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1 Introduction

Artificial boundary conditions (ArBC), defined as Robin boundary conditions (RBC) positioned at a complex space coordinate, are introduced in order to obtain pertinent approximations for the Green's functions in grounded layered media. These artificial boundary conditions include perfectly matched layers (PML) [1]-[2] backed by perfectly electric (PEC) or magnetic conductors (PMC). Since weighted sums of Green's functions are again Green's functions, a weighted mixture of ArBC Green's functions of the PML-PEC and PML-PMC type can be utilized to approximate a given Green's function. The advantage of the approach is that it yields a comprehensive eigenexpansion for the Green's function at hand. An example illustrates the goodness of fit between the approximate Green's function and the Green's function for the uniform halfspace.

2 Artificial boundary conditions

Any laterally symmetric scalar Green's function in a layered medium satisfies the source Helmholtz equation [3]

$$\nabla^2 G(\rho, z, z') + \kappa(z)^2 G(\rho, z, z') = \frac{1}{2\pi\rho} \delta(\rho) \delta(z - z') \quad (1)$$

together with appropriate boundary conditions. With radiation conditions [2] in the lateral direction, we can write this in the Hankel domain as

$$-\gamma^2 \tilde{G}(\gamma, z, z') + \frac{d^2}{dz^2} \tilde{G}(\gamma, z, z') + \kappa(z)^2 \tilde{G}(\gamma, z, z') = \delta(z - z') \quad (2)$$

where $G(\rho, z, z')$ is retrieved by means of the Hankel transform

$$G(\rho, z, z') = \frac{1}{2\pi} \int_0^\infty J_0(\gamma\rho) \tilde{G}(\gamma, z, z') \gamma d\gamma \quad (3)$$

Applying different boundary conditions to (2) will lead to different solutions in the Hankel domain. For example the Green's function for a uniform halfspace ($\kappa(z) = k_0$, where $k_0 = \omega/c$ is the free-space wavenumber, assuming a $e^{i\omega t}$ time dependence) grounded at $z = 0$ is analytically given by

$$G_H(\rho, z, z') = \frac{e^{-ik_0\sqrt{\rho^2+(z+z')^2}}}{4\pi\sqrt{\rho^2+(z+z')^2}} - \frac{e^{-ik_0\sqrt{\rho^2+(z-z')^2}}}{4\pi\sqrt{\rho^2+(z-z')^2}} \quad (4)$$

and transforms in the Hankel domain to [4]

$$\tilde{G}_H(\gamma, z, z') = -\frac{1}{\beta} \sin \beta z_< e^{-i\beta z_>} \quad (5)$$

where

$$\beta = \sqrt{k_0^2 - \gamma^2} \quad z_< = \min(z, z') \quad z_> = \max(z, z') \quad (6)$$

We can write equation (2) with the new parameter β as

$$\beta^2 \tilde{G}(\beta, z, z') + \frac{d^2}{dz^2} \tilde{G}(\beta, z, z') - U(z) \tilde{G}(\beta, z, z') = \delta(z - z'), \quad U(z) = k_0^2 - \kappa(z)^2 \quad (7)$$

Besides the grounded Dirichlet condition $\tilde{G}(\beta, 0, z') = 0$ we must also fix a boundary condition at $z = 1$. To obtain outgoing waves at $z = 1$ we need, in accordance with (5), the outgoing Robin boundary condition (ORBC)

$$\tilde{G}'(\beta, 1, z') = -i\beta \tilde{G}(\beta, 1, z') \quad (8)$$

The problem with the ORBC is that it depends on β and hence on the Hankel transform variable γ . We would be more pleased with a RBC

$$\tilde{G}'(\beta, 1, z') = \tau \tilde{G}(\beta, 1, z') \quad (9)$$

with τ a constant, since an eigensystem expansion — see next section — would then offer an elegant expression for the Green's function. Note that $\tau = 0$ corresponds with Neumann boundary conditions and $\tau = \infty$ with Dirichlet boundary conditions.

Unfortunately, as $\tau \neq -i\beta$, the Green's function thus obtained cannot certainly be expected to be sufficiently close to the Green's function with ORBC. If however we suppose that the Green's function can be analytically extended such that complex z are allowed, then we could impose a constant parameter RBC at $z = b$, where $b \in C$. In other words we require

$$\tilde{G}'(\beta, b, z') = \tau \tilde{G}(\beta, b, z') \quad (10)$$

Let $\tilde{G}_0(\beta, z, z')$ be the exact Green's function with ORBC in the Hankel domain. Then the Green's function in the spatial domain can be found by means of the Hankel transform (3). This would be the end of the story, were it not for the fact that the Hankel transform (3) is notoriously hard to evaluate [2], because the integrand is often highly oscillatory and exhibits singularities of the pole and branchpoint type. Hence our purpose is to avoid the Hankel transform approach altogether and to approximate $G_0(\rho, z, z')$ by a mixture of Green's functions of the ArBC type for which, as will be shown in the next section, explicit eigenexpansions can be written down. Hence we propose the approximation

$$G_0(\rho, z, z') \approx \sum_{k=1}^M q_k G_k(\rho, z, z') \quad (11)$$

where, for this approximation to be a valid Green's function, we must require that the complex coefficients q_k are such that $\sum_{k=1}^M q_k = 1$.

3 Eigenexpansions

Suppose we impose the ArBC

$$\tilde{G}'(\beta, b, z') = \tau \tilde{G}(\beta, b, z') \quad (12)$$

with $b \in C$. Then, taking into account the grounded condition $\tilde{G}(\beta, 0, z') = 0$, it is clear [2]-[3] that solving the eigenvalue problem

$$\frac{d^2}{dz^2} \psi(z) + \kappa(z)^2 \psi(z) = -\lambda \psi(z) \quad (13)$$

with boundary conditions

$$\psi(0) = 0, \quad \psi'(b) = \tau \psi(b) \quad (14)$$

leads to a comprehensive series expansion for the spatial Green's function $G(\rho, z, z')$ with defining equation (1). Using separation of variables [3] we obtain the Green's function series expansion

$$G(\rho, z, z') = \frac{i}{4} \sum_{n=1}^{\infty} H_0^{(2)} \left(-i\rho \sqrt{\lambda_n} \right) \psi_n(z) \psi_n(z') \quad (15)$$

where the $\{\psi_n(z)\}$ are the normalized eigenfunctions. It is readily verified that the eigenfunctions are b -orthonormal in the assumption of non-degeneracy, i.e.,

$$\int_0^b \psi_n(z) \psi_m(z) dz = \delta_{n,m} \quad (16)$$

To find the eigensystem in the general case we utilize the following perturbation approach. Suppose we have obtained the eigenfunctions $\{\chi_n(z)\}$ and the eigenvalues $\{\mu_n\}$ pertaining to the unperturbed eigenvalue problem

$$\frac{d^2}{dz^2} \chi(z) = -\mu \chi(z) \quad (17)$$

with the same boundary conditions

$$\chi(0) = 0, \quad \chi'(b) = \tau \chi(b) \quad (18)$$

If we suppose the $\{\chi_n(z)\}$ complete in $L_2[0,1]$, we can expand any $\psi_n(z)$ of the eigenvalue problem (13)-(14) as

$$\psi_n(z) = \sum_k D_{n,k} \chi_k(z) \quad (19)$$

Utilizing the b -orthonormality (16) we obtain via Galerkin projection

$$-D_{n,m} \mu_m + \sum_k D_{n,k} \int_0^b \chi_m(z) \kappa(z)^2 \chi_k(z) dz = -\lambda_n D_{n,m} \quad (20)$$

Truncating the expansions (15) and (19) to their first N terms, the equations (20) can be written in matrix format as $DY + DP = \Lambda D$, where the entries of the matrix P are given by

$$P_{k,l} = - \int_0^b \chi_k(z) \kappa(z)^2 \chi_l(z) dz = \int_0^1 \chi_k(z) U(z) \chi_l(z) dz - k_0^2 \delta_{k,l} \quad (21)$$

and $Y = \text{diag}(\mu_1, \dots, \mu_N)$, $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_N)$. The eigendecomposition $P + Y = Q \Lambda Q^{-1}$ then provides the eigenvalues λ_n and the matrix $D = Q^{-1}$, assuming P diagonalizable. Since P is complex symmetric the eigenvectors can be chosen to be complex orthonormal [5], i.e., $Q^T Q = I$, in conformity with the b -orthonormality requirement $DD^T = I$.

In the unperturbed Dirichlet case $\tau = \infty$, $U(z) = 0$ we have the Hankel domain Green's function

$$\tilde{G}(\beta, z, z') = - \frac{\sin \beta z < \sin \beta(b - z >)}{\beta \sin \beta b} \quad (22)$$

The corresponding spatial Green's function can be written [2] as

$$G(\rho, z, z') = \frac{i}{4} \sum_{n=1}^{\infty} H_0^{(2)} \left(-i\rho \sqrt{\mu_n - k_0^2} \right) \chi_n(z) \chi_n(z') \quad (23)$$

where the eigensystem $\{\chi_n(z), \mu_n\}$ is given by

$$\chi_n(z) = \sqrt{\frac{2}{b}} \sin \left(\frac{n\pi z}{b} \right), \quad \mu_n = \left(\frac{n\pi}{b} \right)^2 \quad (24)$$

In the unperturbed Neumann case $\tau = 0$, $U(z) = 0$ we have the Hankel domain Green's function

$$\tilde{G}(\beta, z, z') = - \frac{\sin \beta z < \cos \beta(b - z >)}{\beta \cos \beta b} \quad (25)$$

The corresponding spatial Green's function is again given by the expansion (23) where the eigensystem $\{\chi_n(z), \mu_n\}$ is now

$$\chi_n(z) = \sqrt{\frac{2}{b}} \sin\left(\frac{(n-1/2)\pi z}{b}\right), \quad \mu_n = \left(\frac{(n-1/2)\pi}{b}\right)^2 \tag{26}$$

As an example we consider the approximation of the grounded unperturbed halfspace Green's function (4) by means of a Dirichlet-Neumann ArBC mixture ($M = 2$) with $k_0 = 1$, $b_1 = b_2 = 1.3 - 0.5i$ and $q_1 = 1 - q_2 = 0.5 + 0.1i$ and truncation at $N = 32$. The results are shown in Figure 1. It is seen that

Figure 1: The absolute error (in dB) $|G_H(\rho, 1/2, 1/2) - G_{ArBC}(\rho, 1/2, 1/2)|$ in the Dirichlet, Neuman and Mixture cases

the mixture ArBC is better than the Neumann or Dirichlet ArBC's taken separately, especially for small values of ρ .

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