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# Immediate prediction under exchangeability & representation insensitivity

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## 1 The setting

**Sampling:** A subject makes a fixed number $N > 0$ of successive observations, represented by *random variables* $X_1, \dots, X_N$. For example, when drawing coloured balls without replacement from an urn, $X_k$ designates the unknown colour of the $k$-th ball.

**Immediate prediction:** The subject in some way uses zero or more observations $X_1, \dots, X_n$ made previously (so $n$ belongs to $\{0, 1, \dots, N-1\}$), to predict, or make inferences about, the value of the *next* observation $X_{n+1}$.

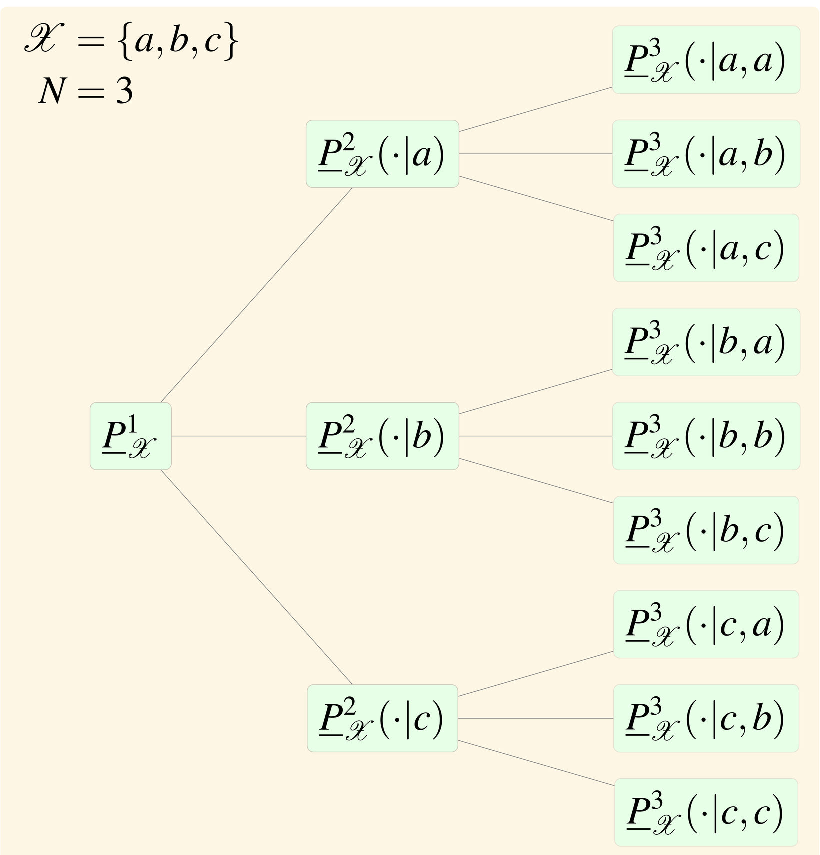
**Families of predictive lower previsions:** The subject can determine, beforehand, a finite and non-empty set $\mathscr{X}$ of possible values, or *categories*, for the random variables.

For each $n$ and each sequence $\boldsymbol{x} = (x_1, \dots, x_n)$ in $\mathscr{X}^n$, she can give a *predictive lower prevision* $\underline{P}^{n+1}_{\mathscr{X}}(\cdot|\boldsymbol{x})$ for $X_{n+1}$, given the values $(X_1, \dots, X_n) = (x_1, \dots, x_n) = \boldsymbol{x}$ of the previous observations. It is defined on the set of all gambles $f$ on $\mathscr{X}$.

| | |
|---|---|
| $\mathscr{X} = \{a,b,c\}$, $N = 3$ | Let $f(a) = 1$, $f(b) = 3$, $f(c) = -2$, then, e.g., $\underline{P}^3_{\mathscr{X}}(f\|c,a) = -\frac{1}{2}$. |
| $n = 2$, $\boldsymbol{x} = (c,a)$ | Let $A = \{a,c\}$, then, e.g., $\underline{P}^3_{\mathscr{X}}(A\|c,a) = \frac{4}{5}$. |

An $\mathscr{X}$*-family* $\sigma^N_{\mathscr{X}}$ of predictive lower previsions is the set formed for all possible observations:

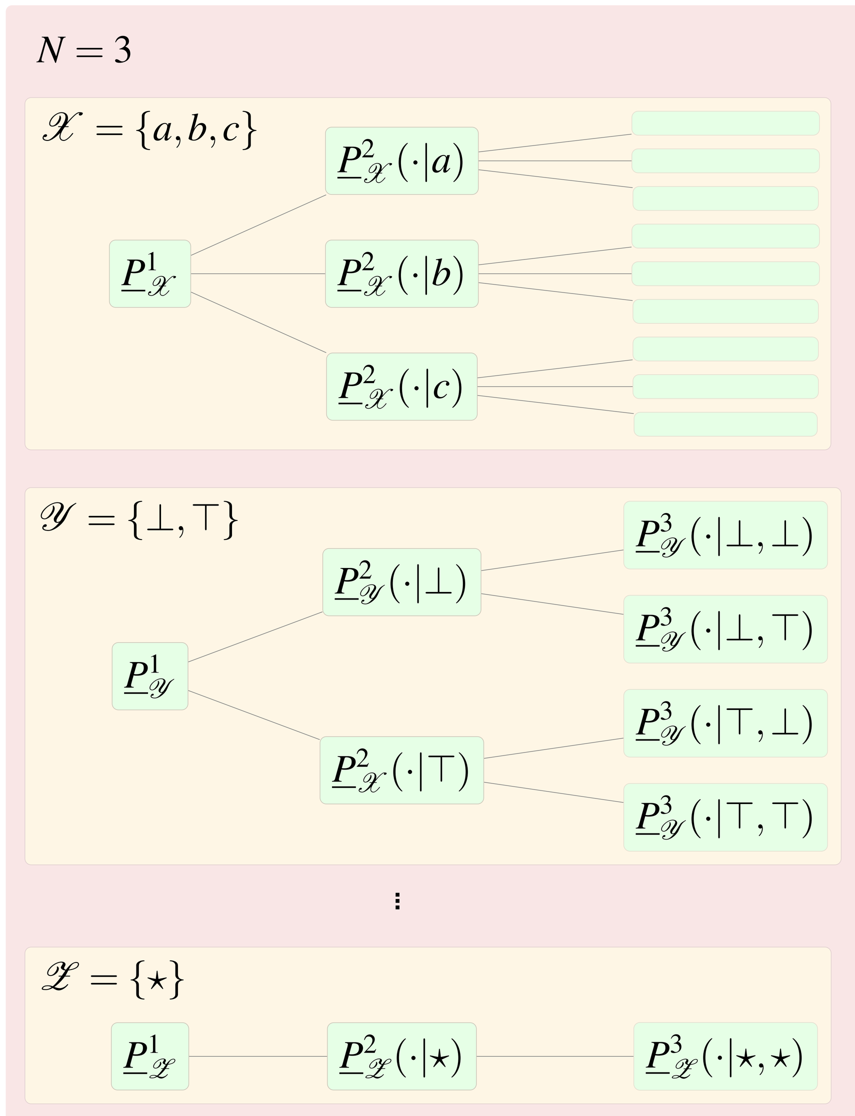
$$\sigma^N_{\mathscr{X}} := \{\underline{P}^{n+1}_{\mathscr{X}}(\cdot|\boldsymbol{x}) \colon \boldsymbol{x} \in \mathscr{X}^n \text{ and } n = 0, 1, \dots, N-1\}.$$




*Precise* predictive families are those that only contain predictive *linear* previsions $P^{n+1}_{\mathscr{X}}(\cdot|\boldsymbol{x})$. With each of these, there corresponds a predictive probability mass function. They in turn allow us, using Bayes's rule, to find the unique joint probability mass functions $p^n_{\mathscr{X}}$ on $\mathscr{X}^n$ and the corresponding *joint linear prevision* $P^N_{\mathscr{X}}$, which models beliefs about the values that the random variables $(X_1, \dots, X_N)$ assume *jointly* in $\mathscr{X}^N$.

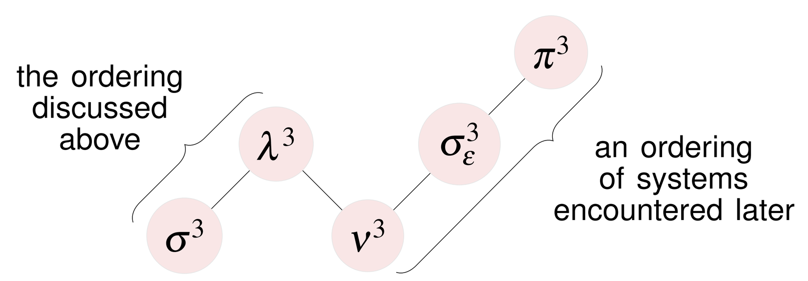
**Systems of predictive lower previsions:** The inferences or predictions of a predictive $\mathscr{X}$-family might depend on the actual choice of $\mathscr{X}$ made. So we let our subject consider predictive families for all conceivable choices of $\mathscr{X}$. We collect these families in a *system* $\sigma^N$ of predictive lower previsions:

$$\sigma^N := \{\sigma^N_{\mathscr{X}} \colon \mathscr{X} \text{ is a finite and non-empty set}\}.$$




*Precise* predictive systems are those that only contain precise predictive families.

Predictive systems can be *partially ordered*: The system $\sigma^N$ is *more conservative* than the system $\lambda^N$, if each predictive lower prevision $\underline{P}^{n+1}_{\mathscr{X}}(\cdot|\boldsymbol{x})$ in $\sigma^N$ is point-wise dominated by the corresponding predictive lower prevision $\underline{Q}^{n+1}_{\mathscr{X}}(\cdot|\boldsymbol{x})$ in $\lambda^N$.




A collection $\{\sigma^N_\gamma \colon \gamma \in \Gamma\}$ of predictive systems may have an infimum with respect to this partial order. Whenever it exists, this infimum system $\sigma^N$ can be seen as a *lower envelope*: each of its predictive lower previsions $\underline{P}^{n+1}_{\mathscr{X}}(\cdot|\boldsymbol{x})$ is defined as the lower envelope $\inf_{\gamma\in\Gamma} \underline{P}^{n+1}_{\mathscr{X},\gamma}(\cdot|\boldsymbol{x})$ of the predictive lower previsions in the predictive systems $\sigma^N_\gamma$.

### Acknowledgements

Jean-Marc Bernard, Frank-Coolen, Thomas Augustin, and the reviewers.

### Selected references

G. de Cooman and E. Miranda. Symmetry of models versus models of symmetry. In W. L. Harper and G. R. Wheeler, editors, *Probability and Inference: Essays in Honor of Henry E. Kyburg, Jr.*, pages 67–149. 2007.

P. Walley and J.-M. Bernard. Imprecise probabilistic prediction for categorical data. Technical Report CAF-9901, Université de Paris 8, 1999.

S. L. Zabell. W. E. Johnson's "sufficientness" postulate. *The Annals of Statistics*, 10:1090–1099, 1982.

## 2 Requirements & Assumptions

**Coherence:** Coherence is a requirement on the individual predictive lower previsions.

A predictive system is called *coherent* if it is the lower envelope of a collection of precise predictive systems. This is equivalent to requiring that all the predictive lower previsions $\underline{P}^{n+1}_{\mathscr{X}}(\cdot|\boldsymbol{x})$ in the system should be separately coherent.

**(Regular) exchangeability:** Exchangeability is an assumption about a family of predictive lower previsions.

A precise predictive system is *exchangeable* if all the associated joint linear previsions $P^N_{\mathscr{X}}$ are exchangeable, i.e., invariant under permutation of the random variables $X_1, \dots, X_N$.

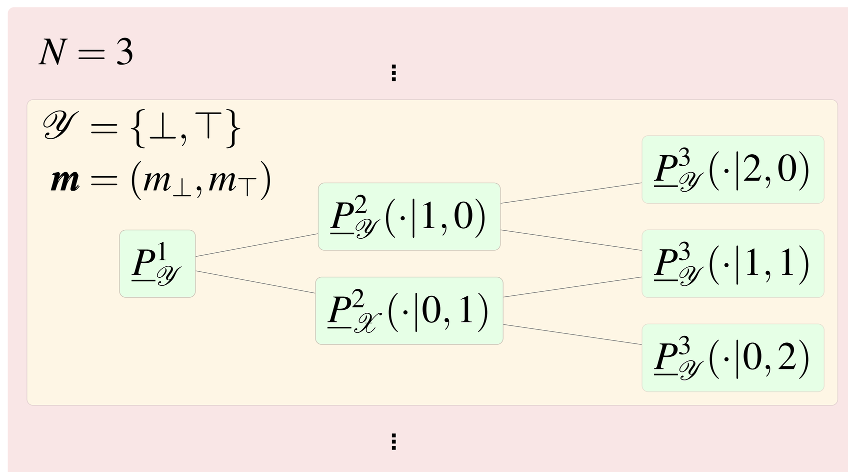
A general predictive system is called *exchangeable* if it is the lower envelope of a collection $\{\sigma^N_\gamma \colon \gamma \in \Gamma\}$ of exchangeable precise predictive systems. It is *regularly exchangeable* if all predictive linear previsions $P^{n+1}_{\mathscr{X},\gamma}(\cdot|\boldsymbol{x})$ in each of these systems $\sigma^N_\gamma$ can be uniquely derived from the joint linear prevision $P^N_{\mathscr{X},\gamma}$ by applying Bayes's rule. (For this, the joint mass functions $p^n_{\mathscr{X},\gamma}$ should be strictly positive for $n < N$.)

## 3 Some results

**From sequences of observations to count vectors:** In any regularly exchangeable predictive system, the predictive lower previsions $\underline{P}^{n+1}_{\mathscr{X}}(\cdot|\boldsymbol{x})$ only depend on the sequence of observations $\boldsymbol{x}$ through its *count vector* $\boldsymbol{m} \in \mathscr{N}^n_{\mathscr{X}}$, with

$$m_z := |\{k \in \{1, \dots, n\} \colon x_k = z\}|,$$
$$\mathscr{N}^n_{\mathscr{X}} := \{\boldsymbol{m} \in \mathbb{N}^{\mathscr{X}}_0 \colon \textstyle\sum_{z\in\mathscr{X}} m_z = n\}.$$

All predictive lower previsions for given sequences with the same count vector $\boldsymbol{m}$ can therefore be written as $\underline{P}^{n+1}_{\mathscr{X}}(\cdot|\boldsymbol{m})$.




So, for regularly exchangeable predictive systems, count vectors are a *sufficient statistic*. From now on, we only consider (possibly non-exchangeable) predictive systems for which this is the case.

**A useful (in)equality** In any regularly exchangeable predictive system, it holds for all gambles $f$ that

$$\underline{P}^{n+1}_{\mathscr{X}}(f|\boldsymbol{m}) \geq \underline{P}^{n+1}_{\mathscr{X}}(\underline{P}^{n+2}_{\mathscr{X}}(f|\boldsymbol{m}+\boldsymbol{e}_{\cdot})|\boldsymbol{m}),$$

where $n \leq N-2$ and $\boldsymbol{e}_x \in \mathscr{N}^1_{\mathscr{X}}$ for $x \in \mathscr{X}$ such that, using the Kronecker delta, $(\boldsymbol{e}_x)_z = \delta_{xz}$. For precise regularly exchangeable predictive systems, this becomes a 'useful equality'.

## 4 Some more requirements

**Representation insensitivity:** Representation insensitivity is a requirement that works between predictive lower previsions for the same number of observations.

It comprises three invariance requirements:

- *pooling invariance*: inferences that do not depend on the distinction between some categories should stay the same when those categories are pooled;
- *renaming invariance*: apart from avoiding confusion, the names of the categories should not matter;
- *category permutation invariance*: in a state of prior ignorance, which we consider here, the subject has no reason to distinguish between the categories, so the inferences should be invariant under a permutation of them.

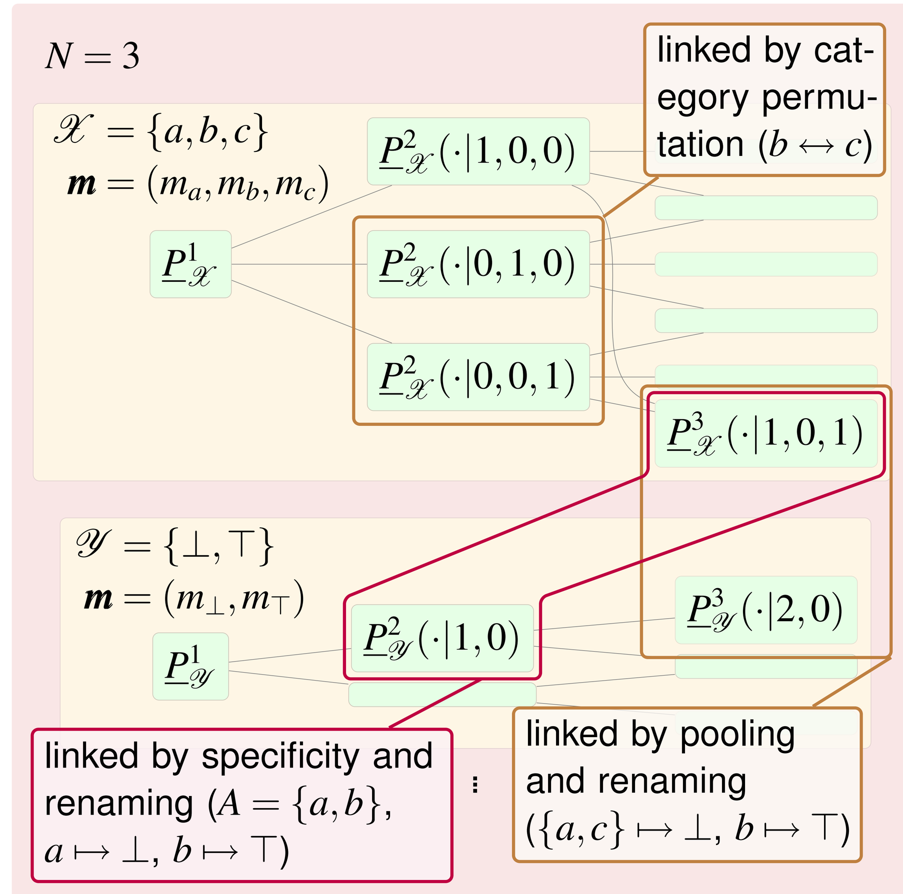
Combining these, we can say a predictive system is *representation insensitive* if for all $n$, for any category sets $\mathscr{X}$ and $\mathscr{Y}$, for any $\boldsymbol{m} \in \mathscr{N}^n_{\mathscr{X}}$ and $\boldsymbol{m}' \in \mathscr{N}^n_{\mathscr{Y}}$, and for any gambles $f$ on $\mathscr{X}$ and $g$ on $\mathscr{Y}$ with identical ranges, the following holds:

$$\boldsymbol{m}^f = \boldsymbol{m}'^g \Rightarrow \underline{P}^{n+1}_{\mathscr{X}}(f|\boldsymbol{m}) = \underline{P}^{n+1}_{\mathscr{Y}}(g|\boldsymbol{m}'),$$

with $m^f_r := \sum_{f(x)=r} m_x$. This means $\underline{P}^{n+1}_{\mathscr{X}}(f|\boldsymbol{m})$ only depends on the values that $f$ may assume, and on the number of times each value has been observed:

$$\underline{P}^{n+1}_{\mathscr{X}}(f|\boldsymbol{m}) = \underline{P}^{n+1}_{f(\mathscr{X})}(\mathrm{id}_{f(\mathscr{X})}|\boldsymbol{m}^f),$$

where $\mathrm{id}_{f(\mathscr{X})}$ is the identity map on the range of $f$.




**Specificity (optional):** Specificity is a requirement that works between predictive lower previsions for a different number of observations related by pooling.

An exchangeable predictive system is *specific* if for all gambles $f$ and all non-trivial events $A$ in $\mathscr{X}$ containing a non-zero number $m_A$ of observations, it holds that

$$\underline{P}^{n+1}_{\mathscr{X}}(f|\boldsymbol{m}, A) = \underline{P}^{m_A+1}_{A}(f_A|\boldsymbol{m}_A),$$

where $f_A$ and $\boldsymbol{m}_A$ are the restriction of $f$ and $\boldsymbol{m}$ to $A$. So, knowing that the $(n+1)$-th observation belongs to $A$ allows you to ignore all the previous observations that lie outside $A$.

## 5 More results

**The lower probability function:** With any predictive system we associate a map $\varphi$ defined for all $n$ and $k \leq n$ by

$$\varphi(n,k) := \underline{P}^{n+1}_{\{0,1\}}(\mathrm{id}_{\{0,1\}}|n-k, k).$$

For representation insensitive systems it fully characterizes all predictive lower probabilities (cfr. *Johnson's sufficientness postulate*) and is therefore called the *lower probability function*; to wit, let $A$ be some event, and $m_A$ the associated number of observations, then

$$\underline{P}^{n+1}_{\mathscr{X}}(A|\boldsymbol{m}) = \underline{P}^{n+1}_{\{0,1\}}(\mathrm{id}_{\{0,1\}}|n-m_A, m_A) = \varphi(n, m_A).$$

It allows us to draw intuitively appealing conclusions, which are valid in any coherent representation insensitive system:

(i) the lower/upper probability of observing an event that has not/always been observed before is zero/one;

(ii) if $n$ remains fixed, then both the lower and upper probability of observing $A$ again do not decrease if $m_A$ increases;

(iii) in systems that are also regularly exchangeable: if $m_A$ remains the same as $n$ increases, then the lower probability for observing $A$ again does not increase.

**Some representation insensitive exchangeable systems:** To start: all the $\underline{P}^1_{\mathscr{X}}$ in a representation insensitive and exchangeable predictive system must be vacuous.

A subject that is too conservative to learn uses the regularly exchangeable *vacuous predictive system* $\nu^N$. All its predictive lower previsions are vacuous, so $\underline{P}^{n+1}_{\mathscr{X}}(f|\boldsymbol{m}) := \min f$.

A subject that believes that categories unobserved in the past remain so in the future, uses the (not regularly) exchangeable *Haldane predictive system* $\pi^N$. For $n > 0$, all its predictive previsions are linear and strongly tied to the observations:

$$P^{n+1}_{\mathscr{X}}(f|\boldsymbol{m}) = S^{n+1}_{\mathscr{X}}(f|\boldsymbol{m}) := \textstyle\sum_{z\in\mathscr{X}} f(z)\frac{m_z}{n}.$$

Other systems can be formed as convex mixtures of the two extreme ones above. We define *mixing predictive systems* $\sigma^N_\varepsilon$ with a $[0,1]$-bounded *mixing sequence* $\varepsilon$ of length $N$ and

$$\underline{P}^{n+1}_{\mathscr{X}}(f|\boldsymbol{m}) := \varepsilon_n S^{n+1}_{\mathscr{X}}(f|\boldsymbol{m}) + (1-\varepsilon_n)\min f;$$

note that implicitly $\varepsilon_0 = 0$. Representation insensitivity is retained after mixing; a sufficient condition for regular exchangeability is the reformulated 'useful inequality' $\frac{\varepsilon_n}{n} \geq \frac{\varepsilon_{n+1}}{n+1}\left(1 + \frac{\varepsilon_n}{n}\right)$.

The lower probability function of a mixing system is given by $\varphi(n,k) = \varepsilon_n\frac{k}{n}$. So, as $\varepsilon_n = n\varphi(n,1)$, a mixing system can be defined by specifying the lower probability of observing any non-trivial event that has been observed once in $n$ trials; it is the most conservative system with these lower probabilities.

**The imprecise Dirichlet-Multinomial model:** Any mixing system that is specific or for which the 'useful equality' holds, is uniquely characterized by some $s > 0$ such that $\varepsilon_n = \frac{n}{n+s}$ and thus

$$\underline{P}^{n+1}_{\mathscr{X}}(f|\boldsymbol{m}) = \frac{n}{n+s} S^{n+1}_{\mathscr{X}}(f|\boldsymbol{m}) + \frac{s}{n+s}\min f.$$

This regularly exchangeable representation insensitive predictive system is related to the imprecise Dirichlet-Multinomial model with hyper-parameter $s$.