A characterization of multiple \((n - k)\)-blocking sets in projective spaces of square order

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Abstract

In [10], it was shown that small \(t\)-fold \((n - k)\)-blocking sets in \(\text{PG}(n,q)\), \(q = p^h\), \(p\) prime, \(h \geq 1\), intersect every \(k\)-dimensional space in \(t \mod p\) points. We characterize in this article all \(t\)-fold \((n - k)\)-blocking sets in \(\text{PG}(n,q)\), \(q\) square, \(q \geq 661\), \(t < c_p q^{1/6}/2\), \(|B| < tq^{n-k} + 2tq^{n-k-1} \sqrt{q}\), intersecting every \(k\)-dimensional space in \(t \mod \sqrt{q}\) points.

1 Introduction

Throughout this paper, \(\text{PG}(n,q)\) will denote the \(n\)-dimensional projective space over the Galois field \(\text{GF}(q)\), where \(q = p^h\), \(p\) prime.

A \(t\)-fold \((n - k)\)-blocking set \(B\) of \(\text{PG}(n,q)\), with \(0 < k < n\), is a set of points of \(\text{PG}(n,q)\) intersecting every \(k\)-dimensional subspace of \(\text{PG}(n,q)\) in at least \(t\) points.

A 1-fold \((n - k)\)-blocking set \(B\) of \(\text{PG}(n,q)\) containing an \(\text{PG}(n - k,q)\) is called trivial.

A point \(r\) of \(B\) is called essential if there is a \(k\)-dimensional subspace through \(r\) intersecting \(B\) in precisely \(t\) points. The \(t\)-fold blocking set \(B\) is called minimal if all of its points are essential. A 1-fold \((n - k)\)-blocking set is also called an \((n - k)\)-blocking set. A \(t\)-fold 1-blocking set in \(\text{PG}(2,q)\) is also called a \(t\)-fold blocking set, or a \(t\)-fold planar blocking set.

These latter \(t\)-fold planar blocking sets have been studied in great detail.

**Theorem 1.1** (Blokhuis et al. [8]) Let \(B\) be a \(t\)-fold blocking set in \(\text{PG}(2,q)\), \(q = p^h\), \(p\) prime, of size \(t(q + 1) + c\). Let \(c_2 = c_3 = 2^{-1/3}\) and \(c_p = 1\) for

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If \( q = p^{2d+1} \) and \( t < q/2 - c_p q^{2/3}/2 \), then \( c \geq c_p q^{2/3} \), unless \( t = 1 \) in which case \( B \) with \( |B| < q + 1 + c_p q^{2/3} \), contains a line.

(2) If \( 4 < q \) is a square, \( t < c_p q^{1/6} \) and \( c < c_p q^{2/3} \), then \( c \geq \sqrt{tq} \) and \( B \) contains the union of \( t \) pairwise disjoint Baer subplanes, except for \( t = 1 \) in which case \( B \) contains a line.

(3) If \( q = p^2 \), \( p \) prime, and \( t < q^{1/4}/2 \) and \( c < p[\frac{1}{4} + \sqrt{\frac{p+1}{2}}] \), then \( c \geq t\sqrt{q} \) and \( B \) contains the union of \( t \) pairwise disjoint Baer subplanes, except for \( t = 1 \) in which case \( B \) contains a line or a Baer subplane.

**Theorem 1.2** (Ball [1]) A \( t \)-fold blocking set in \( PG(2, q) \) which does not contain a line has at least \( tq + \sqrt{tq} + 1 \) points.

If \( B \) is a \( t \)-fold blocking set in \( PG(2, p) \), where \( p > 3 \) is prime, and if \( 1 < t < p/2 \), then \( |B| \geq (t+1/2)(p+1) \), while if \( t > p/2 \), then \( |B| \geq (t+1)p \).

In the theory of 1-fold planar blocking sets, \( 1 \mod p \) results for small 1-fold planar blocking sets play an important role.

**Definition 1.3** A blocking set of \( PG(2, q) \) is called small when it has less than \( 3(q+1)/2 \) points.

If \( q = p^h \), \( p \) prime, \( h \geq 1 \), the exponent \( e \) of the minimal blocking set \( B \) of \( PG(2, q) \) is the maximal integer \( e \) such that every line intersects \( B \) in \( 1 \mod p^e \) points.

**Theorem 1.4** Let \( B \) be a small minimal 1-fold blocking set in \( PG(2, q) \), \( q = p^h \), \( p \) prime, \( h \geq 1 \). Then \( B \) intersects every line in \( 1 \mod p^e \) points, so for the exponent \( e \) of \( B \), we have \( 1 \leq e \leq h \). (Szönyi [18])

In fact, this exponent \( e \) is a divisor of \( h \). (Sziklai [17])

This result was extended by Szönyi and Weiner [19] to 1-fold \((n-k)\)-blocking sets in \( PG(n, q) \).

**Definition 1.5** A 1-fold \((n-k)\)-blocking set of \( PG(n, q) \) is called small when it has less than \( 3(q^{n-k}+1)/2 \) points.

If \( q = p^h \), \( p \) prime, \( h \geq 1 \), the exponent \( e \) of the minimal 1-fold \((n-k)\)-blocking set \( B \) is the maximal integer \( e \) such that every hyperplane intersects \( B \) in \( 1 \mod p^e \) points.

A most interesting question of the theory of blocking sets is to classify the small blocking sets. A natural construction (blocking the \( k \)-subspaces of \( PG(n, q) \)) is a subgeometry \( PG(h(n-k)/e, p^e) \), if it exists (recall \( q = p^h \), so \( 1 \leq e \leq h \) and \( e|h \)).
It is easy to see that the projection of a blocking set, w.r.t. $k$-dimensional subspaces, from a vertex $V$ onto an $r$-dimensional subspace of $\text{PG}(n,q)$, is again a blocking set, w.r.t. the $(k+r-n)$-dimensional subspaces of $\text{PG}(r,q)$ (where $\text{dim}(V) = n - r - 1$ and $V$ is disjoint from the blocking set).

A blocking set of $\text{PG}(r,q)$, which is a projection of a subgeometry of $\text{PG}(n,q)$, is called linear. (Note that the trivial blocking sets are linear as well.) Linear blocking sets were defined by Lunardon, and they were first studied by Lunardon, Polito and Polverino [12], [13].

**Conjecture 1.6 (Linearity Conjecture [17])** In $\text{PG}(n,q)$, every small minimal blocking set, with respect to $k$-dimensional subspaces, is linear.

There are some cases of the Conjecture that are proved already.

**Theorem 1.7** For $q = p^h$, $p$ prime, $h \geq 1$, every small minimal non-trivial blocking set w.r.t. $k$-dimensional subspaces is linear, if

(a) $n = 2$, $k = 1$ (so we are in the plane) and:

(i) (Blokhuis [5]) $h = 1$ (i.e. there is no small non-trivial blocking set at all);

(ii) (Szőnyi [18]) $h = 2$ (the only non-trivial example is a Baer subplane with $p^2 + p + 1$ points);

(iii) (Polverino [14]) $h = 3$ (there are two examples, one with $p^3 + p^2 + 1$ and another with $p^3 + p^2 + p + 1$ points);

(iv) (Blokhuis, Ball, Brouwer, Storme, Szőnyi [6], Ball [2]) if $p > 2$ and there exists a line $\ell$ intersecting $B$ in $|B \cap \ell| = |B| - q$ points (so a blocking set of Rédei type);

(b) for general $k$:

(i) (Szőnyi and Weiner [19]) if $h(n-k) \leq n$, $p > 2$, and $B$ is not contained in an $(h(n-k)-1)$-dimensional subspace;

(ii) (Storme and Weiner [16] (for $k = n - 1$), Bokler [9] and Weiner [20]) $h = 2$, $q \geq 16$;

(iii) (Storme and Sziklai [15]) if $p > 2$ and there exists a hyperplane $H$ intersecting $B$ in $|B \cap H| = |B| - q^{n-k}$ points (so a blocking set of Rédei type).

The following (mod $p$) result is known.
Theorem 1.8 (Szönyi and Weiner [19]) A minimal 1-fold \((n-k)\)-blocking set in \(\text{PG}(n,q)\), \(q = p^h\), \(p > 2\) prime, of size less than \(\frac{3}{2}(q^{n-k} + 1)\) intersects every subspace in zero or in 1 \((\text{mod } p)\) points.

There is an even more general version of the Conjecture. A \(t\)-fold blocking set w.r.t. \(k\)-dimensional subspaces is a point set which intersects each \(k\)-dimensional subspace in at least \(t\) points. Multiple points may be allowed as well.

Conjecture 1.9 (Linearity Conjecture for multiple blocking sets [17])

In \(\text{PG}(n,q)\), any \(t\)-fold minimal blocking set \(B\), with respect to \(k\)-dimensional subspaces, is the union of some (not necessarily disjoint) linear point sets \(B_1, \ldots, B_s\), where \(B_i\) is a \(t_i\)-fold blocking set w.r.t. \(k\)-dimensional subspaces and \(t_1 + \cdots + t_s = t\); provided that \(t\) and \(|B|\) are small enough (\(t \leq T(n,q,k)\) and \(|B| \leq S(n,q,k)\) for two suitable functions \(T\) and \(S\)).

Again, some cases of this conjecture have been proved already; in this paper, we cover many new cases which provide “evidence” to the Linearity Conjecture for multiple blocking sets.

Note that there exists a \((\sqrt[q]{q} + 1)\)-fold blocking set in \(\text{PG}(2,q)\), constructed by Ball, Blokhuis and Lavrauw [3], which is not the union of smaller blocking sets. (This multiple blocking set is a linear point set.)

The 1 \((\text{mod } p)\) result in \(\text{PG}(2,q)\), \(q = p^h\), \(p\) prime, was extended by Blokhuis et al. to a \(t\) \((\text{mod } p)\) result on small minimal \(t\)-fold blocking sets in \(\text{PG}(2,q)\) [7].

Definition 1.10 A \(t\)-fold blocking set of \(\text{PG}(2,q)\) is called small when it has less than \((t + 1/2)(q+1)\) points.

If \(q = p^h\), \(p\) prime, the exponent \(e\) of the minimal \(t\)-fold blocking set \(B\) in \(\text{PG}(2,q)\) is the maximal integer \(e\) such that every line intersects \(B\) in \(t\) \((\text{mod } p^e)\) points.

Theorem 1.11 (Blokhuis et al. [7]) Let \(B\) be a small minimal \(t\)-fold blocking set in \(\text{PG}(2,q)\), \(q = p^h\), \(p\) prime, \(h \geq 1\). Then \(B\) intersects every line in \(t\) \((\text{mod } p)\) points.

Regarding characterization results on small minimal 1-fold \((n-k)\)-blocking sets in \(\text{PG}(n,q)\), we mention the following results.

In the next theorem, \(\theta_m\) denotes the size of an \(m\)-dimensional space \(\text{PG}(m,q)\).
Theorem 1.12 (Bokler [9]) The minimal \((n-k)\)-blocking sets of cardinality at most \(\theta_{n-k} + \sqrt[4]{q}\) in projective spaces \(\text{PG}(n,q)\) of square order \(q\), \(q \geq 16\), are Baer cones with an \(m\)-dimensional vertex \(\text{PG}(m,q)\) and base a Baer subgeometry \(\text{PG}(2(n-k-m-1),\sqrt[4]{q})\), for some \(m\) with \(\max\{-1, n-2k-1\} \leq m \leq n-k-1\).

In the following theorem, \(s(q)\) denotes the size of the smallest blocking set in \(\text{PG}(2,q)\), \(q\) square, not containing a line or Baer subplane.

Theorem 1.13 (Storme and Weiner [16]) Let \(K\) be a minimal 1-blocking set in \(\text{PG}(n,q)\), \(q\) square, \(q = p^h\), \(h \geq 1\), \(p \geq 3\) prime, \(n \geq 3\), with \(|K| \leq s(q)\). Then \(K\) is a line or a minimal planar blocking set of \(\text{PG}(n,q)\).

Theorem 1.14 (Storme and Weiner [16]) In \(\text{PG}(n,q^3)\), \(q = p^h\), \(h \geq 1\), \(p\) prime, \(p \geq 7\), \(n \geq 3\), a minimal 1-blocking set \(K\) of cardinality at most \(q^3 + q^2 + q + 1\) is either:

1. a line;
2. a Baer subplane when \(q\) is a square;
3. a minimal blocking set of cardinality \(q^3 + q^2 + 1\) in a plane of \(\text{PG}(n,q^3)\);
4. a minimal blocking set of cardinality \(q^3 + q^2 + q + 1\) in a plane of \(\text{PG}(n,q^3)\);
5. a subgeometry \(\text{PG}(3,q)\) in a 3-dimensional subspace of \(\text{PG}(n,q^3)\).

The following result was the first characterization result to use the 1 (mod \(p\)) result of Theorem 1.8.

Theorem 1.15 (Weiner [20]) Let \(B\) be a 1-fold \((n-k)\)-blocking set in \(\text{PG}(n,q = p^{2h})\), \(p \geq 2\) prime, \(q \geq 81\), of size \(|B| < 3(q^{n-k} + 1)/2\) and intersecting every \(k\)-space in 1 (mod \(\sqrt[4]{q}\)) points. Then \(B\) is a Baer cone with an \(m\)-dimensional vertex \(\text{PG}(m,q)\) and base a Baer subgeometry \(\text{PG}(2(n-k-m-1),\sqrt[4]{q})\), for some \(m\) with \(\max\{-1, n-2k-1\} \leq m \leq n-k-1\).

Regarding characterizations of small minimal \(t\)-fold \((n-k)\)-blocking sets in \(\text{PG}(n,q)\), we mention the following result.

Theorem 1.16 (Barát and Storme [4]) Let \(B\) be a \(t\)-fold 1-blocking set in \(\text{PG}(n,q)\), \(q = p^h\), \(p\) prime, \(q \geq 661\), \(n \geq 3\), of size \(|B| < tq + c_p q^{2/3} - (t-1)(t-2)/2\), with \(c_2 = c_3 = 2^{-1/3}\), \(c_p = 1\) when \(p \geq 3\), and with \(t < \min(c_p q^{1/6}, q^{1/4}/2)\). Then \(B\) contains a union of \(t\) pairwise disjoint lines and/or Baer subplanes.

Recently, in [10], the following \(t\) (mod \(p\)) result on weighted \(t\)-fold \((n-k)\)-blocking sets in \(\text{PG}(n,q)\) has been obtained.
Theorem 1.17 (Ferret et al. [10]) Let $B$ be a minimal weighted $t$-fold $(n-k)$-blocking set of $\text{PG}(n,q)$, $q = p^h$, $p$ prime, $h \geq 1$, of size $|B| = tq^{n-k}+t+k'$, with $t+k' \leq (q^{n-k}-1)/2$.

Then $B$ intersects every $k$-dimensional space in $t \pmod{p}$ points.

We now use this $t \pmod{p}$ result to characterize multiple blocking sets. We present in this article characterization results on small $t$-fold $(n-k)$-blocking sets in $\text{PG}(n,q)$, $q$ square, intersecting every $k$-dimensional space in $t \pmod{\sqrt{q}}$ points.

2 Intervals for minimal $t$-fold $(n-k)$-blocking sets

The following interval theorems on the size of minimal $t$-fold $(n-k)$-blocking sets in $\text{PG}(n,q)$ will play a crucial role in our arguments.

Theorem 2.1 (Ferret et al. [10]) Let $B$ be a minimal $t$-fold $(n-k)$-blocking set in $\text{PG}(n,q)$, $n \geq 2$, $|B| = tq^{n-k}+t+k'$, with $t+k' \leq (q^{n-k}-1)/2$. Assume that $q = p^h$, $p$ prime, $h \geq 1$, and that $B$ intersects every $k$-dimensional space in $t \pmod{E}$ points, with $E = p^e$, and with $e$ the largest integer for which this is true.

If $2t < E$, then

$$tq^{n-k} + \frac{q^{n-k}}{p^e} + 1 \leq |B| \leq tq^{n-k} + \frac{2tq^{n-k}}{E}.$$  

Theorem 2.2 (Ferret et al. [10]) Let $B$ be a $t$-fold $(n-k)$-blocking set in $\text{PG}(n,q)$. Assume that $q = p^h$, $p$ prime, $h \geq 1$, and that $B$ intersects every $k$-dimensional space in $t \pmod{E}$ points, with $E = p^e$, and with $e$ the largest integer for which this is true.

If $\max\{2t, 4\} < E$, then

$$|B| \leq tq^{n-k} + \frac{2tq^{n-k}}{E} \quad \text{or} \quad |B| \geq Eq^{n-k} + t.$$  

3 $t$-Fold 1-blocking sets

In Theorem 1.16, see also [4], Barát and Storme presented characterization results on $t$-fold 1-blocking sets in $\text{PG}(n,q)$. These results were obtained before the $t \pmod{p}$ results (Theorems 1.11 and 1.17) were known.

Repeating their arguments, but now including the $t \pmod{p}$ results, leads to the following theorem.
Theorem 3.1 Let $B$ be a $t$-fold 1-blocking set in $\text{PG}(n,q)$, $q = p^h$, $p$ prime, $q \geq 661$, $n \geq 3$, of size $|B| < tq + c_p q^{2/3}$, with $c_2 = c_3 = 2^{-1/3}$, $c_p = 1$ when $p > 3$, and with $t < c_p q^{1/6}/2$. Then $B$ contains a union of $t$ pairwise disjoint lines and/or Baer subplanes.

The following result, which relies on the preceding classification of $t$-fold 1-blocking sets, plays an important role in the proofs of the characterization results which will follow.

From now on, let $B$ be a minimal $t$-fold $(n - k)$-blocking set in $\text{PG}(n,q)$, $q = p^h$, $p$ prime, $q \geq 661$, $n \geq 3$, of size $|B| < tq^{n-k} + c_p q^{n-k-1/3}$, with $c_2 = c_3 = 2^{-1/3}$, $c_p = 1$ when $p > 3$, and with $t < c_p q^{1/6}/2$, intersecting every $k$-dimensional space in $t$ (mod $\sqrt{q}$) points.

Lemma 3.2 Let $B$ be a minimal $t$-fold $(n - k)$-blocking set in $\text{PG}(n,q)$, $k \geq 2$, intersecting every $k$-dimensional space in $t$ (mod $\sqrt{q}$) points.

If $\Pi$ is a $(k+1)$-dimensional space intersecting $B$ in a non-minimal $t$-fold 1-blocking set, then $$|\Pi \cap B| \geq q \sqrt{q} + t.$$ 

Proof: Since $\Pi \cap B$ intersects every $k$-dimensional space in $\Pi$ in $t$ (mod $\sqrt{q}$) points, either $|\Pi \cap B| \leq tq + 2t \sqrt{q}$ or $|\Pi \cap B| \geq q \sqrt{q} + t$ (Theorem 2.2). Assume that $|\Pi \cap B| \leq tq + 2t \sqrt{q}$, then by Theorem 3.1, $\Pi \cap B$ contains a union of $t$ pairwise disjoint lines and/or Baer subplanes. Let $S_1$ be the minimal part of $\Pi \cap B$, consisting of those $t$ pairwise disjoint lines and/or Baer subplanes, and let $S_2$ be the remaining part of $\Pi \cap B$.

Let $r \in S_2$. Consider a line $L$ of $\Pi$ through $r$ only intersecting $B$ in $r$. We now prove that it is possible to find a $(k-1)$-dimensional space $\Pi_{k-1}$ of $\Pi$ through $L$ only intersecting $B$ in $r$. This is immediately true for $k = 2$. Let $k \geq 3$, then there are $q^{n-2} + q^{n-3} + \cdots + q + 1$ planes through $L$. Since there are at most $t q^{n-k} + q^{n-k-1/3} < q^{n-2} + \cdots + q + 1$ points in $B$, it is possible to find a plane $\Pi_2$ through $L$ only intersecting $B$ in $r$. Repeating this argument, a 3-dimensional space $\Pi_3$ through $\Pi_2$ only intersecting $B$ in $r$ can be found, a 4-dimensional space $\Pi_4$ through $\Pi_3$ only intersecting $B$ in $r$ can be found, . . . , a $(k-1)$-dimensional space $\Pi_{k-1}$ through $\Pi_{k-2}$ only intersecting $B$ in $r$ can be found since there are $q^{n-k+1} + \cdots + q + 1$ $(k-1)$-dimensional spaces through $\Pi_{k-2}$ and $|B| < tq^{n-k} + q^{n-k-1/3}$.

There are $q + 1$ $k$-dimensional spaces in $\Pi$ through $\Pi_{k-1}$, all intersecting $S_1$ in $t$ (mod $\sqrt{q}$) points. Since these $k$-dimensional spaces intersect $B$ in $t$ (mod $\sqrt{q}$) points, every such hyperplane intersects $S_2$ in $r$ and in at least $\sqrt{q} - 1$ other points. So $|\Pi \cap B| \geq 1 + (q + 1)(\sqrt{q} - 1) + t(q + 1)$. This contradicts $|\Pi \cap B| \leq tq + 2t \sqrt{q}$. □
4 \quad t\text{-Fold 2-blocking sets}

Let $B$ be a minimal $t$-fold 2-blocking set in $\PG(n, q)$ intersecting every $(n-2)$-dimensional space in $t \pmod{\sqrt{q}}$ points. Assume that

$$|B| \leq tq^2 + 2tq\sqrt{q} < tq^2 + c_pq^{5/3},$$

with $q \geq 661$ and with $t < c_pq^{1/6}/2$.

The $t \pmod{\sqrt{q}}$ assumption implies that every $(n-1)$-dimensional space intersects $B$ in at most $tq + 2t\sqrt{q}$ points or in at least $q\sqrt{q} + t$ points (Lemma 3.2).

We will show that $B$ is the union of $t$ pairwise disjoint planes, Baer cones with a point as vertex and a Baer subplane $\PG(2, \sqrt{q})$ as base, or subgeometries $\PG(4, \sqrt{q})$.

Remark 4.1 (1) In this article, when we state that a Baer subline $L$ is contained in $B$, then we mean that this Baer subline is effectively contained in $B$, but that the line $\hat{L}$, defined over $GF(q)$, defined by $L$ is not completely contained in $B$.

(2) In the next lemma, we state that a subset $S$ of points on a line $L$ can be written in a unique way as a union of at most $t$ pairwise disjoint points and Baer sublines. This has the following meaning. If $S$ contains a Baer subline, then, first of all, the $\sqrt{q} + 1$ points of this Baer subline must be considered in this description as a Baer subline and not as $\sqrt{q} + 1$ distinct points, secondly, these Baer sublines and points contained in $S$ are all pairwise disjoint, and thirdly, if you consider the different Baer sublines contained in $S$ and then the remaining points of $S$, the total number of these different Baer sublines and remaining points is at most $t$.

(3) Consider a Baer subline $L$, then $\hat{L}$ will always denote the line, over $GF(q)$, containing the Baer subline $L$.

Lemma 4.2 A line $L$ not contained in $B$ shares at most $t(\sqrt{q} + 1)$ points with $B$. This intersection $L \cap B$ can be written in a unique way as a union of at most $t$ pairwise disjoint points and Baer sublines.

Proof: By using the same arguments as in the proof of Lemma 3.2, it is possible to find an $(n-3)$-dimensional space through $L$ containing no other points of $B$. It is then possible to select an $(n-2)$-dimensional space through this $(n-3)$-dimensional space containing at most $t$ extra points of $B$ since there are $q^2 + q + 1$ $(n-2)$-dimensional spaces through a given $(n-3)$-dimensional space, and $|B| < tq^2 + q^{5/3}$. Similarly, it is then possible to
select a hyperplane \( \pi \) through this \((n-2)\)-dimensional space containing at most \( tq + 2t\sqrt{q} \) other points of \( B \).

Then \( |\pi \cap B| \leq q + t + tq + 2t\sqrt{q} < q\sqrt{q} + t \), so by Theorem 2.2, \( |\pi \cap B| \leq tq + 2t\sqrt{q} \), and then Theorem 3.1 and Lemma 3.2 imply that \( \pi \) intersects \( B \) in a union of \( t \) pairwise disjoint lines and Baer subplanes.

This implies that \( L \) intersects \( B \) in a number of points and/or Baer sublines.

Assume that \( L \) shares at least one Baer subline with \( B \). Since \( t < q^{1/6}/2 \), and since two distinct Baer sublines share at most two points, it is only possible to partition the points of a Baer subline in \( L \cap B \) over other Baer sublines in \( L \cap B \) if \( t \geq (\sqrt{q} + 1)/2 \).

This is not the case, so \( L \cap B \) can be written in a unique way as a union of at most \( t \) pairwise disjoint points and Baer sublines.

\[ \square \]

**Lemma 4.3** Every hyperplane \( \Pi \) intersects \( B \) in a union of \( t \) pairwise disjoint lines and/or Baer subplanes, or intersects \( B \) in at least \( q\sqrt{q} + t \) points.

**Proof:** By Theorem 2.2, since every \((n-2)\)-dimensional space intersects \( B \) in \( t \) \((\mod \sqrt{q})\) points, \( B \) intersects every hyperplane in at most \( tq + 2t\sqrt{q} \) points or in at least \( q\sqrt{q} + t \) points. Assume that a hyperplane \( \Pi \) intersects \( B \) in at most \( tq + 2t\sqrt{q} \) points, then this intersection \( \Pi \cap B \) must be a minimal \( t \)-fold 1-blocking set in \( \Pi \), since for a non-minimal intersection, \( |\Pi \cap B| \geq q\sqrt{q} + t \) (Lemma 3.2).

Since for the case \( |\Pi \cap B| \leq tq + 2t\sqrt{q} \), the intersection must be a minimal \( t \)-fold 1-blocking set, Theorem 3.1 implies that \( B \cap \Pi \) is a union of \( t \) pairwise disjoint lines and/or Baer subplanes.

\[ \square \]

We know from Lemma 4.3 that every hyperplane \( \Pi \) intersects \( B \) in a union of \( t \) lines and/or Baer subplanes, or intersects \( B \) in at least \( q\sqrt{q} + t \) points. Consequently, for every hyperplane \( \Pi \), \( |\Pi \cap B| \geq t(q + 1) \).

Consider an \((n-2)\)-dimensional space \( \Delta \) sharing \( t \) distinct points with \( B \). The \( q + 1 \) hyperplanes through \( \Delta \) all contain at least \( tq + t \) points of \( B \), so if we subtract \((q + 1)tq\) from the size of \( B \), at most \( 2tq\sqrt{q} - tq \) points in \( B \) remain. Dividing this number by \( q\sqrt{q} - tq \) then implies that at most \( 2t \) hyperplanes through \( \Delta \) contain at least \( q\sqrt{q} + t \) points of \( B \). The other, at least \( q + 1 - 2t \), hyperplanes through \( \Delta \) share at most \( tq + 2t\sqrt{q} \) points with \( B \), and therefore intersect \( B \) in a union of \( t \) pairwise disjoint lines and/or Baer subplanes (Lemma 4.3).

This shows that every point of \( \Delta \cap B \) lies on at least \( q + 1 - 2t \) lines and/or Baer subplanes, contained in \( B \).
**Lemma 4.4** Let \( r \in \Delta \cap B \) and suppose that \( r \) lies in two Baer subplanes \( B_1 \) and \( B_2 \), contained in \( B \), in distinct hyperplanes through \( \Delta \).

Then \( B_1 \) and \( B_2 \) define a 4-dimensional Baer subgeometry completely contained in \( B \).

**Proof:** Consider a Baer subline \( L_2 \) of \( B_2 \) through \( r \). Then the line \( \hat{L}_2 \), defined over \( GF(q) \), through \( L_2 \) shares at most \( t(\sqrt{q} + 1) \) points with \( B \) (Lemma 4.2). By using the same arguments as in the proof of Lemma 3.2, it is possible to find an \( (n-3) \)-dimensional space \( \Pi_{n-3} \) through \( L_2 \) containing no other points of \( B \), and intersecting the plane of \( B_1 \) only in \( r \).

There are \( q^2 + q + 1 \) \( (n-2) \)-dimensional spaces through \( \Pi_{n-3} \). Precisely \( q + 1 \) of these \( (n-2) \)-dimensional spaces through \( \Pi_{n-3} \) intersect the plane \( PG(2,q) \) containing the Baer subplane \( B_1 \) in a line through \( r \), so \( q^2 \) of these \( (n-2) \)-dimensional spaces through \( \Pi_{n-3} \) only intersect the plane of \( B_1 \) in \( r \). It is therefore possible to select an \( (n-2) \)-dimensional space \( \Delta' \) through \( \Pi_{n-3} \) containing at most \( t \) extra points of \( B \), and only intersecting the plane of \( B_1 \) in \( r \). Then \( |\Delta' \cap B| \leq t(\sqrt{q} + 1) \) since there are at most \( t(\sqrt{q} + 1) \) points of \( B \) belonging to \( \hat{L}_2 \) (Lemma 4.2).

Since \( |\Delta' \cap B| \equiv t \mod \sqrt{q} \), necessarily \( |\Delta' \cap B| \leq t(\sqrt{q} + 1) \).

Every hyperplane through \( \Delta' \) contains at least \( tq - t\sqrt{q} \) other points of \( B \) since every hyperplane shares at least \( t(q + 1) \) points with \( B \) (Lemma 4.3). If we subtract \( (q + 1)(tq - t\sqrt{q}) \) from the size of \( B \), at most \( 3tq\sqrt{q} - tq + t\sqrt{q} \) points in \( B \) remain. A hyperplane through \( \Delta' \) containing at least \( q\sqrt{q} + t \) points of \( B \) still contains at least \( q\sqrt{q} - tq \) other points of \( B \), so at most \( 3t \) hyperplanes through \( \Delta' \) contain at least \( q\sqrt{q} + t \) points of \( B \).

This implies that at least \( \sqrt{q} + 1 - 3t \) hyperplanes through \( \Delta' \) intersect \( B_1 \) in a Baer subline, and intersect \( B \) in a union of \( t < q^{1/6}/2 \) pairwise disjoint lines and/or Baer subplanes. Since such a hyperplane shares a Baer subline with \( B_1 \) and with \( B_2 \), both passing through the same point \( r \), these two latter Baer sublines must be contained in a Baer subplane contained in \( B \).

The preceding arguments show that at least \( \sqrt{q} + 1 - 3t \) Baer subplanes of the 3-dimensional Baer subgeometry \( \langle B_1, L_2 \rangle \), passing through \( L_2 \), are contained in \( B \).

Assume that the Baer subgeometry \( \langle B_1, L_2 \rangle \) is not contained in \( B \). Select a Baer subline \( N \) of \( \langle B_1, L_2 \rangle \) skew to \( L_2 \) which is not contained in \( B \). Then this Baer subline \( N \) shares at least \( \sqrt{q} + 1 - 3t \) and at most \( \sqrt{q} \) points with \( B \).

By Lemma 4.2, it is possible to describe \( N \cap B \) in a unique way as an union of at most \( t < q^{1/6}/2 \) pairwise disjoint points and Baer sublines.

Since \( \sqrt{q} + 1 - 3t > t \), some of the points of \( N \cap B \) lying in \( \langle B_1, L_2 \rangle \) must lie in Baer sublines contained in \( N \cap B \). Two distinct Baer sublines share...
at most two points. Since $\sqrt{q} + 1 - 3t > 2t$, this is impossible, so the Baer subline $N \cap \langle L_2, B_1 \rangle$ is completely contained in $B$.

This shows that the 3-dimensional Baer subgeometry $\langle L_2, B_1 \rangle$ is completely contained in $B$. By letting vary $L_2$ over all Baer sublines of $B_2$ through $r$, the 4-dimensional Baer subgeometry $\langle B_1, B_2 \rangle$ is completely contained in $B$. By letting vary $L_2$ over all Baer sublines of $B_2$ through $r$, the 4-dimensional Baer subgeometry $\langle B_1, B_2 \rangle$ is completely contained in $B$.

This latter 4-dimensional Baer subgeometry $\langle B_1, B_2 \rangle$ is either a Baer cone with a point as vertex and a Baer subplane as base, or a Baer subgeometry $PG(4, \sqrt{q})$.

In both cases, they are 1-fold 2-blocking sets, and the $t \pmod{\sqrt{q}}$ result implies that $B \setminus \langle B_1, B_2 \rangle$ is a $(t - 1)$-fold 2-blocking set intersecting every $(n - 2)$-dimensional space in $(t - 1) \pmod{\sqrt{q}}$ points.

Since we know from the calculations preceding Lemma 4.4 that every point of $\Delta \cap B$ lies on at least $q + 1 - 2t$ lines or Baer subplanes contained in $B$, the preceding lemma and observations now imply that we can assume that every point of $\Delta \cap B$ lies on at least $q - 2t$ lines contained in $B$. Since $B$ is minimal, it is possible to assume that every point of $B$ lies on at least $q - 2t$ lines of $B$. We now show that there is a plane contained in $B$.

Lemma 4.5 If every point of $B$ lies on at least $q - 2t$ lines contained in $B$, then there is a plane contained in $B$.

Proof: Consider an $(n - 2)$-dimensional space $\Delta$ intersecting $B$ in exactly $t$ points. The calculations preceding Lemma 4.4 indicate that at least $q + 1 - 2t$ hyperplanes through $\Delta$ intersect $B$ in a union of $t$ lines and/or Baer subplanes. But none of the $t$ points of $\Delta \cap B$ lies on two Baer subplanes of $B$ in those hyperplanes. So, at least $q + 1 - 2t - t$ hyperplanes $\Pi$ through $\Delta$ intersect $B$ in $t$ pairwise disjoint lines $L_1, \ldots, L_t$.

Let $r$ be a point of $B \setminus \Pi$. This point $r$ lies on at least $q - 2t$ lines completely contained in $B$. These lines intersect $\Pi$ in a point of $B \cap \Pi = L_1 \cup \cdots \cup L_t$. So at least one of the lines $L_i$ is intersected by at least $(q - 2t)/t$ lines of $B$ passing through $r$.

Then the plane $\langle r, L_i \rangle$ intersects $B$ in at least $(q - 2t)/t$ lines passing through $r$. Then every line of this plane, not passing through $r$, shares already $(q - 2t)/t$ points with $B$. If such a line is not contained in $B$, it shares at most $t(\sqrt{q} + 1)$ points with $B$ (Lemma 4.2).

Since $(q - 2t)/t > t(\sqrt{q} + 1)$, every line of $\langle L_i, r \rangle$, not passing through $r$, is contained in $B$, and so this plane $\langle L_i, r \rangle$ is contained in $B$. \qed
The $t \pmod{\sqrt{q}}$ result again implies that $B \setminus \Pi$, $\Pi$ a plane contained in $B$, is a $(t-1)$-fold blocking set intersecting every $(n-2)$-dimensional space in $(t-1) \pmod{\sqrt{q}}$ points.

Repeating the preceding lemmas for this $(t-1)$-fold blocking set, the following characterization theorem is obtained.

**Theorem 4.6** Let $B$ be a minimal $t$-fold $2$-blocking set, of size at most $tq^2 + 2tq\sqrt{q} < tq^2 + c_p q^{5/3}$, in $\text{PG}(n,q)$, $q \geq 661$, $t < c_p q^{1/6}/2$, intersecting every $(n-2)$-dimensional space in $t \pmod{\sqrt{q}}$ points.

Then $B$ is the union of $t$ pairwise disjoint planes, Baer cones with a point as vertex and a Baer subplane as base, and $4$-dimensional Baer subgeometries $\text{PG}(4,\sqrt{q})$.

5 **$t$-Fold $(n-k)$-blocking sets in $\text{PG}(n,q)$**

We now will present the characterization result on minimal $t$-fold $(n-k)$-blocking sets in $\text{PG}(n,q)$, with $1 \leq k < n-2$, intersecting every $k$-dimensional space in $t \pmod{\sqrt{q}}$ points. The results of the preceding two sections will be the induction bases for the general characterization results.

The general induction hypothesis (IH) we rely on for classifying the minimal $t$-fold $(n-k)$-blocking sets in $\text{PG}(n,q)$, intersecting every $k$-dimensional space in $t \pmod{\sqrt{q}}$ points, is as follows.

**Induction hypothesis (IH):** For $1 \leq j \leq n-k-1$, let $B_j$ be a minimal $t$-fold $(n-k-j)$-blocking set in $\text{PG}(n,q)$, $q$ square, $q \geq 661$, $t < c_p q^{1/6}/2$, of size at most $|B_j| \leq tq^{n-k-j} + 2tq^{n-k-j-1}\sqrt{q} < tq^{n-k-j} + c_p q^{n-k-j-1/3}$, intersecting every $(k+j)$-dimensional space in $t \pmod{\sqrt{q}}$ points.

Then $B_j$ is a union of $t$ pairwise disjoint cones $\langle \pi_{m_i}, \text{PG}(2(n-k-m_i-j-1), \sqrt{q}) \rangle$, $-1 \leq m_i \leq n-k-j-1$, $i = 1, \ldots, t$.

In the above description, if $m_i = n-k-j-1$, then $\langle \pi_{m_i}, \text{PG}(2(n-k-m_i-j-1), \sqrt{q}) \rangle$ is a subspace $\text{PG}(n-k-j,q)$, and if $m_i = -1$, then $\langle \pi_{m_i}, \text{PG}(2(n-k-m_i-j-1), \sqrt{q}) \rangle$ is a Baer subgeometry $\text{PG}(2(n-k-j), \sqrt{q})$.

The goal is to prove the following similar characterization result for $t$-fold $(n-k)$-blocking sets.

Let $B$ be a minimal $t$-fold $(n-k)$-blocking set in $\text{PG}(n,q)$, $q$ square, $q \geq 661$, $t < c_p q^{1/6}/2$, of size at most $|B| \leq tq^{n-k} + 2tq^{n-k-1}\sqrt{q} < tq^{n-k} + c_p q^{n-k-1/3}$, intersecting every $k$-dimensional space in $t \pmod{\sqrt{q}}$ points.
Then $B$ is a union of $t$ pairwise disjoint cones $\langle \pi_{m_i}, \PG(2(n-k-m_i-1), \sqrt{q}) \rangle$, $-1 \leq m_i \leq n-k-1$, $i = 1, \ldots, t$.

So, from now on, we assume that $B$ is a minimal $(n-k)$-blocking set in $\PG(n,q)$, $q$ square, $q \geq 661$, $t < c_p q^{1/6}/2$, of size at most $|B| \leq tq^{n-k} + 2tq^{n-k-1}\sqrt{q} < tq^{n-k} + c_p q^{n-k-1/3}$, intersecting every $k$-dimensional space in $t \pmod{\sqrt{q}}$ points.

We first present some analogous lemmas to lemmas of Section 4.

**Lemma 5.1** Every $(k+1)$-dimensional space $\Pi$ intersects $B$ in a union of $t$ pairwise disjoint lines and/or Baer subplanes, or intersects $B$ in at least $q\sqrt{q} + t$ points.

**Proof:** By Theorem 2.2, since every $k$-dimensional space intersects $B$ in $tq + 2t\sqrt{q}$ points, $B$ intersects every $(k+1)$-dimensional space in at most $tq + 2t\sqrt{q}$ points or in at least $q\sqrt{q} + t$ points. Assume that a $(k+1)$-dimensional space $\Pi$ intersects $B$ in at most $tq + 2t\sqrt{q}$ points, then this intersection $\Pi \cap B$ must be a minimal $t$-fold 1-blocking set in $\Pi$, since for a non-minimal intersection, $|\Pi \cap B| \geq q\sqrt{q} + t$ (Lemma 3.2).

Since for the case $|\Pi \cap B| \leq tq + 2t\sqrt{q}$, the intersection must be a minimal $t$-fold 1-blocking set, Theorem 3.1 implies that $B \cap \Pi$ is a union of $t$ pairwise disjoint lines and/or Baer subplanes.

For the description of the next lemma, we again rely on Remark 4.1 (2).

**Lemma 5.2** A line $L$, not contained in $B$, intersects $B$ in at most $t(\sqrt{q} + 1)$ points. This intersection can be described in a unique way as a union of at most $t$ pairwise disjoint points and Baer sublines.

**Proof:** We know that $|B| \leq tq^{n-k} + 2tq^{n-k-1}\sqrt{q}$.

By using the same arguments as in the proof of Lemma 3.2, it is possible to construct a $(k-1)$-dimensional space $\Pi_{k-1}$ through $L$ containing no other points of $B$. It is then possible to construct a $k$-dimensional space $\Pi_k$ through $\Pi_{k-1}$ containing at most $t$ other points of $B$. So $|\Pi_k \cap B| \leq q + t$.

There are at most $tq^{n-k} + 2tq^{n-k-1}\sqrt{q}$ points in $B$ left. By the induction hypothesis (III), the smallest $t$-fold 1-blocking sets which are the intersection of a $(k+1)$-dimensional space with $B$ are the union of $t$ pairwise disjoint lines, see also Lemma 4.3. Hence, every $(k+1)$-dimensional space through $\Pi_k$ contains at least $(t-1)q$ extra points of $B$. So we observe that at most $tq^{n-k} + 2tq^{n-k-1}\sqrt{q} - (t-1)q(q^{n-k-1} + q^{n-k-2} + \cdots + q + 1)$ other points of $B$ can remain.
If a $(k + 1)$-dimensional space $\Pi_{k+1}$ through $\Pi_k$ contains at least $q\sqrt{q} + t$ points of $B$ (Theorem 2.2 and Lemma 3.2), then it still contains at least $q\sqrt{q} - tq$ other points of $B \setminus \Pi_k$. Since $(q^{n-k-1} + q^{n-k-2} + \cdots + q + 1)(q\sqrt{q} - tq) > tq^{n-k} + 2tq^{n-k-1}\sqrt{q} - (t - 1)q(q^{n-k-1} + q^{n-k-2} + \cdots + q + 1)$, there is at least one $(k + 1)$-dimensional space $\Pi_{k+1}$ through $\Pi_k$ with at most $tq + 2t\sqrt{q} + t$ points of $B$. Then $|\Pi_{k+1} \cap B| \leq tq + 2t\sqrt{q}$ (Theorem 2.2 and Lemma 3.2). This intersection $\Pi_{k+1} \cap B$ is a minimal $t$-fold 1-blocking set in $\Pi_{k+1}$ (Lemma 3.2), so it is a union of $t$ pairwise disjoint lines and/or Baer subplanes (Theorem 3.1). The line $L$ shares zero or one points with the lines of $\Pi_{k+1} \cap B$, and zero, one, or $\sqrt{q} + 1$ points with the Baer subplanes of $\Pi_{k+1} \cap B$. This proves the lemma.

**Lemma 5.3** Let $r$ be a point of $B$ lying on two lines $L_0$ and $L_1$ contained in $B$.

Then the plane $\langle L_0, L_1 \rangle$ is either contained in $B$, or $\langle L_0, L_1 \rangle \cap B$ contains a cone with $r$ as vertex and a Baer subline as base, containing $L_0$ and $L_1$.

**Proof:** Consider the plane $\langle L_0, L_1 \rangle$. Whatever its intersection with $B$ is, the intersection size is at most $q^2 + q + 1$.

By using the same arguments as in the proof of Lemma 3.2, construct a $(k - 1)$-dimensional space $\Pi_{k-1}$ through $\langle L_0, L_1 \rangle$ containing no other points of $B$. Since $|B| < tq^{n-k} + q^{n-k-1}/3$, and since there are $q^{n-k} + \cdots + q + 1$ different $k$-dimensional spaces through $\Pi_{k-1}$, it is possible to construct a $k$-dimensional space $\Pi_k$ through $\Pi_{k-1}$ containing at most $t$ extra points of $B$. Similarly, since $|B| \leq tq^{n-k} + 2tq^{n-k-1}\sqrt{q}$, it is possible to find a $(k + 1)$-dimensional space $\Pi_{k+1}$ through $\Pi_k$ containing at most $tq + 2t\sqrt{q}$ other points of $B$.

So, $|B \cap \Pi_{k+1}| \leq q^2 + q + 1 + tq + 2t\sqrt{q} + t$.

Consider all $q^{n-k-2} + \cdots + q + 1$ $(k + 2)$-dimensional spaces through $\Pi_{k+1}$. Since $|B| < tq^{n-k} + q^{n-k-1}/3$, it is possible to find a $(k + 2)$-dimensional space $\Pi_{k+2}$ through $\Pi_{k+1}$ containing at most $tq^2 + 2tq\sqrt{q}$ other points of $B$. This certainly implies that $|\Pi_{k+2} \cap B| \leq (t + 2)q^2$.

Since $|\Pi_{k+2} \cap B| \leq tq^2 + 2tq\sqrt{q}$ or $|\Pi_{k+2} \cap B| \geq q^2\sqrt{q} + t$ (Theorem 2.2 and Lemma 3.2), necessarily $|\Pi_{k+2} \cap B| \leq tq^2 + 2tq\sqrt{q}$.

Theorem 4.6 implies that $\Pi_{k+2} \cap B$ is a union of $t$ pairwise disjoint planes, cones with a point as vertex and a Baer subplane as base, and Baer subgeometries $PG(4, \sqrt{q})$.

Since $L_0$ and $L_1$ are intersecting lines of this intersection, the plane $\langle L_0, L_1 \rangle$ either is contained in $B$, or its intersection with $B$ contains a cone with $L_0 \cap L_1$ as vertex and a Baer subline as base, which contains the lines.
$L_0$ and $L_1$. \hfill \Box

The following two lemmas are proven in exactly the same way as the preceding lemma. In the following lemma, a Baer cone with vertex $s$ and base the Baer subline $L_2$, $s \notin \hat{L}_2$, is the set of $\sqrt{q} + 1$ lines through the point $s$ and the points of the Baer subline $L_2$. We also recall Remark 4.1 (1); with a Baer subline $L$ contained in $B$, we mean a Baer subline contained in $B$ whose corresponding line $\hat{L}$ over $GF(q)$ is not contained in $B$.

**Lemma 5.4** Suppose that the point $r$ of $B$ lies on a line $L_0$ contained in $B$ and on a Baer subline $L_2$ contained in $B$.

Then there is a Baer cone completely contained in $B$, with a point of $L_0 \setminus \{r\}$ as vertex and with $L_2$ as base.

**Lemma 5.5** Suppose that the point $r \in B$ lies on two Baer sublines $L_0$ and $L_1$ contained in $B$, then the Baer subplane $\langle L_0, L_1 \rangle$ is completely contained in $B$.

**Lemma 5.6** Let $L_2$ be a Baer subline contained in $B$. Let $v$ be a point not lying on the line $\hat{L}_2$, defined over $GF(q)$, by $L_2$. Suppose that the cone with vertex $v$ and with base the Baer subline $L_2$ is contained in $B$.

Let $r$ be a point of $L_2$ and suppose that $L_1$ is an other Baer subline of $B$ through $r$, not lying in the plane $\langle v, L_2 \rangle$.

Then the Baer cone $\Omega$ with vertex $v$ and with base the Baer subplane $\langle L_1, L_2 \rangle$ is contained in $B$.

**Proof:** Let $L'_2$ be a second Baer subline of the Baer cone $\langle v, L_2 \rangle$ passing through $r$. Then the Baer subplane $\langle L_1, L'_2 \rangle$ is contained in $B$ (Lemma 5.5). This Baer subplane $\langle L_1, L'_2 \rangle$ is projected from $v$ onto the Baer subplane $\langle L_1, L_2 \rangle$.

Letting vary $L'_2$ over all Baer sublines of the Baer cone $\langle v, L_2 \rangle$ through $r$, the preceding arguments prove that the Baer cone $\Omega$ with vertex $v$ and with base $\langle L_1, L_2 \rangle$ is completely contained in $B$, up to maybe some points on the line $rv$.

But let $r'$ be an arbitrary point of the line $rv \setminus \{r, v\}$, and let $L_3$ be an arbitrary Baer subline of the Baer cone $\Omega$ through $r'$. This Baer subline is completely contained in $B$, up to maybe the point $r'$. So, $L_3$ contains $\sqrt{q}$ or $\sqrt{q} + 1$ points of $B$. We prove that the Baer subline $L_3$ is completely contained in $B$. Let $\hat{L}_3$ be the line over $GF(q)$ defined by $L_3$, then the intersection of $\hat{L}_3$ with $B$ can be described in a unique way as the union of at most $t$ pairwise disjoint points and Baer sublines (Lemma 5.2). If the Baer
subline \( L_3 \) contains exactly \( \sqrt{q} \) points of \( B \), then these \( \sqrt{q} \) points need to be partitioned over at most \( t < q^{1/6}/2 \) pairwise disjoint points and Baer sublines (Lemma 5.2). Since two distinct Baer sublines share at most two points, this is impossible. So the Baer subline \( L_3 \) is completely contained in \( B \).

This proves that the Baer cone \( \Omega \) with vertex \( v \) and with base the Baer subplane \( \langle L_1, L_2 \rangle \) is completely contained in \( B \).

Consider a point \( r \) from \( B \) and select a subspace \( \Delta_k \simeq \text{PG}(k,q) \) through \( r \) sharing \( t \) points with \( B \).

There is at least one \((k+1)\)-dimensional subspace through \( \Delta_k \) sharing at most \( tq + 2t\sqrt{q} \) points, not in \( \Delta_k \), with \( B \), since these \((k+1)\)-dimensional spaces through \( \Delta_k \) cannot all contain \( q\sqrt{q} \) other points of \( B \) (Theorem 2.2).

Then such a \((k+1)\)-dimensional subspace \( \Delta_{k+1} \) through \( \Delta_k \) shares at most \( tq + 2t\sqrt{q} + t \) points with \( B \). By Theorem 2.2 and Lemma 3.2, \(|\Delta_{k+1} \cap B| \leq tq + 2t\sqrt{q} \). Hence, \( \Delta_{k+1} \) intersects \( B \) in \( t \) pairwise disjoint lines and/or Baer subplanes (Lemma 5.1). So \( \Delta_{k+1} \) shares at most \( tq + t \sqrt{q} \) other points with \( B \). Select \( \Delta_{k+1} \simeq \text{PG}(k+1,q) \) through \( \Delta_k \) sharing at most \( tq + t \sqrt{q} + t \) points with \( B \).

We now prove that we can find an \((n-2)\)-dimensional space \( \Delta_{n-2} \) through \( \Delta_{k+1} \) sharing at most \( t(q^{n-k-2} + q^{n-k-3} \sqrt{q} + q^{n-k-3} + \cdots + \sqrt{q} + 1) \) points with \( B \). We heavily rely on the bounds of Theorem 2.2. Since \( B \) intersects every \( k \)-dimensional space in \( t \pmod{\sqrt{q}} \) points, this theorem states that \( B \) intersects every \((k+i)\)-dimensional space in either at most \( tq^i + 2\sqrt{q}q^{i-1} \) points or in at least \( \sqrt{q}q^i + t \) points. Consider all the \( q^{n-k-2} + \cdots + q + 1 \) different \((k+2)\)-dimensional spaces through \( \Delta_{k+1} \). As \(|\Delta_{k+1} \cap B| \leq tq + t \sqrt{q} + t \), it is impossible that all these \((k+2)\)-dimensional spaces share at least \( \sqrt{q}q^2 + t \) points with \( B \) since \(|B| < tq^{n-k} + q^{n-k-1/3} \), so there is at least one \((k+2)\)-dimensional space \( \Delta_{k+2} \) through \( \Delta_{k+1} \) sharing at most \( tq^2 + 2\sqrt{q}q \) points with \( B \). Repeating this argument by induction on \( i \), it is possible to find a \((k+i)\)-dimensional space \( \Delta_{k+i+1} \), sharing at most \( tq^{i+1} + 2\sqrt{q}q^{i} \) points with \( B \), through a given \((k+i)\)-dimensional space \( \Delta_{k+i} \), sharing at most \( tq^i + 2\sqrt{q}q^{i-1} \) points with \( B \). This leads us eventually to the \((n-2)\)-dimensional space \( \Delta_{n-2} \) through \( \Delta_{k+1} \) sharing at most \( t(q^{n-k-2} + q^{n-k-3} \sqrt{q} + q^{n-k-3} + \cdots + \sqrt{q} + 1) \) points with \( B \). The upper bound on \(|\Delta_{n-2} \cap B| \) follows from the induction hypothesis (IH), which states that \( \Delta_{n-2} \cap B \) is a union of \( t \) pairwise disjoint cones \( (\Pi_{m_i}, \text{PG}(2(n-k-2-m_i-1), \sqrt{q})), -1 \leq m_i \leq n-k-3, i = 1, \ldots, t \).

Now it is possible to find at least two hyperplanes \( H_1, H_2 \) through \( \Delta_{n-2} \) containing at most \( t(q^{n-k-1} + q^{n-k-2} \sqrt{q} + q^{n-k-2} + \cdots + \sqrt{q} + 1) \) points of \( B \), since it is not possible that all hyperplanes through \( \Delta_{n-2} \) share at least \( q^{n-k-1} \sqrt{q} + t \) points with \( B \) (Theorem 2.2). By the induction hypothesis (IH), these two hyperplanes meet \( B \) in a union of \( t \) pairwise disjoint cones
\( (\Pi_{m_i}, \text{PG}(2(n-k-1-m'_i-1), \sqrt{q})) \), with \(-1 \leq m'_i \leq n-k-2, i = 1, \ldots, t.\)

We now prove a first major part of the characterization result for the \( t \)-fold \((n-k)\)-blocking sets in \( \text{PG}(n, q) \). Our goal is to prove that small minimal \( t \)-fold \((n-k)\)-blocking sets in \( \text{PG}(n, q) \), \( q \) square, are a union of \( t \) pairwise disjoint Baer cones \( \langle \pi_m, \text{PG}(2(n-k-m-1), \sqrt{q}) \rangle \), \(-1 \leq m \leq n-k-1.\) For \( m = n-k-1 \), such a Baer cone is in fact an \((n-k)\)-dimensional subspace \( \text{PG}(n-k, q) \), and for \( m < n-k-1 \), such a Baer cone is a cone with an \( m \)-dimensional subspace \( \text{PG}(m, q) \) as vertex and a base \( \text{PG}(2(n-k-m-1) \geq 2, \sqrt{q}) \) which is a non-projected Baer subgeometry. For \( m < n-k-1 \), such a Baer cone contains Baer sublines. The following lemma shows that if there is a line not contained in \( B \), sharing at least one Baer subline with \( B \), then this implies that \( B \) contains a Baer cone \( \langle \pi_m, \text{PG}(2(n-k-m-1), \sqrt{q}) \rangle \), \(-1 \leq m < n-k-1.\)

**Lemma 5.7** Let \( \Delta \) be an \((n-2)\)-dimensional space intersecting \( B \) in a union of \( t \) pairwise disjoint Baer cones \( \langle \Pi_{m_i}, \text{PG}(2(n-k-2-m_i-1), \sqrt{q}) \rangle \), \(-1 \leq m_i \leq n-k-3, i = 1, \ldots, t.\)

Assume that \( m_i < n-k-3 \) for at least one value \( i \in \{1, \ldots, t\}.\)

Then \( B \) contains a Baer cone \( \langle \pi_{m''}, \text{PG}(2(n-k-m''-1), \sqrt{q}) \rangle \), \(-1 \leq m'' < n-k-1.\)

**Proof:** It is possible to find at least two hyperplanes \( H_1, H_2 \) through \( \Delta \) containing at most \( t(q^{n-k-1} + q^{n-k-2}\sqrt{q} + q^{n-k-2} + \cdots + \sqrt{q} + 1) \) points of \( B \), since it is not possible that \( q \) hyperplanes through \( \Delta \) share at least \( q^{n-k-1}\sqrt{q} + t \) points with \( B \) (Theorem 2.2). By the induction hypothesis (IH), these two hyperplanes \( H_1, H_2 \) through \( \Delta \) respectively intersect \( B \) in unions of \( t \) pairwise disjoint cones \( \langle \Pi_{m'_i}, \text{PG}(2(n-k-1-m'_i-1), \sqrt{q}) \rangle \), \(-1 \leq m'_i \leq n-k-2 \), and \( t \) pairwise disjoint cones \( \langle \Pi_{m''_i}, \text{PG}(2(n-k-1-m''_i-1), \sqrt{q}) \rangle \), \(-1 \leq m''_i \leq n-k-2.\)

Since we assume that one of the \( t \) Baer cones of \( \Delta \cap B \) is a cone \( \langle \Pi_m, \text{PG}(2(n-k-2-m-1), \sqrt{q}) \rangle \), with \( m < n-k-3 \), so with base a Baer subspace \( \text{PG}(s = 2(n-k-2-m-1) \geq 2, \sqrt{q}) \), at least one of these Baer cones in \( B \cap \Delta \) contains a Baer subline \( L_2.\)

Then \( H_1 \), and similarly \( H_2 \), shares with \( B \) a cone of type either \( \langle \Pi_m, \text{PG}(s+2, \sqrt{q}) \rangle \) or \( \langle \Pi_{m+1}, \text{PG}(s, \sqrt{q}) \rangle \), intersecting \( \Delta \) in this Baer cone \( \langle \Pi_m, \text{PG}(s, \sqrt{q}) \rangle \). We denote this particular Baer cone in \( H_1 \) by \( \langle \Pi_{m_1}, \text{PG}(s_1, \sqrt{q}) \rangle \), and this particular Baer cone in \( H_2 \) by \( \langle \Pi_{m_2}, \text{PG}(s_2, \sqrt{q}) \rangle \).

Up to equivalence, there are three possibilities. The first possibility is \( m = m_1 = m_2, s_1 = s_2 = s + 2. \) Then \( \langle \Pi_{m_1}, \text{PG}(s_1, \sqrt{q}) \rangle \) and \( \langle \Pi_{m_2}, \text{PG}(s_2, \sqrt{q}) \rangle \) define a Baer cone with vertex \( \Pi_m \) and base \( \text{PG}(s+4, \sqrt{q}) \), or with an
(m + 1)-dimensional vertex and base $\text{PG}(s + 2, \sqrt{q})$. Up to equivalence, the second possibility is $m_1 = m + 1$ and $m_2 = m$, which then means that $s = s_1$ and $s_2 = s + 2$. The smallest Baer cone containing $\langle \Pi_{m_1}, \text{PG}(s_1, \sqrt{q}) \rangle$ and $\langle \Pi_{m_2}, \text{PG}(s_2, \sqrt{q}) \rangle$ is the Baer cone with vertex $\Pi_{m_1}$ and base $\text{PG}(s_2, \sqrt{q})$.

The last possibility is that $m_1 = m_2 = m + 1$ and that $s = s_1 = s_2$. In this case, the smallest Baer cone containing $\langle \Pi_{m_1}, \text{PG}(s_1, \sqrt{q}) \rangle$ and $\langle \Pi_{m_2}, \text{PG}(s_2, \sqrt{q}) \rangle$ is the Baer cone with vertex the $(m + 2)$-dimensional space $\langle \Pi_{m_1}, \Pi_{m_2} \rangle$ and with base $\text{PG}(s, \sqrt{q})$.

The following arguments will show for all three cases that this smallest Baer cone $B_0$ containing $\langle \Pi_{m_1}, \text{PG}(s_1, \sqrt{q}) \rangle$ and $\langle \Pi_{m_2}, \text{PG}(s_2, \sqrt{q}) \rangle$ lies completely in $B$. In the first part of this proof, we prove our crucial result for proving that $B_0$ is contained in $B$.

**Part 1.** Consider a non-singular point $x$ of $\langle \Pi_{m_1}, \text{PG}(s_1, \sqrt{q}) \rangle$, not lying in $\Delta$. Let $L_1 \subset B$ be a Baer subline of $\langle \Pi_{m_1}, \text{PG}(s_1, \sqrt{q}) \rangle$, passing through $x$ and containing a point $r$ of the base $\text{PG}(s, \sqrt{q})$ of the Baer cone $\langle \Pi_m, \text{PG}(s, \sqrt{q}) \rangle$ in $\Delta$. We show that the Baer subgeometry defined by $L_1$ and $\langle \Pi_{m_2}, \text{PG}(s_2, \sqrt{q}) \rangle$ lies completely in $B$.

Lemma 5.6 proves that if you have a Baer cone $\langle v, L_2 \rangle$ of $B$, where $L_2 \simeq \text{PG}(1, \sqrt{q})$, $L_1$ a Baer subline of $B$ not in the plane of $v$ and $L_2$, and $L_1 \cap L_2 \neq \emptyset$, then the cone with vertex $v$ and base $\langle L_1, L_2 \rangle \simeq \text{PG}(2, \sqrt{q})$ lies completely in $B$.

By letting vary $v$ over $\Pi_{m_2}$ and by letting vary $L_2$ over all Baer sublines through $r$ in $\langle \Pi_{m_2}, \text{PG}(s_2, \sqrt{q}) \rangle$, we reach all points of $\langle \Pi_{m_2}, \text{PG}(s_2, \sqrt{q}) \rangle$; the cone with vertex $v$ and base $\langle L_1, L_2 \rangle \simeq \text{PG}(2, \sqrt{q})$ lies in $B$, hence the Baer subgeometry defined by $\Pi_{m_2}$, $L_1$, and $\text{PG}(s_2, \sqrt{q})$ lies completely in $B$.

**Part 2.** The Baer cones $\langle \Pi_{m_1}, \text{PG}(s_1, \sqrt{q}) \rangle$ and $\langle \Pi_{m_2}, \text{PG}(s_2, \sqrt{q}) \rangle$ define a (projected) Baer subgeometry $B_0$ over $\text{GF}(\sqrt{q})$.

Consider in $B_0$ an arbitrary Baer subgeometry $\Omega$ of dimension one larger than the Baer subgeometry $B_0 \cap H_2 = \langle \Pi_{m_2}, \text{PG}(s_2, \sqrt{q}) \rangle$, passing through the Baer subgeometry $B_0 \cap H_2 = \langle \Pi_{m_2}, \text{PG}(s_2, \sqrt{q}) \rangle$. Then $\Omega$ intersects $H_1$ in a Baer cone of dimension at least one larger than $\langle \Pi_m, \text{PG}(s, \sqrt{q}) \rangle$, so $\Omega$ contains points of $B_0$ in $\langle \Pi_{m_1}, \text{PG}(s_1, \sqrt{q}) \rangle$, not lying in $\Delta$. This intersection $\Omega \cap H_1$ contains non-singular points of the Baer cone $\langle \Pi_{m_1}, \text{PG}(s_1, \sqrt{q}) \rangle$, not lying in $\Delta$. For, if there were only such singular points in $H_1 \cap \Omega$, then let $r$ be a point of $\Pi_{m_1} \setminus \Delta$ lying in $\Omega$. Consider the line $rr'$ through $r$ and a point $r'$ of the base $\text{PG}(s, \sqrt{q})$ of the Baer cone $B_0 \cap \Delta = \langle \Pi_m, \text{PG}(s, \sqrt{q}) \rangle$. This line already contains two points $r$ and $r'$ of $\Omega$, so contains at least one Baer subline of $\Omega$. Hence, $\Omega$ contains at least one non-singular point $x$ of $\langle \Pi_{m_1}, \text{PG}(s_1, \sqrt{q}) \rangle$, not lying in $\Delta$. So $\Omega \cap H_1$ intersects $\langle \Pi_{m_1}, \text{PG}(s_1, \sqrt{q}) \rangle$ in a Baer subgeometry of dimension one larger than $\dim(\Pi_m, \text{PG}(s, \sqrt{q}))$. This intersection can be defined uniquely by $\langle \Pi_m, \text{PG}(s, \sqrt{q}) \rangle$ and a Baer subline

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$L_1$ joining $x$ to a non-vertex point in $\langle \Pi_m, \text{PG}(s, \sqrt{q}) \rangle$. We have proven in Part 1 that $L_1$ together with $\langle \Pi_m, \text{PG}(s_2, \sqrt{q}) \rangle$ defines a unique Baer cone, completely lying in $B$. This Baer cone is in fact $\Omega$. Hence, $\Omega$ lies in $B$.

So we conclude that an arbitrary Baer cone in $B_0$, of dimension one larger than $\dim(\Pi_{m_2}, \text{PG}(s_2, \sqrt{q}))$, passing through the Baer subgeometry $B_0 \cap H_2 = \langle \Pi_{m_2}, \text{PG}(s_2, \sqrt{q}) \rangle$, lies completely in $B$. This shows that $B_0 \subseteq B$.

Hence, $B$ contains a Baer cone $B_0 = \langle \pi_{m''}, \text{PG}(2(n - k - m'' - 1), \sqrt{q}) \rangle$, $-1 \leq m'' < n - k - 1$. \qed

Assume that the conditions of the preceding lemma are valid, then using the $t \pmod{\sqrt{q}}$ assumption, $B \setminus B_0$ is a $(t - 1)$-fold $(n - k)$-blocking set, intersecting every $k$-dimensional space in $(t - 1) \pmod{\sqrt{q}}$ points.

Assume that there is a line $L$ defined over $\text{GF}(q)$ intersecting $B$ in a set of at most $t(\sqrt{q} + 1)$ points, containing a Baer subline $L_1$. Then, by using the same arguments as in the proof of Lemma 3.2, it is first of all possible to find a $(k - 1)$-dimensional space $\Pi_{k - 1}$ through $L$ containing no other points of $B$. Since $|B| < tq^{n-k} + q^{n-k-1/3}$, there is a $k$-dimensional space $\Delta_k$ through $\Delta_{k-1}$ containing at most $t$ other points of $B$. Similarly, there is a $(k + 1)$-dimensional space through $\Delta_k$ sharing at most $tq + 2tq$ points with $B$ since it is impossible that all these $(k + 1)$-dimensional spaces through $\Delta_k$ contain at least $\sqrt{q}q + t$ points of $B$ (Theorem 2.2). The same arguments as in the proof of Lemma 5.6 then prove that it is possible to find an $(n - 2)$-dimensional space $\Delta$ through $L$ intersecting $B$ in a union of $t$ pairwise disjoint Baer cones $\langle \pi_{m_i}, \text{PG}(2(n - k - 2 - m_i - 1), \sqrt{q}) \rangle$, $-1 \leq m_i \leq n - k - 3$, $i = 1, \ldots, t$, where for at least one such Baer cone in $\Delta \cap B$, $m_i < n - k - 3$.

Then the conditions of the preceding lemma are met, and it is possible to find a $1$-fold $(n - k)$-blocking set $B_0$ in $B$, such that $B \setminus B_0$ is a $(t - 1)$-fold $(n - k)$-blocking set.

To obtain the complete characterization of $t$-fold $(n - k)$-blocking sets in $\text{PG}(n, q)$ of size at most $tq^{n-k} + 2tq^{n-k-1} \sqrt{q}$, it suffices to consider the case that lines are either completely contained in $B$, or intersect $B$ in at most $t$ distinct points, since it is no longer necessary to assume that Baer sublines are contained in $B$.

We now show that this implies that $B$ contains an $(n - k)$-dimensional space over $\text{GF}(q)$.

Let $\Delta$ be an $(n - 2)$-dimensional space intersecting $B$ in at most $tq^{n-k-2} + 2tq^{n-k-3} \sqrt{q}$ points, so by the induction hypothesis (IH) and also using the fact that there are no Baer sublines contained in $B$, $\Delta$ shares $t$ pairwise disjoint spaces $\text{PG}(n - k - 2, q)$ with $B$. Consider again two hyperplanes $H_1$ and $H_2$ through $\Delta$ intersecting $B$ in at most $tq^{n-k-1} + 2tq^{n-k-2} \sqrt{q}$ points. By
the induction hypothesis, and again using that no Baer sublines are contained in $B$, these two hyperplanes $H_1$ and $H_2$ intersect $B$ in $t$ pairwise disjoint subspaces $\text{PG}(n-k-1, q)$.

Let $\Pi_1$ and $\Pi_2$ be two $(n-k-1)$-dimensional spaces in respectively $H_1$ and in $H_2$, both contained in $B$, and intersecting $\Delta$ in the same $(n-k-2)$-dimensional space $\Pi$. We now show that $\Pi_1$ and $\Pi_2$ define an $(n-k)$-dimensional space $\Pi_{n-k}$ completely contained in $B$.

Let $r$ be a point of $\Pi$ and consider two lines $L_1$ and $L_2$, through $r$, lying in respectively $\Pi_1$ and in $\Pi_2$, but not lying in $\Delta$. Then the plane $\langle L_1, L_2 \rangle$ lies completely in $B$ (Lemma 5.3).

Letting vary the point $r$ in $\Pi$ and letting vary the lines $L_1$ and $L_2$ in $\Pi_1$ and in $\Pi_2$, the $(n-k)$-dimensional space $\Pi_{n-k} = \langle \Pi_1, \Pi_2 \rangle$ lies completely in $B$.

By using the $t \pmod{\sqrt{q}}$ assumption, $B \setminus \Pi_{n-k}$ is a $(t-1)$-fold $(n-k)$-blocking set in $\text{PG}(n,q)$, intersecting every $k$-dimensional space in $(t-1) \pmod{\sqrt{q}}$ points.

The preceding arguments now lead to the desired characterization result.

**Theorem 5.8** Let $B$ be a minimal $t$-fold $(n-k)$-blocking set in $\text{PG}(n,q)$, $q \geq 661$, $t < c_p q^{1/6}/2$, of size at most $|B| \leq tq^{n-k} + 2tq^{n-k-1}\sqrt{q} < tq^{n-k} + c_p q^{n-k-1/3}$, intersecting every $k$-dimensional space in $t \pmod{\sqrt{q}}$ points.

Then $B$ is a union of $t$ pairwise disjoint cones $\langle \pi_{m_i}, \text{PG}(2(n-k-m_i-1), \sqrt{q}) \rangle$, $-1 \leq m_i \leq n-k-1$, $i = 1, \ldots, t$.

**Proof:** Let $\Delta$ be an $(n-2)$-dimensional space intersecting $B$ in at most $tq^{n-k-2} + 2tq^{n-k-3}\sqrt{q}$ points.

The preceding lemma and arguments show that it is possible to find a $1$-fold $(n-k)$-blocking set $B_0$ in $B$ such that $B \setminus B_0$ is a $(t-1)$-fold $(n-k)$-blocking set, intersecting every $k$-dimensional space in $(t-1) \pmod{\sqrt{q}}$ points.

By induction on $t$, this proves the theorem. $\square$

The preceding result is not the end of the classification since such unions of $t \geq 2$ pairwise disjoint cones $\langle \pi_{m_i}, \text{PG}(2(n-k-m_i-1), \sqrt{q}) \rangle$ only exist if $k \geq n/2$.

**Theorem 5.9** Let $B$ be a minimal $t$-fold $(n-k)$-blocking set in $\text{PG}(n,q)$, $q$ square, $t \geq 2$, which is a union of $t$ pairwise disjoint cones $\langle \pi_{m_i}, \text{PG}(2(n-k-m_i-1), \sqrt{q}) \rangle$, max$\{-1, n-2k-1\} \leq m_i \leq n-k-1$. Then $k > n/2$ if
B contains at least one \((n-k)\)-dimensional space \(\text{PG}(n-k,q)\) and \(k \geq n/2\) in the other cases.

**Proof:** If \(B\) contains at least two \((n-k)\)-dimensional spaces \(\text{PG}(n-k,q)\) which are disjoint, then \(k > n/2\). If \(B\) contains an \((n-k)\)-dimensional space and a cone \(\langle \pi_m, \text{PG}(2(n-k-m-1),\sqrt{q}) \rangle\), max\(-1, n-2k-1\) \(\leq m < n-k-1\), then since the Baer cone intersects every \(k\)-dimensional space, necessarily \(n-k < k\), and again \(k > n/2\).

We now assume that \(B\) does not contain \((n-k)\)-dimensional spaces \(\text{PG}(n-k,q)\). A Baer cone \(\langle \pi_m, \text{PG}(2(n-k-m-1),\sqrt{q}) \rangle\), max\(-1, n-2k-1\) \(\leq m < n-k-1\), is in fact a projected Baer subgeometry \(\text{PG}(2n-2k,\sqrt{q})\). This defines a vector space \(V(2n-2k+1,\sqrt{q})\).

The projective space \(\text{PG}(n,q)\) defines a vector space \(V(2n+2,\sqrt{q})\) over \(\text{GF}(\sqrt{q})\). If this \((2n+2)\)-dimensional vector space over \(\text{GF}(\sqrt{q})\) contains two disjoint \((2n-2k+1)\)-dimensional subspaces, necessarily \(2(2n-2k+1) \leq 2n+2\), leading to \(k \geq n/2\).

**Remark 5.10** The lower bound \(k \geq n/2\) is sharp as the following examples of \(t\)-fold \((n-k)\)-blocking sets in \(\text{PG}(n,q)\) show.

Let \(n = 2n'\). Consider \(t\) pairwise disjoint subgeometries \(\text{PG}(n,\sqrt{q})\), \(i = 1, \ldots, t\), of \(\text{PG}(n = 2n',q)\). They are \(t\) pairwise disjoint \(1\)-fold \(n'\)-blocking sets, so they form together a \(t\)-fold \(n'\)-blocking set.

If \(n = 2n' + 1\), then the lower bound on \(k\) is \(k \geq n' + 1\). Consider the example of the preceding paragraph, lying in \(\text{PG}(2n',q)\), and embed this \(2n'\)-dimensional space into a \((2n' + 1)\)-dimensional space. Then the example of the preceding paragraph forms a \(t\)-fold \(n'\)-blocking set in \(\text{PG}(n = 2n' + 1,q)\), so a \(t\)-fold \((n-k)\)-blocking set with \(k = n' + 1\).

**References**


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