Randers spaces with reversible geodesics

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Abstract

A Finsler space is said to have reversible geodesics if for every one of its oriented geodesic paths, the same path traversed in the opposite sense is also a geodesic. The conditions for a Randers space to have reversible geodesics are obtained; this leads to a new simple proof of a well-known theorem giving necessary and sufficient conditions for a Randers space to be Berwald.

A geodesic in a Finsler space (where the Finsler function is \textit{positively} homogeneous) should be thought of as an oriented path, that is, an imbedded one-dimensional submanifold with a sense of direction, or an equivalence class of curves determined up to reparametrization with positive derivative. There is in general no reason why a path which coincides with a geodesic as a point set but is traversed in the opposite direction should be a geodesic. If a Finsler space has the property that all of its geodesics remain geodesics when their orientation is reversed I shall say that the space has reversible geodesics. If the space is such that when \( t \rightarrow x^i(t) \) is a geodesic with constant Finslerian speed then \( t \rightarrow x^i(-t) \) is also a geodesic with constant Finslerian speed then I shall say that the space has strictly reversible geodesics.

A Riemannian space has strictly reversible geodesics, and so more generally has a Finsler space whose Finsler function is absolutely homogeneous. However, these examples do not by any means exhaust the possibilities for Finsler spaces with reversible geodesics. Consider a Randers space, with Finsler function

\[ F(x, y) = \alpha + \beta = \sqrt{a_{ij}y^iy^j} + b_{ij}y^i \]

where \( a_{ij}b^ib^j < 1 \). If \( b_{ik} = b_{kj} \), where the \( b_{ij} \) are the components of the covariant differential of \( b_i \) with respect to the Levi-Civita connection of the Riemannian metric \( a_{ij} \), then the Randers space has reversible geodesics (it is a Douglas space, and its geodesics, as paths, coincide with the geodesics of the Riemannian metric); if \( b_{ij} = 0 \) then its geodesics are strictly reversible (it is a Berwald space, and its geodesics with constant
Finslerian speed coincide with the affinely parametrized geodesics of the Riemannian metric. In fact the equation for geodesics with constant Finslerian speed is [1]

\[ \ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k + a^{ij} (b_{jk} - b_{klj}) \dot{x}^k \alpha(\dot{x}) = \frac{1}{F} \dot{x}^i \left( a^{ij} b_l (b_{jk} - b_{klj}) \dot{x}^k \alpha(\dot{x}) - b_{jk} \ddot{x}^j \dot{x}^k \right) \]

where \( \Gamma^i_{jk} \) are the Riemannian connection coefficients; it is clear that if \( b_{jk} = b_{klj} \), then the Finslerian geodesics are projectively equivalent to the Riemannian ones, while if \( b_{klj} = 0 \) they are identical. One aim of this note is to prove the converse, namely that if a Randers space has reversible geodesics then \( b_{jk} = b_{klj} \), and if it has strictly reversible geodesics then \( b_{klj} = 0 \). These results generalize in some small way the well-known result that the vanishing of \( b_{klj} \) is the necessary and sufficient condition for a Randers space to be Berwald, and enable one to view that result from a new perspective, as well as providing a simple proof of it, different from the one in [3], that requires practically no calculation (which cannot be said of the derivation of the explicit geodesic spray coefficients of a Randers space quoted above).

I shall discuss the reversibility of geodesics in some generality. In fact the definitions of reversibility and corresponding conditions can be formulated for any spray. Consider a spray

\[ \Gamma = y^i \frac{\partial}{\partial x^i} + G^i(x, y) \frac{\partial}{\partial y^i} \]

where the \( G^i \) are positively homogeneous of degree 2 in the \( y^i \). A curve \( t \mapsto x^i(t) \) is a base integral curve of the spray if and only if it satisfies the equations \( \ddot{x}^i = G^i(x, \dot{x}) \). The curve \( t \mapsto x^i(-t) \) is a base integral curve, up to reparametrization, if for some function \( \varphi(t) \), \( \ddot{x}^i = G^i(x, -\dot{x}) + \varphi \dot{x}^i \). Thus the spray is reversible, in the sense that the paths defined by its base integral curves remain so when their orientation is reversed, if and only if

\[ G^i(x, -y) = G^i(x, y) + \lambda(x, y) y^i \]

for all \( y^i \neq 0 \), for some function \( \lambda \), which must clearly be absolutely homogeneous of degree 1 in \( y^i \). The spray is strictly reversible if \( \lambda = 0 \).

We can express the condition for reversibility in a rather more elegant form, as follows. Denote by \( \rho \) the ‘reflection map’ \( (x, y) \mapsto (x, -y) \), and for any spray \( \Gamma \) set \( \overline{\Gamma} = -\rho_\ast \Gamma \). Then

\[ \overline{\Gamma} = y^i \frac{\partial}{\partial x^i} + G^i(x, -y) \frac{\partial}{\partial y^i} \]

so it is natural to call \( \overline{\Gamma} \) the reverse of \( \Gamma \). Then \( \Gamma \) is reversible if and only if it is projectively equivalent to its reverse, and strictly reversible if and only if the two are equal.

The concept of reversibility is a projective one; that is to say, if a spray is reversible so are all sprays projectively equivalent to it. In fact a spray is reversible if and only if its projective equivalence class is invariant under the map which takes a spray to its reverse.
Since we have to deal with projectively equivalent sprays, the following simple observations about the geodesic sprays of Finsler spaces will prove very useful. Let $F$ be a Finsler function — by assumption, positively homogeneous, and strongly convex, so that its fundamental tensor $g_{ij}$ is positive-definite; then a spray $\Gamma$ belongs to the projective equivalence class of geodesic sprays of $F$ if and only if the Euler-Lagrange equation with Lagrangian $F$,

$$\Gamma \left( \frac{\partial F}{\partial y^i} \right) - \frac{\partial F}{\partial x^i} = 0,$$

is satisfied. The geodesic spray $\Gamma$ with constant Finslerian speed is singled out from amongst all those satisfying the Euler-Lagrange equation by the additional condition that $\Gamma(F) = 0$. I shall speak of ‘a geodesic spray’ when I mean any spray of the projective class of solutions of the Euler-Lagrange equation for $F$, and ‘the geodesic spray’ when I mean the one with constant Finslerian speed. With this choice, if $\overline{\Gamma}$ is any projectively equivalent spray, so that $\overline{\Gamma} = \Gamma + \lambda \Delta$ where $\Delta$ is the Liouville vector field and $\lambda$ is homogeneous of degree 1 in $y^i$, then

$$\overline{\Gamma}(F) = \Gamma(F) + \lambda \Delta(F) = \lambda F,$$

so we have an explicit expression for $\lambda$, namely

$$\lambda = \frac{\overline{\Gamma}(F)}{F}.$$

(Though it may not be immediately obvious, these results are essentially equivalent to those given by Shen in [5], Theorem 12.2.6. See also [6] for an intrinsic formulation of this and equivalent conditions, originally due to Rácsák [4].)

It follows that a Finsler space has reversible geodesics if and only if

$$\overline{\Gamma} \left( \frac{\partial F}{\partial y^i} \right) - \frac{\partial F}{\partial x^i} = 0,$$

where $\overline{\Gamma}$ is the reverse of a geodesic spray $\Gamma$; it will be enough to check reversibility when $\Gamma$ is the geodesic spray. Furthermore, the geodesics are strictly reversible if and only if, in addition, $\overline{\Gamma}(F) = 0$.

Now if $F$ is any Finsler function, and $\tilde{F}$ is defined by $\tilde{F}(x, y) = F(x, -y)$ then $\tilde{F}$ is also a Finsler function; it is certainly positively homogeneous in $y^i$, and its fundamental tensor $\tilde{g}_{ij}$ is given by $\tilde{g}_{ij}(x, y) = g_{ij}(x, -y)$ (where $g_{ij}$ is the fundamental tensor of $F$), so $\tilde{g}_{ij}$, like $g_{ij}$, is everywhere positive definite. The geodesic spray $\overline{\Gamma}$ of $\tilde{F}$ is just the reverse of the geodesic spray of $F$.

We can now apply these observations to a Randers space, with

$$F = \alpha + \beta = \sqrt{a_{ij}y^iy^j} + b_iy^i,$$

to show that the necessary and sufficient condition for the space to have reversible geodesics is that $b_{ikj} = b_{ijk}$, and strictly reversible geodesics that $b_{ijk} = 0$. Of course, another way of saying that $b_{ikj} = b_{ijk}$ is that the 1-form $b = b_i dx^i$ is closed. Given
that $b$ is closed, another way of saying that $b_{ijkl} = 0$ is that the function $\beta = b_i y^i$ is a first integral of the geodesic flow of the Riemannian metric $a_{ij}$. So we may equivalently say that the necessary and sufficient condition for the Randers space to have reversible geodesics is that $b$ is closed, and the necessary and sufficient condition for its geodesics to be strictly reversible is that $b$ is closed and $\beta = b_i y^i$ is a first integral of the Riemannian geodesic flow.

These results about reversibility of geodesics in a Randers space are in fact particular cases (though probably the most interesting ones) of more general, but similar, results concerning Randers changes. Let $F_0$ be a Finsler function, and $b = b_i dx^i$ a 1-form on the base manifold such that

$$\sup_{F_0(y) = 1} |b_i y^i| < 1;$$

then $F(x, y) = F_0(x, y) + b_i(x) y^i$ is again a Finsler function, and the process of transforming $F_0$ to $F$ is called a Randers change (see, for example, [5] and [6]). Suppose that $F_0$ is absolutely homogeneous; then the necessary and sufficient condition for $F$ to have reversible geodesics is that $b$ is closed, and the necessary and sufficient condition for the geodesics to be strictly reversible is that $b$ is closed and $\beta = b_i y^i$ is a first integral of the geodesic flow of $F_0$. I shall devote the rest of this note to proving these assertions.

The necessary and sufficient condition for $F$ to have reversible geodesics is that

$$\bar{\Gamma} \left( \frac{\partial F}{\partial y^i} \right) - \frac{\partial F}{\partial x^i} = 0$$

where $\bar{\Gamma}$ is the reverse of $\Gamma$, the geodesic spray of $F$; moreover, $\bar{\Gamma}$ is the geodesic spray of $\bar{F}$. The necessary and sufficient condition for $F$ to have strictly reversible geodesics is that, in addition, $\bar{\Gamma}(F) = 0$. Now $F = F_0 + \beta$ where $F_0$ is absolutely homogeneous. Then $\bar{F} = F_0 - \beta$, so $F = \bar{F} + 2\beta$. Since

$$\bar{\Gamma} \left( \frac{\partial F}{\partial y^i} \right) - \frac{\partial F}{\partial x^i} = 0,$$

we have

$$\bar{\Gamma} \left( \frac{\partial F}{\partial y^i} \right) - \frac{\partial F}{\partial x^i} = 2 \left( \bar{\Gamma} \left( \frac{\partial \beta}{\partial y^i} \right) - \frac{\partial \beta}{\partial x^i} \right)$$

$$= 2 \left( \bar{\Gamma} \left( b_i - \frac{\partial b_j}{\partial x^j} y^j \right) \right)$$

$$= 2 \left( \frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i} \right) y^j.$$

It follows that if $F$ is obtained by a Randers change from an absolutely homogeneous Finsler function then it is geodesically reversible if and only the 1-form defining the Randers change is closed.

Notice that for any spray $\bar{\Gamma}$,

$$\bar{\Gamma} \left( \frac{\partial \beta}{\partial y^i} \right) - \frac{\partial \beta}{\partial x^i} = \left( \frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i} \right) y^j.$$
So for the geodesic spray $\Gamma_0$ of the ‘reference’ Finsler function $F_0$

$$\Gamma_0 \left( \frac{\partial F}{\partial y'} \right) - \frac{\partial F}{\partial x'} = \Gamma_0 \left( \frac{\partial \beta}{\partial y'} \right) - \frac{\partial \beta}{\partial x'} = \left( \frac{\partial b_i}{\partial x'} - \frac{\partial b_i}{\partial x^j} \right) y^i,$$

from which it follows that the closure of $b$ is also the necessary and sufficient condition for $\Gamma_0$ to be projectively equivalent to $\Gamma$; if $b$ is closed we have $\Gamma = \Gamma_0 + \lambda \Delta$ with $\lambda = \Gamma_0(F)/F = \Gamma_0(\beta)/(F_0 + \beta)$, and similarly $\Gamma = \Gamma_0 - \mu \Delta$ with $\mu = -\Gamma_0(\beta)/(F_0 - \beta)$.

Thus given that $b$ is closed, the condition for $F$ to have strictly reversible geodesics, so that $\Gamma = \Gamma$, is that $\Gamma_0(\beta)/(F_0 + \beta) = -\Gamma_0(\beta)/(F_0 - \beta)$, or $\Gamma_0(\beta) = 0$.

In fact, when $b$ is closed the geodesic sprays of both $F$ and $\tilde{F}$ are projectively equivalent to the (strictly reversible) geodesic spray of $F_0$; and when $\Gamma_0(\beta) = 0$ the two geodesic sprays coincide with the geodesic spray of $F_0$. (Projective equivalence under a Randers change is discussed in [5] and [6]. The condition on $b$ was originally found by Hashiguchi and Ichijyo [2].)

The necessary and sufficient conditions for a Randers space to be Douglas or Berwald are simple corollaries of the results just obtained. Those results apply of course to a Randers space, with $F_0$ the Riemannian Finsler function. If a Randers space is a Douglas space, so that its geodesic spray is projectively equivalent to an affine spray, then the geodesics of the Randers space must be reversible, so $b$ must be closed. If a Randers space is Berwald, so that its geodesic spray is affine, its geodesics must be strictly reversible, so $\beta$ must be a first integral of the Riemannian geodesic flow. In each case, the affine spray is the Riemannian geodesic spray.

Finally, I shall point out how an example of Shen’s [5] provides a memorable illustration of a Randers space with non-reversible geodesics. We start with the spray $\Gamma$ on $\mathbb{R}^2$ given by

$$\Gamma = u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} - \alpha \left( v \frac{\partial}{\partial u} - u \frac{\partial}{\partial v} \right), \quad \alpha = \sqrt{u^2 + v^2},$$

where now $(x, y)$ are the base coordinates and $(u, v)$ the fibre coordinates. This spray is manifestly non-reversible. Its base integral curves are in fact circles of constant radius 1, traversed in the anti-clockwise sense. To see this, note first that $\Gamma(u^2 + v^2) = 0$, which means that $\dot{x}^2 + \dot{y}^2$ is constant on any base integral curve. For a point describing the circle with centre $(a, b)$ and radius 1, with constant speed $\alpha$ in the anti-clockwise sense, we have $(x - a)^2 + (y - b)^2 = 1$; $\dot{x} = \alpha \dot{y} - \dot{y} (y - b) = 0$; $\dot{x} = -\alpha \dot{y}$, $\dot{y} = \alpha (x - a)$ with $\alpha = \sqrt{\dot{x}^2 + \dot{y}^2}$ constant — note that at $x = a + 1, y = b$ we have $\dot{x} = 0, \dot{y} = \alpha > 0$ as is required for the motion to be anti-clockwise; and finally $\ddot{x} = -\alpha \dot{y}, \ddot{y} = \alpha \dot{x}$, so the circle is indeed a base integral curve of $\Gamma$. Consider now the function

$$F(x, y, u, v) = \sqrt{u^2 + v^2} + \frac{1}{2}(yu - xv) = \alpha + \beta.$$

I show that $\Gamma$ is a geodesic spray of this function, by calculating the Euler-Lagrange expressions, using the fact that (due to rotational symmetry)

$$\left( v \frac{\partial}{\partial u} - u \frac{\partial}{\partial v} \right) (\alpha) = 0;$$

$$5$$
we easily find that

$$\Gamma \left( \frac{\partial F}{\partial u} \right) - \frac{\partial F}{\partial x} = \Gamma \left( \frac{u}{\alpha} + \frac{1}{2} y \right) + \frac{1}{2} v = -v + \frac{1}{2} v + \frac{1}{2} v = 0$$

$$\Gamma \left( \frac{\partial F}{\partial v} \right) - \frac{\partial F}{\partial y} = \Gamma \left( \frac{v}{\alpha} - \frac{1}{2} y \right) - \frac{1}{2} u = u - \frac{1}{2} u - \frac{1}{2} u = 0.$$

Now $F$ is a Finsler function on the open disc $x^2 + y^2 < 4$; so we have here an example of a Finsler function whose geodesics are unit circles — but always traversed in the anti-clockwise sense.

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