worden ook nog octonenvlakken genoemd, verwijzend naar het feit dat ze gecoördinatiseerd worden door een octonendelingsalgebra $\mathcal{O}$. De puntenverzameling $\mathcal{P}$ bestaat uit drie verschillende types punten. Punten van het eerste type zijn elementen van de vorm (a, b) met $a, b \in \mathcal{O}$, punten van het tweede type zijn (c) met $c \in \mathcal{O}$. Tenslotte is er nog een laatste type bestaande uit slechts één punt, namelijk (∞).

Volledig analoog zijn er drie types rechten. Een eerste type is van de vorm $[m, k]$ met $m, k \in \mathcal{O}$, rechten van het tweede type zijn rechten van de vorm $[l]$ met $l \in \mathcal{O}$ en als laatste één rechte $[\infty]$ van het derde type.

De incidentierelatie $*$ tussen punten en rechten is als volgt:

$$\begin{aligned}
(a,b) * [m,k] &\quad \Longleftrightarrow \quad ma + b = k,\\
(a,b) * [l] &\quad \Longleftrightarrow \quad a = l,\\
(c) * [m,k] &\quad \Longleftrightarrow \quad c = m,\\
(c) * [\infty] &\quad \text{voor alle } c \in \mathcal{O},\\
(\infty) * [l] &\quad \text{voor alle } l \in \mathcal{O},\\
(\infty) * [\infty]. &
\end{aligned}$$

B.2.2 Constructie Moufang verzamelingen

Stel Ψ een polariteit van een Moufang vlak $\mathbb{P}^2(\mathcal{O})$ met minstens drie absolute punten en G de *kleine projectieve groep* van $\mathbb{P}^2(\mathcal{O})$, i.e. de deelgroep van $\mathrm{Aut}(\mathbb{P}^2(\mathcal{O}))$ voortgebracht door de wortelgroepen (of equivalent, de groep voortgebracht door alle elaties). We beschrijven een methode om uit $\mathbb{P}^2(\mathcal{O})$ en Ψ een Moufang verzameling te construeren.

We starten met het bepalen van de verzameling X van abolute vlaggen en de deelgroep $C_G(\Psi)$ van G, i.e. de groep bestaande uit alle elementen van G die commuteren met Ψ. We merken op dat voor elk element $\sigma \in C_G(\Psi)$ en voor elke $x \in X$, de vlag $\sigma(x)$ opnieuw absoluut is. Op deze manier verkrijgen we een natuurlijke actie van $C_G(\Psi)$ op de verzameling van absolute vlaggen van de polariteit Ψ. Het is juist deze actie die een Moufang verzameling bepaalt. We bewijzen volgend lemma in de meer algemene context van Moufang veelhoeken:

Lemma B.2.1. *Stel Δ een Moufang n-gon en G een deelgroep van* $\mathrm{Aut}(\Delta)$ *die alle wortelgroepen bevat. Veronderstel verder dat Ψ een polariteit is op Δ en X de corresponderende verzameling absolute vlaggen van Ψ is. Neem tenslotte aan dat $|X| \geq 3$. Dan vormt X een Moufang verzameling met als kleine projectieve groep een quotiëntgroep van een deelgroep van $C_G(\Psi)$. De wortelgroepen van X corresponderen met de doorsnede van $C_G(\Psi)$ met het unipotente radicaal $U_+ = U_1 \cdots U_n$ van Δ ten opzichte van een vast appartement van Δ dat twee absolute vlaggen bevat.*

B.2.3 Classificatie polariteiten en corresponderende Moufang verzamelingen

Uit de artikels [18, 19] destilleren we de verschillende types van polariteiten met minstens drie absolute punten die kunnen bestaan over een veld met willekeurige karakteristiek. We vinden dat elke polariteit kan bekomen worden uit een *standaard polariteit* samengesteld met een collineatie. Verder mogen we zonder verlies van algemeenheid veronderstellen dat elke collineatie wordt geïnduceerd door een automorfisme van de octonendelingsalgebra $\mathcal{O}$ (zie [18, Stelling 3.6] voor een bewijs). De standaard polariteit π_0 is

$$\begin{aligned}(a,b) &\leftrightarrow [\overline{a}, -\overline{b}]\\ (c) &\leftrightarrow [\overline{c}]\\ (\infty) &\leftrightarrow [\infty]\end{aligned}$$

met $\overline{\cdot}$ de standaard involutie op $\mathcal{O}$. We vinden dus dat elke polariteit van $\mathbb{P}^2(\mathcal{O})$ toegevoegd is aan een polariteit van de volgende vorm voor een $\eta \in \mathrm{Aut}(\mathcal{O})$:

$$\begin{aligned}(a,b) &\leftrightarrow [\eta(\overline{a}), -\eta(\overline{b})]\\ (c) &\leftrightarrow [\eta(\overline{c})]\\ (\infty) &\leftrightarrow [\infty].\end{aligned}$$

Het volstaat daarom om de automorfismen van $\mathcal{O}$ te overlopen. We vinden uiteindelijk vier verschillende types polariteiten, waarbij we met verschillend bedoelen dat ze aanleiding geven tot een Moufang verzameling van een verschillend type:

(I) de standaard polariteit;
(II) een polariteit die enkel bestaat als het centrum E van $\mathcal{O}$ een separabele kwadratische uitbreiding is van een kleiner veld k;
(III) een polariteit in karakteristiek verschillend van 2 komend van een automorfisme dat een quaternionendelingsalgebra van $\mathcal{O}$ fixeert;
(IV) een polariteit in karakteristiek gelijk aan 2 komend van een automorfisme dat een 4-dimensionaal deelveld van $\mathcal{O}$ fixeert.

We bepalen voor elk van de hierboven beschreven polariteiten de corresponderende Moufang verzameling. Gebruik makend van ondermeer de methode beschreven in de vorige subsectie vinden we telkens een Moufang verzameling met

$$U = \big\{(a,b) \in \mathcal{O} \times \mathcal{O} \mid \eta(\overline{a}) \cdot a + \eta(\overline{b}) + b = 0\big\},$$

met groepsbewerking + op U gegeven door

$$(a,b) + (c,d) = \big(a + c, b + d - \eta(\overline{c}) \cdot a\big)$$

voor alle $(a,b),(c,d) \in U$. Als permutatie τ vinden we

$$\tau(a,b) = (-ab^{-1}, b^{-1})$$

voor alle $(a,b) \in U^*$.

Om te bepalen welke types Moufang verzamelingen we op deze manier bekomen, rest ons enkel nog de kleine projectieve groep te bepalen van elk type Moufang verzameling. We vinden dat de eerste polariteit aanleiding geeft tot een Moufang verzameling van type F_4, het derde type is een hermitische Moufang verzameling terwijl de laatste een Moufang deelverzameling is van een projectieve Moufang verzameling $\mathbb{M}(\mathcal{O})$ over $\mathcal{O}$. Bij de tweede polariteit vinden we echter dat de Moufang verzameling als kleine projectieve groep een lineaire algebraïsche groep van type ${}^2\mathsf{E}_{6,1}^{29}$ heeft. Deze klasse Moufang verzamelingen werd nog niet eerder in de literatuur beschreven. We hebben dus met andere woorden via onze polariteit een mooie beschrijving gevonden voor alle Moufang verzamelingen corresponderend met dit type van lineaire algebraïsche groep. Daarom hebben we volgende stelling:

Stelling B.2.1. *De Moufang verzameling $\mathbb{M}(U,\tau)$ bekomen uit een polariteit van type II is een Moufang gebouw geassocieerd aan een lineaire algebraïsche groep van type ${}^2\mathsf{E}_{6,1}^{29}$. Omgekeerd kan elke Moufang verzameling corresponderend een dergelijke groep bekomen worden uit een polariteit van het tweede type.*

B.3 Moufang verzamelingen van gemixt type F_4

In dit gedeelte bespreken we een constructie van een nieuw type Moufang verzamelingen uit gemixte groepen van type F_4. Deze groepen bestaan enkel over velden van karakteristiek 2 en zijn gedefinieerd over velden (K, L) zodanig dat $L^2 \leq K \leq L$.

Stel k en ℓ twee velden zodanig dat $\ell^2 \leq k \leq \ell$. Veronderstel verder dat $\delta \in k$ zodanig dat $x^2 + x + \delta$ irreduciebel is over k. Veronderstel dat γ de oplossing is van $x^2 + x = \delta$, en stel $K = k(\gamma)$ en $L = \ell(\gamma)$. Dan zijn K en L separabele kwadratische extensies van k en ℓ respectievelijk. De standaard involutie op L en K corresponderend met γ noteren we met $x \mapsto \overline{x}$.

Stel Φ een wortelsysteem van type F_4 met als fundamentele wortels $\Pi := \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Ten opzichte van een orthonormale basis $\{e_1, e_2, e_3, e_4\}$ van $\mathbb{R}^4$ kunnen deze laatste geschreven worden als $\alpha_1 = \frac{1}{2}(-e_1 - e_2 - e_3 + e_4)$, $\alpha_2 = e_3$, $\alpha_3 = e_2 - e_3$, $\alpha_4 = e_1 - e_2$ en het volledige wortelsysteem wordt gegeven door

$$\Phi = \begin{cases} \pm e_i \pm e_j \text{ met } 1 \leq i < j \leq 4, \\ \pm e_i \text{ met } 1 \leq i \leq 4, \\ \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4). \end{cases}$$

Zij $\mathsf{F}_4(L)$ de standaard Chevalley groep over L met wortelsysteem Φ, dan is

$$\mathsf{F}_4(K,L) = \Big\langle \{u_r(s) \mid r \in \Phi_\ell, s \in K\} \cup \{u_r(t) \mid r \in \Phi_s, t \in L\} \Big\rangle$$

de gemixte groep van type F_4 corresponderend met de velden (K, L).

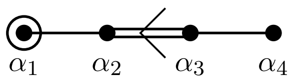
Met elke Moufang verzameling correspondeert op natuurlijke wijze een gespleten BN-paar van rang 1 en omgekeerd correspondeert met elk dergelijk BN-paar een Moufang verzameling. In wat volgt proberen we een gespleten, gesatureerd BN-paar te construeren en daaruit een voorstelling te vinden voor de Moufang verzameling in termen van een groep $(U, +)$ en een permutatie τ.

De groep $\mathsf{F}_4(K, L)$ heeft op natuurlijke wijze een BN-paar. We construeren een involutie σ op $\mathsf{F}_4(K, L)$ zodanig dat we een deelgroep vinden gefixeerd door σ die een gespleten, gesatureerd BN-paar van rang 1 heeft. De corresponderende Moufang verzameling noemen we dan een Moufang verzameling van gemixt type F_4.

Definieer de involutie σ als

$$\sigma : \mathsf{F}_4(K,L) \to \mathsf{F}_4(K,L) \colon u_r(t) \mapsto u_{\sigma(r)}(c_r\, \bar{t})$$

met c_r nog te bepalen constanten. De corresponderende actie op het wortelsysteem wordt ook met σ genoteerd. Analoog als in het algebraïsche geval, kiezen we de actie van σ op het wortelsysteem op een zodanige manier dat de corresponderende Tits index er als volgt uitziet:



In het niet-gemixte geval weten we dat we de actie van σ op het B_3-deelgebouw op een zodanige manier moeten kiezen dat de groep gefixeerd onder σ isomorf is met een $\mathsf{PGO}(Q)$, waarbij Q een anisotrope kwadratische vorm van dimensie 7 en triviale Hasse invariant is. Elke dergelijke vorm kan bekomen worden als restrictie tot het spoor-nul deel van een 8-dimensionale normvorm van een octonendelingsalgebra. Volledig analoog moet de fixpuntverzameling hier isomorf zijn met $\mathsf{PGO}(q)$, waarbij q het spoor-nul deel van de 'gemixte norm' van een octonendelingsalgebra is.

Een gemixte norm ziet er op $\mathcal{O}_{\text{mixed}} := \ell \oplus K \oplus K \oplus K \subseteq \mathcal{O}_\ell$ als volgt uit:

$$q : \mathcal{O}_{\text{mixed}} \to k; (y_1, y_2, y_3, y_4) \mapsto y_1^2 + \alpha y_2 \overline{y_2} + \beta y_3 \overline{y_3} + \alpha\beta y_4 \overline{y_4}$$

met constanten $\alpha, \beta \in k^*$.

Om $\mathsf{PGO}(q)$ als fixpuntverzameling te bekomen, moet de actie van σ op Π er als volgt uitzien:

$$\begin{cases} \alpha_1 \mapsto \frac{1}{2}(e_1+e_2+e_3+e_4) = \alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4 \\ \alpha_2 \mapsto -\alpha_2 \\ \alpha_3 \mapsto -\alpha_3 \\ \alpha_4 \mapsto -\alpha_4. \end{cases}$$

De constanten c_{α_i} horende bij de fundamentele wortels moeten gelijk zijn aan $c_{\alpha_1} = \alpha^{-1}\beta^{-1}$, $c_{\alpha_2} = \alpha\beta$, $c_{\alpha_3} = \alpha^{-1}$ en $c_{\alpha_4} = \alpha\beta^{-1}$.

We steunen voor de effectieve constructie van het gespleten, gesatureerd BN-paar sterk op een artikel van A. Steinbach [29]. In dit artikel wordt getoond hoe je via een goed gekozen involutie op een Chevalley groep een nieuw BN-paar kan construeren. We vertalen deze resultaten naar resultaten voor gemixte Chevalley groepen en vinden een BN-paar van rang 1 met als corresponderende Moufang verzameling $\mathbb{M}(U,\tau)$:

$$U := \big\{(a,b) \in \mathcal{O}_\ell \oplus \mathcal{O}_{\text{mixed}} \mid \mathrm{N}(a) + \mathrm{T}(b) = 0\big\},$$

met als groepsbewerking:

$$(a,b) + (c,d) := (a+c, b+d+g(a,c))$$

voor alle $(a,b),(c,d) \in U$. Hier is

$$g\colon \mathcal{O}_\ell \times \mathcal{O}_\ell \to L\colon \big((a_1,a_2,a_3,a_4),(b_1,b_2,b_3,b_4)\big) \mapsto \overline{a_1}b_1 + \alpha\overline{a_2}b_2 + \beta\overline{a_3}b_3 + \alpha\beta\overline{a_4}b_4.$$

De permutatie τ op U^* wordt gegeven door

$$\tau(a,b) := \Big(a \cdot \big(b+f(a)\big)^{-1}, \big(b+f(a)\big)^{-1} + f\big(a\cdot(b+f(a))^{-1}\big)\Big)$$

voor alle $(a,b) \in U$ met

$$f\colon \mathcal{O}_\ell \to \mathcal{O}_\ell\colon (a_1,a_2,a_3,a_4) \mapsto \big(\mathrm{N}(a_1,a_2,a_3,a_4), a_1a_2 + \beta\overline{a_3}a_4,\ a_1a_3 + \alpha\overline{a_2}a_4,\ a_2a_3 + \overline{a_1}a_4\big).$$

Opmerking B.3.1. We merken op dat als $k = l$, de gevonden Moufang verzameling $\mathbb{M}(U,\tau)$ isomorf is met een algebraïsche Moufang verzameling van type F_4.

B.4 Moufang deelverzamelingen van algebraïsche Moufang verzamelingen

In het laatste hoofdstuk van deze thesis bestuderen we algebraïsche Moufang verzamelingen in karakteristiek verschillend van twee. Meer algemeen stellen we ons de vraag of een Moufang deelverzameling van een algebraïsche Moufang verzameling opnieuw algebraïsch is.

Stel $\mathbb{M}(U, \tau)$ een Moufang verzameling en stel $(T, +)$ een deelgroep van $(U, +)$. Veronderstel bovendien dat T gesloten is onder de μ-afbeeldingen μ_x met x een element van T, dan noemen we $\mathbb{M}(T, \mu_x)$ een *Moufang deelverzameling* van $\mathbb{M}(U, \tau)$. Door gebruik te maken van de definitie van de μ-afbeeldingen is het eenvoudig in te zien dat het voldoende is om te eisen dat T gesloten is onder μ_x voor een $x \in T$.

B.4.1 Niet-abelse Moufang verzamelingen corresponderend met compositiedelingsalgebra's

We bestuderen Moufang verzamelingen van type ${}^2\mathsf{A}_3$, C_3, ${}^2\mathsf{A}_5$, F_4 en ${}^2\mathsf{E}_6$. We geven eerst een gemeenschappelijke beschrijving van elk van deze Moufang verzamelingen. Stel $E = k$ of een separabele kwadratische uitbreiding van k zodanig dat $\tilde{\ }$ het niet-triviale element is van $\mathrm{Gal}(E/k)$. Stel A_k een kwadratische separabele uitbreiding van k, een quaternionendelingsalgebra over k of een octonendelingsalgebra over k en veronderstel verder dat $A = A_k \otimes_k E$ een delingsalgebra blijft over E. Dan is $\tilde{\ }$ een involutie op E, die uit te breiden is tot een niet-lineair automorfisme $\tilde{\ }$ op A door de involutie toe te passen op elke coëfficiënt met betrekking tot een vaste basis van A_k. Stel vervolgens

$$U := \{(a, b) \in A \times A \mid \eta(\overline{a})a + \eta(\overline{b}) + b = 0\}$$

met η de identiteit of $\tilde{\ }$ op A. De verzameling U met optelling

$$(a, b) + (c, d) := (a + b, c + d - \eta(\overline{c})a)$$

voor alle $(a, b), (c, d) \in U$ vormt een groep. We definiëren volgende permutatie τ op U^*

$$\tau(a, b) := (-ab^{-1}, b^{-1})$$

voor alle $(a, b) \in U^*$. Dan vormt $\mathbb{M}(U, \tau)$ een algebraïsche Moufang verzameling van type ${}^2\mathsf{A}_3$, C_3, ${}^2\mathsf{A}_5$, F_4 of ${}^2\mathsf{E}_6$.

We bekomen uiteindelijk volgend resultaat:

Stelling B.4.1. *Stel $\mathbb{M}(U,\tau)$ een Moufang verzameling van type $^2\mathsf{A}_3$, C_3, $^2\mathsf{A}_5$, F_4 of $^2\mathsf{E}_6$ in karakteristiek verschillend van twee, dan is elke Moufang deelverzameling ook van dit type of gelijk aan een Moufang verzameling $\mathbb{M}(J)$ over een kwadratische Jordan delingsalgebra J.*

B.4.2 Hermitische Moufang verzamelingen

Een tweede type Moufang verzamelingen waarvan we de Moufang deelverzamelingen bestuderen, zijn de hermitische Moufang verzamelingen. Voor een beschrijving van deze Moufang verzamelingen verwijzen we naar Definitie 2.3.4, waarbij we K_σ^- door $K_0 := \{x \in K \mid x = x^\sigma\}$ en de hermitische vorm door een scheef-hermitische vorm $h : L_0 \times L_0 \to K$ vervangen. Verder stellen we

$$\tau(a,t) = (-a \cdot t^{-1}, -t^{-1}).$$

Voor dit type Moufang verzamelingen hebben we nog geen volledige classificatie van de Moufang deelverzamelingen bekomen. Meer algemeen weten we dus ook nog niet of elke Moufang deelverzameling van een hermitische Moufang verzameling van algebraïsche aard is. We vatten kort de reeds bekomen resultaten samen.

Stel $\mathbb{M}(U', \tau|_{U'})$ een Moufang deelverzameling van $\mathbb{M}(U,\tau)$, dan stellen we $V_0 := \{a \in L_0 \mid \exists t \in K : (a,t) \in U'\}$ en noteren we met D_0 de deelverzameling van K_0 corresponderend met $Z(U) \cap U'$. Verder voeren we volgende definitie in:

Definitie B.4.2. De deelverzameling V_0 *bewaart de vectorruimtestructuur van* L_0 als voor elke $(a,t) \in U'$ geldt dat $b \cdot t \in V_0$ voor elke $b \in V_0$.

We tonen vervolgens aan dat als V_0 deze vectorruimtestructuur bewaart dat $\mathbb{M}(U', \tau|_{U'})$ ook hermitisch is. Hieruit volgt direct al volgend deelresultaat:

Stelling B.4.3. *Als K een veld is, dan is elke Moufang deelverzameling (met V_0 en D_0 niet triviaal) opnieuw hermitisch.*

Ten slotte kunnen we ook het volgende aantonen:

Stelling B.4.4. *Elke Moufang deelverzameling (met V_0 en D_0 niet triviaal) waarvoor D_0 geen veld is, is opnieuw hermitisch.*

Als V_0 of D_0 triviaal is, dan is de corresponderende Moufang verzameling een Moufang verzameling corresponderend met een kwadratische Jordan delingsalgebra.

Concreet hebben we dus voor elke Moufang deelverzameling, uitgezonderd wanneer D_0 een veld is en K geen veld, gevonden dat de Moufang deelverzamelingen opnieuw hermitisch zijn of isomorf met $\mathbb{M}(J)$ waarbij J een kwadratische Jordan delingsalgebra is.

Bibliography

[1] P. Abramenko, *Lineare algebraische Gruppen: Eine elementare Einführung*, Johann Wolfgang Goethe-Universität, Frankfurt am Main, 1990.

[2] P. Abramenko and K.S. Brown, *Buildings: Theory and applications*, Graduate texts in Mathematics **248**, Springer, New York, 2008.

[3] A. Borel, Linear algebraic groups, *in* "Algebraic Groups and Discontinuous Groups," Boulder 1965, *Proc. Sympos. Pure Math.* **9** (1966), 3–19.

[4] A. Borel and J. Tits, Groupes réductifs, *Inst. Hautes Etudes Sci. Publ. Math.* **27** (1965), 55–150.

[5] E. Callens and T. De Medts, Moufang sets arising from polarities of Moufang planes over octonion division algebras, *Manuscripta Mathematica*, to appear, `http://dx.doi.org/10.1007/s00229-013-0622-1`.

[6] E. Callens and T. De Medts, Moufang sets of mixed type F_4, submitted.

[7] E. Callens and T. De Medts, web page "Moufang sets of mixed type F_4", `http://java.ugent.be/~tdemedts/mixedF4`.

[8] R. Carter, *Simple groups of Lie type.* Pure and Applied Math., Vol.28. John Wiley & Sons, London-New York-Sydney, 1972.

[9] B. Conrad, O. Gabber and G. Prasad, *Pseudo-reductive Groups*, New Mathematical Monographs **17**, Cambridge University Press, 2010.

[10] T. De Medts, Algebraic inclusions of Moufang polygons, *Forum Math.* **17** (2005), no. 6, 921–957.

[11] T. De Medts, Skew fields inside quaternion division algebras, *Mathoverflow*, `http://mathoverflow.net/questions/120777`.

[12] T. De Medts and Y. Segev, A course on Moufang sets, *Innov. Incidence Geom.* **9** (2009), 79–122.

[13] T. De Medts and H. Van Maldeghem, Moufang sets of type F_4, *Math. Z.* **265** (2010), no. 3, 511–527.

[14] T. De Medts and R.M. Weiss, Moufang sets and Jordan division algebras, *Math. Ann.* **335** (2006), no. 2, 415–433.

[15] F. Murnaghan, Linear algebraic groups, *Clay Math. proc.* **4** (2005), 379–391.

[16] J.R. Faulkner, Octonion planes defined by quadratic Jordan algebras, *Mem. Amer. Math. Soc.* **104**, no. 71, 1970.

[17] S. Garibaldi, Kneser–Tits for a rank 1 form of E_6 (after Veldkamp), *Compos. Math.* **143** (2007), 191–200.

[18] N. Knarr and M. Stroppel, Polarities and planar collineations of Moufang planes, *Monatsh. Math.* **169** (2013), no. 3-4, 383–395.

[19] N. Knarr and M. Stroppel, Baer involutions and polarities in Moufang planes of characteristic two, *Adv. Geom.*, to appear, `http://dx.doi.org/10.1515/advgeom-2012-0016`.

[20] B. Mühlherr and H. Van Maldeghem, Moufang sets from groups of mixed type, *J. Algebra* **300** (2006), no. 2, 820–833.

[21] K. McCrimmon, *A Taste of Jordan Algebras*, Springer-Verlag, Berlin, Heidelberg, New York, 2004.

[22] H.P. Petersson, The structure group of an alternative algebra, *Abh. Math. Sem. Univ. Hamburg* **72** (2002), 165–186.

[23] R. Ree, A family of simple groups associated with the simple Lie algebra of type (F_4), *Amer. J. Math.* **83** (1961), 401–420.

[24] M. Selbach, *Klassifikationstheorie halbeinfacher algebraischer Gruppen*, Bonner Mathematische Schriften, No. 83, 1976.

[25] J.-P. Serre, *Galois cohomology* (Translated from the French by Patrick Ion and revised by the author), Springer-Verlag, Berlin, 1997.

[26] T.A. Springer, *Linear Algebraic Groups,* Second edition, Progress in Mathematics **9**, Birkhäuser Boston, Inc., Boston, MA, 1998.

[27] T.A. Springer and F.D. Veldkamp, *Octonions, Jordan Algebras and Exceptional Groups*, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2000.

[28] W.A. Stein et al., *Sage Mathematics Software* (Version 4.6.1), The Sage Development Team, 2011, `http://www.sagemath.org`.

[29] A. Steinbach, Some twisted variants of Chevalley groups, *Bull. Belg. Math. Soc. Simon Stevin* **12** (2005), 491–503.

[30] A. Steinbach, Realizing Moufang quadrangles of type E_n inside Chevalley groups, preprint.

[31] J. Tits, Classification of algebraic semisimple groups, *in* "Algebraic Groups and Discontinuous Groups," Boulder 1965, *Proc. Symp. Pure Math.* **9** (1966), 33–62.

[32] J. Tits, *Buildings of spherical type and finite BN-pairs*, Lecture Notes in Math. **386**, Springer-Verlag, 1974.

[33] J. Tits, Twin buildings and groups of Kac–Moody type, in *Groups, combinatorics & geometry (Durham, 1990)*, 249–286, London Math. Soc. Lecture Note Ser. **165**, Cambridge Univ. Press, Cambridge, 1992.

[34] J. Tits and R.M. Weiss, *Moufang Polygons*, Springer Monographs in Mathematics. Springer-Verlag, Berlin, Heidelberg, New York, 2002.

[35] H. Van Maldeghem, *Generalized Polygons*, Monographs in Mathematics. Birkhäuser Verlag, Basel, 1998.

[36] F.D. Veldkamp, Unitary groups in octave planes, *Compos. Math.* **19** (1968), 213–258.