A 2D differential surface admittance operator approach to model the skin effect

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Abstract — In this paper a new exact differential surface admittance approach is put forward to model the skin effect in multiconductor lines from DC to the high-frequency regime. The important practical case of conductors with rectangular cross-section is treated in detail. Numerical results concentrate on the determination of inductance and resistance matrices.

1 INTRODUCTION

The International Technology Roadmap for Semiconductors (ITRS) predicts that the smallest on-chip features will shrink from 150 nm in 2003 to 50 nm by 2012 while the clock rate will increase from 1.5 GHz to 10 GHz. The correct assessment of signal integrity for such high clock rates crucially depends on the correct modeling of the skin-effect. Consider the current distribution in the cross-section of a conductor of width $w$, thickness $t$ ($w \geq t$) and conductivity $\sigma$. In the low frequency range, the skin depth $\delta = \frac{2}{\omega \sigma}$, with $\omega = 2\pi f$ is much larger than both $w$ and $t$. The current distribution in the conductor is then governed by the solution of the Laplace equation. With increasing frequency, inductive effects come into play, pushing the currents towards the surface of the conductor and leading to an increase of the resistance and a decrease of the internal inductance. This is the case when the skin depth becomes comparable to the (smallest) dimension of the conductor’s cross-section (at intermediate frequencies). Only for the highest frequencies and provided the skin depth becomes much smaller than both $w$ and $t$, the well-known skin-effect occurs. In this case the current is flowing in a small surface layer and the behaviour of the conductor is usually described in terms of the surface impedance

$$Z_s = \sqrt{\frac{j\omega \sigma}{\mu}} = \frac{(1+j)}{\delta^2}.$$  

It is clear that accurate electromagnetic modeling tools need to correctly account for the redistribution of the conductor current. This has of course been recognised by many authors and numerous publications address this so-called current crowding problem, see e.g. [1], [2], [3], [4]. Similar to the approach in [4], the purpose of this paper is to provide a surface admittance operator description of the conductor. At each frequency, this description associates a fictitious electric surface current density $\mathbf{J}_e(r, \omega)$ at each point $r$ on the surface of the conductor to the tangential electric fields $\mathbf{E}_{\text{tan}}(r, \omega)$ at every other point on the surface. The surface admittance operator allows to replace each conductor by equivalent surface currents and to replace the conductor medium by the medium of the material layer it is embedded in. The remaining field problem can then be solved by solely considering the interactions between the equivalent surface currents.

In this paper, we restrict ourselves to two-dimensional configurations and to the TM-case, i.e., the configuration is invariant in the $z$-direction and currents are flowing in this direction. Section 2 sketches the theoretical background and provides an expression for the surface admittance operator in terms of the Dirichlet eigenfunctions of the conductor’s cross-section. In Section 3, a discretised form of the surface admittance operator (a surface admittance matrix) for conductors of rectangular cross-section is obtained through a Method of Moments (MoM) technique. Section 4 shows how this matrix can be used to determine the resistance (R) and inductance (L) matrices of a set of parallel conductors by means of an electric field integral equation (EFIE). A more detailed analysis of the problem can be found in [5]. Finally, Section 5 gives two examples of inductance and resistance matrices of 2D interconnect structures illustrating the correctness and versatility of the new technique.

2 THE SURFACE ADMITTANCE AND ITS DIRICHLET REPRESENTATION

Consider, in the case of time-harmonic ($e^{j\omega t}$ dependence) TM-polarisation, the electric field $\mathbf{E}_z$ inside a conducting non-magnetic cylinder with homogeneous cross-section $S$ as in Fig. 1a. Further suppose that the conductor is embedded in a planar stratified medium. The particular layer the conductor is embedded in, is characterised by the constitutive parameters $\varepsilon_{\text{out}}$, $\mu_0$ and $\sigma_{\text{out}}$. On the boundary $c$ of $S$ we now have that

$$H_t = \frac{1}{j \omega \mu_0} \partial_n E_z = \frac{1}{j \omega \mu_0} D_k \cdot E_z,$$  

(1)
with the index \( t \) referring to the tangential component of the magnetic field. The expression \( \partial_n E_z \) stands for the limit of the normal derivative of the electric field tending from the inside of the cylinder to \( c \). \( \mathcal{D}_k \) is the Dirichlet-Neumann operator, mapping the values of the field on \( c \) to the values of the normal derivatives of the field on \( c \). Now suppose that the constitutive parameters of the conducting cylinder are replaced by those of the medium outside the conductor, in particular of the material layer the conductor is embedded in. On the boundary \( c \) of \( S \) we now have that

\[
H_{\theta} = \frac{1}{j \omega \mu_0} \partial_n E_{z0} = \frac{1}{j \omega \mu_0} \mathcal{D}_{k_{\text{out}}} \cdot E_{z0}. \tag{2}
\]

If we want to replace the conductor by the medium of its surrounding layer, in this way restoring the planar stratified nature of the medium and undoing the discontinuity in conductivity and permittivity due to the conductor’s presence, it suffices to introduce an equivalent surface current density \( J_{sz} \), related to the value of the field \( E_{z0} \) on the boundary, by means of the differential surface admittance operator \( \mathcal{Y} \) given by

\[
J_{sz} = \mathcal{Y} E_{z0} = \frac{1}{j \omega \mu_0} [\mathcal{D}_k - \mathcal{D}_{k_{\text{out}}}] \cdot E_{z0}. \tag{3}
\]

This is depicted in Fig. 1b. When solving the field problem of Fig. 1b, the obtained result is only identical to the one for the original configuration of Fig. 1a, taken outside the conductor. Inside the conductor a fictitious field is obtained. However, in order to obtain relevant data such as total Joule losses, total conduction current or inductance and resistance matrices, the sole knowledge of the surface current density \( J_{sz} \) suffices. Note that on the boundary \( c \) and only on \( c \), \( E_{z0} = E_z \).

One possible way to obtain the operator \( \mathcal{Y} \) is to use the Dirichlet eigenfunctions of the cross-section \( S \). Calculations, the details of which are given in [5], show that

\[
J_{sz} = \mathcal{Y} E_z = \tau \sum_{m=1}^{\infty} \frac{\partial_n \xi_m}{(k_{\text{out}}^2 - \lambda_m)(k^2 - \lambda_m)} \int \mathcal{E}_z \partial_n \xi_m \, dc \tag{4}
\]

with \( \tau = [\sigma - \sigma_{\text{out}} + j \omega (\epsilon - \epsilon_{\text{out}})] \) and where \( k \) resp. \( k_{\text{out}} \) is the wavenumber of the conducting cylinder, resp. of the medium replacing the material of the cylinder. The \( \xi_m \) are the Dirichlet eigenfunctions of the cross-section \( S \) with corresponding eigenvalues \( \lambda_m \).

### 3 SURFACE ADMITTANCE MATRIX FOR A RECTANGLE

For a rectangular conductor (\( 0 \leq x \leq a \) and \( 0 \leq y \leq b \)), the Dirichlet eigenfunctions and eigenvalues are

\[
\xi_{mn} = \frac{2}{\sqrt{ab}} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right), \tag{5}
\]

with \( \lambda_{mn} = (\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2 \). In [5] it is shown that an analytical expression for \( J_{sz} \) can be obtained by expanding \( E_z \) on each side of the rectangle in an appropriate Fourier sine series. E.g. for \( y = 0 \) and \( 0 \leq x \leq a \) this series is

\[
E_z = \sin(m \pi x/a) \tag{6}
\]

However, when we want to use \( E_z \) and \( J_{sz} \) in a Galerkin MoM approach to obtain \( R \)- and \( L \)-matrices, it is indicated to use pulse basis functions. Hence, we rewrite \( E_z \) as

\[
E_z = \sum_{j=1}^{M} E_j t_j(x), \tag{7}
\]

where \( t_j = 1 \) for \( x_{j-1} < x \leq x_j \) and zero elsewhere, with \( x_0 = 0 \) and \( x_M = a \) and with \( M \) the number of pulse basis functions along the considered side. A similar expression can be put forward for \( J_{sz} \), replacing the coefficients \( E_j \) by \( J_j / (x_j - x_{j-1}) \) (the factor \( (x_j - x_{j-1}) \) is necessary in view of the Galerkin testing). We can now collect all the pulse basis amplitudes \( E_j \), on all of the four sides, into a vector \( \mathbf{E} \) and similarly all \( J_j \)'s into a vector \( \mathbf{J} \). Tedium, but completely analytical calculations [5] allow to obtain the discretised form of \( \mathcal{Y} \) as

\[
\mathbf{J} = \mathbf{Y}_s \cdot \mathbf{E}. \tag{8}
\]

\( \mathbf{Y}_s \) is the \( M \times M \) differential surface admittance matrix (all entries of \( \mathbf{Y}_s \) have dimension \( \Omega^{-1} \)).
4 R- AND L-MATRICES FOR MULTI-CONDUCTOR LINES

The relevant EFIE, valid as long as the cross-sectional dimensions of the conductors remain small with respect to the free space wavelength, is [2]

\[ E_z(r) = -j\omega A_z(r) - \frac{\partial V(r)}{\partial z}. \]  

(9)

\( A_z \) is the vector potential, \( V \) is the scalar potential. Using the differential surface admittance, the conductors can be replaced by equivalent surface currents \( J_{sz} \). The vector potential of these currents in free space is given by

\[ A_z(r) = -\mu_0 \int_{\mathcal{C}} J_{sz}(r') \frac{1}{2\pi} \ln |r - r'| dc(r'). \]  

(10)

Further suppose we have a system of \( N \) conductors. The following relationship then holds

\[ \frac{\partial V}{\partial z} = -(R + j\omega L)I. \]  

(11)

In (11), \( V \) is a \( N \times 1 \) column vector formed by the constant potentials \( V_j \) of each conductor cross-section, with \( p = 1, 2, ..., N \). \( I \) is also a \( N \times 1 \) column vector formed by the total currents \( I_p \) through each conductor \( p \) and \( R \) and \( L \) respectively represent the \( N \times N \) R- and L-matrix per unit of length. Following the approach explained in Section 3, the surface current on the circumference of each conductor is discretised using a pulse basis, leading to a total of \( T \) pulses for the \( N \) conductors. The same is done for \( E_z \). Combining (9) and (11) then leads to

\[ E_i + j\omega \mu_0 \sum_{j=1}^{T} G_{ij} J_j = \sum_{q=1}^{N} (R_{pq} + j\omega L_{pq}) I_q. \]  

(12)

\( E_i \) is the amplitude of basis function \( i \) with \( i = 1, 2, ..., T \). \( G_{ij} \) results from the discretisation of the boundary integrals in (10) and expresses the interaction between pulse \( i \) and pulse \( j \). Further suppose that pulse \( i \) is located on conductor \( p \). In the right hand member of (12), the summation runs over all conductors and involves the self coupling and mutual coupling resistances and inductances between conductor \( p \) and all other conductors. Finally, by invoking the relationship (8) for each conductor, the \( E_i \)'s in (12) can be expressed in terms of the \( J_j \)'s. The values of the elements of the resistance and inductance matrices can now be obtained by solving (12) \( N \) times, enforcing the fact that for each of these solutions the total current running through one of the conductors is equal to unity, while all other total currents remain zero.

5 NUMERICAL EXAMPLES

As a first example we consider the copper two-conductor system \((\sigma = 5.6 \times 10^7 (\Omega m)^{-1})\) depicted in Fig. 2 [1], [4]. We consider both the 2mm × 0.2mm case (case 1) and the 2mm × 2mm case (case 2) for separation distances \( s = 0.5 \text{mm}, 1 \text{mm} \) and 2mm. The number of pulses is 20 on each side. Fig. 3 shows the resistance results between 100 Hz and 10 GHz. Fig. 4 shows the corresponding results for the inductance. Observe that the inductance results are much more sensitive to the distance between the conductors. For case 1, the separation distance used in [1] is \( s = 0.8 \text{mm} \). The inductance result for this distance is also shown on the plot (dashed line).

Figure 2: Two pairs of copper signal lines.

Figure 3: Resistance for the example of Fig. 2.

As a second example consider the configuration depicted in Fig. 5. It is the cross-section of a coaxial line with two copper signal conductors surrounded by a copper outer conductor \((\sigma = 5.8 \times 10^7 (\Omega m)^{-1})\). All dimensions are in units of 0.1mm and a discretisation of 4 divisions per 0.1mm is
used. As our method can only handle rectangular conductors, the outer conductor is subdivided into 4 separate conductors, as indicated on the figure. However, the gap between the conductors is kept extremely small (less than a tenth of a µm and we verified numerically that for such small gaps stable numerical results are obtained). The total number of segments amounts to 528. We assign a zero

**Figure 4:** Inductance for the example of Fig. 2.

**Figure 5:** Copper coaxial line with 2 signal lines.

reference potential to the four conductors forming the outer coaxial shield and determine the $2 \times 2$ resistance and inductance matrix of the resulting configuration as shown in Fig. 6. The DC value for $R_{11} = R_{22}$ is $1.73408 \Omega/m$, i.e. the sum of the DC resistance of the outer coaxial shield and one of the signal conductors. The DC value for $R_{12} = R_{21}$ is $2.9829 \times 10^{-2} \Omega/m$, i.e. the DC resistance of the outer coaxial shield. Further remark that the low-frequency value of $L_{12} = L_{21}$ is negative. This is allowed as long as the $2 \times 2$ inductance matrix remains positive-definite. This positive-definite nature has been verified for the complete frequency range and implies that the magnetic energy always remains positive whatever the currents used to excite the configuration. We have also determined the inductance matrix for the case of perfect conductors, using a finite difference technique to solve Laplace’s equation for the capacitance problem and using the fact that the product of the inductance and capacitance matrix equals $1/(\epsilon_0 \mu_0)$ . The obtained results are: $L_{11} = 157.4 \text{ nH/m}$ and $L_{12} = 12.06 \text{ nH/m}$, while the present technique yields $L_{11} = 158.62 \text{ nH/m}$ and $L_{12} = 12.11 \text{ nH/m}$.

**Figure 6:** (2x2) R- and L-matrix for Fig. 5.

Acknowledgments

We gratefully acknowledge the cooperation with and financial support of Agilent Technologies.

References


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