An exact 2-dimensional model for a periodic circular array of head-to-head permanent magnets.

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Abstract

An analytical expression is derived for the magnetic field of a 2-dimensional periodic circular array of head-to-head permanent magnets, assuming there is either no gap between the magnets or that they are separated by a non-magnetic pole-piece. Some more general symmetry properties are described. The force experienced by the magnets is calculated and an analytical approximation is derived. We comment on the published dipole-approximation used for dealing with this kind of problem and correct also some mistakes.

1 Introduction

In a recent paper [1] a model is presented of a device constructed with permanent magnets and intended for separating and transporting dia- and para-magnetic particles [2]. The basic outline of the separating device considered resembles the one given in [3]. Its main component is a periodic circular arrangement of $2N$ permanent magnets, magnetized along the circumference and positioned head to head (Fig.1).

The model proposed in [1] is based on the assumption that the field of a permanent magnet is equivalent with that of a magnetic dipole. Although sufficiently far away this will become a good approximation, at close distances the approximation will break down. Also one should be careful in the choice of the equivalent dipole moment, as will be shown further.

Figure 1: Periodic arrangement of magnets, placed head to head and separated by soft ferromagnetic pole pieces.

In the device considered in [3] the particles are transported with a belt covering (part of) the outer circumference of the assembly. On the contrary the device proposed in [1] consists of 2 concentric assemblies of magnets as in Fig.1 but without pole-pieces and with slightly different number of magnets (e.g. $2N_i = 16$ for the inner and $2N_o = 18$ for the outer assembly), and in addition the inner assembly rotates (with angular velocity ω)¹. For realistic velocities the resulting field can be calculated with the usual quasi-stationary approximation, meaning that the static field just rotates together with the magnets and radiation effects, as considered in [1], will be negligible. The so-called “novelty of the device” can therefore be understood without any detailed field calculations. Due to the periodic arrangement of the magnets the magnetic field (of a static assembly) at any radius r , will show the same periodicity:

$$H(r, \varphi) = H_a(r) \sin(N\varphi) \quad (1)$$

where for simplicity the higher harmonics are not taken into account and H is any component of the magnetic field. Supposing the fields of the 2 assemblies can be superposed the resulting field of 2 static assemblies is then given by:

$$H(r, \varphi) = H_i(r) \sin(N_i\varphi) + H_o(r) \sin(N_o(\varphi + \Delta\varphi)) \quad (2)$$

In general the field patterns of both assemblies will interfere. If e.g. the 2 assemblies differ by 1 period ($N_o - N_i = 1$) then the envelope of the resulting interference pattern will be maximal where the inner and outer magnets are aligned “parallel” and it will be minimal on the opposite side, where the inner and outer

¹In a following step also more complicated structures are considered by placing several such modules in a sequence.

magnets are aligned “anti-parallel”. Rotating the inner assembly over 1 period (2 magnets) causes the envelope to make one complete turn and therefore the envelope of the field pattern rotates at a speed N_i times the speed of the inner assembly. As already noted, this is a purely quasi-static result, which has nothing to do with radiation effects and it is misleading to speak of “travelling magnetic waves” and “magnetic field propagation”[1]. There remains the problem of calculating the basic amplitudes $H_i(r), H_o(r)$ and, if desired, higher order amplitudes.

In the next section we summarize the proper quasi-stationary equations for dealing with this kind of problem. In section 3 we consider a 2-dimensional model which allows for an analytical solution if there are no gaps between the magnets or if the pole-pieces are non-magnetic. In the next section we calculate the forces on the magnets for the same 2-dimensional cases and in the last section we analyse the dipole approximation.

2 Basic equations

The basic problem to be solved is the calculation of the fields of a set of moving permanent magnets, possibly in the presence of soft ferromagnetic pole-pieces. We will follow standard electromagnetic theory as can be found in many textbooks (e.g. [4]). We will assume that the permanent magnets are ideal, with a fixed magnetization^{2 3} $\overline{\mathcal{M}}$. The problem becomes much more difficult if soft ferromagnetic pole-pieces are present, since simple super position of the fields of 2 assemblies can now not longer be applied. In particular the fields of 2 assemblies as described are not longer independent but depend on their relative position.

From the magnetization in the local co-moving reference frame $\overline{\mathcal{M}}$, the magnetization $\overline{\mathbf{M}}$ and the polarization $\overline{\mathbf{P}}$ in the laboratory frame can be found as:

$$\overline{\mathbf{M}} \approx \overline{\mathcal{M}} \quad \overline{\mathbf{P}} \approx \frac{\overline{\mathbf{v}} \times \overline{\mathcal{M}}}{c^2} \quad (3)$$

where we have assumed that $|\overline{\mathbf{v}}|^2 \ll c^2$, $\overline{\mathbf{v}}$ being the material velocity and c the velocity of light. The problem can then be reduced to the calculation of the electric field $\overline{\mathbf{E}}$ and the magnetic induction $\overline{\mathbf{B}}$ generated by the following effective charge and current densities:

$$\begin{aligned} \rho_{\text{eff}} &= -\frac{1}{c^2} \nabla \cdot (\overline{\mathbf{v}} \times \overline{\mathcal{M}}) \\ \overline{\mathbf{J}}_{\text{eff}} &= \nabla \times \overline{\mathcal{M}} \end{aligned} \quad (4)$$

² $\overline{\mathcal{M}}$ represents the magnetization measured in a local co-moving reference frame.

³Throughout this paper vectors are denoted with an overbar, except in the drawings where vectors are indicated in bold.

For our intended purpose we can simplify the problem by assuming that $\bar{v}/\bar{\mathcal{M}}$ so that $\rho_{\text{eff}} = 0$.

Expressing the electric field and the magnetic induction in terms of the vector potential $\bar{A}$:

$$\bar{B} = \nabla \times \bar{A} \quad \bar{E} = -\frac{\partial \bar{A}}{\partial t} \quad (5)$$

the latter obeys the following wave-equation (assuming the gauge condition $\nabla \cdot \bar{A} = 0$):

$$\nabla^2 \bar{A} - \frac{1}{c^2} \frac{\partial^2 \bar{A}}{\partial t^2} = -\mu_0 \bar{J}_{\text{eff}} \quad (6)$$

Using the well-known retarded potentials the solution of (6) is given by:

$$\bar{A}(\bar{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\bar{J}_{\text{eff}}(\bar{r}', t - \frac{|\bar{r} - \bar{r}'|}{c})}{|\bar{r} - \bar{r}'|} dv' \quad (7)$$

For calculating the radiated power a Taylor series is introduced leading to:

$$\lim_{r \rightarrow \infty} \bar{A}(\bar{r}, t) = \frac{\mu_0}{4\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \nabla^n : \frac{1}{r} \int \bar{r}'^n \bar{J}_{\text{eff}}(\bar{r}', t_e) dv' \quad (8)$$

where the origin of the coordinate system has been chosen at a suitable reference point within the effective current density distribution, r is the distance from the observation point to this reference point and $t_e = t - \frac{r}{c}$ is the time of emission for that reference point. Finally the 3 dots stand for an n -fold scalar product. The gradient should only be applied to the r occurring in t_e leading to:

$$\lim_{r \rightarrow \infty} \bar{A}(\bar{r}, t) = \frac{\mu_0}{4\pi r} \sum_{n=0}^{\infty} \frac{1}{c^n} \frac{\partial^n}{\partial t^n} \frac{1}{n!} \int (\bar{r}' \cdot \bar{1}_r)^n \bar{J}_{\text{eff}}(\bar{r}', t_e) dv' \quad (9)$$

where $\bar{1}_r$ is the unit vector along $\bar{r}$. The integrands in (9) can be expressed in terms of the conventional magnetic multipole moments. Consider now a ring of magnets as described (see Fig.1). The z -axis is an N -fold symmetry axis of the effective current density and therefore the lowest non-zero moment in (9) is at least of order N . As a consequence the dominant contribution to the vector potential at infinity will be proportional to k^N , where $k = \frac{\omega}{c}$. The power radiated will then at least be proportional with $(kr_0)^{2N+1}$, where r_0 is a characteristic dimension of the ring. Although the approximation by magnetic dipoles should be justified in this situation (large distance), a k^5 dependence was found in [1] probably due to an error in the calculation. For all practical purposes the radiated power is extremely small.

Figure 2: Cross-section of a circular array of $2N$ magnets with indication of the coordinate system used. (in this example $N = 4$).

For calculating the field in the neighbourhood of the magnets (instead of the power radiated) a quasi-stationary approximation will usually be sufficient. The fields will change periodically with angular frequency $N\omega$. Since the waveform will not be purely sinusoidal higher harmonics with angular frequency $(2m + 1)N\omega$ must be taken into account. But if the corresponding wavelength is much larger than the largest dimension considered, then a quasi-stationary approximation is sufficient and the 2nd order time derivatives in (6) can be neglected, or equivalently c can be made infinite. These conditions are easily met for realistic conditions.

3 Field of a 2-dimensional assembly without pole-pieces or with non-magnetic pole-pieces.

We assume that the magnetization is given by:

$$\overline{\mathcal{M}} = M_0 \frac{r_0}{r} f(\varphi) \bar{1}_\varphi \quad (10)$$

where (r, φ) are polar coordinates, $\bar{1}_\varphi$ is the unit vector tangential to the circle $r = cte$, M_0 is a constant, and $r_0 = \sqrt{r_1 r_2}$ where r_1 and r_2 are the inner and outer radius, and $f(\varphi)$ is a periodic function (see Fig.3) with values ± 1 and which can be decomposed as:

$$f(\varphi) = \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\sin(N(2m+1)\varphi)}{2m+1} \quad (11)$$

Figure 3: Definition of $f(\varphi)$.

with N the number of magnet-pairs. Although one might expect as the ideal case a constant magnetization within a single permanent magnet, for a true cylindrical arrangement the r^{-1} dependence assumed in (10) is more appropriate: starting from the non-magnetized state one could in principle magnetise the magnets in Fig.2 with a line-current I in the centre (or any equivalent current distribution with rotational symmetry). Due to the rotational symmetry and Ampère's law $H_\varphi = \frac{I}{2\pi r}$ everywhere. For an isotropic material B_φ and $\mathcal{M}_\varphi$ will then show the same r -dependence. Once magnetised one would reverse half of the magnets to obtain the desired head-to-head arrangement.

Due to this r^{-1} dependence, $\bar{\mathbf{J}}_m = \nabla \times \bar{\mathcal{M}} = 0$, and the effective current density reduces to a surface current density $\bar{\mathbf{K}}_m = -\bar{\mathbf{n}} \times \bar{\mathcal{M}}$ on the inner and outer surfaces of the magnets, where $\bar{\mathbf{n}}$ is the outward pointing normal unit vector:

$$\bar{\mathbf{K}}_m = M_0 f(\varphi) \bar{\mathbf{1}}_z \begin{cases} \frac{r_0}{r_1} & r = r_1 \\ -\frac{r_0}{r_2} & r = r_2 \end{cases} \quad (12)$$

The vector potential is then also directed along the z -axis. The z -component A_z , subsequently notated as A , obeys the Laplace-equation in the 3 distinct regions (see Fig.2):

$$r \frac{\partial}{\partial r} \left(r \frac{\partial A_i}{\partial r} \right) + \frac{\partial^2 A_i}{\partial \varphi^2} = 0 \quad (13)$$

($i = 1, 2, 3$) and is continuous on the 2 boundaries:

$$\begin{aligned} A_1(r_1, \varphi) &= A_2(r_1, \varphi) \\ A_2(r_2, \varphi) &= A_3(r_2, \varphi) \end{aligned} \quad (14)$$

whereas the 2nd boundary condition is given by:

$$\begin{aligned}\left.\frac{\partial A_1}{\partial r}\right|_{r=r_1} - \left.\frac{\partial A_2}{\partial r}\right|_{r=r_1} &= \mu_0 \frac{r_0}{r_1} M_0 f(\varphi) \\ \left.\frac{\partial A_2}{\partial r}\right|_{r=r_2} - \left.\frac{\partial A_3}{\partial r}\right|_{r=r_2} &= -\mu_0 \frac{r_0}{r_2} M_0 f(\varphi)\end{aligned}\quad (15)$$

One can verify that the problem shows inversion symmetry with respect to the radius $r_0 = \sqrt{r_1 r_2}$, which means that in corresponding points for which $rr' = r_0^2$ the vector potential $A(r', \varphi) = -A(r, \varphi)$.

The potential problem is easily solved by super position and “separation of variables” (see e.g. [4], [5]). The proper separated solutions to consider in this case are $r^k \sin(k\varphi)$ and $r^{-k} \sin(k\varphi)$. Using the decomposition (11) the result can be written as:

$$\begin{aligned}A &= \frac{2}{\pi} \mu_0 M_0 \frac{r_0}{N} \sum_{m=0}^{\infty} \frac{\sin(N(2m+1)\varphi)}{(2m+1)^2} \left(\left[g\left(\frac{r}{r_1}\right) \right]^{N(2m+1)} - \left[g\left(\frac{r}{r_2}\right) \right]^{N(2m+1)} \right) \\ g(u) &= \begin{cases} u & u \leq 1 \\ \frac{1}{u} & u \geq 1 \end{cases}\end{aligned}\quad (16)$$

The radial component of the magnetic induction follows from:

$$\begin{aligned}B_r &= \frac{1}{r} \frac{\partial A}{\partial \varphi} \\ &= \frac{2}{\pi} \mu_0 M_0 \frac{r_0}{r} \sum_{m=0}^{\infty} \frac{\cos(N(2m+1)\varphi)}{2m+1} \left(\left[g\left(\frac{r}{r_1}\right) \right]^{N(2m+1)} - \left[g\left(\frac{r}{r_2}\right) \right]^{N(2m+1)} \right)\end{aligned}\quad (17)$$

Using the series[6]:

$$\sum_{k=0}^{\infty} \frac{s^{2k+1} \cos((2k+1)x)}{2k+1} = \frac{1}{4} \ln \left(\frac{1+s^2+2s\cos x}{1+s^2-2s\cos x} \right) \quad 0 < x < 2\pi, s^2 \leq 1 \quad (18)$$

the solution for B_r can be written in closed form:

$$B_r = \frac{\mu_0 M_0 r_0}{\pi r} \left(F\left(\left(\frac{r}{r_1}\right)^N, N\varphi\right) - F\left(\left(\frac{r}{r_2}\right)^N, N\varphi\right) \right) \quad (19)$$

with the definition:

$$F(s, x) = \ln \sqrt{\frac{1+s^2+2s\cos x}{1+s^2-2s\cos x}} \quad (20)$$

and where we've also used that $F(s^{-1}, x) = F(s, x)$. One can verify that due to the inversion symmetry $r'B_r(r', \varphi) = -rB_r(r, \varphi)$, where $rr' = r_0^2$.

The tangential component can be calculated in a similar way:

$$\begin{aligned} B_\varphi &= -\frac{\partial A}{\partial r} \\ &= \frac{2}{\pi} \mu_0 M_0 \frac{r_0}{r} \sum_{m=0}^{\infty} \frac{\sin(N(2m+1)\varphi)}{2m+1} (\pm \left[g\left(\frac{r}{r_1}\right) \right]^{N(2m+1)} \mp \left[g\left(\frac{r}{r_2}\right) \right]^{N(2m+1)}) \end{aligned} \quad (21)$$

where the upper sign holds for $u \leq 1$ and the lower sign for $u \geq 1$, where $u = \frac{r}{r_i}$, $i = 1, 2$. Using now the series[6]:

$$\sum_{k=0}^{\infty} \frac{s^{2k+1} \sin((2k+1)x)}{2k+1} = \frac{1}{2} \arctan\left(\frac{2s \sin x}{1-s^2}\right) \quad 0 < x < 2\pi, s^2 \leq 1 \quad (22)$$

B_φ can be written in closed form as:

$$B_\varphi = -\frac{\mu_0 M_0 r_0}{\pi r} (G\left(\left(\frac{r}{r_1}\right)^N, N\varphi\right) - G\left(\left(\frac{r}{r_2}\right)^N, N\varphi\right)) \quad (23)$$

with the definition:

$$G(p, x) = \arctan\left(\frac{2s \sin x}{1-s^2}\right) \quad (24)$$

and where we've also used that $G(s^{-1}, x) = -G(s, x)$. Finally one can verify that $r'B_\varphi(r', \varphi) = rB_\varphi(r, \varphi)$, where $rr' = r_0^2$.

The closed form solutions (19) and (23) can be understood as follows. Consider a single period of the configuration ($0 < \varphi < \frac{2\pi}{N}$); with the transformation⁴ $\left(\frac{r}{r_0}\right)^N \rightarrow \frac{r'}{r'_0}$, $N\varphi \rightarrow \varphi'$ this 1-period segment is stretched so as to fill 2π again. This is in fact a conformal transformation and both problems are therefore equivalent. Since under this transformation $\frac{1}{N} \frac{\partial}{\partial \varphi} \rightarrow \frac{\partial}{\partial \varphi'}$ and $\frac{r}{N} \frac{\partial}{\partial r} \rightarrow r' \frac{\partial}{\partial r'}$ it follows that the magnetic inductions in both problems are related by $\frac{r}{N} \bar{B} \rightarrow r' \bar{B}'$. and $\frac{r_0}{N} M_0 \rightarrow r'_0 M'_0$. The field of a ring of $2N$ magnets [described by the coordinates (r, φ)] can thus be found by transforming the field of just 2 semi-circular magnets⁵ [described by the coordinates (r', φ')]. For the latter case the results (19) and (23) have a simple geometric interpretation (see Appendix A). Using the results from the Appendix the solution for the more general case where the magnets are separated by a gap (see Fig.4) can also be written down analytically. Since the expressions are rather cumbersome they are omitted here.

Figure 4: The first period of a ring of $2N$ magnets separated by a gap. The gap length is expressed as a fraction γ of the length of $1/2$ period $\frac{\pi}{N}$.

Figure 5: The radial component of the magnetic induction B_r (a) for the case without gap ($\gamma = 0$) and along the line $\varphi = 0$ (top curve in full) and (b) for the case with a gap ($\gamma = 0.1$) along respectively $\varphi = 0$ (lower curve in full) and $\varphi = \gamma\frac{\pi}{2N}$ (dashed). The other parameters are $N = 9$, $r_0 = 1$ and $r_1 = 0.9$ and B_r is normalized with $\mu_0 M_0$.

Figure 6: The magnitude of the magnetic induction along the circle $r = r_2 + 0.01$ without and with a gap between the magnets. Other parameters and conditions as in Fig.5.

An example of the field pattern in the neighbourhood of the edge between 2 magnets is shown in Fig.5 and Fig.6. As is clear from (A-7) the radial component of the magnetic induction tends to infinity at the corners of the magnets.

4 Force on the magnets.

Due to the anti parallel arrangement of the magnets they will experience a radial outwards oriented force. The resulting force on a single magnet F can conveniently be expressed as an equivalent pressure p which should be applied at the geometric mean radius r_0 and which results in the same total force. One easily finds that F and p are related by:

$$F = 2pr_0 \sin\left(\frac{\pi}{2N}\right) \quad (25)$$

⁴Although we use again accented coordinates one should not confuse this conformal transformation with the previous mentioned inversion symmetry operation.

⁵This particular result remains true as long as the N -fold symmetry is preserved and it is also valid if e.g. the magnets are separated by a gap or by soft magnetic pole-pieces.

We've calculated the force using the Maxwell stress:

$$\vec{t}_M = (\vec{n} \cdot \vec{B}) \frac{\vec{B}}{\mu_0} - \frac{B^2}{2\mu_0} \vec{n} \quad (26)$$

which should be integrated over an arbitrary surface enclosing just one magnet and where $\vec{n}$ is the outward oriented unit vector normal to this surface. We'll use the surface bounded by $\varphi = 0$ and $\varphi = \frac{\pi}{N}$. Along these lines $\vec{B}$ is oriented radial and thus tangential to the surface. The Maxwell stress is therefore a pressure with magnitude $\frac{B_r^2}{2\mu_0}$. The resulting force is then given by:

$$F = \frac{1}{\mu_0} \sin\left(\frac{\pi}{2N}\right) \int_0^\infty B_r^2(r, 0) dr \quad (27)$$

or with (25):

$$p = \frac{1}{2\mu_0 r_0} \int_0^\infty B_r^2(r, 0) dr \quad (28)$$

Using the inversion symmetry the integration over $[r_0, \infty]$ can be reduced to an integration over $[0, r_0]$ with the final result:

$$p = \frac{1}{2\mu_0 r_0} \int_0^{r_0} \left(1 + \left(\frac{r}{r_0}\right)^2\right) B_r^2(r, 0) dr \quad (29)$$

which is obviously always positive. Some results are shown in Fig.7. The pressure increases with increasing relative thickness of the magnets and with decreasing gap width. Note that for $\mu_0 M_0 = 1$ Tesla the pressure is in units of approximately 800 kPa or 8 bar. For the case without air gaps the integrand can be simplified in the limit for large N leading to the following expression (see Appendix B):

$$\frac{p}{\mu_0 M_0^2} = \frac{1}{2\pi^2} \left(\frac{r_0}{r_1} + \frac{r_1}{r_0}\right) \left(\frac{7\zeta(3)}{N} - 8\left(\frac{1}{N} - \ln\left(\frac{r_1}{r_0}\right)\right) \left(\frac{r_1}{r_0}\right)^{2N}\right) \quad (30)$$

where $\zeta(3) = 1.202$ is the zeta-function.

5 Dipole approximation.

Having found an analytical solution it is instructive to analyse how far we can get with the dipole approximation used in [1]. If we consider a single permanent magnet of the set then we know that at a sufficiently large distance its field will

Figure 7: The internal pressure in a circular array of N magnet pairs normalized with $\mu_0 M_0^2$ for $r_1 = \frac{r_0}{\sqrt{2}}$ (3 upper curves) and $r_1 = 0.9r_0$ (3 lower curves) and for $\gamma = 0$ (diamonds), $\gamma = 0.05$ (squares), $\gamma = 0.1$ (circles). For $\gamma = 0$ and N sufficiently large the pressure can be approximated well by (30) (dashed curves).

behave as that of an elementary dipole with a dipole moment equal to the integrated magnetization. It may seem logical then to replace each permanent magnet by such an elementary dipole with dipole moment:

$$m_{naive} = 2M_0 r_0 (r_2 - r_1) \sin\left(\frac{\pi}{2N}\right) \quad (31)$$

which is found by integrating (10) over the area of a single magnet. However due to the symmetric position of the $2N$ magnets the far field (due to the inversion symmetry the limit $r \rightarrow \infty$ is equivalent with $r \rightarrow 0$) is no longer a dipole field. In fact from (19) and (23) one obtains the following limits for $r \rightarrow 0$ and $r \rightarrow \infty$:

$$\begin{aligned} \lim_{r \rightarrow 0} \begin{bmatrix} B_r \\ B_\phi \end{bmatrix} &= \mu_0 M_0 \frac{2}{\pi} \left(\frac{r}{r_0}\right)^{N-1} \left(\left(\frac{r_0}{r_1}\right)^N - \left(\frac{r_0}{r_2}\right)^N \right) \begin{bmatrix} \cos(N\phi) \\ -\sin(N\phi) \end{bmatrix} \\ \lim_{r \rightarrow \infty} \begin{bmatrix} B_r \\ B_\phi \end{bmatrix} &= \mu_0 M_0 \frac{2}{\pi} \left(\frac{r_0}{r}\right)^{N+1} \left(\left(\frac{r_0}{r_1}\right)^N - \left(\frac{r_0}{r_2}\right)^N \right) \begin{bmatrix} \cos(N\phi) \\ \sin(N\phi) \end{bmatrix} \end{aligned} \quad (32)$$

which correspond with the field of a 2^N -pole. If we replace each magnet by an elementary dipole, which we position on the radius r_0 , then the limits of their field will also be that of a 2^N -pole, but it is obvious that the equivalent dipole moment given in (31) will not result in the correct 2^N -pole moment, since e.g. there appears no $\sin\left(\frac{\pi}{2N}\right)$ factor in (32). In other words the magnetic dipole moment m of the equivalent elementary dipoles should be chosen so as to obtain the same 2^N -pole moment as the original configuration and this will be different from the naive value in (31). To obtain the field of the $2N$ dipoles we could sum the $2N$ individual dipole fields which however will be rather tedious. It is much easier to exploit the symmetry and derive the field from the special case ($N = 1$) and the conformal transformation formulas given in the text after (24) ⁶. One then finds:

$$\frac{B_{r,dipole}}{\mu_0 M_0} = \frac{N^2 m}{\pi r r_0} \left(\frac{r}{r_0}\right)^N \left(1 - \left(\frac{r}{r_0}\right)^{2N}\right) \frac{\left(\frac{r}{r_0}\right)^{4N} + 2(2 - \cos(2N\phi)) \left(\frac{r}{r_0}\right)^{2N} + 1}{\left(\left(\frac{r}{r_0}\right)^{4N} + 2\cos(2N\phi) \left(\frac{r}{r_0}\right)^{2N} + 1\right)^2} \cos(N\phi) \quad (33)$$

$$\frac{B_{\phi,dipole}}{\mu_0 M_0} = -\frac{N^2 m}{\pi r r_0} \left(\frac{r}{r_0}\right)^N \left(1 + \left(\frac{r}{r_0}\right)^{2N}\right) \frac{\left(\frac{r}{r_0}\right)^{4N} - 2(2 + \cos(2N\phi)) \left(\frac{r}{r_0}\right)^{2N} + 1}{\left(\left(\frac{r}{r_0}\right)^{4N} + 2\cos(2N\phi) \left(\frac{r}{r_0}\right)^{2N} + 1\right)^2} \sin(N\phi) \quad (34)$$

⁶Under this transformation dipole moments transform according to $\frac{m'}{r_0'} = N \frac{m}{r_0}$. Note also that (31) is not compatible with this transformation which is another indication that this expression cannot be correct.

Taking the limit for $r \rightarrow 0$ and comparing with (32) one arrives at the correct equivalent dipole moment:

$$m = \frac{2M_0 r_0^2}{N^2} \left(\left(\frac{r_0}{r_1} \right)^N - \left(\frac{r_0}{r_2} \right)^N \right) \quad (35)$$

For $N = 1$ the whole configuration behaves as a dipole and only for this case (31) gives the same result as (35).

Fig.8 gives an idea of the accuracy of the dipole approximation. For large N the field decays rather rapidly with r and the dipole approximation gives reasonable results up to fairly close to the magnet boundaries. Of course the approximation breaks down inside the magnets and also the singularity in B_r is absent. For smaller values of N the approximation becomes progressively worse (not shown).

6 Discussion.

We have presented an analytical expression for the magnetic field of a periodic circular array of head-to-head permanent magnets, which finds application in conventional (and maybe less conventional) systems for separating particles based on their magnetic properties. In doing so we also found some symmetry properties which may have more widespread application. In order to arrive at the analytical result we made some simplifications. The analysis was limited to 2 dimensions. For the conventional device as in [3] this is no severe restriction since the length of the device is much larger than is diameter. However in this device the array of magnets only covers part of the circumference. It is to be expected that the field pattern nearby the gap between consecutive magnets (which matters must) will not change appreciably by this. Of course the far away field will change and will become a dipole field now. If desired the exact field could be calculated using the formulas given in the Appendix A. For the more exotic devices presented in [2] the 2-dimensional approximation will probably be too rough.

Another more severe restriction is the absence of soft ferromagnetic pole-pieces, which are considered in [3]. It is an experimental fact that the repulsive force between 2 head-to-head permanent magnets may be overcome by separating them by a sufficiently large soft ferromagnetic plate. However in the present case it is to be expected that due to the curvature the pressure will remain directed outward although it will become smaller. This has been verified to some extent using a commercial electromagnetics software package. The presence of ferromagnetic pole-pieces will of course strongly influence the field pattern near the gap. The force on a small spherical particle (radius R , relative permeability μ) is given by:

$$F = 2\pi\mu_0 R^3 \frac{\mu - 1}{\mu + 2} \nabla(H^2) \quad (36)$$

Figure 8: Comparison between the exact field components (full lines) and the dipole approximation (dashed lines). Shown are the radial component B_r along $\varphi = 0$ (a) and the angular component B_ϕ along $\varphi = \frac{\pi}{2N}$ (b). The components are normalized with $\mu_0 M_0$ and $N = 9$, $r_0 = 1$, $r_1 = \frac{1}{\sqrt{2}}$.

where H is the magnetic field produced by the separating device. It is not immediately obvious that the presence of a soft ferromagnetic pole-piece will lead to a larger gradient of the field squared and thus the force, and it would certainly be interesting to find the optimal configuration. The 2-dimensional problem where the gaps are filled with soft ferromagnetic material can also be solved using the “Separation by variables” method, but requires solving a set of linear equations with the expansion coefficients as unknowns. Since a large number of coefficients may be required in order to properly resolve the singularities in the field, this may not be the most attractive method and probably a numerical procedure will be more appropriate.

References

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APPENDIX A: direct calculation for $N = 1$.

Consider the magnetic field $d\vec{H}$ at a point P from an infinitesimal current sheet Kdl situated at Q (see Fig.9). One can always suppose that dl is situated on a circle with midpoint O and radius R , so that $dl = Rd\psi$, where $d\psi$ is the angle subtended by dl at O. Taking O as the centre of polar coordinates (r, φ) , the polar components of the magnetic field are given by:

$$dH_r = \frac{KRd\psi}{2\pi r'} \sin \theta \quad dH_\varphi = \frac{KRd\psi}{2\pi r'} \cos \theta \quad (\text{A-1})$$

Figure 9: Coordinates used for calculating the field of a 2-dimensional current sheet (left) and of a cylindrical current sheet (right). Note that the begin- (A) and end-point (B) of the arc are chosen according to the direction of φ .

where r' is the distance between P and Q and θ the angle subtended by OQ at P. Using the sine-rule:

$$\sin \theta = \frac{R}{r} \sin \alpha \quad (\text{A-2})$$

where α is defined in Fig.9, and $dr' = Rd\psi \sin \alpha$, the radial component is written as:

$$dH_r = \frac{KR^2 d\psi}{2\pi r r'} \sin \alpha = \frac{K}{2\pi} \frac{R}{r} \frac{dr'}{r'} \quad (\text{A-3})$$

With $r'd\theta = Rd\psi \cos \alpha$ the tangential component is first rewritten as:

$$dH_\varphi = \frac{K d\theta \cos \theta}{2\pi \cos \alpha} \quad (\text{A-4})$$

But differentiating (A-2) we find (r and R are to be considered constant):

$$\cos \theta d\theta = \frac{R}{r} \cos \alpha d\alpha \quad (\text{A-5})$$

and therefore:

$$dH_\varphi = \frac{K}{2\pi} \frac{R}{r} d\alpha = \frac{K}{2\pi} \frac{R}{r} (d\psi + d\theta) \quad (\text{A-6})$$

Figure 10: Application of Fig.9 to the special case $N = 1$ and no air gap. The coordinates r_+ , α_+ , r_- and α_- are shown for the upper current sheet.

For a cylindrical current sheet (A-3) and (A-6) are easily integrated and we then find for the field components:

$$H_r = \frac{K R}{2\pi r} \ln \left(\frac{r_+}{r_-} \right) \quad H_\phi = \frac{K R}{2\pi r} (\alpha_+ - \alpha_-) = \frac{K R}{2\pi r} (\psi + \theta) \quad (\text{A-7})$$

where (r_+, α_+) are the coordinates (r', α) of the end point B and similarly (r_-, α_-) those of the begin point A, and ψ and θ are the angles subtended by the arc AB at respectively the centre O and the observation point P. If one moves the observation point P continuously without crossing the current sheet and takes into account the sign reversal of θ when crossing the line through A and B (but only outside the circle) then these expressions are valid everywhere.

With (A-7) the 2-dimensional problems treated in the main text can be solved. Consider the case without air gap and for $N = 1$. It suffices to consider e.g. the current distribution on the inner sides of the 2 magnets (Fig.10).

With:

$$r_\pm^2 = r^2 + R^2 \pm 2rR \cos \varphi \quad (\text{A-8})$$

the radial component of the magnetic field due to the upper semi-circular arc is given by:

$$H_{r,upper} = \frac{K R}{2\pi r} \ln \sqrt{\frac{r^2 + R^2 + 2rR \cos \varphi}{r^2 + R^2 - 2rR \cos \varphi}} \quad (\text{A-9})$$

For the lower semi-circular arc we have to switch the sign of K but at the same time the roles of r_+ and r_- are reversed meaning another switch of sign. Therefore

the contributions of the 2 halves to H_r are exactly equal and the total H_r is given by twice the expression in (A-9). With $R = r_1$, $K = M_0 \frac{r_0}{r_1}$ this is in agreement with (19) in the main text.

A similar reasoning can be followed for the φ -component. The contribution from the 2 halves are respectively:

$$H_{\varphi,upper} = \frac{K}{2\pi} \frac{R}{r} (\pi + \theta) \quad H_{\varphi,lower} = -\frac{K}{2\pi} \frac{R}{r} (\pi - \theta) \quad (\text{A-10})$$

and therefore the total component is given by:

$$H_{\varphi} = \frac{K}{\pi} \frac{R}{r} \theta \quad (\text{A-11})$$

The angle θ can with the help of Fig.10 be expressed as:

$$\theta = \arctg\left(\frac{y}{x-R}\right) - \arctg\left(\frac{y}{x+R}\right) = \arctg\left(\frac{2Rr \sin \varphi}{r^2 - R^2}\right) \quad (\text{A-12})$$

where x, y are the coordinates of the observation point. substituting in (A-11) one finds an expression in agreement with (23) in the main text.

APPENDIX B: proof of Eq. (29).

The equivalent pressure in (29) with $B_r(r, 0)$ given by (19) and (20) becomes:

$$\frac{p}{\mu_0 M_0^2} = \frac{1}{2\pi^2} \int_0^{r_0} \left(1 + \left(\frac{r_0}{r}\right)^2\right) \left(F\left(\left(\frac{r}{r_1}\right)^N, 0\right) - F\left(\left(\frac{r}{r_2}\right)^N, 0\right)\right)^2 d\left(\frac{r}{r_0}\right) \quad (\text{B-1})$$

where:

$$F(s, 0) = \ln\left(\frac{1+s}{|1-s|}\right) \quad (\text{B-2})$$

Using the new variable $\left(\frac{r}{r_1}\right)^N = u$ and the notation $\frac{r_1}{r_0} = \frac{r_0}{r_2} = \alpha$ the integral becomes:

$$\frac{p}{\mu_0 M_0^2} = \frac{1}{2\pi^2} \frac{1}{N} \int_0^{\alpha^{-N}} \left(\alpha u^{\frac{1}{N}} + \frac{1}{\alpha} u^{-\frac{1}{N}}\right) (F(u, 0) - F(u\alpha^{2N}, 0))^2 \frac{1}{u} du \quad (\text{B-3})$$

We note that $F(u, 0)$ shows a singularity for $u = 1$. Therefore we can approximate the first factor under the integral by its value for $u = 1$. Secondly, for not too small values of N , $u\alpha^{2N} \leq \alpha^N \ll 1$ and therefore we can make the approximation:

$$F(u\alpha^{2N}, 0) \approx 2u\alpha^{2N} \quad (\text{B-4})$$

and thus:

$$\frac{p}{\mu_0 M_0^2} \approx \frac{1}{2\pi^2} \frac{1}{N} \left(\alpha + \frac{1}{\alpha} \right) \int_0^{\alpha^{-N}} \left(\ln \left(\frac{1+u}{|1-u|} \right) - 2u\alpha^{2N} \right)^2 \frac{1}{u} du \quad (\text{B-5})$$

Expanding the integrandum we get:

$$\frac{p}{\mu_0 M_0^2} \approx \frac{1}{2\pi^2} \frac{1}{N} \left(\alpha + \frac{1}{\alpha} \right) \left(\int_0^{\alpha^{-N}} \ln \left(\frac{1+u}{|1-u|} \right)^2 \frac{1}{u} du - 4\alpha^{2N} \int_0^{\alpha^{-N}} \ln \left(\frac{1+u}{|1-u|} \right) du + 2\alpha^{2N} \right) \quad (\text{B-6})$$

The remaining integrals are calculated keeping at most terms of first order in α^{2N} :

$$\begin{aligned} \int_0^{\alpha^{-N}} \ln \left(\frac{1+u}{|1-u|} \right) du &= \int_0^1 \ln \left(\frac{1+u}{1-u} \right) du + \int_1^{\alpha^{-N}} \ln \left(\frac{u+1}{u-1} \right) du \\ &= 2 - 2N \ln \alpha - \frac{1}{3} \alpha^{2N} + O(\alpha^{4N}) \end{aligned} \quad (\text{B-7})$$

$$\begin{aligned} \int_0^{\alpha^{-N}} \ln \left(\frac{1+u}{|1-u|} \right)^2 \frac{1}{u} du &= \int_0^1 \ln \left(\frac{1+u}{1-u} \right)^2 \frac{1}{u} du + \int_1^{\alpha^{-N}} \ln \left(\frac{u+1}{u-1} \right)^2 \frac{1}{u} du \\ &= 7\zeta(3) - 2\alpha^{2N} + O(\alpha^{4N}) \end{aligned} \quad (\text{B-8})$$

Collecting all terms we find:

$$\frac{p}{\mu_0 M_0^2} \approx \frac{1}{2\pi^2} \left(\alpha + \frac{1}{\alpha} \right) \left(\frac{7\zeta(3)}{N} - 8\alpha^{2N} \left(\frac{1}{N} - \ln \alpha \right) + O(\alpha^{4N}) \right) \quad (\text{B-9})$$