lengths of fiber ranging from 50–500 m at a data rate of 370 KBs per second. The data rate along with signal strength and quality was monitored using software supplied with Wi-Fi cards. RF amplification in both sides of the link was required and the ranges achieved were still quite short, typically 5–10 cm. The laptops were approximately 30-m apart physically and thus care had to be taken in order to ensure that a conventional wireless link was not being formed between them. This is easily confirmed by breaking one arm of the optical link and observing the breaking of the wireless optical link.

3. CONCLUSION

This paper has shown the integration of both a VCSEL and a photodiode with microstrip patch antennas. They were individually shown to have sufficient bandwidth to operate as part of a 2.4-GHz wireless-over-fiber link. The VCSEL antenna has been incorporated into a fully bidirectional link and data have been successfully transmitted between two wireless-enabled laptops. It is hoped that with optimization of the VCSEL and photodiode position on the patch antennas, the inclusion of matching circuits, and the use of MMF for improved coupling efficiency, increased range can be achieved. The use of an integrated VCSEL driver and a transimpedance amplifier after the photodiode is also being studied.

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A NORMALIZED PLANE-WAVE METHOD FOR 2D HELMHOLTZ PROBLEMS

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ABSTRACT: A new technique to diagonalize the 2D Green's function is presented. The new method, called the normalized plane-wave method, is an extension of the plane-wave method into the low-frequency regime. As such, it provides a way to achieve $\mathcal{O}(\mathcal{N})$ complexity for both low- as well as high-frequency problems without having to resort to inhomogeneous plane-wave methods. The accuracy of the proposed method is thoroughly tested and it is shown that it can be made as high as one may reasonably want in any of the frequency regimes. Furthermore, it is shown to be faster than the traditional fast-multipole method (FMM) for low-frequency problems, by virtue of diagonal translations. © 2005 Wiley Periodicals, Inc. *Microwave Opt Technol Lett* 48: 237–243, 2006; Published online in Wiley InterScience (www.interscience.wiley.com). DOI 10.1002/mop.21316

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1. INTRODUCTION

The basic problem addressed in this paper is the 2D scattering of electromagnetic waves, that is, a 2D Helmholtz problem. This scattering problem can always be written as an integral equation in which a convolution occurs with the 2D Green's function. All sources and fields are assumed time-harmonic; temporal dependencies $e^{j\omega t}$ are suppressed. Therefore, the 2D Green's function is given by the cylindrical Hankel function of the 2nd kind and 0th order. The method of moments (MoM) reduces the continuous integral equation to a linear system of equations of dimension $\mathcal{N}$. However, often problems involve so many unknowns that direct methods to solve the linear system, for example, LU decomposition, are too slow because of their $\mathcal{O}(\mathcal{N}^3)$ computational complexity. The brute-force application of iterative solvers, for example, the biconjugate gradient method, permits us to solve the linear system in $\mathcal{O}(P\mathcal{N}^2)$ operations, where P is the number of iterations required by the iterative solver in order to reduce the residual error below a preset threshold. The computational cost can be reduced dramatically by applying a fast multiplication scheme, for example, the multilevel fast multipole algorithm [1, 2], which is the multilevel version of the fast-multipole method (FMM). At this point, a distinction has to be made between low-frequency problems (for example, microwave integrated circuits), which are physically small compared to the wavelength but require many unknowns because of the small geometrical details, and high-frequency problems (for example, antenna arrays), which are large in comparison to the wavelength.

Consider the configuration depicted in Figure 1, comprising a source and an observation point residing at ρ_i and ρ_j , which are part of square source and observation boxes centered about ρ_l and ρ_{ll} , respectively. The core of the 2D FMM is the multipole expansion of the Hankel function [2]:

$$H_0^{(2)}(\gamma\rho_{ji}) \approx \sum_{n=-K}^K \sum_{m=-K}^K [\bar{\beta}(\rho_{jl})]_{0n} [\bar{\alpha}(\rho_{li})]_{nm} [\bar{\beta}(\rho_{li})]_{m0}, \quad (1)$$

Figure 1 Arrangement of the position vectors for two interacting groups

where ρ_{ab} is a shorthand notation for $\rho_b - \rho_a$, ρ and ϕ are defined by $\rho = \rho \cos \phi u_x + \rho \sin \phi u_y$, and $\gamma = \omega \sqrt{\epsilon \mu}$ is the wavenumber; K is the multipole order above which the series is truncated. The $\bar{\alpha}$ and $\bar{\beta}$ matrices are defined in terms of Hankel and Bessel functions, respectively, as

$$[\bar{\alpha}(\rho)]_{nm} = H_{n-m}^{(2)}(\gamma\rho) e^{j(n-m)\phi}, \quad (2a)$$

$$[\bar{\beta}(\rho)]_{nm} = J_{n-m}(\gamma\rho) e^{j(n-m)\phi}. \quad (2b)$$

Eq. (1) can be generalized to the expansion of higher order Hankel functions as follows:

$$\bar{\alpha}(\rho_{ji}) = \underbrace{\bar{\beta}(\rho_{jl})}_{\text{Disaggregation}} \cdot \underbrace{\bar{\alpha}(\rho_{li})}_{\text{Translation}} \cdot \underbrace{\bar{\beta}(\rho_{il})}_{\text{Aggregation}} \quad (3)$$

For low-frequency problems, the implementation of Eq. (3) into a multilevel scheme immediately leads to a matrix-vector multiplication with a computational complexity that scales as $\mathcal{O}(\mathcal{N})$. For very-low-frequency problems, that is, very small $\gamma\rho$, the Hankel and Bessel functions in Eqs. (2a) and (2b) become very large and very small, respectively, thus resulting in overflow and underflow errors. However, this problem can be easily solved using an appropriate normalization [2]. Although the computational cost scales as $\mathcal{O}(\mathcal{N})$, this scheme is “slow,” that is, has a large pre-factor, because translations and (dis)aggregations require matrix-vector multiplications with dense matrices $\bar{\alpha}$ and $\bar{\beta}$.

For the same reason, the application of Eq. (3) does not lead to a lower computational complexity for high-frequency problems. Indeed, for these problems, the required number of multipoles increases for higher levels according to the formula $K \approx \gamma R + B$, where R is the radius of the circumscribed circle of the square boxes on that level and B is a constant. The cost needed to multiply the dense $\bar{\alpha}$ or $\bar{\beta}$ matrices with a vector is proportional to the square of the number of multipoles. As a consequence, the computational complexity remains $\mathcal{O}(\mathcal{N}^2)$ [2]. To obtain a lower computational complexity for high-frequency problems, it is necessary to use diagonal aggregation, translation, and disaggregation matrices. This can be achieved by expanding the Hankel function into plane waves (that is, the plane wave method [1, 3]):

$$H_0^{(2)}(\gamma\rho_{ji}) = \frac{1}{2Q+1} \sum_{q=-Q}^Q e^{j\gamma\rho_{jl}\cos(\alpha_q - \phi_{jl})} T_q(\gamma, \rho_{li}, \phi_{li}) e^{j\gamma\rho_{il}\cos(\alpha_q - \phi_{il})}, \quad (4)$$

$$T_q(\gamma, \rho, \phi) = \sum_{p=-Q}^Q H_p^{(2)}(\gamma\rho) e^{jp(\phi + (\pi/2) - \alpha_q)}, \quad (5)$$

$$\alpha_q = \frac{2\pi q}{2Q+1}. \quad (6)$$

Similarly to the number of multipoles, the number of plane waves Q is related to the box size R through $Q \approx 2\gamma R + B'$, where B' is another constant.

To reduce the prefactor for low-frequency problems, it would be very convenient to have diagonal operators for that case as well. Unfortunately, the plane-wave method (4) breaks down at low frequencies because of a numerical instability caused by roundoff errors. Indeed, $H_p^{(2)}(\gamma\rho)$ diverges exponentially as a function of $|p|$ if $|p| > \gamma\rho$. Hence, in Eq. (5), very large numbers, that is, $H_p^{(2)}(\gamma\rho)$ for large $|p|$, are added to numbers of the order of unity, that is, $H_p^{(2)}(\gamma\rho)$ for small $|p|$, and the information given by the smallest numbers is lost. In [4, 5], this problem was tackled using inhomogeneous plane waves. In this paper, an alternative and conceptually simpler scheme is presented. In addition, the proposed scheme uses a lot of the machinery from the original low-frequency FMM, which means that existing low-frequency FMM solvers can be easily modified.

2. THE PLANE-WAVE METHOD

In this section, an alternative formulation of the plane-wave method is derived, which allows us to remove the numerical instability mentioned at the end of section 1. This formulation starts directly from the truncated form of the addition theorem given by Eq. (1). A translation in the FMM corresponds to calculating $\forall n \in [-K, K]$ somewhat of the following form:

$$\begin{aligned} \Phi_n &= \sum_{m=-K}^K [\bar{\alpha}(\rho_{li})]_{nm} [\bar{\beta}(\rho_{il})]_{m0} \\ &= \sum_{m=-K}^K H_{n-m}^{(2)}(\gamma\rho_{li}) e^{j(n-m)\phi_{li}} J_m(\gamma\rho_{il}) e^{jm\phi_{il}}. \end{aligned} \quad (7)$$

As the matrix $\bar{\alpha}(\rho_{li})$ clearly is Toeplitz, the required matrix-vector multiplication can be performed using fast Fourier transforms (FFTs). To show this, consider the following identities:

$$1 \equiv \sum_{k=-2K}^{2K} \delta_{k,n-m} \quad \forall n, m \in [-K, K], \quad (8)$$

$$\delta_{k,n-m} \equiv \frac{1}{4K+1} \sum_{q=-2K}^{2K} e^{-j(2\pi q/4K+1)[k-(n-m)]} \quad \forall n, m \in [-K, K], \quad (9)$$

with $\delta_{i,j}$ the Kronecker delta. By means of Eq. (8), Eq. (7) can be re-expressed as

$$\Phi_n = \sum_{m=-K}^K \sum_{k=-2K}^{2K} \delta_{k,n-m} H_k^{(2)}(\gamma\rho_{li}) e^{jk\phi_{li}} J_m(\gamma\rho_{il}) e^{jm\phi_{il}}. \quad (10)$$

Replacing $\delta_{k,n-m}$ by the sum in Eq. (9) and exchanging some summation signs, a diagonalized form for the translation operator is obtained:

$$\Phi_n = \sum_{q=-2K}^{2K} \left[\frac{1}{4K+1} \sum_{k=-2K}^{2K} H_k^{(2)}(\gamma\rho_{r'l}) e^{jk\phi_{r'l}} e^{-j(2\pi qk/4K+1)} \right] \times \left[\sum_{m=-K}^K J_m(\gamma\rho_{li}) e^{jm\phi_{li}} e^{-j(2\pi qm/4K+1)} \right] e^{j(2\pi qn/4K+1)}. \quad (11)$$

All the sums in Eq. (11) can be calculated using FFTs and, what is even more important, the translations themselves are pointwise multiplications of two functions of q . It is worthwhile to point out that formula (11) has the same physical meaning and computational cost as the plane-wave method, but that it retains multipoles rather than plane waves as the basis functions for the radiation patterns.

However, this alternative formulation of the plane-wave expansion still suffers from the same numerical instability as the original version (4). When looking at the summation over k in Eq. (11), it is obvious that, as $H_k^{(2)}(\gamma\rho)$ becomes exponentially larger as a function of k , the lower-order terms become swamped in the higher-order terms. A similar swamping, but this time of the higher-order multipoles, occurs in the sum over m . These numerical instabilities eventually lead to the complete failure of Eq. (11) for low frequencies. In the original formulation of the plane-wave method (4), this cannot be corrected [3]. However, the new formulation will allow us to derive an expansion that is diagonal and stable at low frequencies.

3. A NORMALIZED PLANE-WAVE METHOD

Next we will show how the translation operator can be stably diagonalized in the low-frequency case. Identity (8) can be rewritten as

$$1 \equiv \sum_{k=-2K}^{-1} \delta_{k,n-m} t^{|k|} t^{(n-m)} + \sum_{k=0}^{2K} \delta_{k,n-m} t^{|k|} t^{-(n-m)} \quad \forall n, m \in [-K, K], \quad (12)$$

where t is still a freely chosen number called the normalization factor. This new form of Eq. (8) has been opted for because of the factor $t^{|k|}$ that also occurs in the asymptotic behavior of $H_k^{(2)}(x)$ if x is small ($x < 1$):

$$|H_k^{(2)}(x)| \approx \begin{cases} \frac{2^{|k|}(|k|-1)!}{\pi x^{|k|}} & \forall k \neq 0 \\ \frac{2}{\pi} \ln x & k = 0 \end{cases}. \quad (13)$$

Following the same steps as in the previous section with (12) instead of (8), the following expression for the matrix-vector multiplication is found:

$$\Phi_n = \sum_{q=-2K}^{2K} \left[\sum_{k=-2K}^{-1} \frac{1}{4K+1} H_k^{(2)}(\gamma\rho_{r'l}) t^{|k|} e^{jk\phi_{r'l}} e^{-j(2\pi qk/4K+1)} \right] \times \left[\sum_{m=-K}^K J_m(\gamma\rho_{li}) e^{jm\phi_{li}} t^{-m} e^{-j(2\pi qm/4K+1)} \right] t^n e^{j(2\pi qn/4K+1)}$$

$$+ \sum_{q=-2K}^{2K} \left[\sum_{k=0}^{2K} \frac{1}{4K+1} H_k^{(2)}(\gamma\rho_{r'l}) t^{|k|} e^{jk\phi_{r'l}} e^{-j(2\pi qk/4K+1)} \right] \times \left[\sum_{m=-K}^K J_m(\gamma\rho_{li}) e^{jm\phi_{li}} t^m e^{-j(2\pi qm/4K+1)} \right] t^{-n} e^{j(2\pi qn/4K+1)}. \quad (14)$$

This new form (14) can be made numerically stable with an appropriate choice of the normalization factor t . Indeed, every Hankel function is accompanied by a factor $t^{|k|}$, thus allowing it to be normalized. A remarkable feature of (14) is the fact that a good normalization for $H_k^{(2)}(x)$ also normalizes $J_m(x)$ relatively well. Consider the asymptotic expansion of the Bessel function for small x :

$$|J_m(x)| \approx \frac{x^{|m|}}{2^{|m|}|m|!}. \quad (15)$$

In the first term of expression (14), the summation over m is normalized for $m \geq 0$. The $J_m(x)$ for which $m < 0$, are “anti-normalized,” resulting in their rapid loss. However, as can be seen in (7), the terms for which $m < 0$ also very rapidly lose importance as $|m|$ increases. A detailed analysis shows that the loss of the $J_m(x)$ is approximately compensated by a comparable loss of importance of these terms in (7). Here, “approximately” means that the net loss does not depend on $\gamma\rho$. In the second term of (14), the $J_m(x)$ values for which $m \leq 0$ are normalized and the others are anti-normalized. For the same reason, this does not result in a dramatic loss of precision and the numerical instability associated with the summations over m in (14) is avoided.

In addition, the matrix-vector-multiplications associated with the aggregation and disaggregation steps in a multilevel scheme can also be diagonalized in a similar way by using two variations of (8) as follows.

For the aggregations, the following kind of matrix-vector multiplication must be done:

$$\Phi_n = \sum_{m=-K}^K [\bar{\beta}(\rho_{r'l})]_{nm} [\bar{\beta}(\rho_{li})]_{m0}, \quad (16)$$

$$= \sum_{m=-K}^K J_{n-m}(\gamma\rho_{r'l}) e^{j(n-m)\phi_{r'l}} J_m(\gamma\rho_{li}) e^{jm\phi_{li}}. \quad (17)$$

The $\bar{\beta}(\rho_{r'l})$ matrices are also Toeplitz, but the Bessel functions decay exponentially as their order increases. Because of this, the powers of t have to be rearranged as follows:

$$1 \equiv \begin{cases} t^{|n|} \sum_{k=-2K}^{2K} \delta_{k,n-m} t^{-(k+m)} & \forall n \geq 0 \\ t^{|n|} \sum_{k=-2K}^{2K} \delta_{k,n-m} t^{(k+m)} & \forall n < 0 \end{cases}, \quad (18)$$

which yields the following diagonal formula:

$$\Phi_n = \begin{cases} t^{|n|} \sum_{q=-2K}^{2K} \left[\sum_{k=-2K}^{2K} \frac{1}{4K+1} J_k(\gamma \rho_{rl}) t^{-k} e^{jk\phi_{rl}} e^{-j(2\pi qk/4K+1)} \right] \\ \times \left[\sum_{m=-K}^K J_m(\gamma \rho_{ll}) e^{jm\phi_{ll}} t^{-m} e^{-j(2\pi qm/4K+1)} \right] e^{j(2\pi qn/4K+1)} & \forall n \geq 0 \\ t^{|n|} \sum_{q=-2K}^{2K} \left[\sum_{k=-2K}^{2K} \frac{1}{4K+1} J_k(\gamma \rho_{rl}) t^k e^{jk\phi_{rl}} e^{-j(2\pi qk/4K+1)} \right] \\ \times \left[\sum_{m=-K}^K J_m(\gamma \rho_{ll}) e^{jm\phi_{ll}} t^m e^{-j(2\pi qm/4K+1)} \right] e^{j(2\pi qn/4K+1)} & \forall n < 0 \end{cases} \quad (19)$$

For the disaggregations, a similar approach can be applied. Since the multipole expansions that are to be disaggregated increase with growing multipole order, another form of (8) is in order:

$$1 \equiv \begin{cases} \sum_{k=-2K}^{2K} \delta_{k,n-m} t^{-(n-k)} t^{|m|} & \forall m \geq 0 \\ \sum_{k=-2K}^{2K} \delta_{k,n-m} t^{(n-k)} t^{|m|} & \forall m < 0 \end{cases} \quad (20)$$

4. DETERMINATION OF THE OPTIMAL NORMALIZATION FACTOR

Having derived normalized diagonal forms for the aggregation, translation, and disaggregation operators, the task ahead is the selection of a suitable, if not optimal, normalization factor t . First, we consider the following sum, arising in the formula (14) for the diagonal translation:

$$\sum_{k=0}^{2K} H_k^{(2)}(\gamma \rho) t^{|k|} e^{jk\phi} e^{-j(2\pi k/4K+1)}. \quad (21)$$

In comparison to the other sums, this sum is much more prone to numerical instability, mostly because the Hankel functions $H_k^{(2)}(\gamma \rho)$ increase with increasing k . As a consequence, the lowest-order terms which contribute the most in the addition theorem (3) are lost in the higher-order terms. This suggests that (21) is the most "dangerous" summation, and thus the one upon which to base the derivation of t . By choosing t as

$$t = \left| \frac{H_0^{(2)}(\gamma \rho)}{H_{2K}^{(2)}(\gamma \rho)} \right|^{1/2K}, \quad (22)$$

the highest- and lowest-order terms in (21) are of the same magnitude. It is worthwhile to point out that the asymptotic expression (13) for $H_k^{(2)}(\gamma \rho)$ contains the factorial $(|k| - 1)!$. A factorial never can be completely compensated with an exponential function and this implies that $t^{|k|} H_k^{(2)}(\gamma \rho)$ possesses a minimum at $k = \pm k_{min}$. A straightforward calculation shows that this minimum is located at:

$$k_{min} = \left\lceil \frac{\gamma \rho}{2t} \right\rceil \quad (23)$$

for small $\gamma \rho$ ($\gamma \rho < 1$). Here, $\lceil \cdot \rceil$ is the ceiling function. Since the largest term in (21) is $H_0^{(2)}(\gamma \rho)$, which is of about the same magnitude as unity, the ratio of the smallest and largest term is approximately equal to

$$t^{\lceil \gamma \rho / 2t \rceil} H_{\lceil \gamma \rho / 2t \rceil}^{(2)}(\gamma \rho) \approx \sqrt{\frac{2}{\pi}} \frac{\gamma \rho}{2t} e^{-(\gamma \rho / 2t)}. \quad (24)$$

To obtain (24), expression (13) was used, together with Stirling's formula and the fact that $\lceil \gamma \rho / 2t \rceil \approx \gamma \rho / 2t$. Also, expression (22) for t can be rudely approximated as (again for small $\gamma \rho$ and large K):

$$t \approx \frac{e \gamma \rho}{4K}, \quad (25)$$

where e is Euler's number. From this it is clear that the ratio of the smallest and largest terms, and thus the loss of precision, depends only upon the number of multipoles used and not on the argument $\gamma \rho$:

$$\sqrt{\frac{2}{\pi}} \frac{\gamma \rho}{2t} e^{-(\gamma \rho / 2t)} \approx \sqrt{\frac{e}{\pi K}} e^{-(2K/e)}. \quad (26)$$

For K equal to 25, this yields a maximum loss of precision of about 10 digits for the multipole orders $k = \pm k_{min}$. This seems to be quite a lot but, as will be shown further on, the loss of precision for the addition theorem (1) will turn out to be much less than this because the higher-order terms (the ones close to k_{min}) contribute much less to the 2D Green's function than the lower-order terms. Hence they do not require the same accuracy.

Now, if the same value for t is used on an entire level, which is desirable because it eliminates the need to do the FFTs for every separate translation, it cannot be expected that all of the diagonal operators are fully normalized. For example, consider the situation depicted in Figure 2. Translations 1 and 2 translate the multipole pattern over a distance ρ_1 and ρ_2 , respectively. These distances satisfy the following relation:

Figure 2 The two most extreme translations in 2D FMM

Figure 3 Geometry for accuracy testing

$$\frac{\rho_1}{\rho_2} = \frac{\sqrt{2}}{3}. \quad (27)$$

If t is chosen to be optimal for translation 2 according to formula (22), then t will not be optimal for translation 1. At this point, it is important to realize that translation 2 requires less multipoles than translation 1 to achieve the same accuracy. This fact can be exploited by choosing t as optimal for the translation over the shortest translation distance, that is, translation 1. According to (22), this normalization factor will be too small for translation 2, resulting in an “overnormalized” translation operator of which the multipole components decay with increasing multipole order. However, the loss of the higher-order terms does not result in a loss of precision because we do not need as many multipoles for translation 2 as for translation 1.

This same value for t can also be used for the aggregations (18) and disaggregations (20). Since all translations on one level are over a distance at least twice as large as the radius of a box on that level, the diagonal aggregation and disaggregation operators will be undernormalized, again resulting in the favorable situation of decaying multipole components. Due to the fact that a value for t can be found which turns all multipole patterns occurring on a certain level in mildly decaying multipole patterns, the error is controllable over a wide interval of possible values for K . This is illustrated in the next section.

5. RESULTS

The method described above has been numerically tested on the situation sketched in Figure 3. This figure shows a two-level interaction between ρ_i and ρ_j , corresponding to the following expansion of the Green's function:

$$H_0^{(2)ADD}(\gamma\rho_{ji}) \approx \sum_{n=-K}^K \sum_{m=-K}^K [\bar{\beta}(\rho_{jd})]_{0n} [\bar{\beta}(\rho_{dc}) \cdot \bar{\alpha}(\rho_{cb}) \cdot \bar{\beta}(\rho_{ba})]_{nm} [\bar{\beta}(\rho_{ai})]_{m0}. \quad (28)$$

As all three stages of the FMM occur in this expansion, this configuration is ideal for testing the accuracy of the normalized plane wave method (NPWM) with the diagonal aggregation, translation, and disaggregation stages. Figures 4 and 5 show the relative error between the direct application of (28) and the normalized plane-wave method as a function of the normalization factor t for $K = 10$ and $K = 25$. The functions $\Delta(\gamma\rho, t)$ and $\theta(\gamma\rho)$ are relative errors, defined as

$$\Delta(\gamma\rho, t) = \frac{H_0^{(2)NPWM}(\gamma\rho, t)}{H_0^{(2)ADD}(\gamma\rho)} - 1, \quad (29)$$

Figure 4 Comparison between the addition theorem and the diagonal form of the addition theorem as a function of the normalization factor t for $K = 10$

$$\theta(\gamma\rho) = \frac{H_0^{(2)ADD}(\gamma\rho)}{H_0^{(2)}(\gamma\rho)} - 1. \quad (30)$$

In both figures, the errors for the shortest and the longest translation (as in Fig. 2) are plotted. The shortest translation corresponds to translation 1 ($\rho_1 = 3.18 \times 10^{-3}\lambda$) and longest translation corresponds to translation 2 ($\rho_2 = 6.75 \times 10^{-3}\lambda$), with an additional diagonal aggregation and disaggregation. The two horizontal lines give the relative errors between the addition theorem (truncated at K multipoles) and the Hankel function for the shortest and longest translations. The vertical line is the optimal value for t obtained with (22) and the “shortest translation” prescription. From this, it is clear that the proposed formula for t works very well, since it gives us the almost exact location of the minimum in the error curve. Figure 5 also demonstrates that the achievable accuracy of the normalized plane-wave method is high. Indeed, with $K = 25$ both Δ and θ reach 10^{-10} . As a consequence, the error between the normalized plane-wave method and the Hankel function is on the order of 10^{-10} .

In the high-frequency regime, t becomes approximately equal to 1, thus rendering the method equivalent to the plane-wave method. Since the normalized plane-wave method entails only diagonal operators and FFTs, the method also has the same complexity, $\mathcal{O}(\mathcal{N})$, for the high-frequency regime as the plane wave method. However, in the region between the HF and LF regimes,

Figure 5 Comparison between the addition theorem and the diagonal form of the addition theorem as a function of the normalization factor t for $K = 25$

Figure 6 Minimal error achievable with the normalized plane-wave method as a function of the translation distance

some care is required. The reasoning which lead to the conclusion that the accuracy of the normalized plane-wave method is high supposes that $\gamma\rho < 1$ in order to make sure that the asymptotic forms (13) and (15) can be used. In the intermediate frequency zone this is no longer true, and error estimates cannot be easily made. Therefore, we resorted to numerical testing. Figure 6 shows Ψ , the best accuracy possible (optimized towards the number of multipoles K) with the normalized plane-wave method, as a function of the translation distance $\gamma\rho$:

$$\Psi(\gamma\rho) = \frac{H_0^{(2)NPWM}(\gamma\rho, t_{optimal})}{H_0^{(2)}(\gamma\rho)} - 1. \quad (31)$$

This calculation was performed for a geometry similar to the one depicted in Figure 3, but scaled up or down according to the translation distance. It is clear that the method's accuracy is high

Figure 7 Geometry of the simulated structure

TABLE 1 Cost of the Translation Stage of the Two Methods and for Different Multipole Orders

K	FMM	NPWM
11	0.18	0.08
15	0.30	0.23
20	0.53	0.15

for both the HF and LF regimes, and that the accuracy is slightly more limited in the intermediate region, but not dramatically.

The normalized plane-wave method described above has been implemented in a multilevel scheme. For simplicity, only perfectly electrically conducting (PEC) targets were used, illuminated by a TM_z -polarized wave (for which the electric field is directed parallel with the targets). The simplest integral equation describing this problem is the well-known electric-field integral equation (EFIE), which states that the total electric field vanishes on the surface of the targets:

$$\lim_{\rho \rightarrow C} \left[E_z^i(\rho) - \frac{\omega\mu}{4} \oint_C H_0^{(2)}(\gamma|\rho - \rho'|) \mathcal{J}_z(\rho') dc' \right] = 0. \quad (32)$$

Here, C is the integration path on the surface of the PEC targets. The unknown quantity is the surface current $\mathcal{J}_z$.

The simulated structure consists of a dense arrangement of P^2 PEC cylinders inside a square with size 1 m, as shown in Figure 7. For all the simulations to follow, $P = 51$. The surface current is discretized in terms of a set of $\mathcal{N}$ pulse functions. The number of unknowns $\mathcal{N}$ then amounts to 65025. The wavelength is 6000 m. The normalized plane-wave method is compared to a multipole solver, which acts as a reference. The multipole solver itself is also optimized in the sense that translations that are further away are done with less multipoles than translations that are closer. Table 1 shows the CPU time for the translation stage of both the FMM and the normalized plane-wave method for different multipole orders K . This table shows that, for $K = 15$, the cost of the translation stage is even larger than the cost with $K = 20$. This can be explained by looking at the corresponding dimension $4K + 1$ of the FFTs that are required:

$$4 \cdot 11 + 1 = 45 = 3 \cdot 3 \cdot 5 \quad \text{and} \quad 4 \cdot 20 + 1 = 81 = 3 \cdot 3 \cdot 3 \cdot 3.$$

These are both products of small primes, but $4 \cdot 15 + 1 = 61$ is prime, so the FFTs will be significantly slower if $K = 15$. The FFTs were done with the FFTW library [6]. Table 2 shows the cost of the FFTs and the translations [pointwise multiplications as in (14)]. This demonstrates that the cost of the translations is indeed rising linearly with the multipole order K . It has to be recognized that the cost associated with the FFTs can vary wildly as a function of K , but even if a value for K is chosen such that $4K + 1$ is prime, the diagonal method is still faster than the traditional FMM.

TABLE 2 Cost of the FFTs and the Diagonal Transfers for Different Multipole Orders

K	FFTs	Transfers
11	0.025	0.06
15	0.15	0.08
20	0.045	0.105

6. CONCLUSION

In this paper, a normalized diagonal form of the addition theorem for multipoles in 2D has been derived. This approach has several merits, such as being more efficient than the traditional FMM in the low-frequency regime and having the same complexity as the plane-wave method in the high-frequency regime. The normalized plane-wave method should thus be interpreted as a generalization of the plane-wave method to the intermediate- and low-frequency regimes. Moreover, existing multipole solvers easily can be modified to use this new method, since multipoles remain an important part of the normalized plane-wave method. The multilevel algorithm is implemented and the computer labor for the translation stage is shown to decrease considerably, compared to an already optimized multipole solver.

The extension of the normalized plane-wave method to three dimensions is a possible area for future research. However, a similar normalization and diagonalization of the addition theorem as described for 2D in this paper may not be as straightforward in 3D.

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A METHOD TO GENERATE A SPARSE MATRIX FOR THIN-WIRE STRUCTURES USING THE METHOD OF MOMENTS

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ABSTRACT: In this work, a numerical procedure is presented to obtain a sparse moment matrix for thin-wire electromagnetic scattering problems. The numerical procedure involves developing a set of basis functions spanning several subdomains, as opposed to spanning one or two subdomains in the conventional method of moments (MoM) procedure. The numerical results are presented for various wire structures (including wire junctions) and compared using the traditional MoM procedure. © 2005 Wiley Periodicals, Inc. *Microwave Opt Technol Lett* 48: 243–246, 2006; Published online in Wiley InterScience (www.interscience.wiley.com). DOI 10.1002/mop.21317

Key words: sparse matrix; wire structures; method of moments

1. INTRODUCTION

Computational electromagnetics (CEM) has undergone evolutionary developments in the last 40 years. In this scenario, wires represent a special class of conducting objects because of the simplifications they offer, and because so many actual problems do involve wires. The analysis of radiation and scattering from thin conducting wires is one of the active areas of research in CEM [1, 2].

For frequency-domain radiation and scattering problems involving conducting wires, the method of moments (MoM) is perhaps the most widely used numerical technique for analyzing both thin-wire structures and wire-grid models of solid-surface structures [3–5]. However, the limitation with the traditional MoM is that for electrically large bodies, the moment matrix generated is very large in size and fully populated. To overcome this limitation,

Figure 1 Implementation of a junction